The Image of J as a Space Mod p

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The J-homomorphism of Whitehead [10] was originally defined for homotopy groups, but it has an important interpretation in terms of fibrations. The (classical) classification theorem for vector bundles reads:

<u>Theorem.</u> The equivalence classes of n-plane bundles over a CW-complex X are in one-to-one correspondence with the set of homotopy classes $[X, BO(n)]$.

Two vector bundles over X are called fibre homotopy equivalent if their sphere bundles are homotopy equivalent via fibre wise maps and homotopies. Let $G(n)$ denote the topological monoid of homotopy equivalences $S^n \longrightarrow S^n$ and let $F(n)$ denote the submonoid of base point preserving ones. Dold and Lashof showed:

<u>Theorem.</u> Fibre homotopy equivalence classes of n-plane bundles over a CW-complex X are in one-to-one correspondence with the image of $[X, BO(n)] \longrightarrow [X, BG(n-1)]$.

As for $[X, BG(n)]$, I have shown it classifies spherical fibrations $S^n \longrightarrow E \longrightarrow X$.

Suspension induces a map $G(n) \longrightarrow F(n+1)$. Taking $X = S^q$ and noting $F(n+1)$ is two components of $\Omega^{n+1}S^{n+1}$ we can analyze the above map.

$$\begin{array}{ccccc} \pi_q(BO(n)) & \longrightarrow & \pi_q(BG(n-1)) & & \\ \wr\wr & & \wr\wr & & \\ \pi_{q-1}(O(n)) & \longrightarrow & \pi_{q-1}(G(n-1)) & \longrightarrow & \pi_{q-1}(F(n)) \approx \pi_{q+n-1}(S^n) . \end{array}$$

The lower composite is Whitehead's original J-homomorphism [10].

For q small with respect to n, the groups and homomorphism are independent of n. This is the stable J-homomorphism

$$J: \pi_{q-1}(0) \longrightarrow \pi^s_{q-1}, \text{ the stable } (q-1)\text{-stem.}$$

Except for a factor of two in certain dimensions, the image of $\pi_{q-1}(0)$ is known. It is in fact a direct summand. There is some evidence to indicate that this image is realized by a subspace of $BG = BF$.

Conjecture: There exists a space BJ and maps $BO \longrightarrow BJ \longrightarrow BF$ such that

1) the composite is the usual map
2) $\pi_i(BO) \longrightarrow \pi_i(BJ)$ is onto
3) $\pi_i(BJ) \longrightarrow \pi_i(BF)$ is mono.

Since $\pi_i(BF)$ is finite for each i, we can define spaces BF_p, p prime, such that BF_p has only p-primary homotopy and $BF \simeq \prod_p BF_p$. If BJ existed, it would have the same sort of decomposition, so we attach the conjecture in p-primary parts.

Conjecture mod p. There exists a space BJ_p and maps $BO \longrightarrow BJ_p \longrightarrow BF_p$ such that

1) $\pi_i(BO) \longrightarrow \pi_i(BJ_p)$ is onto
2) $\pi_i(BJ_p) \longrightarrow \pi_i(BF_p)$ is mono
3) $\pi_i(BO) \longrightarrow \pi_i(BF_p)$ is the p-primary component of the J-homomorphism.

Since this research was initiated, Clough has attacked the case $p = 2$ [3]. This paper is primarily concerned with $p > 2$, but an effort is made to indicate how much of the argument is the same for $p = 2$.

<u>Theorem 1.</u> For $p > 2$, there exists BJ_p and a map $BO \longrightarrow BJ_p$ such that $\pi_i(BO)$ maps onto $\pi_i(BJ_p)$ which is isomorphic with the p-primary part of $\mathrm{Im}(\pi_i(BO) \longrightarrow \pi_i(BF))$.

These particular spaces BJ_p were suggested to me by Frank Adams.

This theorem depends on our complete knowledge of the p-primary component of the image of the J-homomorphism. Our next concern is to discover the cohomology of this space, which turns out to have a particularly simple structure related to known characteristic classes of spherical fibrations.

For any orientable spherical fibration $S^{n-1} \longrightarrow E \xrightarrow{\pi} B$, we have a Thom isomorphism $\phi\colon H^i(B) \longrightarrow H^{i+n}(CE \cup_\pi B)$. With Z_p coefficients, $p > 2$, there are characteristic Wu classes $q_i \in H^{2i(p-1)}(B;Z_p)$, analogous to Stiefel-Whitney classes, defined by

$$q_i = \phi^{-1} \mathcal{P}^i \phi(1)$$

where $\mathcal{P}^i$ is the i-th Steenrod reduced power operation. In particular q_i is defined in terms of the universal example as a class in $H^*(BSF;Z_p)$ or $H^*(BSO;Z_p)$. Here the letter S is used to refer to the component of the identity of O or F. With Z_p coefficients, $H^*(BSF;Z_p) \approx H^*(BF;Z_p)$, as remarked by Milnor [5], since $BF \simeq BSF \times K(Z_2,1)$. Finally we have need of the Bockstein (connecting) homomorphism $\beta: H^i(;Z_p) \longrightarrow H^{i+1}(;Z_p)$ derived from the coefficient sequence

$$0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0 .$$

Theorem 2. For $p > 2$ and $i > 0$ there are non-trivial classes $r_i \in H^{2i(p-1)}(BJ_p;Z_p)$ with non-trivial Bocksteins βr_i such that r_i maps to q_i in BSO and

$$H^*(BJ_p;Z_p) \approx Z_p[r_i] \otimes E(\beta r_i) .$$

We also describe the k-invariants of BJ_p and relate them to a certain cyclic exact sequence of Toda [9, p.140].

§1. Localization: The p-primary parts of a space.

Given a space X with $\pi_i(X)$ finite for each i, there are spaces X_p for each prime p, such that $X = \prod_p X_p$ and $\pi_i(X_p)$ is isomorphic to the p-primary part of $\pi_i(X)$. These spaces can be defined functorially as follows:

For any homotopy associative, homotopy commutative H-space X , the set of homotopy classes $[\ ,X]$ is a representable functor, and it follows easily that $[\ ,X] \otimes Q_p$ is also where Q_p denotes the subgroup of the rationals consisting of those which expressed in lowest terms have deonominators prime to p . Let X_p represent $[\ ,X] \otimes Q_p$. The stated properties follow easily if $\pi_i(X)$ is finite.

Another case of interest is BO_p . For p odd, there is a space W_p such that $BO_p \simeq \prod_{i=o}^{\frac{p-1}{2}} \Omega^{4i} W_p$ [7, p.299] and $H^*(W_p;Z_p) \approx Z_p[q_i | t \geq 1]$.

§2. The Adams maps and BJ_p for all p.

The space BO represents $\widetilde{KO}$-theory, at least on finite complexes. J. F. Adams has described operations ψ^k in KO-theory which are represented by maps ψ^k: BO $\longrightarrow$ BO . We are interested in the map ψ^k-1 or rather would be interested in a desuspension mapping BBO $\longrightarrow$ BBO , except that such does not exist. By Bott periodicity, we know BBO exists, namely it can be defined to be Ω^7 of a suitably connected covering of BO . Adams shows that under Bott periodicity we have

$$\begin{array}{ccc} KO(S^8X) & \xrightarrow{\psi^k} & KO(S^8X) \\ \wr\wr & & \wr\wr \\ KO(X) & \xrightarrow{k^4\psi^k} & KO(X) \end{array} ,$$

which means that $k^4\psi^k$: BO $\longrightarrow$ BO can be "delooped" eight times. Turning to $\psi^k \otimes 1$: $BO_p \longrightarrow BO_p$, if k is prime to p , we see that it can be delooped eight times to a map induced by $\psi^k \otimes 1/k^4$.

Now let f: $BBO_p \longrightarrow BBO_p$ be induced by $\Omega^7(\psi^k \otimes 1/k^4 - 1)$ and let BJ_p be the fibre of f for k a primitive root of unity mod p such that $k^{p-1} \not\equiv 1(p^2)$. BJ_p is up to homotopy independent of the choice of such k .

Equivalently we can regard BJ_p as induced over BBO_p by f from the path space fibration over BBO_p :

$$\begin{array}{ccc} BO_p & = & BO_p \\ \downarrow & & \downarrow \\ BJ_p & \longrightarrow & \mathcal{L}BBO_p \\ \downarrow & & \downarrow \\ BBO_p & \xrightarrow{\ f\ } & BBO_p \end{array}$$

Thus we have our map $BO \longrightarrow BO_p \longrightarrow BJ_p$.

From Adams [2] we learn:

$$\pi_{4i}(BO) \xrightarrow{\psi^k - 1} \pi_{4i}(BO)$$

is multiplication by $(k^{2i}-1)$.

Now let p be an odd prime. If k is a primitive root of unity mod p and $k^{p-1} \not\equiv 1\ (p^2)$, we have $\pi_i(BJ_p) \approx Z_{p^s}$ where p^{s-1} is the largest power of p dividing $i/2(p-1)$. This is precisely the p-primary component of the image of J [6, 1]. This establishes Theorem 1.

For $p=2$, a space BJ_2 is defined by the same method, but the homotopy groups are not the 2-component of the image of J . However, this discrepancy is necessary, except in very low dimensions, as indicated by Clough.

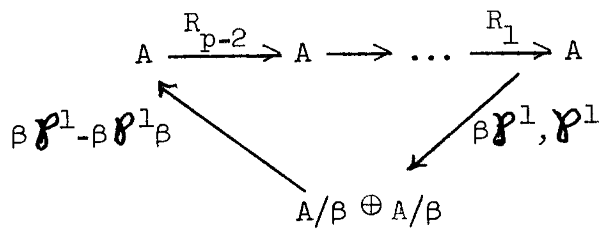
§3. The Postnikov system of BJ.

Next we study the k-invariants of BJ_p. For any space X, let $X[0,n)$ denote the (n-1)st stage of the Postnikov system of X; i.e. there is a map $X \longrightarrow X[0,n)$ which induces $\pi_i(X) \approx \pi_i(X[0,n))$ for $i < n$ and $\pi_i(X[0,n)) = 0$ for $i \geq n$. We are interested in the k-invariants $k \in H^{n+1}(X[0,n),\pi_n(X))$ which determine the fibrations $X[0,n+1) \longrightarrow X[0,n)$. In particular for $X = BJ_p$, we have $k_i \in H^{2i(p-1)+1}(BJ_p[0,2i(p-1)),Z_{p^{\mu(i)}})$ where $\mu(i)$ is one more than the power of p in i. Let $r = 2(p-1)$.

Since $BJ_p[0,ir) = BJ_p[0,(i-1)r+1)$, we can define classes

$$\ell_{i+1} \in H^{(i+1)r+1}(Z_{p^{\mu(i)}},ir;Z_p)$$

by restricting k_{i+1} to the fibre of $BJ_p[0,(i+1)r) \longrightarrow BJ_p[0,ir)$, and reducing mod p. The sequence $\{\ell_i\}$ realizes a certain exact sequence of Toda [9, p.140] in a sense we will make precise. The exact sequence in question is:



with $R_i(a) = a[(i+1)\mathcal{P}^1\beta - i\beta\mathcal{P}^1]$. Toda uses R_i for $1 \leq i \leq p-2$; we will also refer to R_{p-1} with the same definition and for $i > p$, will regard R_i as the same as R_j where $i \equiv j(p)$.

Theorem 3. For $i \not\equiv 0(p)$, we have $\ell_{i+1} = \lambda R_i(\iota_{ir})$ for some $\lambda \neq 0 \in Z_p$. For $i \equiv 0(p)$, we have $\ell_{i+1} = \lambda(\beta \mathcal{P}^1 \pm \mathcal{P}^1 \beta')\iota_{ir}$ where β' is the higher order Bochstein which is non-zero on ι_{ir}.

Proof. The k-invariants are determined only up to automorphisms of the system so the coefficient λ is unimportant. If we consider $BU \longrightarrow BJ_p$, then ℓ_{i+1} will map to a corresponding class

$$\ell^U_{i+1} \in H^*(Z, ir; Z_p)$$

which, according to Milnor [5] or Singer [7], is $\beta \mathcal{P}^1 \iota_{2r}$ up to such a constant λ. Thus ℓ_{i+1} must be of the form $R_j \iota_{ir}$ for some j, up to such a coefficient λ. (For $i = 1$, we eliminate a possible term $\iota\beta\iota$ by recognizing that BJ_p is a loop space.)

Consider first $i = p-1$. Since $\pi_{pr} \approx Z_{p^2}$, we must have $\beta\ell_p = 0$ which implies $\ell_p = \lambda\beta \mathcal{P}^1 \iota_{(p-1)r} = \lambda R_{p-1} \iota_{(p-1)r}$. Thus for k_p to exist, we must have $\beta \mathcal{P}^1 \ell_{p-1} = 0$. Since we are in the stable range of $H^*(Z_p, ir; Z_p)$ for $i > 1$, Toda's exact sequence, together with the fact that $\beta \mathcal{P}^1 R_{p-1} \neq 0$, implies k_p could exist only if ℓ_{p-1} were a multiple of $R_{p-2} \iota_{(p-2)r}$, and we know ℓ_{p-1} cannot be zero. By recursion we establish the theorem for all $i < p$.

To handle ℓ_{p+1}, notice that for $i \equiv 0(p)$, in the stable range, $H^*(Z_{p^{\mu(i)}}, ir; Z_p)$ is isomorphic to $A/\beta \oplus A/\beta$ where one copy is $(A/\beta)\iota_{ir}$ and the other is $(A/\beta) \dfrac{\beta}{p^{\mu(i)-1}} \iota_{ir}$. Since ℓ_{p+1} is a non-zero class of which the transgression restricts to zero in $K(Z_p, (p-1)r)$, Toda's exact sequence implies ℓ_{p+1} is a non-zero multiple of $(\beta \mathcal{P}^1 \pm \mathcal{P}^1 \frac{\beta}{p^{\mu(i)-1}}) \iota_{pr}$.

Repetition of this argument and the similar one for $i \not\equiv 0(p)$ establishes the Theorem.

Clough has analyzed the case $p = 2$.

§4. Computation of $H^*(BJ_p;Z_p)$ for $p > 2$.

From the definition of BJ_p , its cohomology is intimately related to that of BSO and BBSO . For $p > 2$ however the same space may be obtained by replacing BSO by BU and BBSO by BBU . Recall that $H^*(BU;Z) \approx Z[c_i | i \geq 1]$ where c_i is the i-th Chern class. By Bott periodicity $H^*(BBU;Z) \approx H^*(SU;Z)$, which is $E(y_i | i \geq 1)$ where $y_i \in H^{2i+1}(SU;Z)$. We need a more detailed result.

Let s_i be the symmetric polynomial $ic_i + \ldots \pm c_1^i$ which is primitive. If we write c_i formally as the i-th elementary symmetric function $\sigma_i(t_1,\ldots,t_n)$ $n \geq i$ then $s_i = \Sigma t_j^i$.

<u>Theorem 4.</u> $H^*(BBU;Z) \approx E(d_i | i \geq 1)$ where d_i represents τs_i .

Proof. Consider the Eilenberg-Moore spectral sequence with $E_2 = \mathrm{Ext}_{H_*(BU)}(Z,Z)$ which converges to $E_0 H^*(BBU)$. $H_*(BU;Z)$ is isomorphic to $Z[y_i]$ where y_i is dual to s_i . A "little" resolution of Z over $Z[y_i]$ is $Z[y_i] \otimes E(z_i)$ with $\partial z_i = y_i$. Thus $\mathrm{Ext}_{H_*(BU)}(Z,Z)$ is additively isomorphic with the dual of $E(z_i)$. Since the E_2 term of the Eilenberg-Moore spectral sequence is a Hopf algebra, it must, as an algebra, be an exterior algebra. The spectral sequence collapses since the chains of BU can be given by a complex with trivial differential. In homology the classes z_i survive to E^∞ as representatives of σy_i, so in cohomology, E_∞ is an exterior algebra on d_i. Since d_i is odd dimensional, E_∞ is free commutative and there is no extension problem; the theorem follows.

Now to compute $H^*(BJ_p;Z_p)$ we need to know the effect of ψ^k-1 on $H^*(BU)$. From Adams we learn $(\psi^k-1)^* s_i = (k^i-1)s_i$. Thus in the fibration $BU_p \longrightarrow BJ_p \longrightarrow BBU_p$ we have $\tau s_i = (k^i-1)d_i$.

We also need to know the behavior of the Wu classes. Let $q_i \in H^*(BU;Z_p)$ denote the pullback of the i-th Wu class into BU. Since the Wu classes can be expressed in terms of the Pontrjagin classes reduced mod p [4, p.120], the q_i in BU can similarly be expressed in terms of the Chern classes. We find $q_i = \lambda c_{i(p-1)}$ modulo decomposable elements with $\lambda \not\equiv 0(p)$. Thus, $A = Z_p[q_i]$ is a sub Hopf algebra and we can write $H_*(BU;Z_p) \approx A^* \otimes Z_p[y_j | j \not\equiv 0(p-1)]$ where $*$ denotes the dual Hopf algebra. Since $k^i \equiv 1(p)$ iff $i \equiv 0(p-1)$ we obtain $H_*(BJ_p;Z_p) \approx A^* \otimes E(d^*_{i(p-1)})$ [8, p.130] and $H^*(BJ_p;Z_p) \approx Z_p[q_i] \otimes E(d_{i(p-1)})$, where projection onto $Z_p[q_i]$ is induced by pulling back to BU.

Next we wish to show, by a change of basis, the generators $d_{i(p-1)}$ can be replaced by βr_i where r_i are classes which pull back to q_i in BU. For this purpose we look at the Eilenberg-Moore spectral sequence for the principal fibration $BU_p \longrightarrow BU_p \longrightarrow BJ_p$. We have $E_2 \approx \mathrm{Ext}_{H_*(BU;Z_p)}(H_*(BU;Z_p),Z_p)$ where $H_*(BU;Z_p)$ is a module over itself via $(\psi^k-1)_*$. The spectral sequence again collapses and E_∞ being free graded commutative will be isomorphic to $H^*(BJ_p;Z_p)$.

In general if $A \otimes \bar{R}$ is a resolution of M over an algebra A and $f: A \longrightarrow A'$ is a homomorphism, then we can compute $\mathrm{Tor}_A(A',M)$ from $A' \otimes \bar{R}$ where if $\partial(1\otimes r) = \Sigma\, a_i \otimes r_i$ then $\partial'(1\otimes r) = \Sigma f(a_i)\otimes r_i$. Since $H_*(BU) \otimes H_*(BBU)$ with $\partial d_i{*} = y_i$ is a resolution of Z over $H_*(BU)$, we can compute $\mathrm{Ext}_{H_*(BU)}(H_*(BU),Z_p)$ as the Z_p-cohomology of $H_*(BU) \otimes H_*(BBU)$ with $\partial d_i{*} = (k^i-1)y_i$. Since $k^i \equiv 1(p)$ iff $i \equiv 0(p-1)$, we get $A^* \otimes E(d_{i(p-1)}{}^*)$. Dually we have $Z_p[q_i]\otimes E(d_i)$ with $\delta s_i(q_1,\ldots,q_n) = (k^{i(p-1)} - 1)d_i$. This is trivial mod p , but we must look at it to determine the Bochsteins we need. We can write $\delta q_i = \lambda d_i$ modulo decomposable terms. On the other hand $s_i(q_1,\ldots,q_n) = iq_i$ + decomposable terms, so we have $i\lambda = (k^{i(p-1)} - 1)$. Now there is one more power of p in $k^{i(p-1)} - 1$ than in i [1, Lemma (2.12)] so p divides λ but p^2 does not. Thus d_i can be replaced as a generator of $H^*(BJ_p;Z_p)$ by βr_i where r_i pulls back to q_i .

An argument for $p = 2$ similar to that just given is possible for a complex analogue of BJ_2 , called BJC_2 . We find that $H^*(BJC_2;Z_2) \approx Z_2[c_i] \otimes E(d_i)$ where d_i can be chosen to be βc_i if i is odd and $\frac{\beta}{2}\, c_i$ if i is even. This difference occurs because (k^i-1) is divisible by 2 but not 4 if i is odd, while $k^{2i}-1$ is divisible by one more power of 2 than $2i$ is, i.e. by 2 more powers of 2 than i is divisible by.

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