Differentiable embeddings of S^n in S^{n+q} for $q>2$

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This paper can be considered as a complement to the fundamental paper of J. Levine [10]. Instead of studying the group θ_n^q of isotopy classes of embedded homotopy n-spheres in S^{n+q}, we are interested here in the group C_n^q of isotopy classes of embeddings of the usual n-sphere S^n in S^{n+q}. Our main result is the isomorphism of C_n^q with the triad homotopy group $\pi_{n+1}(G; \mathrm{SO}, G_q)$ for $q > 2$, where G_q is the space of maps of degree one of S^{q-1} onto itself, and G its stable suspension. This isomorphism was suggested to us (cf. 4.12) by the results of Levine [10]. We use essentially an extension of the main idea of Levine, namely the existence of the homomorphism of θ_n^q in $\pi_n(G_q, \mathrm{SO}_q)$ (cf. [10]). Nevertheless this paper is written so that it is independent of the papers of Kervaire-Milnor [9] and of Levine [10], except to indicate the relations with these works. On the other hand, we use one of the main result of Smale [14] to translate the problem of isotopy into a problem of concordance (Th. 1.2), and throughout the paper, the theory of handle decomposition.

We first define in § 1 the group C_n^q, and we indicate its relation with the group θ_n^q. In § 2 and § 3, we define and prove the isomorphism $C_n^q = \pi_{n+1}(G; \mathrm{SO}, G_q)$ for $q>2$. By using one of the homotopy exact sequences of the triad $(G; \mathrm{SO}, G_q)$ and the paper of Smale [13], we theoretically solve in § 4 a problem posed by Smale in [13]: what are the immersions of S^n in S^{n+q} which are regularly homotopic to an embedding? In § 5 we study the group of framed embeddings of S^n in S^{n+q} and we show its relation with the classification of handlebodies.

Let F_q be the space of maps of degree one of S^q onto S^q with a common fixed point; by suspension, G_q is identified to a subspace of F_q. We prove in § 6 that the suspension homomorphism $C_n^q \to C_n^{q+1}$ finds its place in an exact sequence:

$$\cdots \longrightarrow \pi_{n+1}(F_q, G_q) \longrightarrow C_n^q \longrightarrow C_n^{q+1} \longrightarrow \pi_n(F_q, G_q) \longrightarrow \cdots .$$

In § 7 and § 8, we prove geometrically the other main result of this paper, namely the isomorphism $\pi_n(F_q, G_q) = \pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$ for $n \leqq 3q - 6$. This establishes the link with our first paper on knots [3] and we use extensively here the technique of framed spherical modifications. It is to be expected that this last result can be recovered by pure algebraic topology. Our method gives, in the above range, an explicit construction (cf. 8.12–13) of those embedded spheres whose suspension is trivial.

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Terminology

0.1. D^n is the subspace of the Hilbert space H formed by the vectors $x = (x_1, \cdots, x_i, \cdots)$ such that $x_i = 0$ for $i > n$ and $|x| \leqq 1$. The n-sphere $S^n = \partial D^{n+1}$ is the subspace of vectors of norm 1 in D^{n+1}. The interior of $D^n = \{x \mid x \in D^n, |x| < 1\}$ is denoted by $\operatorname{int} D^n$.

We define

$$D^n_+ = \{x \mid x \in S^n, x_1 \geqq 0\}$$

and

$$D^n_- = \{x \mid x \in S^n, x_1 \leqq 0\} .$$

The natural basis of the Hilbert space H is denoted by $e_1, \cdots, e_i, \cdots$. Everything has the riemannian metric induced from Hilbert space metric.

The *suspension* of a map $f: D^n \to D^m$ is the map $Ef: D^{n+1} \to D^{m+1}$ mapping the arc of circle going from e_{n+1}, by $x \in D^n$, to $-e_{n+1}$ on the arc of circle going from e_{m+1}, by $f(x)$, to $-e_{m+1}$, and commuting with the projections on $\mathbf{R}$ given respectively by the $n+1$ and $m+1$ coordinate. Note that, if f is differentiable at $x \in D^n$, so is Ef at $x \in D^{n+1}$. The suspension of a map f of S^n in S^m is defined in the same way. The iterated suspension is defined by induction.

0.2. We shall not be afraid of meeting manifolds with corners like $D^n \times I$, or $D^n \times D^q \times I$, and most of the time, we shall not round them. An n-manifold V with boundary will be locally diffeomorphic to an open subset of the subspace $\mathbf{R}^n_+$ of $\mathbf{R}^n$ defined by $x_i \geqq 0, i = 1, \cdots, n$. The points of V which are images, by local charts, of points of the boundary of $\mathbf{R}^n_+$ form the boundary ∂V of V. An open q-face of V is an union of connected components of the set of points which are images, by local charts, of those points of $\mathbf{R}^n_+$ defined by $x_1 > 0, \cdots, x_q > 0, x_i = 0$ for $i > q$. A q-face of V is the closure of an open q-face. In what follows, we shall only consider $(n-1)$-faces which are also manifolds.

A p-submanifold M of a manifold V will be locally diffeomorphic to the subspace of $\mathbf{R}^n_+$ defined by $x_{p+1} = \cdots = x_n = 1$, in the case where $\partial M \subset \partial V$. We shall also consider p-submanifolds M of V with a free face, namely a $(p-1)$-face not contained in ∂V; in that case, M will be locally diffeomorphic to the subspace of $\mathbf{R}^n_+$ defined by $x_p \geqq 1, x_{p+1} = \cdots = x_n = 1$.

0.3. A *framed p-submanifold* M of an n-manifold V is a submanifold with a differentiable framing $f = (f_1, \cdots, f_{n-p})$, i.e. $n-p$ independent differentiable vector fields $f_1, \cdots, f_{n-p}$ along M and complementary to M. At a point of ∂M, we shall assume that the framing is the image, by a suitable local chart, of the standard framing $e_{p+1}, \cdots, e_n$ of the subspace $x_i = 0$ for $i > p$. If $\partial M \subset \partial V$,

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each open $(p-1)$-face of M is naturally a framed submanifold of ∂V. If M has a free face A in V, then A will be considered as a framed submanifold of V, with the framing $(\nu, f_1, \cdots, f_{n-p})$, where ν is normal to A in V and pointing outside. We shall denote by $-A$ the submanifold A with the framing $(-\nu, f_1, \cdots, f_{n-p})$.

The natural framing of D^n in D^{n+q} is $(e_{n+1}, \cdots, e_{n+q})$. The suspension in $V \times D^{n+q}$ of a framed submanifold M of $V \times D^n$ is the submanifold M with its framing completed by $e_{n+1}, \cdots, e_{n+q}$ as last vectors.

0.4. A continuous map g of a manifold V in a manifold X is regular on a point $x \in X$ (or x is a regular value of g) if g is differentiable on a neighborhood of $g^{-1}(x)$ and if, at each point of $g^{-1}(x)$, the differential of g (and of its restriction to each face of V) is surjective. If $\varepsilon_1, \cdots, \varepsilon_n$ is a frame at x, then the submanifold $g^{-1}(x)$ will be framed by vector fields $f_1, \cdots, f_n$ such that the differential of g maps f_i on ε_i. For instance, if $0 \in D^n$ is a regular value of a map $g: V \to D^n$, then $g^{-1}(0)$ is a framed submanifold of V (it is understood that we take the standard frame at 0).

Relative Thom construction. Let M be a framed submanifold of V such that $\partial M \subset \partial V$. Let $g: \partial V \to S^q$ be a map regular on $x \in S^q$ and such that $g^{-1}(x)$ is ∂M with the given framing. Then, by [15], there is a map $G: V \to S^q$, regular on x, such that $G \mid \partial V = g$ and $G^{-1}(x)$ is the framed submanifold M (it is again understood that a frame giving the positive orientation of S^q is given).

1. Definition of the group C_n^q

1.1. Two differentiable embeddings f_0 and f_1 of a differentiable manifold V in a differentiable manifold X are *concordant*, if there is an embedding $F: V \times I \to X \times I$ such that $F(x, i) = (f_i(x), i)$ for $i = 0, 1$; the map F is called a concordance connecting f_0 to f_1. If, moreover, F is level preserving, i.e., of the form $F(x, t) = (f_t(x), t)$ for each $t \in I$, then f_0 and f_1 are isotopic, and f_t is an isotopy connecting f_0 to f_1.

The concordance relation is an equivalence relation, and an equivalence class will be called a concordance class. The set of concordance classes of embeddings of S^n in S^{n+q} will be denoted by C_n^q.

Two embeddings which are isotopic are of course concordant; it follows from a result of Smale [14] that a partial converse is true.

1.2. THEOREM. *For $q > 2$, two embeddings of S^n in S^{n+q} which are concordant are also isotopic.*

PROOF. Let $F: S^n \times I \to S^{n+q} \times I$ be a concordance connecting two embeddings f_0 and f_1. For $q > 2$ and $n < 2$, all embeddings are isotopic. For

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$q > 2$ and $n \geqq 2$, Smale proves in [14, Cor. 3.2], the existence of a diffeomorphism H of $S^{n+q} \times I$ preserving orientation and such that

$$H(f_0 x, t) = F(x, t) \qquad \text{for } (x, t) \in S^n \times I .$$

Let H_1 be the diffeomorphism of S^{n+q} defined by $(H_1 y, 1) = H(y, 1)$. We have $H_1 f_0 = f_1$. Let h be the restriction of H_1 to the complement D of the interior of a small $(n + q)$-disk which does not intersect $f_0(S^n)$. Then $hf_0 = f_1$. As h is isotopic to the identity, f_0 and f_1 are isotopic.

Group structure on C_n^q. The following is a consequence of the tubular neighborhood theorem.

1.3. Lemma. (a) *Any embedding of S^n in S^{n+q} $(q > 0)$ is isotopic to an embedding f such that*

(i) *$f \mid D_-^n$ is the identity map*

(ii) $f(\text{int } D_+^n) \subset \text{int } D_+^{n+q}$

(b) *If $f_0, f_1 \colon S^n \to S^{n+q}$ are two concordant embeddings satisfying* (i) *and* (ii) *of* (a), *there is a concordance F connecting f_0 to f_1 such that*

(i) *$F \mid D_-^n \times I$ is the identity*

(ii) $F(\text{int } D_+^n \times I) \subset \text{int } D_+^n \times I$.

1.4. *Definition of the sum.* Let R_t be the rotation of the Hilbert space whose restriction to the plane generated by e_1 and e_2 is a rotation of angle πt and which leaves fixed its orthogonal complement. For any embedding $f \colon S^n \to S^{n+q}$, the embeddings $R_{-t} f R_t$ and f are isotopic.

Let α and β be two elements of C_n^q. We can represent them by embeddings f and g resp. which satisfy condition (i) and (ii) of Lemma 1.4 (a). By definition, the class $\alpha + \beta$ will be represented by the embedding $f + g$ defined by

$$(f + g)(x) = \begin{cases} f(x) & \text{for } x \in D_+^n \\ R_1 g R_1(x) & \text{for } x \in D_-^n . \end{cases}$$

According to Lemma 1.3 (b), this definition is independent of the particular choice of f and g.

This sum operation is *commutative*, because $f + g$ is isotopic to $R_1(f + g)R_1$, which is equal to $g + f$.

To prove *associativity*, let us represent three elements α_i, $i = 1, 2, 3$, by embeddings f_i whose restrictions to the subspace $x_1 \leqq 0$ or $x_2 \leqq 0$ of S^n are the identity, and which map its complement in the part of S^{n+q} defined by $x_1 > 0$ and $x_2 > 0$. One has

$$\begin{aligned} &f_1 + R_{-1/2}(f_2 + R_{-1/2} f_3 R_{1/2}) R_{1/2} \\ &\qquad = R_{1/2}[R_{-1/2}(f_1 + R_{-1/2} f_2 R_{1/2}) R_{1/2} + R_{-1/2} f_3 R_{1/2}] R_{-1/2} . \end{aligned}$$

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The first expression is a representative of $\alpha_1 + (\alpha_2 + \alpha_3)$; and the second one, of $(\alpha_1 + \alpha_2) + \alpha_3$.

1.5. It is clear that the identity map of S^n in S^{n+q} represents a unit element. The following lemma is easy to prove.

LEMMA. *An embedding $f: S^n \to S^{n+q}$ is concordant to the identity map if and only if there is an embedding F of D^{n+1} in D^{n+q+1} which is an extension of f.*

1.6. Each element α of C_n^q has an inverse $-\alpha$. Let f be an embedding representing α and satisfying conditions (i) and (ii) of Lemma 1.3 (a). Let σ_i be the symmetry of Hilbert space with respect to the hyperplane $x_i = 0$. Then $\sigma_2 f \sigma_2$ represents $-\alpha$. Indeed the map $f + \sigma_2 f \sigma_2$ can be extended to a differentiable embedding of D^{n+1} in D^{n+q+1} by mapping linearly the segment $[x, \sigma_1 x]$ onto the segment $[fx, \sigma_1 fx]$, $x \in D_+^n$.

We have proved

1.7. THEOREM. *With the sum operation defined in* 1.4, C_n^q *is an abelian group.*

1.8. *Relations with the group* θ_n^q. Let us recall that two embedded oriented homotopy n-spheres K_0^n and K_1^n in S^{n+q} are h-cobordant if there is an oriented submanifold W of $S^{n+q} \times I$ such that

(i) $\partial W = K_1^n \times 1 - K_0^n \times 0$

(ii) the inclusions $K_i^n \times i \to W$, $i = 0, 1$, are homotopy equivalences.

These h-cobordism classes form a group θ_n^q (cf. [3]) where the sum operation can be defined as in 1.4. The group θ_n of h-cobordism classes of homotopy n-spheres (cf. [9]) is isomorphic to θ_n^q for q large enough. According to Smale [14], the elements of θ_n^q correspond bijectively to the isotopy classes of embedded homotopy n-spheres in S^{n+q}, if $q > 2$ and $n \geqq 5$.

The groups C_n^q, θ_n^q and θ_n are related by the following exact sequence, valid at least for $n \geqq 5$:

$$C_n^q \longrightarrow \theta_n^q \longrightarrow \theta_n \longrightarrow C_{n-1}^q \longrightarrow \cdots. \tag{1.9}$$

The homomorphism $C_n^q \to \theta_n^q$ maps the concordance class of $f: S^n \to S^{n+q}$ on the cobordism class of $f(S^n)$. The homomorphism $\theta_n^q \to \theta_n$ is obvious.

The third one $\partial: \theta_n \to C_{n-1}^q$ is defined by using the fact that, at least for $n \geqq 5$, each element of θ_n is represented by a manifold K^n obtained in glueing two n-disk along their boundaries by a diffeomorphism h (cf. Smale [14]). The image by ∂ of the diffeomorphism class of K^n is the concordance class of the embedding $i \circ h$, where i is the natural inclusion of S^{n-1} in S^{n-1+q}.

Proving exactness is easy, if one changes the definition of θ_n^q and θ_n as

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follows. We consider embeddings $f: D^n \to D^{n+q}$ with $f(\partial D^n) = S^{n-1}$; two such embeddings are concordant if there is an embedding $F: D^n \times I \to D^{n+q} \times I$ with $F(\partial D^n \times I) = S^{n-1} \times I$ which relate them. The concordance classes of such embeddings form a group isomorphic to θ_n^q if $n \geqq 5$. Also the group of concordance classes of diffeomorphisms of degree one of S^{n-1} is isomorphic to θ_n for $n \geqq 5$ (cf. Smale [14]). With θ_n and θ_n^q replaced by these groups, the exact sequence (1.9) is valid for all $n > 0$ and $q > 0$.

2. Construction of the homomorphism $\psi: C_n^q \to \pi_{n+1}(G; \mathrm{SO}, G_q)$.

2.1. *The group* $\pi_{n+1}(G; \mathrm{SO}, G_q)$ (cf. [1]). We shall denote by G_q the space of maps of S^{q-1} onto itself of degree one. Suspension defines a natural inclusion of G_q in G_{q+1}, and G will denote the inductive limit of the G_q under iterated suspensions. The image of G_q in G_{q+N} by N-fold suspension will still be denoted by G_q.

SO_q is the space of rotations of D^q (or S^{q-1}) and the inductive limit of the SO_q by suspensions is denoted by SO. As above, SO_q is identified to the subgroup of SO_{q+N} leaving fixed the orthogonal complement of R^q.

An element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$ is represented by a continuous map $f: D^{n+1} \times S^{N-1} \to S^{N-1}$, for some N large enough, having the following properties: (for $x \in D^{n+1}$, $f_x: S^{N-1} \to S^{N-1}$ is the map defined by $f_x(y) = f(x, y)$)

(i) for $x \in D_-^n$, $f_x \in \mathrm{SO}_N$

(ii) for $x \in D_+^n$, $f_x \in G_q$

(iii) for $x \in D^{n+1}$, $f_x \in G_N$.

Note that the suspension $Ef: D^{n+1} \times S^N \to S^N$ of f defined by $Ef_x =$ suspension of f_x for each $x \in D^{n+1}$, also satisfies (i), (ii), (iii).

Two such maps $f: D^{n+1} \times S^{N-1} \to S^{N-1}$ and $f': D^{n+1} \times S^{N'-1} \to S^{N'-1}$ will represent the same element of $\pi_{n+1}(G; \mathrm{SO}, \mathrm{SO}_q)$ if there is a map $F: D^{n+1} \times S^{M-1} \times I \to S^{M-1}$ for some $M \geqq N, N'$, such that the map $f_t: D^{n+1} \times S^{M-1} \to S^{M-1}$ defined by $f_t(x, y) = F(x, y, t)$ satisfies (i), (ii) and (iii) (with N replaced by M) and that f_0, f_1 are suspensions of f and f' respectively.

To define the sum of two elements of $\pi_{n+1}(G; \mathrm{SO}, G_q)$, one can represent the first (resp. the second) by a map f (resp. g) such that f_x (resp. g_x) is the identity for x with $x_2 \leqq 0$ (resp. $x_2 \geqq 0$); then the sum will be represented by the map h defined by

$$h_x = \begin{cases} f_x & \text{for } x \text{ with } x_2 \geqq 0 \\ g_x & \text{for } x \text{ with } x_2 \leqq 0\,. \end{cases}$$

From known elementary stability properties of G_N and SO_N, we could choose throughout a fixed $N > n + 2$.

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2.2. Each element of C_n^q can be represented by a map $f: S^n \to S^{n+q}$ such that $f \mid D_-^n$ is the identity and $f(\operatorname{int} D_+^n) \subset \operatorname{int} D_+^{n+q}$ (cf. Lemma 1.3). We can even choose f such that $f(S^n)$ is contained in the subspace of S^{n+q} defined by $(x_{n+2})^2 + \cdots + (x_{n+q+1})^2 \leqq 1/2$. Indeed, F does not meet the $(q-1)$-sphere dual to S^n defined by $x_1 = \cdots = x_{n+1} = 0$, and by radial expansion we can push f outside of the tubular neighborhood defined by $(x_1)^2 + \cdots + (x_{n+1})^2 \leqq 1/2$. The same remarks apply to concordance.

From now on, we shall identify the subspace of S^{n+q} defined by

$$(x_{n+2})^2 + \cdots + (x_{n+q+1})^2 \leqq 1/2$$

to $S^n \times D^q$ by the diffeomorphism mapping $x = (x_1, \cdots, x_{n+q+1})$ on

$$\left(y/|y|, \sqrt{2}\,z\right) \in S^n \times D^q ,$$

where $y = (x_1, \cdots, x_{n+1})$ and $z = (x_{n+2}, \cdots, x_{n+q+1})$. Hence with this identification, we can always represent the elements of C_n^q by embeddings $f: S^n \to S^n \times D^q$ such that $f \mid D_-^n$ is the natural inclusion $D_-^n \to D_-^n \times 0$ and that

$$f(\operatorname{int} D_+^n) \subset \operatorname{int}(D_+^n \times D^q) .$$

The similar statement is also true for concordance.

The inclusion $D^n \subset D^N$, $N \geqq q$, induces an inclusion

$$S^n \times D^q \subset D^{n+1} \times D^q \subset D^{n+1} \times D^N .$$

If N is large enough (in fact $> n + 2$ by Whitney [16]), f can be extended to an embedding $\bar{f}: D^{n+1} \to D^{n+1} \times D^N$ which is orthogonal to $\partial(D^{n+1} \times D^N)$ along $f(S^n)$.

2.3. THEOREM. *There is a natural homomorphism* $\psi: C_n^q \to \pi_{n+1}(G; \mathrm{SO}, G_q)$ *characterized by the following property. Let* α *be an element of* C_n^q *represented by an embedding* $f: S^n \to S^n \times D^q$ *as in* 2.2, *and let* $\bar{f}: D^{n+1} \to D^{n+1} \times D^N$ *be an embedding which extends* f (cf. 2.2). *For* N *large enough, a map* $\varphi: D^{n+1} \times S^{N-1} \to S^{N-1}$ *represents* $\psi(\alpha)$ *if* φ *admits an extension* $\phi: D^{n+1} \times D^N \to D^N$ *such that*:

(i) ϕ *is regular on* $0 \in D^N$ *and* $\phi^{-1}(0) = \bar{f}(D^{n+1})$

(ii) $\phi_x \in \mathrm{SO}_N$ *for* $x \in D_-^n$

(iii) ϕ_x *is the suspension of a map* $D^q \to D^q$ *for* $x \in D_+^n$.

We first prove the

2.4. LEMMA. *Let* $g: D^r \to D^r \times D^k$ *be a differentiable embedding such that* $g(S^{r-1}) \subset S^{r-1} \times \operatorname{int} D^k$, *together with a framing of the submanifold* $\Delta^r = g(D^r)$. *Let* $G_0: S^{r-1} \times D^k \to D^k$ *be a map such that*

(i) $G_0(S^{r-1} \times S^{k-1}) \subset S^{k-1}$

(ii) G_0 *is regular on* $0 \in D^k$, *and* $G_0^{-1}(0) = \partial\Delta^r$ *as a framed submanifold.*

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Then there is an extension $G: D^r \times D^k \to D^k$ *of* G_0 *satisfying conditions* (i) *and* (ii) *with* S^{r-1} *replaced by* D^r *and* $\partial\Delta^r$ *by* Δ^r.

PROOF. Let $D^k_\varepsilon = \{x \in D^k \text{ with } |x| \leqq \varepsilon\}$. Using the framing of $g(D^r)$ and the uniqueness of tubular neighborhood, for ε small enough, we can construct an embedding $\tau: D^r \times D^k_\varepsilon \to D^r \times D^k$ with $\tau(x, 0) = g(x)$ and $G_0\tau(x, y) = y$ for $x \in \partial D^r$. We can extend G_0 on $T = \tau(D^r \times D^k)$ by defining $G_1 = G_0$ on $\partial D^r \times D^k$ and $G_1\tau(x, y) = y$; we have $G_1^{-1}(0) = \Delta^r$. The map G_1, restricted to

$$B = \partial D^r \times D^k \cup (T - \text{int } T) ,$$

can be extended as a map G_2 of $A = D^r \times D^k - \text{int } T$ in $D^k - 0$; this is because the possible obstructions should lie in $H^i(A; B) = H^i(D^r \times D^k; \Delta^r) = 0$ for all i. We can also make $G_2(D^r \times \partial D^k) \subset \partial D^k$. Define G to be equal to G_1 on T and to G_2 on the complement.

2.5. *Proof of the theorem.* We first prove that, given f, we can construct ϕ. Let us construct a framing of $\bar{f}(D^{n+1})$ such that, along $f(D^n_+)$, the first q vectors are contained in $D^n_+ \times D^q$ and the last ones are the restrictions of the natural framing of D^q in D^N. Along $f(D^n_-)$ we assume that the framing is orthonormal and gives the natural orientation of D^N. The restriction ϕ_- of ϕ to $D^n_- \times D^N$ is uniquely defined by the condition (ii) and the condition that $\phi_-^{-1}(0)$ is the framed submanifold $f(D^n_-)$. The restriction ϕ_+ of ϕ to $D^n_+ \times D^q$ will be the $(N - q)$-fold suspension of a map of $D^n_+ \times D^q \to D^q$ constructed by using Lemma 2.4 with the given framing on $f(D^n_+)$ and $G_0 = \phi_- \mid D^n_+ \times D^q$. Finally, using the lemma again, we extend $\phi_- \cup \phi_+$ to get a map ϕ verifying (i)-(iii).

Let $f_0, f_1: S^n \to S^n \times D^q \subset S^{n+q}$ be two concordant embeddings; we want to prove that two maps $\varphi_0, \varphi_1: D^{n+1} \times S^{N-1} \to S^{N-1}$ with extensions ϕ_0, ϕ_1 verifying (i)-(iii), represent the same element of $\pi_{n+1}(G; \text{SO}, G_q)$. We first construct a concordance $F: S^n \times I \to S^n \times D^q \times I$ connecting f_0 to f_1, such that $F \mid D^n_- \times I$ is standard and $F(\text{int } D^n_+ \times I) \subset \text{int } (D^n_+ \times D^q) \times I$. Let

$$\bar{F}: D^{n+1} \times I \longrightarrow D^{n+1} \times D^N \times I$$

be an embedding which is an extension of F, $\bar{f}_0$ and $\bar{f}_1$. Let us also construct a framing of $\bar{F}(D^{n+1} \times I)$ such that $\bar{F}(D^n_+ \times I)$ with this framing is an $(N - q)$-fold suspension and $\bar{F}(D^{n+1} \times i) = \phi_i^{-1}(0) \times i$ as framed submanifolds for $i = 0, 1$. Using 2.4 twice, a map $\phi: D^{n+1} \times D^N \times I \to D^N$ can be constructed satisfying (i)-(iii) with D^{n+1} replaced by $D^{n+1} \times I$, and extending $\phi_i \times i$. The restriction of ϕ to $D^{n+1} \times S^{N-1} \times I \to S^{N-1}$ gives a homotopy connecting φ_0 to φ_1.

The fact that ψ is a homomorphism is verified by taking representative for f and φ which are standard on the parts of S^n or D^{n+1} defined by $x_2 \geqq 0$ or $x_2 \leqq 0$.

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3. The homomorphism $C_n^q \to \pi_{n+1}(G; \mathrm{SO}, G_q)$ is an isomorphism for $q > 2$

3.1. *Cobordism interpretation of* $\pi_{n+1}(G; \mathrm{SO}, G_q)$. An element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$ can be represented by a map $\varphi: D^{n+1} \times S^{N-1} \to S^{N-1}$ as in 2.1 which is regular on $e_1 = (1, 0, \cdots, 0) \in S^{N-1}$. Hence $\varphi^{-1}(e_1)$ is a framed submanifold V of $D^{n+1} \times S^{N-1}$ such that

(a) $V \cap (D_-^n \times S^{N-1}) = \partial V^-$ is the graph of a map $s: D_-^n \to S^{N-1}$ and we can choose the framing at points (x, sx) to be contained in $x \times S^{N-1}$ and orthonormal,

(b) $V \cap (D_+^n \times S^{N-1}) = \partial V^+$ is the suspension of a framed submanifold in $D_+^n \times S^{q-1}$.

If $\varphi_1, \varphi_2: D^{n+1} \times S^{N-1} \to S^{N-1}$ represent the same element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$ and are regular on e_1, then $V_1 = \varphi_1^{-1}(e_1)$ and $V_2 = \varphi_2^{-1}(e_1)$ are framed submanifolds satisfying (a) and (b). We can choose a homotopy $\varphi: D^{n+1} \times S^{N-1} \times I \to S^{N-1}$ connecting φ_0 to φ_1 and which is regular on e_1. Hence $\varphi^{-1}(e_1) = Z$ is a framed submanifold of $D^{n+1} \times S^{N-1} \times I$ with $\partial Z = (V_0 \times 0) \cup (V_1 \times 1) \cup X$, where X is the framed submanifold $Z \cap (D^{n+1} \times S^{N-1} \times I)$ and satisfies

(a′) $X \cap (D_-^n \times S^{N-1} \times I)$ is the graph of a map $s: D_-^n \times I \to S^{N-1}$ with the same condition on the framing as in (a),

(b′) $X \cap (D_+^n \times S^{N-1} \times I)$ is the suspension of a framed submanifold in $D_+^n \times S^{q-1} \times I$.

Conversely, let V be a framed submanifold of $D^{n+1} \times S^{N-1}$ satisfying (a) and (b). There is a map $\varphi: D^{n+1} \times S^{N-1} \to S^{N-1}$ representing an element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$, which is regular on e_1, and such that $\varphi^{-1}(e_1)$ is equal to V as a framed submanifold. Indeed, on $D_-^n \times S^{N-1}$, φ is defined uniquely by condition 2.1 (i); on $D_+^n \times S^{q-1}$, we construct φ by relative Thom construction (cf. 0.4) and on $D_+^n \times S^{N-1}$ by suspension; to get φ on $D^{n+1} \times S^{N-1}$, we again apply Thom construction.

Let Z be a framed submanifold of $D^{n+1} \times S^{N-1} \times I$ as above and let φ_i be maps of $D^{n+1} \times S^{N-1} \to S^{N-1}$ as in 2.1 with

$$\varphi_i^{-1}(e_1) \times i = V_i \times i = Z \cap (D^{n+1} \times S^{N-1} \times i) .$$

The same argument shows the existence of a map $\varphi: D^{n+1} \times S^{N-1} \times I \to S^{N-1}$ which is a homotopy connecting φ_0 to φ_1, which is regular on e_1 and such that $\varphi^{-1}(e_1) = Z$.

As a conclusion, we can represent elements of $\pi_{n+1}(G; \mathrm{SO}, G_q)$ by framed submanifolds V satisfying (a) and (b); two such framed submanifolds V_0, V_1 represent the same element if there is a framed submanifold Z satisfying (a′) and (b′).

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3.2. With this interpretation, Theorem 2.3 can be restated as follows. A framed submanifold $V \subset D^{n+1} \times S^{N-1}$ verifying (a) and (b) of 3.1 represents the element $\psi(\alpha)$ if there is a framed submanifold $W \subset D^{n+1} \times D^N$ such that

(i) $\partial W \cap (D^{n+1} \times S^{N-1}) = V$,

(ii) ∂W has a free face in $D^{n+1} \times D^N$, namely $\bar{f}(D^{n+1})$,

(iii) $\partial W \cap (D^n_- \times D^N)$ is the radial extension of $V \cap (D^n_- \times S^{N-1}) = \partial V^-$, i.e., the set of points $(x, ts(y))$ with $0 \leqq t \leqq 1$ and $(x, s(x)) \in \partial V^-$ (cf. 3.1, (a)),

(iv) $\partial W \cap (D^n_+ \times D^N)$ is the $(N-q)$-fold suspension of a framed submanifold in $D^n_+ \times D^q$.

Indeed, let $\varphi\colon D^{n+1} \times S^{N-1} \rightarrow S^{N-1}$ be a map representing $\psi(\alpha)$, regular on e_1 and such that $\varphi^{-1}(e_1) = V$. We can construct $\phi\colon D^{n+1} \times D^N \rightarrow D^N$ like in 2.3 and such that ϕ, restricted to the complement of $\bar{f}(D^{n+1})$, and composed with the map $y \rightarrow y/|y|$ of $D^N - 0$ on S^{N-1}, is regular on e_1. Then the inverse image by ϕ of the radius te_1, $0 \leqq t \leqq 1$, is a framed submanifold W which satisfies (i)-(iv).

Conversely, if there is a framed submanifold W which satisfies (i)-(iv), then using relative Thom construction, we can construct a map as in Theorem 2.3.

3.3. Two framed submanifolds V_0 and V_1, with the same boundary, in the manifold M are framed cobordant, if there is a framed submanifold W in $M \times I$ such that

$$\partial W = (V_0 \times 0) \cup (V_1 \times 1) \cup (\partial V_0 \times I)$$

with their framings.

LEMMA. *Let V_0 be a compact framed submanifold of dimension n in a manifold M of dimension $n + q$. We assume that M is $(n/2 - 1)$-connected and that $q > n/2$. Then V_0 is framed cobordant to a framed submanifold which is the union of an n-disk and of handles of indices $> n/2 - 1$.*

PROOF. V_0 can be represented as union of handles of increasing indices (cf. Smale, Ann. of Math., 74 (1961), 391-406). Assume inductively that this handle decomposition of V_0 has r handles of indices $\leqq n/2 - 1$. Let V' be the union of the first handle D^n and of the second one $D^k \times D^{n-k}$, where $k \leqq n/2-1$.

We prove that V' is diffeomorphic to $S^k \times D^{n-k}$. First, V' is an $(n-k)$-disk bundle over S^k (cf. Smale, *loc. cit.*). The attaching embedding $f\colon \partial D^k \times D^{n-k} \rightarrow D^n$ of the handle is isotopic to an embedding g such that $g \mid \partial D^k \times 0$ is the natural inclusion in ∂D^n, because $2(k-1) + 1 < n - 1$; moreover, the standard tubular neighborhood of S^{k-1} in S^{n-1} being identified with $S^{k-1} \times D^{n-k}$ as in 2.2, we can assume (by the tubular neighborhood theorem) that g maps each $x \times D^{n-k}$ isometrically on $x \times D^{n-k}$. Hence W is obtained by glueing

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two copies of the trivial $(n-k)$-disk bundle $D^k \times D^{n-k}$ with g which is fiber preserving. But this disk bundle is trivial. Indeed, it is isomorphic to the normal bundle ν of the zero section; the tangent bundle of M, restricted to this zero section, is trivial, because M is k-connected, and it is the direct sum of ν and of a trivial bundle (V' is framed). Hence ν is trivial, because it is characterized by an element of the stable group $\pi_{k-1}(\mathrm{SO}_{n-k})$ whose suspension is trivial.

We now perform on V_0 a framed spherical modification of index $k+1$ whose aim is to replace one handle of index k by one of index $n-k-1$ (see [18]). Following [3] (see also 8.3), this modification will be defined by an embedding ϕ of $D^{k+1} \times D^{n-k}$ in M, with a normal framing $F_2, \cdots, F_q$ such that:

(i) $\phi(\partial D^{k+1} \times D^{n-k})$ is the subspace $V' = S^k \times D^{n-k}$ of V_0 and the image of ϕ does not meet V_0 elsewhere,

(ii) along $\phi(\partial D^{k+1} \times D^{n-k})$, ϕ is tangent to the first vector f_1 of the framing $(f_1, \cdots, f_q)$ of V and $f_i = F_i$ for $i \geqq 2$.

The edge is suitably smoothed along $\phi(\partial D^{k+1} \times \partial D^{n-k})$ (cf. [3] and 8.3). It is possible to construct $\phi \mid D^{k+1} \times 0$ because M is k-connected and $n+k+1 < n+q$, and to construct the framing $F_2, \cdots, F_q$ because $2k < n$ (cf. [8.2]).

$V_1 = (V_0 - V') \cup \phi(D^{k+1} \times \partial D^{n-k})$ is a framed submanifold, framed cobordant to V_0 (cf. [3]). As $D^{k+1} \times S^{n-k-1}$ is diffeomorphic to the union of D^n and a handle of index $n-k-1$, V_1 will admit a handle decomposition with $r-1$ handles of indices $\leqq n/2 - 1$.

Hence, after $r-1$ such modifications, V will be transformed in a framed submanifold satisfying the conclusion of the lemma.

3.4. MAIN THEOREM. *The homomorphism* $\psi\colon C_n^q \to \pi_{n+1}(G; \mathrm{SO}, G_q)$ *is an isomorphism for* $q > 2$.

3.5. *Proof of surjectivity.* Given an element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$, we can represent it by a framed submanifold $V \subset D^{n+1} \times S^{N-1}$ satisfying 3.1 (a), (b), and which is, by 3.3, the union of an $(n+1)$-disk Δ and of handles of indices $\leqq (n-1)/2$. Hence, by looking at the dual decomposition, V minus the interior Δ_0 of Δ is diffeomorphic to a tubular neighborhood $\partial V \times I$ of ∂V with handles of indices $\leqq n/2 + 1$ attached. We can assume that these handles do not touch $\partial V^- \times 1$.

We want to construct a framed submanifold W as in 3.2.

Let $\mu\colon V - \Delta_0 \to [0, 1]$ be a differentiable function, with gradient non-zero on $\partial(V - \Delta_0)$ and such that $\mu^{-1}(0) = \partial V$ and $\mu^{-1}(1) = \partial\Delta$; moreover for $(x, t) \in \partial V^- \times I$, we assume $\mu(x, t) = t$. Let Z be the submanifold of $I \times V$ defined by

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$$Z = \{(t, x) \mid x \in V - \Delta_0 \text{ and } t \leqq \mu(x), \text{ or } x \in \Delta \text{ and } t \in [0, 1]\} .$$

The boundary ∂Z is made up of three faces: $0 \times V$ which will be identified to V, $1 \times \Delta$ and a face diffeomorphic to $V - \Delta_0$ by the map $x \to (\mu x, x)$.

The theorem will be proved if we can construct an embedding ρ of Z in $D^{n+1} \times D^N$, with a framing, such that the framed submanifold $W = \rho(Z)$ satisfy all conditions (i)-(iv) of 3.2. This will be done essentially by general position argument.

We shall denote by V_r (resp. V_r^+) the union of $\partial V \times I$ (resp. $\partial V^+ \times I$) with the first r handles. $\bar{V}_r$ (resp. $\bar{V}_r^+$) will be the image of V_r (resp. V_r^+) by the map $x \to (\mu x, x)$; Z_r is the set of points $(t, x) \in Z$ with $x \in V_r$ and $t \leqq \mu(x)$.

V_r is obtained by attaching to V_{r-1} a handle $D^k \times D^{n+1-k}$ with an embedding $g_r^0\colon \partial D^k \times D^{n+1-k} \to \partial V_{r-1}$ and rounding corners along $g_r^0(\partial D^k \times \partial D^{n+1-k})$ (cf. 8.3). Also Z_r is diffeomorphic to Z_{r-1} with $I \times D^k \times D^{n+1-k}$ attached with an embedding $g_r\colon I \times \partial D^k \times D^{n+1-k} \to \partial Z_r$ defined by $g_r(t, z) = (\mu(z)t, g_r^0(z))$, where $z \in \partial D^k \times D^{n+1-k}$. By this diffeomorphism, $I \times D^k \times D^{n+1-k}$ will be identified with a subspace of Z_r.

3.6. We shall construct, by induction on r, an embedding $\rho_r\colon Z_r \to D^{n+1} \times D^N$, with a framing, such that $\rho_r \mid V_r$ is the identity, $\rho_r(\partial V^- \times I)$ is the radial extension of ∂V^-, and $\rho_r(V_r^+)$ is the $(N - q)$-fold suspension of a framed submanifold in $D_+^n \times D^q$. The framing on $\rho_r(V_r)$ will be the given one.

The embedding ρ_0 is easily constructed. Suppose ρ_{r-1} satisfying the preceding conditions has been already constructed. The extension ρ_r of ρ_{r-1} will be done in three steps.

(1) We want to construct an extension of ρ_{r-1} on $I \times D^k \times 0$. We first construct an embedding $\varphi_1\colon 1 \times D^{k-1} \times 0 \to D_+^n \times D^q$ which extends ρ_{r-1}: near the boundary of $1 \times D^k \times 0$, this is always possible and also on the remainder by the general position argument of Whitney because $2k < n + q$ (cf. [16]). On the other hand, we can assume that $\varphi_1(1 \times \operatorname{int} D^k \times 0)$ does not intersect the image of ρ_{r-1}; we argue by induction on the number of handles of V_{r-1}: if $D^s \times D^{n+1-s}$ is a handle of V_{r-1}, we can arrange, by general position argument because $k + 1 < n + q$, that $\varphi_1(1 \times \operatorname{int} D^k \times 0)$ does not meet $\rho_{r-1}(1 \times D^s \times 0)$ and also $\varphi_{r-1}(1 \times D^s \times D^{n+1-s})$ by radial expansion.

As N is big enough ($>n + 2$), we can construct an embedding

$$\varphi\colon I \times D^k \times 0 \longrightarrow D^{n+1} \times D^N$$

which is an extension of ρ_{r-1} and such that $\varphi \mid (1 \times D^k \times 0) = \varphi_1$, $\varphi \mid (0 \times D^k \times 0) =$ identity and $\varphi(I \times \operatorname{int} D^k \times 0) \cap \rho_{r-1}(Z_{r-1}) = \emptyset$.

(2) *Construction of the framing along* $\varphi(I \times D^k \times 0)$. We can always construct a field of N-frames $f = (f_1, \cdots, f_N)$ along $\varphi(I \times D^k \times 0)$, transversal

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to this submanifold, and which coincide with the framing already given on $\varphi[(I \times \partial D^k \times 0) \cup (0 \times D^k \times 0)]$. But on $\varphi(1 \times D^k \times 0)$, we want the last $N-q$ vector fields to form the restriction of the standard framing $e_{q+1}, \cdots, e_N$ of $D^{n+1} \times D^q$ in $D^{n+1} \times D^N$. This will be possible if there is a homotopy in the normal bundle of $\varphi(1 \times D^k \times 0)$ in $D^n_+ \times D^N$, fixed on $\varphi(1 \times \partial D^k \times 0)$, carrying the framing $f_{q+1}, \cdots, f_N$ on $e_{q+1}, \cdots, e_N$. As the rank of this bundle is $N + n - k$, the possible obstruction would be an element of $\pi_k(V_{N+n-k, N-q})$, where $V_{p+s,p}$ is the Stiefel manifold of p-frames in $\mathbf{R}^{p+s}$. This group is trivial if $k < n - k + q$. We can extend this homotopy to the framing $f_1, \cdots, f_N$; hence we can assume that $f_i = e_i$ for $q < i \leqq N$ on $\varphi(1 \times D^k \times 0)$.

(3) Using existence and uniqueness of tubular neighborhood, we can construct an embedding ρ_r of Z_r in $D^{n+1} \times D^N$ which is an extension of ρ_{r-1}, φ and the identity map on $0 \times D^k \times D^{n+1-k}$, and such that $\rho_r(I \times D^k \times D^{n+1-k})$ is transversal to the framing $f_1, \cdots, f_N$ along $\varphi(I \times D^k \times 0)$ and that

$$\rho_r(1 \times D^k \times D^{n+1-k}) \subset D^n_+ \times D^q .$$

There is no obstruction to extending the framing on $\rho_r(I \times D^k \times D^{n+1-k})$.

Finally ρ can be constructed on $I \times \Delta$, because $N > n + 2$. The concordance class of the embedding $x \rightarrow (1, x)$ of $\partial\Delta = S^n$ in $S^n \times D^q \subset S^{n+q}$ is mapped by ψ on the element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$ represented by V.

3.7. *Proof of injectivity.* Let $f\colon S^n \rightarrow S^n \times D^q$ be an embedding as in 2.2, and let W be a framed submanifold of $D^{n+1} \times D^N$ as in 3.2 with

$$V = W \cap (D^{n+1} \times S^{N-1}) .$$

We assume that V represents the trivial element of $\pi_{n+1}(G; \mathrm{SO}, G_q)$. Hence we can choose W such that V is $D^{n+1} \times e_1 \subset D^{n+1} \times S^{N-1}$. After rounding the corners along $S^n \times S^{N-1}$, we can replace $D^{n+1} \times D^N$ by D^{n+N+1}, and W will now be a framed $(n + 2)$-submanifold of D^{n+N+1} such that $W \cap S^{n+N} = V'$ is the $(N - q)$-fold suspension of a framed submanifold contained in S^{n+q} and whose boundary is $f(S^n)$; moreover W has a free face contained in D^{n+N+1} equal to $\bar{f}(D^{n+1})$.

After framed spherical modifications (cf. 3.3), we can assume that W is diffeomorphic to $V' \times I$ with handles of index $\leqq n/2 + 3/2$ attached. By the same method as in 3.6, we can construct an embedding ρ of W in D^{n+q+1} which is the identity on V'. Then $\rho\bar{f}$ will be an embedding of D^{n+1} in D^{n+q+1} which is an extension of f.

3.8. *Remark.* It follows from a recent work of Kervaire (cf. [8]) that the homomorphism $\psi\colon C^2_n \rightarrow \pi_{n+1}(G; \mathrm{SO}, G_2) = \pi_{n+1}(G; \mathrm{SO})$ is also an isomorphism for n even.

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4. Immersions and embeddings of spheres in spheres

4.1. Let Im_n^q be the group of concordance classes of immersions of S^n in S^{n+q}, $q > 0$; concordance and group structure are defined as for C_n^q, except that embedding is replaced by immersion. It follows from Smale-Hirsch classification theorem [5] that concordance classes and regular homotopy classes of immersions of S^n in S^{n+q} are the same for all n and $q > 0$.

4.2. We now describe a homomorphism ϕ of Im_n^q in $\pi_n(\mathrm{SO}, \mathrm{SO}_q)$. Let $f: S^n \to S^{n+q}$ be an immersion; we can extend it as an immersion $\bar{f}: D^{n+1} \to D^{n+N+1}$ for N big (in fact $N > n + 2$). We choose a trivialization of the normal bundle of $\bar{f}$; with respect to it, the map associating to $x \in S^n$ the $(N-q)$-frame $e_{n+q+2}, \cdots, e_{N+q+1}$ defines a map of S^n in the Stiefel manifold $V_{N,N-q}$; its homotopy class $\in \pi_n(V_{N,N-q}) = \pi_n(\mathrm{SO}, \mathrm{SO}_q)$ depends only on the concordance class of f and defines a homomorphism ϕ of Im_n^q in $\pi_n(\mathrm{SO}, \mathrm{SO}_q)$.

4.3. THEOREM (Smale). *The homomorphism ϕ: $\mathrm{Im}_n^q \to \pi_n(\mathrm{SO}, \mathrm{SO}_q)$ is an isomorphism.*

This theorem follows easily from the classification theorem for immersions (cf. [5], [13]) and from the fact that $\pi_n(V_{n+q+1,n+1})$ is isomorphic to $\pi_n(\mathrm{SO}, \mathrm{SO}_q)$ for $q > 0$. The details will be left to the reader.

4.4. We shall be interested in the natural homomorphism of C_n^q in Im_n^q (an embedding is also an immersion). It is easy to check that the following diagram commutes up to sign:

$$(4.5) \qquad \begin{array}{ccc} \pi_{n+1}(G; \mathrm{SO}, G_q) & \xrightarrow{\partial} & \pi_n(\mathrm{SO}, \mathrm{SO}_q) \\ \psi \uparrow & & \uparrow \phi \\ C_n^q & \longrightarrow & \mathrm{Im}_n^q \end{array}$$

where ∂ is the boundary homomorphism in the following exact sequence associated to the triad $(G; \mathrm{SO}, G_q)$ (cf. [1]):

$$(4.6) \qquad \longrightarrow \pi_{n+1}(G, G_q) \longrightarrow \pi_{n+1}(G; \mathrm{SO}, G_q) \longrightarrow \pi_n(\mathrm{SO}, \mathrm{SO}_q) \longrightarrow \pi_n(G, G_q) \ .$$

Note that $G_q \cap \mathrm{SO} = \mathrm{SO}_q$.

As a consequence of 3.4, 4.3, 4.5 and 4.6, we obtain:

4.7. THEOREM. *For $q > 2$, an immersion $f: S^n \to S^{n+q}$ corresponding to $\alpha \in \pi_n(\mathrm{SO}, \mathrm{SO}_q)$ by the isomorphism ϕ, is regularly homotopic to an embedding if and only if the image of α in $\pi_n(G, G_q)$ is zero. The isotopy classes of embeddings of S^n in S^{n+q} which are trivial as immersions correspond bijectively to the cokernel of the homomorphism $\pi_{n+1}(\mathrm{SO}, \mathrm{SO}_q) \to \pi_{n+1}(G, G_q)$.*

4.8. *Remark.* One can define the notion of combinatorial or piecewise

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linear immersion of S^n in S^{n+q} and define as in § 1 the group Plim_n^q of concordance classes of such immersions. In a forthcoming paper, we shall prove that this group is isomorphic, for $q > 2$, to the group $\pi_n(G, G_q)$, and that the exact sequence 4.6 is isomorphic to the geometric exact sequence:

$$\longrightarrow C_n^q \longrightarrow \mathrm{Im}_n^q \longrightarrow \mathrm{Plim}_n^q \xrightarrow{\partial} C_{n-1}^q \longrightarrow \tag{4.9}$$

where the homomorphism $\mathrm{Im}_n^q \to \mathrm{Plim}_n^q$ associates to the class of the differentiable immersion f, the class of a piecewise linear immersion which is piecewise differentiably isotopic to f; the homomorphism ∂ measures the obstruction to smoothing a piecewise linear immersion.

4.10. *Relations with Kervaire-Milnor and Levine exact sequences.* To the triad $(G; \mathrm{SO}, G_q)$ is also associated the exact sequences

$$\pi_{n+1}(G, \mathrm{SO}) \longrightarrow \pi_{n+1}(G; \mathrm{SO}, G_q) \longrightarrow \pi_n(G_q, \mathrm{SO}_q) \longrightarrow \pi_n(G, \mathrm{SO}) \longrightarrow , \tag{4.11}$$

which is related to the exact sequences of Kervaire-Milnor and Levine [10] as follows.

The Levine exact sequence (cf. [10])

$$\longrightarrow P_{n+1} \longrightarrow \theta_n^q \longrightarrow \pi_n(G_q, \mathrm{SO}_q) \longrightarrow P_n \longrightarrow \qquad (n \geqq 5)$$

is mapped by stable suspension in the Kervaire-Milnor exact sequence

$$\longrightarrow P_{n+1} \longrightarrow \theta_n \longrightarrow \pi_n(G, \mathrm{SO}) \longrightarrow P_n \longrightarrow$$

where $P_n = 0$ for n odd, Z_2 for $n = 2 \pmod 4$, Z for $n = 0 \pmod 4$.

These sequences together with 1.9 and 4.11 (where we have replaced $\pi_{n+1}(G; \mathrm{SO}, G_q)$ by C_n^q) form a diagram commutative up to sign, valid for $n > 4$ and $q > 2$:

$$\begin{array}{ccccccc} C_n^q & & \longrightarrow & \pi_n(G_q, \mathrm{SO}_q) & \longrightarrow & & P_n \\ & \searrow & \nearrow & & \searrow & \nearrow & \\ & & \theta_n^q & & \pi_n(G, \mathrm{SO}) & & \\ & \nearrow & \searrow & & \nearrow & \searrow & \\ P_{n+1} & & \longrightarrow & \theta_n & \longrightarrow & & C_{n-1}^q \end{array} \tag{4.12}$$

Checking commutativity will be left to the reader.

5. Framed embeddings of S^n in S^{n+q}

5.1. A framed embedding of S^n in S^{n+q} is a differentiable embedding $f\colon S^n \times D^q \to S^{n+q}$ preserving orientation. By Smale [14], concordance classes of such embeddings coincide with isotopy classes for $q > 2$. We can always change f by an isotopy so that $f \mid D_-^n \times D^q$ is the standard embedding in D_-^{n+q} (cf. 2.2.) and $f(D_+^n \times D^q) \subset D_+^{n+q}$. Hence we can define a sum operation as in § 1 and prove that *the concordance classes of framed embeddings of S^n in S^{n+q} form an abelian group denoted by FC_n^q.*

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We could also have defined a framed embedding of S^n in S^{n+q} as an embedding $f_0: S^n \to S^{n+q}$ with a framing of $f_0(S^n)$ giving the right orientation.

5.2. On the other hand, let us consider simply connected oriented compact differentiable manifolds V of dimension $n+q+1$, with boundary, such that $H_i(V) = Z$ for $i = 0, n+1$ and $= 0$ otherwise. According to Smale [14], these manifolds, for $n+q+1 \geqq 6$, are handlebodies obtained in glueing to D^{n+q+1} a handle $D^{n+1} \times D^q$ with an embedding $f: D^{n+1} \times D^q \to D^{n+q+1}$.

We consider couples (V, γ), where V is such a handlebody and γ is a generator of $H_{n+1}(V)$; we identify (V, γ) and (V', γ') if there is a diffeomorphism of V on V', preserving orientation, and carrying γ on γ'. In this way we get the *set* $\mathcal{H}_{n+1}^{n+q+1}$ *of diffeomorphism classes of oriented handlebodies of dimension* $n+q+1$, *with one handle of index* $n+1$ *and a preferred basis.*

Each framed embedding $f: S^n \times D^q \to S^{n+q}$ defines such a handlebody; two framed embeddings, whose class in FC_n^q are the same, define the same element of $\mathcal{H}_{n+1}^{n+q+1}$, for $q > 2$; this is because concordance = isotopy and we can apply the theorem of extension of isotopy.

5.3. THEOREM. *The map* $FC_n^q \to \mathcal{H}_{n+1}^{n+q+1}$ *defined above is bijective for* $q > 2$.

PROOF. Surjectivity is obvious. Injectivity is proved as follows. Let $V_i = D^{n+q+1} \bigcup_{f_i} (D_i^{n+1} \times D_i^q)$, $i = 0, 1$, be two handlebodies, where f_0, f_1 are framed embeddings of S^n in S^{n+q}. Let h be an orientation preserving diffeomorphism which carries the generator of $H_{n+1}(V_0) = H_{n+1}(V_0, D^{n+q+1})$ represented by $D_0^{n+1} \times 0$ on the generator of $H_{n+1}(V_1) = H_{n+1}(V_1, D^{n+q+1})$ represented by $D_1^{n+1} \times 0$. We have to prove that f_0 and f_1 are concordant.

Let $\frac{1}{2} D^{n+q+1}$ be the disk formed by the points $x \in D^{n+q+1}$ with $|x| \leqq 1/2$. We can assume that $h \mid \frac{1}{2} D^{n+q+1}$ is the identity. Let Δ be the $(n+1)$-disk in V_0, union of $D^{n+1} \times 0$ with the annulus formed by the points $x \in D^{n+q+1}$ such that $x/|x| \in f_0(\partial D_0^{n+1} \times 0)$ and $1/2 \leqq |x| \leqq 1$. The intersection number of $h(\Delta)$ with $0 \times D_1^q$ is one. Applying the process of Whitney to eliminate the double points [17], after an isotopy, we can assume that the restriction of h to $D_0^{n+1} \times \frac{1}{2} D_0^q$ is the natural diffeomorphism on $D_1^{n+1} \times \frac{1}{2} D_1^q$, and that $h(x) \in D^{n+q+1}$ if $1/2 \leqq |x| \leqq 1$ and $x/|x| \in f_0(\partial D_0^{n+1} \times D_0^q)$.

Let $g: D^{n+q+1} - \text{int}\, \frac{1}{2} D^{n+q+1} \to S^{n+q} \times I$ be defined by $g(tx) = (x, 2t-1)$, $x \in \partial D^{n+q+1}, t \in [1/2, 1]$. Then the map $gh\left[\frac{(t+1)}{2} f_0(x, y)\right]$ of $S^n \times \frac{1}{2} D^q$ in $S^{n+q} \times I$ is a concordance connecting the restrictions of f_0 and f_1 to $S^n \times \frac{1}{2} D^q$. It is then immediate that f_0 and f_1 are also concordant.

5.4. *Remark.* The same argument and theorem is valid in the *combi-*

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natorial case. One has to replace everywhere differentiable by piecewise linear. The group of concordance classes of framed piecewise linear embeddings of S^n in S^{n+q} is isomorphic, for $q > 2$ and $n > 4$, to the group $F\theta_n^q$ of framed isotopy classes of homotopy n-spheres in S^{n+q} (cf. Levine [10]). This follows easily from Cairns-Hirsch theorem. One has the following exact sequence, analogous to 1.9:

$$(5.5) \qquad \cdots \longrightarrow FC_n^q \longrightarrow F\theta_n^q \longrightarrow \theta_n \longrightarrow FC_{n-1}^q \longrightarrow \cdots .$$

5.6. *Remark.* The same method can be applied to classify handlebodies with more than one handle. One has to classify framed links of spheres; this will be done in a forthcoming paper.

Computation of FC_n^q. With a few minor modifications of § 2 and § 3, one proves the following theorem.

5.7. THEOREM. *There is a natural homomorphism* $\tilde{\Psi}$ *of* FC_n^q *into the group* $\tilde{\pi}_{n+1}(G; \mathrm{SO}, G_q)$ *of homotopy classes of maps* $g: D^{n+1} \to G$ *such that* $g(D_-^n) \subset \mathrm{SO}$, $g(D_+^n) \subset G_q$ *and* $g(\partial D_-^n = \partial D_+^n) =$ *identity. For* $q > 2$, *this homomorphism is an isomorphism.*

$\tilde{\Psi}$ is defined as in § 2. Let $f: S^n \times D^q \to S^{n+q}$ be a framed embedding; after an isotopy, and identification of the canonical tubular neighborhood of $S^n \subset S^{n+q}$ with $S^n \times D^q$ (cf. 2.2), we can consider f as an embedding of $S^n \times D^q$ in $S^n \times D^q$ and assume that $f \mid D_-^n \times D^q =$ identity. Let $f_0: S^n \to S^{n+q}$ be defined by $f_0(x) = f(x, 0)$. In Theorem 2.3, one has to replace f by f_0 and ϕ must also verify the condition $\phi f(x, y) = y$ for $x \in D_+^n$.

5.8. Let $(A; B, C)$ be a triad, where A is a topological space containing B and C and let $x \in B \cap C$ be a base point. We denote by $\tilde{\pi}_{n+1}(A; B, C)$ the group of homotopy classes of maps $g: D^{n+1} \to A$ such that

$$g(D_-^n) \subset B , \quad g(D_+^n) \subset C , \quad g(\partial D_-^n) = x .$$

We have three exact sequences which are easy to establish (cf. [1]):

$$\longrightarrow \pi_n(B \cap C) \longrightarrow \tilde{\pi}_{n+1}(A; B, C) \longrightarrow \pi_{n+1}(A; B, C) \longrightarrow \pi_{n-1}(B \cap C) \longrightarrow ;$$

$$\longrightarrow \pi_n(A, B) \longrightarrow \tilde{\pi}_{n+1}(A; B, C) \longrightarrow \pi_n(C) \longrightarrow \pi_n(A, B) \longrightarrow \cdots .$$

The third one is obtained by exchanging B and C.

Hence if we identify FC_n^q to $\tilde{\pi}_{n+1}(G; \mathrm{SO}, G_q)$ by $\tilde{\Psi}$, we get:

5.9. COROLLARY. *For* $q > 2$, *we have three exact sequences*

$$(5.10) \qquad \longrightarrow \pi_n(\mathrm{SO}_q) \longrightarrow FC_n^q \longrightarrow C_n^q \longrightarrow \pi_{n-1}(\mathrm{SO}_q) \longrightarrow \cdots ;$$

$$(5.11) \qquad \longrightarrow \pi_{n+1}(G, \mathrm{SO}) \longrightarrow FC_n^q \longrightarrow \pi_n(G_q) \longrightarrow \pi_n(G, \mathrm{SO}) \longrightarrow \cdots ;$$

$$(5.12) \qquad \longrightarrow \pi_{n+1}(G, G_q) \longrightarrow FC_n^q \longrightarrow \pi_n(\mathrm{SO}) \longrightarrow \pi_n(G, G_q) \longrightarrow \cdots .$$

The geometric meaning of 5.10 is clear. The homomorphism $C_n^q \to \pi_{n-1}(\mathrm{SO}_q)$

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associates to the class of $f: S^n \to S^{n+q}$ the obstruction to trivializing the normal bundle of $f(S^n)$. The homomorphism $\pi_n(\mathrm{SO}_q) \to FC_n^q$ associates to the homotopy class of $r: S^n \to \mathrm{SO}_q$ the framed embedding obtained by composition of the diffeomorphism $(x, y) \to (x, r(x)y)$ of $S^n \times D^q$ with the natural inclusion in S^{n+q}.

The homomorphism $FC_n^q \to \pi_n(G_q)$ in 5.11 (i.e., the composition of $\tilde{\Psi}$ with the homomorphism $\pi_{n+1}(G, \mathrm{SO}, G_q) \to \pi_n(G_q)$) is described as follows. An element $\alpha \in FC_n^q$ can be represented by an embedding $f: S^n \times D^q \to S^n \times D^q$ which commutes homotopically with the projection on S^n (cf. 2.2). The natural projection $f(x, y) \to y$ of $f(S^n \times S^{q-1}) \to S^{q-1}$ can be extended as a map $g: S^n \times D^q - f(S^n \times 0) \to S^{q-1}$. The restriction of g to $S^n \times S^{q-1}$ represents the image of α in $\pi_n(G_q)$. This homomorphism can also be described as follows (cf. Levine [10]). Let $f: S^n \times D^q \to S^{n+q}$ be a framed embedding and $e \in S^n$. The map $g: S^{q-1} \to S^{n-q} - f(S^n \times 0)$ defined by $g(y) = f(e, y)$ is a homotopy equivalence; let h be a homotopic inverse of g. Then $hf \mid S^n \times S^{q-1} \to S^{q-1}$ represents the image, up to sign, of the class of f.

Finally we can get the homomorphism $FC_n^q \to \pi_n(\mathrm{SO})$ up to sign as follows. Let $\bar{f}: D^{n+1} \times D^q \to D^{n+q+1}$ be defined by $\bar{f}(tx, y) = tf(x, y)$, where $x \in S^n$, $y \in D^q$ and $t \in [0, 1]$. We define a map φ of S^n in the general linear group GL_{n+q+1} by taking the image by the differential of $\bar{f}$ along $S^n \times 0$, of the constant field $e_1, \cdots, e_{n+q+1}$. The map φ defines an element of $\pi_n(\mathrm{GL}_{n+q+1}) = \pi_n(\mathrm{SO}_{n+q+1}) = \pi_n(\mathrm{SO})$.

Remark. The exact sequences 5.10, 5.11, 4.11 and the homotopy exact sequence of the pair (G_q, SO_q) form a diagram, commutative up to signs, which is analogous to diagram 5 of 2.2 of Levine [10].

(5.13)
$$\begin{array}{ccccc} \pi_n(\mathrm{SO}_q) & \longrightarrow & \pi_n(G_q) & \longrightarrow & \pi_n(G, \mathrm{SO}) \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & FC_n^q & & \pi_n(G_q, \mathrm{SO}_q) & \\ & \swarrow \quad \searrow & & \nearrow \quad \searrow & \\ \pi_{n+1}(G, \mathrm{SO}) & \longrightarrow & C_n^q & \longrightarrow & \pi_{n-1}(\mathrm{SO}_q)\,. \end{array}$$

We also have the diagram made up of the sequences 5.10, 5.12, 4.6 and of the homotopy exact sequence of the pair $(\mathrm{SO}, \mathrm{SO}_q)$:

(5.14)
$$\begin{array}{ccccc} \pi_n(\mathrm{SO}_q) & \longrightarrow & \pi_n(\mathrm{SO}) & \longrightarrow & \pi_n(G, G_q) \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & FC_n^q & & \pi_n(\mathrm{SO}, \mathrm{SO}_q) & \\ & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\ \pi_{n+1}(G, G_q) & \longrightarrow & C_n^q & \longrightarrow & \pi_{n-1}(\mathrm{SO}_q)\,. \end{array}$$

With 5.11, 5.12 and the homotopy exact sequences of the pairs (G, G_q) and (G, SO), we have

(5.15)
$$\begin{array}{ccccc} \pi_{n+1}(G, G_q) & \longrightarrow & \pi_n(G_q) & \longrightarrow & \pi_n(G, \mathrm{SO}) \\ & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & FC_n^q & & \pi_n(G) & \\ & \nearrow \quad \searrow & & \nearrow \quad \searrow & \\ \pi_{n+1}(G, \mathrm{SO}) & \longrightarrow & \pi_n(\mathrm{SO}) & \longrightarrow & \pi_{n-1}(G, G_q)\,. \end{array}$$

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5.16. *Computation of C_3^3 and FC_3^3.* We want to give an explicit construction of generators of these groups.

We have the following diagram:

$$\begin{array}{ccccccccc} & & & & \pi_3(G_3, F_2) = \pi_3(S^2) = \mathbf{Z} & & & & \\ & & & & \uparrow \gamma & & & & \\ 0 \longrightarrow & \pi_4(G, G_3) & \xrightarrow{\alpha} & \tilde{\pi}_4(G; \mathrm{SO}, G_3) & \xrightarrow{\beta} & \pi_3(\mathrm{SO}) & \longrightarrow & \pi_3(G, G_3) & \longrightarrow 0 . \\ & \| & & \| & & \| & & \| & \\ & \mathbf{Z} & & \mathbf{Z} + \mathbf{Z} & & \mathbf{Z} & & \mathbf{Z}_2 & \end{array}$$

The horizontal sequence is exact (cf. 5.12); the composition $\gamma\alpha$ is a multiplication by 6.

We identify FC_3^3 with $\tilde{\pi}_4(G; \mathrm{SO}, G_3)$ by the isomorphism $\tilde{\psi}$ of Theorem 5.7.

The first generator a of FC_3^3 will be represented by the embedded 3-spheres in S^6 described in [3, 4.1], with the framing obtained by taking the standard one on each of the three components S_1, S_2, S_3. This element a generates the image of α. It is clear that a is in the kernel of β. On the other hand, $\gamma(a)$ is obtained up to sign, by computing the element of $\pi_3(S^2)$ represented in $S^6 - S$ by S pushed along the first vector of the framing; we can check that we obtain 6 times a generator of $\pi_3(S^2)$. Hence a is the image by α of a generator of $\pi_4(G, G_3)$.

The other generator b of FC_3^3 is represented by the standard S^3 in S^6 with the framing obtained from the natural one by a twist representing the generator of $\pi_3(\mathrm{SO}_3) = \mathbf{Z}$. The element a is the image of this generator by the homomorphism $\tau: \pi_3(\mathrm{SO}_3) \to FC_3^3$ defined in 5.10. As $\beta\tau$ is the stable suspension which is a multiplication by 2, the element $\beta(b)$ generates the image of β.

Diagram 5.14 shows that C_3^3 is isomorphic to $\mathbf{Z}$ and a generator is represented by an embedding f whose image is S.

The exact sequence 4.11

$$\begin{array}{ccccc} C_3^3 & \longrightarrow & \pi_3(G_3, \mathrm{SO}_3) & \longrightarrow & \pi_3(G, \mathrm{SO}) \\ \| & & \| & & \| \\ \mathbf{Z} & & \mathbf{Z}_2 & & 0 \end{array}$$

shows that the generator of C_3^3 has a non-trivial image in $\pi_3(G_3, \mathrm{SO}_3)$. This implies, by Levine [10], that S does not bound in S^6 a framed submanifold. Hence S is not isotopic to the suspension (cf. 6) of a knotted sphere in S^5, because in codimension 2, any knotted sphere is the boundary of a framed submanifold.

On the other hand, any even multiples of S is the boundary of a framed submanifold V in S^6; using the argument of Wall (*Bull. Amer. Math. Soc.*, 71 (1965), 566), we can assume, after framed spherical modifications, that V is union of handles of index $\leqq 2$. Hence it is possible to compress V by an isotopy

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in S^5, so that its boundary is isotopic to S'. We have proved:

5.17. THEOREM. *$\mathbf{F}C_3^3 = \mathbf{Z} + \mathbf{Z}$ and $C_3^3 = \mathbf{Z}$ with generator the one described in* [3]. *The elements of C_3^3 which are the suspension of elements of C_3^2 are exactly the even multiples of the generator.*

6. The suspension sequence

6.1. The *suspension homomorphism* $C_n^q \to C_n^{q+1}$ is defined by associating to the class of $f: S^n \to S^{n+q}$ the class of $i \circ f: S^n \to S^{n+q+1}$, where i is the natural inclusion of S^{n+q} in S^{n+q+1}. *Via* the isomorphism ψ of § 2 and § 3, for $q > 2$, we have to study the homomorphism $\pi_{n+1}(G; \mathrm{SO}, G_q) \to \pi_{n+1}(G; \mathrm{SO}, G_{q+1})$ induced by the inclusion of G_q in G_{q+1}.

6.2. LEMMA. *One has the following exact sequence*:

$$\begin{aligned} &\pi_{n+1}(G_{q+1}; \mathrm{SO}_{q+1}, G_q) \longrightarrow \pi_{n+1}(G; \mathrm{SO}, G_q) \\ &\qquad \longrightarrow \pi_{n+1}(G; \mathrm{SO}, G_{q+1}) \longrightarrow \pi_n(G_{q+1}; \mathrm{SO}_{q+1}, G_q) \ . \end{aligned}$$

PROOF. This sequence, together with the exact sequences of the triads $(G_{q+1}; \mathrm{SO}_{q+1}, G_q)$, $(G; \mathrm{SO}, G_q)$ and $(G; \mathrm{SO}, G_{q+1})$ form the following diagram, commutative up to sign:

$$\begin{array}{ccccccc}
\pi_{n+1}(G_{q+1}; \mathrm{SO}_{q+1}, G_q) & \longrightarrow & & \pi_n(G_q, \mathrm{SO}_q) & & \longrightarrow & \pi_n(G, \mathrm{SO}) \\
& \searrow & \nearrow & & \searrow & \nearrow & \\
& & \pi_{n+1}(G; \mathrm{SO}, G_q) & & \pi_{n+1}(G_{q+1}, \mathrm{SO}_{q+1}) & & \\
& \nearrow & \searrow & & \nearrow & \searrow & \\
\pi_{n+1}(G, \mathrm{SO}) & \longrightarrow & & \pi_{n+1}(G; \mathrm{SO}, G_{q+1}) & & \longrightarrow & \pi_{n+1}(G_{q+1}, \mathrm{SO}_{q+1}, G_q) \ .
\end{array}$$

The exactness of 6.2 follows from the exactness of the three other sequences and the fact that the composion of two consecutive homomorphisms is zero.

6.3. LEMMA. *Let F_q be the space of maps of degree one of S^q onto itself with a fixed point e. One has the following isomorphisms*:

$$\begin{aligned} &\pi_n(F_q, \mathrm{SO}_q) = \pi_n(G_{q+1}, \mathrm{SO}_{q+1}) \\ &\pi_n(F_q, G_q) = \pi_n(F_q; \mathrm{SO}_q, F_{q-1}) = \pi_n(G_{q+1}; \mathrm{SO}_{q+1}, G_q) \ . \end{aligned}$$

PROOF. In the exact sequence associated to the triad $(G_q, \mathrm{SO}_q, F_{q-1})$:

$$\longrightarrow \pi_{n+1}(G_q; \mathrm{SO}_q, F_{q-1}) \longrightarrow \pi_n(\mathrm{SO}_q, \mathrm{SO}_{q-1}) \longrightarrow \pi_n(G_q, F_{q-1}) \longrightarrow \cdots ,$$

the second homomorphism is an isomorphism. Hence $\pi_{n+1}(G_q, \mathrm{SO}_q, F_{q-1}) = 0$. The other homotopy exact sequence of this triad gives the isomorphism

$$\pi_n(F_{q-1}, \mathrm{SO}_{q-1}) = \pi_n(G_q, \mathrm{SO}_q) \ .$$

The exact sequence

$$\longrightarrow \pi_{n+1}(G_q; \mathrm{SO}_q, F_{q-1}) \longrightarrow \pi_{n+1}(F_q; \mathrm{SO}_q, F_{q-1}) \longrightarrow \pi_{n+1}(F_q, G_q) \longrightarrow \cdots$$

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gives the second isomorphism of 6.3.

Exactness of this last sequence is proved by forming a diagram similar to the preceding one (6.2) made up of this sequence, of the exact sequences of the triads $(G_q; SO_q, F_{q-1})$, $(F_q; SO_q, F_{q-1})$, and of the exact sequence of the triple F_q, G_q, SO_q.

The inclusion of the triad $(F_q; SO_q, F_{q-1})$ in $(G_{q+1}; SO_{q+1}, G_q)$ induces a homomorphism of the exact sequences:

$$\begin{array}{ccccc} \pi_{n+1}(F_q; SO_q, F_{q-1}) & \longrightarrow & \pi_n(F_{q-1}, SO_{q-1}) & \longrightarrow & \pi_n(F_q, SO_q) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{n+1}(G_{q+1}; SO_{q+1}, G_q) & \longrightarrow & \pi_n(G_q, SO_q) & \longrightarrow & \pi_n(G_{q+1}, SO_{q+1}) \ . \end{array}$$

The last two vertical homomorphisms are isomorphisms. Hence by the five lemma, so is the first one.

By Lemmas 6.2, 6.3 and Theorem 3.4, we get:

6.4. THEOREM. *One has the following suspension exact sequence* $(q > 2)$

$$\pi_{n+1}(F_q, G_q) \longrightarrow C_n^q \longrightarrow C_n^{q+1} \longrightarrow \pi_n(F_q, G_q) \longrightarrow \cdots \ .$$

Remark. The same sequence is valid if we replace C by θ, namely

(6.5) $$\pi_{n+1}(F_q, G_q) \longrightarrow \theta_n^q \longrightarrow \theta_n^{q+1} \longrightarrow \pi_n(F_q, G_q) \longrightarrow \cdots .$$

Here the first homomorphism is the composition $\pi_{n+1}(F_q, G_q) \to C_n^q \to \theta_n^q$. The last one is the composition

$$\theta_n^{q+1} \longrightarrow \pi_n(G_q, SO_q) \longrightarrow \pi_n(G_{q+1}; SO_{q+1}, G_q) = \pi_n(F_q, G_q) \ .$$

To prove the exactness of 6.5, we consider the following diagram, commutative up to sign, made up of 6.5, 6.4 and 1.9 for q and $q + 1$:

$$\begin{array}{ccccccc} \pi_{n+1}(F_q, G_q) & & \longrightarrow & \theta_n^q & \longrightarrow & & \theta_n \\ & \searrow & & \nearrow \quad \searrow & & \nearrow & \\ & & C_n^q & & \theta_n^{q+1} & & \\ & \nearrow & & \searrow \quad \nearrow & & \searrow & \\ \theta_{n+1} & & \longrightarrow & C_n^{q+1} & \longrightarrow & & \pi_n(F_q, G_q) \ . \end{array}$$

The exactness of 6.5 follows from the exactness of the three other sequences and because the composition $\theta_n^q \to \theta_n^{q+1} \to \pi_n(F_q, G_q)$ is zero; indeed it is equal, up to sign, to the composition

$$\theta_n^q \longrightarrow \pi_n(G_q, SO_q) \longrightarrow \pi_n(G_{q+1}, SO_{q+1}) \longrightarrow \pi_n(G_{q+1}, SO_{q+1}, G_q) = \pi_n(F_q, G_q) \ .$$

The argument used at the end of 5.16 is valid in general and shows that *an element of C_n^3 is the suspension of an element of C_n^2 if and only if its image in $\pi_n(G_3, SO_3)$ is zero.* Note that $\pi_n(G_3, SO_3) = \pi_n(F_2, G_2) = \pi_{n+2}(S^2)$ for $n > 2$.

6.6. COROLLARY (cf. [2]). $C_n^q = 0$ *for* $n < 2q - 3$.

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This follows from the fact that $\pi_{n+1}(F_q, G_q) = 0$ for $n < 2q - 3$ (cf. James [6]; for a geometrical proof, see 7.7) and that $C_n^q = 0$ for q large.

6.7. COROLLARY (*compare Levine* [10; 6.7]). *C_n^q is finite except that C_{4k-1}^q has a* $\mathbf{Z}$*-component for $q \leqq 2k+1$. The suspension tensored by the rationals* $\mathbf{Q}$: $C_{4k-1}^{q-1} \otimes \mathbf{Q} \to C_{4k-1}^q \otimes \mathbf{Q}$ *is an isomorphism for $q \leqq 2k+1$.*

We may check, from the exact sequence

$$\pi_n(F_q, G_q) \longrightarrow \pi_{n-1}(G_q, F_{q-1}) \longrightarrow \pi_{n-1}(F_q, F_{q-1}) \longrightarrow \cdots,$$

and from the known properties of finiteness of $\pi_n(S^{q-1})$, that $\pi_n(F_q, G_q)$ is finite, except that $\pi_{2(q-1)}(F_q, G_q)$ has one free component of rank one for q odd.

6.8. *Remark.* It is easy to check, from the exact sequence 4.6, that the immersion class of any element of infinite order in C_{4k-1}^q is non-trivial for $q < 2k+1$.

6.9. THEOREM. *The elements of C_n^q which are in the kernel of the suspension, are trivial as immersions* ($q > 2$).

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} \pi_{n+1}(G_{q+1}; \mathrm{SO}_{q+1}, G_q) & \longrightarrow & \pi_{n+1}(G; \mathrm{SO}, G_q) = C_n^q \\ \partial \downarrow & & \downarrow \\ \pi_n(\mathrm{SO}_{q+1}, \mathrm{SO}_q) & \xrightarrow{j} & \pi_n(\mathrm{SO}, \mathrm{SO}_q) \quad = \mathrm{Im}_n^q . \end{array}$$

By Lemma 6.2 and 6.1, it is sufficient to prove that $j\partial = 0$. In fact ∂ is already zero. Indeed, in the homotopy exact sequence

$$\longrightarrow \pi_{n+1}(G_{q+1}, \mathrm{SO}_{q+1}, G_q) \longrightarrow \pi_n(\mathrm{SO}_{q+1}, \mathrm{SO}_q) \longrightarrow \pi_n(G_{q+1}, G_q) \longrightarrow \cdots,$$

the second homomorphism is injective, because its composition with the homomorphism $\pi_n(G_{q+1}, G_q) \to \pi_n(G_{q+1}, F_q)$ is an isomorphism.

6.10. COROLLARY (Kervaire [7]). *For $n < 2q - 1$, any element of C_n^q is trivial as immersion.*

Indeed in that range, the suspension of any element of C_n^q is trivial (cf. 6.6).

6.11. *Remark.* For framed embeddings, we have the exact sequence:

$$\longrightarrow \pi_{n+1}(G_{q+1}, G_q) \longrightarrow FC_n^q \longrightarrow FC_n^{q+1} \longrightarrow \pi_n(G_{q+1}, G_q) \longrightarrow \cdots.$$

From Lemma 6.3, we see that $\pi_{n+1}(G_{q+1}, G_q) = \pi_{n+1}(F_q, G_q) + \pi_{n+1}(S^q)$.

7. A cobordism interpretation of $\pi_n(F_q, G_q)$

7.1. *Definition of the groups P_n^q, P_n and Q_n^q.* We consider framed n-submanifolds V of D^{n+q} such that $\partial V \subset \partial D^{n+q}$ is a homotopy sphere. Two such

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framed submanifolds V_0 and V_1 are cobordant if there is a framed submanifold W in $D^{n+q} \times I$ such that:

$$\partial W = (V_0 \times 0) \cup (V_1 \times 1) \cup X ,$$

where X is a framed submanifold of $S^{n+q+1} \times I$ and $V_i \times i$, $i = 0, 1$, is a deformation retract of X. The cobordism classes of such submanifolds form an abelian group with respect to the following sum operation.

We can choose a representative V_0 such that its intersection with the half-ball defined by $\{x \mid x \in D^{n+q}$, with $x_1 \leqq 0\}$ is the half n-ball defined by $\{x \mid x \in D^n, x_1 \leqq 0\}$ with the standard framing; we denote by V_0^+ the intersection of V_0 with the half-ball $\{x \mid x \in D^{n+q}, x_1 \geqq 0\}$. We assume that V_1 satisfies the same conditions. Then we define the sum of the cobordism classes of V_0 and V_1 to be the cobordism class of the framed submanifold V which is the union of V_0^+ with $R_1 V^+$, where R_1 is the rotation defined in 1.4. The fact that this sum operation is well defined for cobordism classes, is commutative and associative, is proved as in § 1.

The zero element is the class of the standard D^n in D^{n+q}. A framed submanifold $V \subset D^{n+q}$ is cobordant to zero if there is, in the half-ball defined by $\{x \mid x \in D^{n+q+1} : x_{n+q+1} \geqq 0\}$, a framed submanifold W such that its boundary is the union of V with a homotopy n-disk in the northern hemisphere of S^{n+q}.

The inverse of the class of $V \subset D^{n+q}$ is the cobordism class of σV, where σ is the symmetry of D^{n+q} with respect to the hyperplane $x_1 = 0$.

This group will be denoted by P_n^q. By natural inclusion of D^{n+q} in D^{n+q+1}, we have the suspension homomorphism $P_n^q \rightarrow P_n^{q+1}$. The limit of P_n^q by iterated suspension is, by definition, the group P_n. The elements of P_n can also be interpreted as cobordism classes of framed submanifold V of $\mathbf{R}^{n+N}$, $N > n + 2$, without the condition that ∂V is contained in ∂D^{n+N}.

The group P_n has been computed by Kervaire-Milnor. We shall not need their results because we shall be interested in the kernel of the stable suspension homomorphism $P_n^q \rightarrow P_n$, which will be denoted by Q_n^q.

We have the exact sequence

$$0 \longrightarrow Q_n^q \longrightarrow P_n^q \longrightarrow P_n \longrightarrow 0 ,$$

and it is well known that this sequence splits.

7.2. *The homomorphism* $P_n^q \rightarrow \pi_n(F_q, G_q)$. We denote by $2D^q$ the disk defined by $|x| \leqq 2$ in $\mathbf{R}^q$. An element of $\pi_n(F_q, G_q)$ can be represented by a map $f: D^n \times 2D^q \rightarrow S^q$, such that, if $f_x: 2D^q \rightarrow S^q$ is defined by $f_x(y) = f(x, y)$,

(i) $f_x(\partial 2D^q) =$ north pole e_{q+1} of S^q,

(ii) for $x \in \partial D^n$, f_x is the radial extension of a map of $\partial D^q = S^{q-1}$ in the equatorial sphere S^{q-1} of S^q,

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(iii) the point e_1 of S^q is a regular value of f.

Then $f^{-1}(e_1)$ is a framed submanifold V' of $D^n \times 2D^q$ such that $\partial V' \subset \partial D^n \times \partial D^q$, the first vector of the framing on $\partial V'$ being normal to $D^n \times S^{q-1}$ and pointing outside.

Now we can identify $D^n \times 2D^q$ to D^{n+q} by a diffeomorphism (except along the edge $\partial D^n \times \partial 2D^q$) whose restriction to $\partial D^n \times D^q$ is the map described in 2.2.

Under this identification, V' will be a framed submanifold of D^{n+q} whose boundary $\partial V'$ is contained in $\partial D^n \times \partial D^q \subset D^{n+q}$, the first vector of the framing on $\partial V'$ being normal to $S^{n-1} \times S^{q-1}$ in S^{n+q-1} and pointing outside of $S^{n-1} \times D^q$.

Conversely, any such framed submanifold of D^{n+q} represents an element of $\pi_n(F_q, G_q)$. Moreover two such framed submanifolds V_0', V_1' of D^{n+q} will represent the same element if there is a framed submanifold W in $D^{n+q} \times I$ such that $\partial W = (V_0' \times 0) \cup (V_1' \times 1) \cup X$, where X is a framed submanifold of $S^{n-1} \times S^{q-1} \times I$.

7.3. The elements of P_n^q can be represented by framed submanifolds V of D^{n+q} such that $\partial V \subset S^{n-1} \times \operatorname{int} D^q \subset \partial D^{n+q}$.

The homomorphism $P_n^q \to \pi_n(F_q, G_q)$ will associate to the cobordism class of V the element represented by a framed submanifold V' in D^{n+q}, like in 7.2, which is characterized by the following condition: there is a framed submanifold W of D^{n+q} such that ∂W is made up of a framed submanifold X in $\partial D^n \times D^q$, of $-V$ and V' (cf. 0.4).

To prove the existence of V, we first construct, as in 2.4–5, the submanifold X in $\partial D^n \times D^q$ such that ∂X is the union of ∂V and of a framed submanifold Y of $\partial D^n \times \partial D^q$. Then W will be the submanifold of D^{n+q} generated by $X \cup V$ pushed inside D^{n+q}, namely V along the first vector of the framing and X along the normal to ∂D^{n+q} in D^{n+q}, Y remaining fixed.

The uniqueness of the class of V' is proved similarly.

7.4. *Remark.* There is a natural homomorphism of P_{n+1}^q in $F\theta_n^q$ (framed homotopy n-spheres in S^{n+q}) obtained by taking the boundary ∂V of the framed submanifold V representing an element of P_{n+1}^q.

We also have a homomorphism of Q_{n+1}^q in FC_n^q, for $n \geqq 5$, defined as follows. A representative V of an element of Q_{n+1}^q is stably cobordant to a framed $(n+1)$-disk Δ^{n+1} in D^{n+N}, N large, and $\partial\Delta^{n+1} = \partial V$. As $\partial\Delta^{n+1}$ is naturally diffeomorphic to S^n up to concordance for $n \geqq 5$ (by a diffeomorphism which can be extended to the interiors of Δ^{n+1} and D^{n+1}), it will define, with its framing, a framed embedding of S^n in S^{n+q}.

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The following diagram is commutative:

$$\begin{array}{ccc} \pi_{n+1}(F_q, G_q) & \longrightarrow & \tilde{\pi}^{n+1}(G; \mathrm{SO}, G_q) \\ \psi \uparrow & & \tilde{\psi} \uparrow \\ Q^q_{n+1} & \longrightarrow & FC^q_n \end{array}$$

where ψ is the restriction to Q^q_{n+1} of the homomorphism $P^q_n \to \pi_n(F_q, G_q)$ defined in 7.3.

7.5. THEOREM. *The homomorphism $\psi: Q^q_n \to \pi_n(F_q, G_q)$ is an isomorphism for $q > 2$ and $n \geqq 5$.*

Proof of surjectivity. Let $V' \subset D^{n+q}$ be a framed submanifold as in 7.2 representing an element of $\pi_n(F_q, G_q)$. We can consider V' as a framed submanifold of D^{n+N} by suspension, N large. By framed spherical modifications (cf. 3.3), which do not touch $\partial V'$, we can construct a framed submanifold V'' of D^{n+N} with $\partial V''$ which is the union of a tubular neighborhood $\partial V'' \times I$ of $\partial V''$, of handles of indices $\leqq (n+1)/2$ and of an n-disk Δ^n. Let V''_0 be the complement in V'' of the interior of Δ^n. For $q > 2$, by general position as in 3.5, we can construct an isotopy g_t: $V''_0 \to D^{n+N}$ connecting the inclusion to an embedding on a submanifold X of $\partial D^n \times D^q$, g_t being fixed on V''. Moreover, g_t can be extended to the framings so that we get on X a framing $f'_1, \cdots, f'_N$, where f'_1 is normal to S^{n+N-1} inside D^{n+N}, $f'_2, \cdots, f'_q$ is the framing of X as a submanifold of $\partial D^n \times D^q$, and $f'_{q+j} = e_{n+q+j}$. The boundary of X is the union of $\partial V'$ and of an $(n-1)$-sphere $g_1(\partial\Delta^n)$.

If we push $V' \cup X$ inside D^{n+q} without moving $g_1(\Delta^n)$, we get a framed submanifold V in D^{n+q} representing an element of P^q_n whose image by the homomorphism ψ is the class of V'. Moreover the class of V is in Q^q_n, because V is stably cobordant to the n-disk which is the union of Δ^n with the cylinder described by $\partial\Delta^n$ during the isotopy g_t.

7.6. *Proof of injectivity.* Let V be a framed submanifold of D^{n+q} representing an element of Q^q_n whose image in $\pi_n(F_q, G_q)$ is zero. Hence if $W \subset D^{n+q}$ is as in 7.3 with $\partial W = V' \cup (-V) \cup X$, there is in $D^{n+q} \times I$ a framed submanifold W' such that $\partial W'$ is the union of a framed submanifold X' in $\partial D^n \times \partial D^q \times I$, of $V' \times 0$ and of the disk $\Delta^n = D^n \times e_1 \times 1$ in

$$D^n \times D^q \times 1 = D^{n+q} \times 1 .$$

We identify $D^{n+q} \times 0$ with D^{n+q} and we consider $Y = X \cup X' \cup \Delta^n$ as a framed submanifold of $(\partial D^{n+q} \times I) \cup (D^{n+q} \times 1) \simeq D^{n+q}$ (we complete the framing of X by adding as first vector the outside normal to $S^{n+q+1} \times I$). The union $W \cup W'$ establishes a cobordism between Y and V, hence Y is stably cobordant to a disk. This means that, in $D^{n+q} \times I^N$, for N large, there is a framed

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submanifold M such that ∂M is the union of the suspension of Y and a disk. We can also assume that M is the union of $Y \times I$ with handles of indices $\leqq n/2 + 1$ attached to $(Y - \Delta^n) \times 1$. Again as in 3.5, we can construct an embedding $g: M \to (D^{n+q} \times 1) \cup (\partial D^n \times D^q \times I)$ with a framing such that

(i) g maps $\Delta^n \times I$ in $D^{n+q} \times 1 = D^n \times D^q \times 1$ by $g(x, e_1, t) = (x, te_1, 1)$

(ii) g maps $M - \Delta^n \times I$ in $\partial D^n \times D^q \times I$

(iii) $g \mid Y$ is the identity and along $g(Y)$ the framing is what is already given.

By pushing slightly $W \cup W' \cup g(M)$ inside $D^{n+q} \times I$ without moving its boundary, we see that V is cobordant in P_n^q to the standard D^n.

7.7. *Remark.* From this isomorphism, we can deduce easily the result of James [6], namely $\pi_n(F_q, G_q) = 0$ for $q > n/2 + 1$.

We have to prove that $Q_n^q = 0$ for $q > n/2 + 1$. Consider a framed submanifold $V \subset D^{n+q}$ representing an element of Q_n^q. By hypothesis there is a framed submanifold W of $D^{n+N} \times I$, N large, such that ∂W is the union of $V \times 0$, $\partial V \times I$ and of a disk in $D^{n+N} \times 1$. We can assume (cf. 3.3) that W is the union of $V \times I$ with handles of indices $\leqq n/2 + 1$. By general position as in 3.5, we can construct an embedding of W on a framed submanifold of $D^{n+q} \times I$ connecting V to a disk in $D^{n+q} \times 1$. Equivalently, V can be transformed in a disk by a sequence of framed spherical modifications of indexes $\leqq n/2 + 1$ in D^{n+N} (cf. 8.4). But all these modifications can be actually performed in D^{n+q} (cf. 8.3).

8. The isomorphism $Q_n^q \approx \pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$ for $n \leqq 3q - 6$.

8.1. *The homomorphism* λ. Let $V \subset D^{n+q}$ be a framed submanifold of dimension n, whose boundary is in ∂D^{n+q}. Let $j: S^{q-1} \to D^{n+q} - V$ be the inclusion of S^{q-1} as the boundary of a fiber of a tubular neighborhood of V, with the orientation given by the framing. The map j induces a homomorphism $j: H_r(S^{q-1}) \to H_r(D^{n+q} - V)$. By Alexander duality, we have

$$H_r(D^{n+q} - V) = H_{r-q+1}(V, \partial V)$$

for $r > 0$. Hence if V and ∂V are k-connected and $q > 2$, then j induces an isomorphism $j_*: \pi_i(S^{q-1}) \to \pi_i(D^{n+q} - V)$ for $i \leqq k + q - 1$.

When V and ∂V are k-connected and $q > 2$, we can define the homomorphism $\lambda: \pi_i(V) \to \pi_i(S^{q-1})$, for $i \leqq k + q - 1$, as follows. Let ν be the map which pushes V along the first vector f_1 of the framing (i.e., $\nu(x) = x + \varepsilon f_1(x)$, where ε is small); ν induces a homomorphism $\nu_*: \pi_i(V) \to \pi_i(D^{n+q} - V)$. We define λ by $\lambda = j_*^{-1} \circ \nu_*$.

8.2. *The function* ξ. Let V be as before, and let $f_1, \cdots, f_q$ be the

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framing of V. We consider an embedding $f\colon S^r \to V$ representing an element $\alpha \in \pi(V)$. Let $\xi \in \pi_r(V_{n+q,r+q})$ be represented by the following map (cf. [3]): to each point $x \in f(S^r)$, we associate the frame $\varepsilon_1(x), \cdots, \varepsilon_{r+1}(x), f_2(x), \cdots, f_q(x)$, where $\varepsilon_1(x), \cdots, \varepsilon_{r+1}(x)$ is the natural trivialization of the vector bundle generated by the tangent bundle of $f(S^r)$ and the field f_1 restricted to $f(S^r)$.

This element ξ is the obstruction to constructing an immersion $\varphi\colon D^{r+1} \to D^{n+q}$ together with a normal framing $F_2, \cdots, F_q$ such that $\varphi = f$ on S^r along $f(S^r)$, φ is tangent to f_1 and $f_i = F_i$, $i \geqq 2$.

If the manifold V is $(2r - n + 2)$-connected, and if $2n \geqq 3r + 3$, then any element of $\pi_r(V)$ is represented by an embedding, and two such embeddings are also regularly homotopic (cf. [2]). Hence under these conditions, the element ξ depends only on the homotopy class of f and we get a map

$$\xi\colon \pi_r(V) \longrightarrow \pi_r(V_{n+q,r+q}) = \pi_r(\mathrm{SO}, \mathrm{SO}_{n-r}) \ .$$

This map is not a homomorphism. In fact we have the same formula as for the function α in Theorem 1 of the paper: C.T.C. Wall, *Classification of handlebodies*, Topology 2 (1963), 253–261. We shall not need it here.

We shall note that ξ is stable, i.e., independent of q.

8.3. *Spherical modifications of framed submanifolds.* Let

$$P\colon D^k \times D^{n+1-k} \longrightarrow D^k \times D^{n+1-k}$$

be the injective map defined by $P(x, y) = (x\delta(y^2), y\delta(x^2))$, where $\delta(t)$ is an even function differentiable, such that $\delta(0) = 1/2$, $\delta(t) = 1$ for $t \geqq 1$, $\delta'(t) > 0$ for $0 < t < 1$. Except on the edge $\partial D^k \times \partial D^{n+1-k}$, P is differentiable of rank $n + 1$; its restriction to $\partial D^k \times D^{n+1-k}$ or to $D^k \times \partial D^{n+1-k}$ is a differentiable embedding. We shall denote by ν (resp. ν') the field of unit normal vectors along $P(\partial D^k \times D^{n+1-k})$ (resp. $P(D^k \times \partial D^{n+1-k})$) pointing inside (resp. outside) the image of P.

Let V be a framed n-submanifold of D^{n+q} with a framing $f = f_1, \cdots, f_q$. Let ϕ be a map of $D^k \times D^{n+1-k}$ in the interior of D^{n+q} which is the composition of the map P with a differentiable embedding ϕ_0 of $D^k \times D^{n+1-k}$ in D^{n+q}; suppose that

(i) $\phi(\partial D^k \times D^{n+1-k}) \subset V$,

(ii) $\phi(\operatorname{int} D^k \times D^{n+1-k}) \cap V = \varnothing$,

(iii) the field f_1 along $\phi(\partial D^k \times D^{n+1-k})$ is the image of ν by the differential of ϕ_0.

We suppose moreover that a framing $F = F_2, \cdots, F_q$ of $\phi(D^k \times D^{n+1-k})$ is given such that:

(iv) $F_i = f_i$ along $\phi(\partial D^k \times D^{n+1-k})$.

The couple (ϕ, F), or simply ϕ with F understood, is called *a handle of in-*

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dex k attached to the framed submanifold V, or a framed spherical modification of V of index k, killing the element of $\pi_{k-1}(V)$ represented by $\phi(S^{k-1} \times 0)$.

Let V' be the submanifold $[V - \phi(\partial D^k \times D^{n+1-k})] \cup \phi(D^k \times \partial D^{n+1-k})$ and let $f' = f'_1, \cdots, f'_q$ be the framing of V' equal to f on $V \cap V'$, such that $f'_i = F_i$, $i > 1$, on $\phi(D^k \times \partial D^{n+1-k})$ and that f'_1 is the image of ν' by the differential of ϕ. Note that everything is smooth along $\phi(\partial D^k \times \partial D^{n+1-k})$.

We shall say that the framed submanifold V' is obtained from V by a framed spherical modification of index k defined by the handle ϕ.

Note that V is obtained from V' by a spherical modification of index $n + 1 - k$, also defined by ϕ.

It is clear that V and V' are framed cobordant (cf. [3]).

If $\alpha \in \pi_{k-1}(V)$, and if $\xi(\alpha)$ and $\lambda(\alpha)$ are defined and equal to zero, then we have seen in [3] that it is always possible to perform a framed spherical modification on V killing α.

8.4. Finally we indicate the relations between spherical modifications and handle decomposition. Let W be a framed submanifold in $D^{n+N} \times I$, N large, such that ∂W is the union of $V \times 0$, $V' \times 1$ and $\partial V \times I$, where V and V' are framed submanifolds of D^{n+N}. Assume that W is obtained from $V \times I$ by attaching s handles of indexes k_i. Then V' is isotopic, with its framing, to a manifold obtained from V by a corresponding sequence of s framed spherical modifications of indexes k_i. Indeed we can construct an embedding f of W on a framed submanifold W' of D^{n+N} such that $f \mid V \times 0$ is the natural map on V, that $f(\partial V \times I) \subset \partial D^{n+N}$ and that the suspension of $W' \times 0$ in $D^{n+N} \times I$ is isotopic to W with its framing. As N is large, $f(V' \times 1)$, with its framing completed by the exterior normal to W' as first vector, is isotopic to V' with its given framing. On the other hand, it is clear that we pass from $f(V \times 1)$ to $f(V' \times 1)$ by a sequence of spherical modifications defined by the handles of the decomposition of W.

8.5. LEMMA 1. *For $q - 1 \leqq n/2 \leqq 2q - 3$, each element of Q^q_n can be represented by a framed submanifold $V \subset D^{n+q}$ whose stable suspension can be transformed in a disk by just one framed spherical modification of index q which kills an element $\alpha \in \pi_{q-1}(V)$ with $\lambda(\alpha) = 1$.*

PROOF. After framed spherical modifications, we can assume that V is $(q-2)$-connected. Moreover we can assume that $\lambda \colon \pi_{q-1}(V) \to \pi_{q-1}(S^{q-1}) = \mathbf{Z}$ is surjective, after an eventual framed spherical modification of index $q - 1$. Indeed we can attach trivially to V a handle ϕ such that $\phi(D^{q-1} \times 0)$ is obtained by joining, with a small tube, V to a $(q-1)$-sphere in $D^{n+q} - V$ with linking number 1 with V.

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As V is stably cobordant to D^n, there exists a framed submanifold W of D^{n+N} such that $\partial W = V \cup n$-disk. After spherical modifications, we can assume $\pi_i(W) = 0$ for $i \leqq (n-1)/2$. Hence by Smale [14] and 8.4, we can transform $V \subset D^{n+N}$ to an n-disk by a sequence of $s+1$ framed spherical modifications of indexes r, with $q \leqq r \leqq n/2 + 1$. We can also assume that the handle ϕ defining the first modification is attached by a map $S^{q-1} \times D^{n-q+1} \to V$ which defines an element $\alpha \in \pi_{q-1}(V)$ with $\lambda(\alpha) = 1$.

Now we argue by induction on the number s. It will be sufficient to prove that V is cobordant to V', where V' verifies the same conditions as V, but s being replaced by $s-1$ (for $s > 1$).

Let $V_0 = V - \phi(S^{q-1} \times \operatorname{int} D^{n+1-q})$ and $\bar{V} = V_0 \cup \phi(D^q \times \partial D^{n-q+1})$. We want to prove that we have a split exact sequence:

$$0 \longrightarrow \pi_i(S^{q-1}) \longrightarrow \pi_i(V_0) \longrightarrow \pi_i(\bar{V}) \longrightarrow 0 \tag{8.6}$$

for $i \leqq 2q - 3$, where the first homomorphism is induced by the map $x \to \phi(x, e)$ of S^{q-1} in V_0, $e \in \partial D^{n-q+1}$, and the second one by inclusion. For that, we note that the pair $(D^q, \partial D^q)$ is mapped by ϕ in the pair $(\bar{V}, V_0)$. Hence we have the commutative diagram:

$$\begin{array}{ccccccc}
\pi_{i+1}(\bar{V}) & \longrightarrow & \pi_{i+1}(\bar{V}, V_0) & \longrightarrow & \pi_i(V_0) & \longrightarrow & \pi_i(\bar{V}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\pi_{i+1}(D^q) & \longrightarrow & \pi_{i+1}(D^q, \partial D^q) & \longrightarrow & \pi_i(\partial D^q) & \longrightarrow & \pi_i(D^q) \, .
\end{array}$$

The homomorphism $\pi_{i+1}(D^q, \partial D^q) \to \pi_{i+1}(\bar{V}, V_0)$ is surjective for $i \leqq 2q - 3$, because $\pi_{i+1}(\bar{V}, V_0, D^q) = 0$, as the pairs $(V_0, \partial D^q)$ and $(D^q, \partial D^q)$ are $(q-1)$-connected (cf. [1]). On the other hand, the homomorphism $\pi_i(\partial D^q) \to \pi_i(V_0)$ is injective for $i \leqq 2q - 3$, because its composition with λ is the identity.

The second handle ϕ' is attached to $\bar{V}$ by the embedding $\phi'\colon \partial D^k \times D^{n-k+1} \to \bar{V}$, and $k \leqq n/2 + 1$. By 8.6, we can assume, after an isotopy, that $\phi'(\partial D^k \times D^{n+1-k}) \subset V_0$; this implies that the spherical modifications ϕ and ϕ' can be exchanged stably. Moreover we can assume, by 8.6, that the element of $\pi_{k-1}(V)$ represented by $\phi'(\partial D^k \times 0)$ is in the kernel of λ. Hence we can construct an embedding $\varphi\colon D^k \to D^{n+q}$ such that $\varphi(x) = \varphi'(x, 0)$ for $x \in \partial D^k$, $\varphi(\operatorname{int} D^k) \cap V = \emptyset$, and φ is tangent along $\varphi(\partial D^k)$ to the first vector of the framing of V. We can construct an isotopy $\phi_t\colon D^k \times D^{n+1-k} \to D^{n+N}$ of the second handle ϕ', fixed on $\partial D^k \times D^{n+1-k}$, such that ϕ_1 is the suspension of a handle in D^{n+q}, i.e., $\phi_1(D^k \times D^{n+1-k}) \subset D^{n+q}$ and the last $N - q$ vectors of the framing are the restrictions of the natural framing of D^{n+q} in D^{n+N}. First we construct the isotopy on $D^k \times 0$ so that $\phi_1(x, 0) = \varphi(x)$ for $x \in D^k$. Then we extend it to the normal framing as in 3.5, and finally to $D^k \times D^{n+1-k}$.

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Now the framed submanifold V' of D^{n+q} deduced from V by the spherical modification ϕ_1 satisfies the same conditions as V, but s is replaced by $s-1$. Note that the value of λ on the element of $\pi_{q-1}(V')$ represented by $\phi(S^{q-1}\times 0)$ is still one.

8.7. LEMMA 2. *Let V be a framed submanifold of D^{n+q} satisfying the conditions of Lemma 1 and representing the zero element of Q^q_n. Then there is a framed submanifold W of the half-ball D^{n+q+1}_N defined by $\{x \mid x\in D^{n+q+1}, x_{n+q+1}\geqq 0\}$ whose boundary is the union of V with a disk in the northern hemisphere of S^{n+q}, and such that the map $S^{q-1}\to V\subset W$ representing α is a homotopy equivalence of S^{q-1} with W.*

PROOF. The existence of $W\subset D^{n+q+1}_N$, without this last condition, follows from the vanishing of the class of V in Q^q_n. If we consider W as a framed submanifold of R^{n+N}, N large, by the hypothesis of Lemma 1, we can glue a handle $\phi: D^q\times D^{n+1-q}\to R^{n+N}$ to W along V so that $W\cup\phi(D^q\times D^{n+1-q})$ is a framed submanifold $\bar{W}$ of R^{n+N} whose boundary is a homotopy n-sphere.

We can assume that $\bar{W}$ is cobordant to 0 as an element of P_{n+1}, or equivalently that there is a sequence of s spherical modifications of indexes $r\leqq(n+3)/2$ transforming $\bar{W}$ in an $(n+1)$-disk. If this would not be the case, we could change W as follows. Let W_- be a framed submanifold of $\mathbf{R}^{n+N}$ with boundary a homotopy n-sphere and which represents in P_{n+1} the opposite of the element represented by $\bar{W}$. We can choose W_- such that $\pi_i(W_-)=0$ for $i\leqq(n-1)/2$. Hence by the general position argument we often use here, W_- is framed isotopic in R^{n+N} to a submanifold which is the suspension of a framed submanifold W' in D^{n+q+1}_N, such that $W\cap W'=\emptyset$ and $\partial W'$ is contained in the northern hemisphere of S^{n+q}. We can obtain the new W as the connected sum of W and W' by means of a small half-tube (cf. for instance [3, p. 463]).

The rest of the proof is very similar to the proof of Lemma 1. We have a homomorphism $\Lambda:\pi_i(W)\to\pi_i(S^{q-1})$ defined as in 8.1, and λ is the composition of the homomorphism $\pi_i(V)\to\pi_i(W)$ with Λ. We also have an exact sequence

$$0\longrightarrow\pi_i(S^{q-1})\longrightarrow\pi_i(W)\longrightarrow\pi_i(\bar{W})\longrightarrow 0\ ,$$

where the first homomorphism is induced by the map $x\to\phi(x,0)$ of S^{q-1} in $V\subset W$ and is an inverse of Λ. From that we see that the stable framed spherical modifications of $\bar{W}$ can be performed on W itself in D^{n+q+1}_N. Finally W will be such that $W\cup\phi(D^q\times D^{n+1-q})$ is an $(n+1)$-disk. Hence W has the homotopy type of S^{q-1}.

8.8. THEOREM. *For $n\leqq 3q-6$, there is a homomorphism*

$$\Xi: Q^q_n\longrightarrow\pi_{n-q+1}(\mathrm{SO},\mathrm{SO}_{q-1})\ .$$

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PROOF. We consider a framed submanifold V of D^{n+q} as in Lemma 1. We first prove that there is an element $\beta \in \pi_{n-q+1}(V)$ such that $\lambda(\beta) = 0$, and β has intersection number $+1$ with some element $\alpha \in \pi_{q-1}(V)$ for which $\lambda(\alpha) = 1$. The element β is unique, except if $n/2 = q - 1$, where it is defined up to sign.

In the case $q - 1 < n/2$, we have the split exact sequence:

$$0 \longrightarrow \pi_{n-q+1}(S^{q-1}) \longrightarrow \pi_{n-q+1}(V) \longrightarrow H_{n-q-1}(V) \longrightarrow 0 ,$$

where the first homomorphism is induced by an embedding $j: S^{q-1} \to V$ representing the unique element $\alpha \in \pi_{q-1}(V)$ for which $\lambda(\alpha) = 1$. The second one is the Hurewicz homomorphism. Indeed, if we write the homotopy exact sequence of the pair $(V, j(S^{q-1}))$, we have:

$$0 \longrightarrow \pi_i(S^{q-1}) \longrightarrow \pi_i(V) \longrightarrow \pi_i(V, j(S^{q-1})) \longrightarrow 0 ,$$

for $i \leqq 2q - 3$, because λ is an inverse for j_*. Moreover $H_i(V, j(S^{q-1})) = 0$ for $i < n - q + 1$ and isomorphic to $H_{n-q+1}(V) = \mathbf{Z}$ for $i = n - q + 1$. Hence this group is isomorphic to $\pi_i(V, j(S^{q-1}))$ for $i \leqq n - q + 1$. The element β is then the unique element which is in the kernel of λ and whose image by the Hurewicz homomorphism is the dual of α.

When $q - 1 = n/2$, then $\pi_{q-1}(V) = \pi_{n-q+1}(V) = H_{n/2}(V) = \mathbf{Z} + \mathbf{Z}$, and the existence of β follows from the fact that $\lambda: \pi_{q-1}(V) \to \pi_{q-1}(S^{q-1}) = \mathbf{Z}$ is surjective.

According to 8.2, for $n \leqq 3q - 6$, we define

$$\xi(V) = \xi(\beta) \in \pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1}) .$$

This element if well defined, because $\xi(\beta) = \xi(-\beta)$ when $q - 1 = n/2$. This can be checked directly from the definition of ξ.

8.9. *If V is cobordant to zero, then* $\xi(V) = 0$. Indeed in Lemma 2, the element $\beta \in \pi_{n-q+1}(V)$ has a trivial image in $\pi_{n-q+1}(W)$, because in the commutative diagram

$$\begin{array}{ccc} \pi_{n-q+1}(V) & \searrow^{\lambda} & \\ \downarrow & & \pi_{n-q+1}(S^{q-1}) \\ \pi_{n-q+1}(W) & \nearrow_{\Lambda} & \end{array}$$

Λ is an isomorphism and β is in the kernel of λ. An embedding φ representing β can be extended as an immersion $\bar{\varphi}: D^{n-q+2} \to W$. We can see it by [2] or as follows. We have $H_i(W, V) = 0$ for $i \neq n - q + 2$ and

$$H_{n-q+2}(W, V) = \pi_{n-q+2}(W, V) = \mathbf{Z} .$$

The element β is a generator of the kernel of λ, hence it is the image of a generator of the kernel of λ, hence it is the image of a generator γ of $\pi_{n-q+2}(W, V)$.

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By Smale, W is diffeomorphic to $V \times I$ with a handle $\phi: D^{n-q+2} \times D^{q-1} \rightarrow W$, where $\phi(D^{n-q+2} \times 0)$ represents β; this gives the embedding $\bar{\varphi}$. Now consider the field of $(n+2)$-frames $\varepsilon_1, \cdots, \varepsilon_{n-q+2}, f_1, \cdots, f_q$ along $\bar{\varphi}(D^{n+q+2})$ in $\mathbf{R}^{n+q+1}$, where $\varepsilon_1, \cdots, \varepsilon_{n-q+2}$ is the image, by the differential of $\bar{\varphi}$, of the tangent framing of D^{n-q+2}, and $f_1, \cdots, f_q$ the normal framing of W. It is easy to check that this map of D^{n-q+2} in the Stiefel manifold $V_{n+q+1,n+2}$ restricted to ∂D^{n-q+2} represents, up to sign, the suspension of $\xi(\beta)$.

8.10. To prove that $\xi(V)$ depends only on the cobordism class of V and gives a homomorphism Ξ of Q_n^q into $\pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$, it will be sufficient to show the following: *if V_1 and V_2 are as in Lemma* 1, *then $V_1 + V_2$ is cobordant to a framed submanifold V as in Lemma* 1, *and such that*

$$\xi(V) = \xi(V_1) + \xi(V_2) .$$

Let $\alpha_i \in \pi_{q-1}(V_i)$ and $\beta_i \in \pi_{n-q+1}(V_i)$, $i = 1, 2$, such that $\lambda(\alpha_i) = 1$, $\xi(\alpha_i) = 0$, $\lambda(\alpha_i) = 0$, and $(\alpha_i, \beta_i) = 1$ (for $\mu \in \pi_r(V)$ and $\nu \in \pi_{n-r}(V)$, the integer (μ, ν) denotes the intersection of the homology classes represented by them). The group $\pi_k(V_1) + \pi_k(V_2)$ is naturally isomorphic to $\pi_k(V_1 + V_2)$. We can perform a framed spherical modification ϕ of index q on $V_1 + V_2$ which kills $\alpha_1 - \alpha_2$, because $\lambda(\alpha_1 - \alpha_2) = \lambda(\alpha_1) - \lambda(\alpha_2) = 0$, and $\xi(\alpha_1 - \alpha_2) = \xi(\alpha_1) - \xi(\alpha_2) = 0$. We obtain a framed submanifold $V = (V_1 + V_2) - \phi(D^q \times \operatorname{int} D^{n+1-q})$ which satisfies the conditions of Lemma 1. Indeed we can represent α_1 by an embedded sphere in $V_0 = (V_1 + V_2) - \phi(\partial D^q \times \operatorname{int} D^{n+1-q})$ because $(\alpha_1, \alpha_1 - \alpha_2) = 0$ ($\xi(\alpha_1) = 0$ implies $(\alpha_1, \alpha_1) = 0$). This sphere represents an element $\alpha \in \pi_{q-1}(V)$ such that $\lambda(\alpha) = 1$, and $\xi(\alpha) = 0$. We can also represent $\beta_1 + \beta_2$ by an embedded sphere in V_0, because $(\beta_1 + \beta_2, \alpha_1 - \alpha_2) = 0$. This sphere represents an element $\beta \in \pi_{n-q+1}(V)$ such that $(\alpha, \beta) = 1$. It is easy to check that $\lambda(\beta) = 0$ because $\lambda(\beta) = \lambda(\beta_1 + \beta_2)$. Moreover, $\xi(\beta) = \xi(\beta_1) + \xi(\beta_2)$; indeed the sphere which represents β is regularly homotopic in $V_1 + V_2$ to a sphere obtained by joining with a tube two embedded spheres representing β_1 and β_2.

8.11. THEOREM. *The homomorphism $\Xi: Q_n^q \rightarrow \pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$ defined above for $n \leqq 3q - 6$ is an isomorphism. Hence, by* 7.5,

$$\pi_n(F_q, G_q) = \pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1}) \qquad \textit{for } n \leqq 3q - 6 .$$

PROOF. It is immediate to prove that Ξ is injective. Indeed let $V \subset D^{n+q}$ be as in Lemma 1, and suppose that $\xi(\beta) = 0$. As $\lambda(\beta) = 0$, we can perform a spherical modification on V defined by a handle which kills β (cf. [3]). The manifold V' we obtain is an n-disk, because its homology is trivial.

To prove surjectivity, we construct an explicit framed submanifold V in D^{n+q} such that $\xi(V)$ is a given element of $\pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$.

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8.12. We give a construction in a more general setting, because it is useful in the study of links. Let a, b, c be three positive integers. Let D_1 and D_2 be two disks of dimension a embedded in D^{a+1}, such that D_1 and D_2 are in D^{a+1}, and D_1 and D_2 intersect orthogonally along a $(q-1)$-sphere S in the interior or D^{a+1}. Let Δ_i be the a-disk in D_i bounded by S, $i = 1, 2$.

We can represent any element of $\pi_a(\mathrm{SO}_{b+c}, \mathrm{SO}_b)$ by a differentiable map $\zeta\colon D_1 \to \mathrm{SO}_{b+c}$ such that $\zeta(x) \in \mathrm{SO}_b$ for $x \in D_1 - \Delta_1$. Let D_1^{a+b} be the image of $D_1 \times D^b$ in $D^{a+1} \times D^{b+c}$ by the embedding P defined by $P(x, y) = (x, \zeta(x)y)$, where D^b is identified to the disk in D^{b+c} defined by $x_{b+1} = \cdots = x_c = 0$; $\zeta(x)y$ is the natural action of SO_{b+c} on D^{b+c}.

On the other hand, let D_2^{a+c} be the image of $D_1^a \times D^c$ in $D^{a+1} \times D^{b+c}$ by the embedding $Q(x, y) = (x, y)$, where D^c is identified here to the disk in D^{b+c} defined by $x_1 = \cdots = x_b = 0$.

D_1^{a+b} is a twisted band and D_2^{a+c} is a straight band in $D^{a+1} \times D^{b+c}$; they intersect transversally along $S \times 0$. The boundaries of these two bands are two spheres S_1^{a+b-1} and S_2^{a+b-1} disjointly embedded in $S^{a+b+c} = \partial(D^{a+1} \times D^{b+c})$. The reader may check that S_2^{a+c-1} represents in $S^{a+b+c} - S_2^{a+c-1} \approx S^b$ the image, up to sign, of $\partial\gamma \in \pi_{a-1}(\mathrm{SO}_b)$ by the J-homomorphism: $\pi_{a-1}(\mathrm{SO}_b) \to \pi_{a+b-1}(S^b)$, where γ is the element of $\pi_a(\mathrm{SO}_{b+c}, \mathrm{SO}_b)$ represented by ζ.

8.13. Let ν_1 be the unit vector field normal to D_1 in D^{a+1} and pointing inside Δ_2 along S; same definition for ν_2, with 1 and 2 interchanged. We consider D_1^{a+b} as a framed submanifold, with framing $f_1, \cdots, f_{c+1}$, where

$$f_1(x, y) = \big(\nu_1(x), 0\big)$$
$$f_{i+1}(x, y) = \big(0, \zeta(x)e_{b+i}\big)$$

(as before, $e_1, \cdots, e_k$ is the natural basis of R^k).

We perform a framed spherical modification on D_1^{a+b} defined by a handle $\phi\colon \Delta_2 \times D^{b+1} \to D^{a+1} \times D^{b+c}$ such that

(i) $\phi \mid \Delta_2 \times 0 =$ identity and $\phi(\Delta_2 \times D^{b+1}) \cap (D^{a+1} \times 0) = \phi(\Delta_2 \times D^1)$,

(ii) the normal framing $F_2, \cdots, F_{c+1}$ along $\phi(\Delta_2 \times 0)$ is defined by

$$F_{i+1}(x, 0) = (0, e_{b+i}) ,$$

(iii) $D_1^{a+b} - \phi(\Delta_2 \times \operatorname{int} D^{b+1}) = V_1$ has an empty intersection with D_2^{a+b}.

After this framed spherical modification, we get a framed submanifold V_1. Let $\alpha \in \pi_b(V_1)$ be represented by $\phi(x_0 \times \partial D^{b+1})$; this sphere pushed along the first vector of the framing bounds a disk which does not intersect V_1 and does intersect D_2^{a+b} in one point and transversally. Let $\beta \in \pi_a(V_1)$ be represented by the intersection with V_1 of the $(n+1)$-disk in D^{a+1} bounded by $\Delta_1 \cup \Delta_2$. It is clear that this intersection, pushed away from V_1 along the first vector of the

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framing, bounds a disk which does not meet V_1 and D_2^{a+c}. The element $\xi(\beta)$, whenever it is defined, is represented by the map of $\Delta_1 \cup \Delta_2$ into the Stiefel manifold of $(a + c + 1)$-frames in $\mathbf{R}^{a+1} \times \mathbf{R}^{b+c}$ associating to $x \in \Delta_1$ the frame $(e_1, 0), \cdots, (e_{a+1}, 0), (0, \zeta(x)e_{b+1}), \cdots, (0, \zeta(x)e_{b+c})$, and to $x \in \Delta_2$ the frame

$$(e_1, 0), \cdots, (e_{a+1}, 0), (0, e_{b+1}), \cdots, (0, e_{b+c}) .$$

It follows that $\xi(\beta) \in \pi_a(V_{a+b+c+1, a+c+1}) = \pi_a(\mathrm{SO}, \mathrm{SO}_b)$ is equal, up to an automorphism, to the suspension of the element of $\pi_a(\mathrm{SO}_{b+c}, \mathrm{SO}_b)$ represented by ζ. Hence the element $\xi(\beta)$ can be given in advance.

Now choose $a = n - q + 1$, $b = c = q - 1$, and assume $n \leqq 3q - 6$, so that $\xi(\beta)$ is well defined. Let V be the framed submanifold in D^{n+q} obtained by joining V_1 to D_2^{a+c} with a small half-tube along a path in ∂D^{n+q}. After corners have been rounded, it is clear that V satisfies the conditions of Lemma 1, and that $\xi(V)$ is a given element of $\pi_{n-q+1}(\mathrm{SO}, \mathrm{SO}_{q-1})$.

Recall that the boundary of V is an $(n-1)$ sphere embedded in S^{n+q-1} and whose suspension is trivial (cf. 7.4 and 6.4).

8.14. COROLLARY. *For $d \geqq 2$, then*

$$C_{2d-1}^{d+1} = \begin{cases} \mathbf{Z} & d \text{ even} \\ \mathbf{Z}_2 & d \text{ odd} . \end{cases}$$

This follows from 6.4, 6.6, 8.11 for $d > 2$, because $\pi_d(\mathrm{SO}, \mathrm{SO}_d) = \mathbf{Z}$ or $\mathbf{Z}_2$ according that d is even or odd, and $\pi_i(\mathrm{SO}, \mathrm{SO}_d) = 0$ for $i < d$. For the case $d = 2$, see 5.16.

8.15. COROLLARY (cf. [4]). *For $d \geqq 3$, one has a surjective homomorphism*

$$\pi_{d+1}(\mathrm{SO}, \mathrm{SO}_d) \longrightarrow C_{2d}^{d+1} \longrightarrow 0 .$$

In particular

$$C_{4k-2}^{8k} = 0 \qquad \text{for all } k .$$

For $d > 4$, this follows from 6.4, 6.6, 8.11. Recall that $\pi_{d+1}(\mathrm{SO}, \mathrm{SO}_d) = 0$, $\mathbf{Z}_2$, $\mathbf{Z}_2 + \mathbf{Z}_2$ or $\mathbf{Z}_4$, according that $d \equiv -1, +1, 0$ and $2 \pmod 4$ (see for instance Paechter [12]).

For the case $d = 3$, we check that $\pi_7(F_4, G_4) = 0$ by an easy direct verification, so that $C_6^4 = 0$.

Note that $C_4^3 = \mathbf{Z}_{12}$.

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