Topological data analysis of exclusion zones

Proposal to the Alan Turing Institute for a meeting in Edinburgh
by G.Carlsson, S.Ramamoorthy, A.Ranicki and U.Tillmann

An important mathematical problem arising in modern engineering is that of understanding and computing with regions that are defined by a "negative" condition, that is that they consist of the set of points which do not belong to a set that is defined in an explicit manner. Here are two important examples.

Sensor nets are collections of large numbers of sensors with very primitive capabilities, distributed over a domain. The idea is to use these sensors to detect the presence of an intruder within the domain. The sensors do not have GPS capability, so are not able to recognize their absolute position, but they are able to detect an intruder at a fixed threshold radius R. They are also able to recognize the presence of another sensor of the same type, also at a distance R. One can then speak of the covered region as the union of balls of radius R with centers at the positions of the sensors. An important question to ask is whether or not these balls cover the domain, or equivalently whether or not the covered region is the entire domain. If that is the case, then one knows that if an intruder is present in the domain, it will be detected by one of the sensors. It turns [1] out that there are topological methods for determining whether or not the domain is covered, from knowing only the information about which sensors are within a distance R of each other. There are numerous extensions of this question. For example, one might include a time parameter, and ask whether an intruder can move from one point on the boundary of the domain to another over times, as the sensors move [2],[3]. In addition, one might want to learn the structure of the uncovered region in the case where one does not have coverage, so as to understand how to navigate while avoiding sensors.

A central problem in robotics is motion planning in domains where there are obstacles. In this case, one may have explicit knowledge of the positions of the obstacles and their shapes, but is interested in that part of the domain not included in the obstacles. One might have a complicated set of obstacles, and nevertheless want to obtain an understanding of the complement of the collection of obstacles (the "feasible region"), so as to be able to understand the motion paths through it. The potential of topology is to determine if a path exists, and if so, to yield a coarse classification of such paths, up to the homotopy relation. Once such a classification exists, one can then imagine optimizing for various objective functions within the equivalence class. A desirable feature for such a scheme would be to develop it without having to parametrize the feasible region, but only using explicit understanding of the infeasible region, i.e. the union of the various obstacles.

Idealized versions of this problem have been developed within algebraic topology. For example, Alexander duality shows that the homology of the complement of a finite subcomplex X in Euclidean space is completely determined by the homology of X. More generally, the Spanier-Whitehead duality theorem shows that the stable homotopy type of the complement of X is determined by the stable homotopy type of X. However, it is also clear that the unstable homotopy type of the complement is not determined by the unstable homotopy type of X, since it is well known that different knotted circles in three-dimensional space produce complements with non-isomorphic fundamental groups. The key piece of missing information to determine the unstable homotopy type of the complement is the specific embedding of X in Euclidean space, much as a knot is determined by an explicit embedding of a circle in three-dimensional Euclidean space. Embedding questions have been studied extensively in topology, particularly through the seminal work of Haefliger [4],[5]. More recently, the idea of embedding calculus has been introduced in order to study the classification of embeddings of one manifold in another [6], [7]. This work seems well suited to the study of the problems described above, but in order to become applicable, it must be generalized in two directions.

The manifold hypothesis must be removed, since the regions defined as the covered regions of families of sensors and the obstacles in robotics typically are not manifolds. The calculus needs to be extended to the situation of spaces over a fixed base space. For example, in the case of sensors or obstacles that are moving in time, on must study the problem over the base space consisting of the real line, which is tracking the time variable.

The above generalizations are a critical first step. Once there is some understanding of how these extensions work, it will then be important to make them more directly computational in real situations, and also to identify standard unstable homotopy theoretic invariants from the embedding space calculations.

We propose a small focused meeting to address the required generalizations. We would bring together investigators in the actual embedding calculus, as well as engineers and topologists working in computational topology.

References

[1] V. de Silva and R. Ghrist, Coverage in sensor networks via persistent homology, Algebraic and Geometric Topology 7 (2007), 339-358.

[2] H. Adams and G. Carlsson, Evasion paths in mobile sensor networks, International Journal of Robotics Research, vol. 34, (1), (2015), 90-104,.

[3] R. Ghrist and S. Krishnan, Positive Alexander duality for pursuit and evasion, arXiv:1507.04741, (2015).

[[4] A. Haefliger and M. Hirsch, On the existence and classification of differentiable embeddings, Topology, vol. 2, issues 1-2, 1963.

[5] A. Haefliger, Differentiable embeddings of Sn in Sn+q for q ¿2, Annals of Mathematics, vol. 83, no.3, (1966), 402-436.

[6] M. Weiss, Calculus of embeddings, Bull. Amer. Math. Soc. 33, (1996), 177-187.

[7] T. Goodwillie, J. Klein, M. Weiss, Spaces of smooth embeddings, disjunction, and surgery, in S. Cappell, A. Ranicki, J. Rosenberg, Surveys in Surgery Theory, vol. 2, Annals of Mathematics Studies 149, Princeton University Press, Princeton N.J., 2000.