PROBLEMS PROPOSED AT THE CONFERENCE

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Problem sessions were held as part of the Conference on Algebraic Topology at Chicago Circle in June, 1968. Some problems proposed were recollections of known problems, many of the problems were phrased in the context of the talks given. The presentation of these problems here also assumes that context as little background or explanation of notation is given. Where significant progress has been made since the conference, comments are added so that the list reflects the state of the subject in late 1968.

1. The Structure of BF and related spaces.

Milgram: $H^*(BSF;Z_2) \approx Z_2[W_i] \otimes E(e_{i,i}, Q^I W_i)$.

Here the $e_{i,i} \in H^{2i+1}$ and the Q^I are certain formal Dyer-Lashof operations. Milgram, moreover, has the structure as an $_2$-algebra.

Peterson-Toda: $H^*(BSF;Z_p) \approx Z_p[q_i] \otimes E(\beta q_i) \otimes C$ as Hopf algebras over the Steenrod algebras.

<u>Conjecture 1</u>. $C \approx E(Q^I q_i) \otimes T(Q^J q_i)$ where Q^I, Q^J are certain Dyer-Lashof operations of odd and even degree respectively and T denotes a truncated polynomial algebra, truncated at height p.

<u>Conjecture 2</u>. C_* the dual of C, is free as an algebra.

Problem 3. Give a nice description of $\quad_p$-algebra structure; mod 2, nice enough to exhibit a copy of $H^*(BBSO;Z_2)$ in $H^*(BSF;Z_2)$.

Problem 4. (Milgram) Boardman has shown F is an infinite loop space with respect to composition. Describe Dyer-Lashof operations in F with respect to the composition structure and relate to Dyer-Lashof operations in QS^o = the base point component of $\lim \Omega^n S^n$.

Conjecture 5. $BF \simeq BJ \times B \operatorname{Cok} J$

Sullivan: F is mod p equivalent to $J \times \operatorname{Cok} J$ for $p > 2$.

Problem 7. (Peterson) $H^*(BJ;Z_p) \approx Z_p[q_i] \otimes E(\beta q_i)$ but in terms of $BF \simeq BJ \times B \operatorname{Cok} J$, not all the q_i sit in BJ. Describe precisely how do they lie.

Problem 8. (Stasheff) Describe a set of generators for $H^*(BSF)$ in terms of operations on the Thom class. Peterson thus describes classes $e_i \in H^{p^i r-1}$ which might be an independent set of generators as an $\quad_p$-algebra. Mod 2, Mahowald has an alternate description which agrees with Milgram's $e_{2^i-1,2^i-1}$ modulo decomposables and Peterson's e_2 is $e_{3,3}$ mod decomposables.

Problem 9. (Peterson) Describe corresponding characteristic classes for Poincare duality complexes by internal methods like Wu's characterization of the S-W classes of a manifold.

Problem 10. (Peterson) Find the smallest integer a_i such that for the Pontrjagin class $p_i \in H^{4i}(BSO;Z)$, we have that $a_i p_i$ + decomposables pulls back from BSPL.

A partial solution has been obtained by Brumfield, and the odd factors are claimed by Sullivan.

<u>Problem 11</u>. (<u>Sullivan</u>) Is $K(\text{Cok } J) = 0$ or at least small? Analyze $[X, \text{Cok } J]$ and $[\text{Cok } J, X]$ for reasonable X.

<u>Problem 12</u>. (<u>Mahowald-Peterson</u>) Form B^n_k from B^n_{k-1} by killing a basis for the intersection of the kernels for all maps $M^n \longrightarrow B^n_{k-1}$ with $B^n_0 = BO$. Compute $H^*(B^n_\infty)$. Does $H^*(BO)$ map onto it?

<u>Conjecture 13</u>. $M^n \subseteq R^{2n-\alpha(n)}$ where $\alpha(n)$ = the number of 1's in a dyadic expansion of n.

<u>Problem 14</u>. (<u>Stasheff</u>). If $\mathfrak{F}$ is a twisted operation based on a relation R in $_p$, and Φ is the untwisted operation based on R, then in a space where the twisted action of agrees with the untwisted, i.e., all $q_i = 0$, describe $\mathfrak{F} - \Phi$. Mosher has an example to show $\mathfrak{F} \neq \Phi$.

II. Cobordism.

<u>Problem 15</u>. What does $\pi_*(MF)$ signify as a cobordism theory? Computation of $\pi_*(MF)$ follows from knowledge of $H^*(BF)$.

<u>Problem 16</u>. Let Ω^P denote cobordism of complexes satisfying Poincare duality. Compute Ω^P.

Sullivan: It is not a homology theory. [Levitt has since conjectured an exact sequence

$\ldots \pi_n(F/PL) \longrightarrow \Omega^P_n \longrightarrow \pi_n(MSF) \longrightarrow \ldots]$

<u>Conjecture 17</u>. (<u>Peterson</u>) Ker: $\longrightarrow H^*(MSPL; Z_p)$ is (Q_0, Q_1).

Problem 18. (Sullivan) Which connective spectra are retracts of Thom spectrum? For example, $K(Q_p)$ and $(bO)_2$ are, but not $(bO)_{p>2}$.

Conjecture 19. (Peterson) $H^*(MSPL;Z_p)$ can be filtered so the associated graded is a direct sum of -modules of the form / (B) where $B \subset E(Q_0,Q_1)$.

Problem 20. (Peterson) Find characteristic numbers giving a complete set of invariants for Ω^{PL}.

Problem 21. (Peterson) Find generators for Ω^{Spin}.

Problem 22. (Peterson) Describe the Arf invariant for Spin manifolds in terms of KO-characteristic numbers.

Conjecture 22. (Barratt) The Arf invariant of a framed manifold can be made zero by a suitable combination of surgery and reframing. (Verified for dim ≤ 61.)

Problem 24. (Brown) Find a good cobordism theory in which to compute the ARF invariant.

III. Steenrod Algebra.

Problem 25. (Steenrod) If $H^*(X;Z_p) = Z_p[x_i]$, what dimensional restrictions are imposed on x_i? Extensive results are due to Hubbuck, complete with Z_p replaced by Q, no 2-torsion, and number of generators ≤ 5.

Problem 26. (May) Introduce Steenrod operations into the Eilenberg-Moore spectral sequences. Puppe has done this completely for the spectral sequence going from $H^*(X)$ to $H^*(\Omega X)$.
[Solutions have since been obtained independently by Madsen and Rector.]

Problem 27. (Milgram) Solve the extension problem as -modules for the E_∞-term of the E-M spectral sequences.

Problem 28. (Smith) If X,Y are H-spaces and $f:X \longrightarrow Y$ is an H-map, then the induced fibre space $\Omega Y \longrightarrow E \longrightarrow X$ is an H-space. Find effective computable invariants to determine $H_*(E)$ as an extension of Hopf algebras (over). [Milgram has done the case $X = K(Z_2,n)$, $f = Sq^{n-i}$ for certain i.]

Conjecture 29. (Stasheff) If $\beta\ {}^{n}x_{2n+1} = 0$, then $\frac{\beta}{p}\ \ pn\ \ {}^{n}x_{2n+1}$ is contained in the p^2-fold Massey product $\langle x,\ldots,x\rangle$.

Problem 30. (Milgram) Find all the relations between the Steenrod algebra and Massey products. Some relations such as ${}^{i}\langle x_1,\ldots,x_n\rangle$ are known by Milgram and May.

Conjecture 31. (Cohen) Given a Z_p-algebra R and Z_p-modules F, M, all connected, then there is a spectral sequence of R-modules $F \otimes R \Rightarrow M$ if and only if there is a spectral sequence $\mathrm{Tor}_R(M,Z_p) \Rightarrow F$.

Problem 32. (Kraines) If M is an algebra over R, relate the change of rings spectral sequence converging to $\mathrm{Tor}_R(M,Z_p)$ to the $F \otimes R \Rightarrow M$ spectral sequence.

IV. H-spaces and Lie groups.

Conjecture 33. (Smith) G simply connected, finite dimensional H-space. If $x \in QH^{2n}(G;Z_p)$ and $\beta x \neq 0$ then $\exists y$ such that $x = {}^{t}y$. This will imply ΩG is torsion free.

Problem 34. (Mimura) Classify H-spaces E which are 3-sphere fiberings over S^7. ($H^*(E) = E(x_3,x_7)$. Homotopy associativity implies ${}^1x_3 = x_7$.) Hilton and Roitberg have exhibited one which is not of the homotopy type of a Lie group. [Since the conference, Stasheff has used methods of Zabrodsky to show E is an H-space if classified by $n^{\omega} \in \pi_6(S^3)$, $n \neq 2$ (4).]

Problem 36. (Mimura) Estimate the mapping degree of the standard map $S^{n_1}x\ldots xS^{n_r} \longrightarrow G$ for exceptional Lie groups G.

Problem 37. (Smith) If G,H are Lie, when is $f: BG \longrightarrow BH$ homotopic to Bp for some homomorphism p? E.g. $G = H = S^3$.

Conjecture 38. (Smith) $BS^3 \longrightarrow BS^3$ restricted to S^4 has degree 0 or ± 1. [Berstein, Cooke, Smith, Stong: degree must be 0 or odd square.]

Conjecture 39. (May) Hodgkins' K theoretic Eilenberg-Moore sequence for $G/_H \longrightarrow B_H \longrightarrow B_G$ collapses.
$K(G/_H) \approx K(B_H) \otimes_{K(B_G)} K(*)$.

Problem 40. (Smith-Mimura) Find the least integer $N(t)$ such that $N(t)x$ is spherical for all $x \in PH_t(G;Z)$/torsion.

Problem 41. (Smith) Find the least integer $n(t)$ such that $n(t)|\sigma$ for all spherical $\sigma \in H_*(G;Z)$/torsion.

Conjecture 42. (Atiyah-Mimura) $x \in PH_*(G)$ is spherical if and only if $\langle x, ch\xi \rangle \in Z$ for all $\xi \in K^*(G)$.