HIGHER PRODUCT OPERATIONS

Gerald J. Porter *

It has been recognized for some time that the Massey product operation in cohomology and the Toda bracket operation in stable homotopy were in some sense similar operations. However, the Massey product has always been defined in an algebraic category and the Toda bracket in a topological category. We define a universal topological higher product such that the associated internal product is the Massey product and the external product is the stable Toda bracket. This construction extends the definition of Massey products to cohomology theories arising from associative ringed spectra. We sketch the construction and give the properties of such products. The details will appear in [4].

DEFINITION OF HIGHER PRODUCTS

Definition 1: A ringed set of topological spaces, $\mathcal{R} = \{R_i, i \in I, \mu\}$, is a collection of based topological spaces and maps, $\mu: R_i \times R_j \to R_k$ (some $k \in I$), such that $\mu(r, *) = \mu(*, r) = *$ for all $r \in R$. We say μ is associative if

$$\mu(1 \times \mu) = \mu(\mu \times 1) : R_i \times R_j \times R_k \to R(i,j,k)$$

We denote the range of μ on $R_i \times R_j$ by $R(i,j)$ and assume that all possible pairings on $R_{j_1} \times \cdots \times R_{j_n}$ have $R(j_1, \ldots, j_n)$ as their

*The author was supported in part by NSF GP-6969.

common range.

<u>Example 1</u>: $\mathcal{R} = \{S^p,\ p \geq 0,\ \mu\}$ where $\mu: S^p \times S^q \to S^{p+q}$ is the quotient map.

<u>Example 2</u>: $\mathcal{R} = \{K(\pi,\ n),\ n \geq 0,\ \mu,\ \pi$ a commutative ring with $1\}$ $\mu: K(\pi,\ n) \times K(\pi, m) \to K(\pi,\ m+n)$ is the map which induces the cup product. There are many models for $K(\pi,\ n)$ and many maps μ. However it is possible to choose models and a map μ such that μ is associative.

<u>Example 3</u>: For any ringed set, $\mathcal{R}$, the associated ringed set $\Sigma\mathcal{R} = \{\Sigma^n R_i,\ R_i \in \mathcal{R},\ n \geq 0,\ \bar{\mu}\}$ where

$$\bar{\mu}:\ \Sigma^n R_i \times \Sigma^m R_j \to \Sigma^{n+m} R_{i+j}$$

is defined by

$$\bar{\mu}((t_1,\ldots,t_n,\ r),(t_{n+1},\ldots,\ t_{n+m},\ s)) = (t_1,\ldots,\ t_{m+n},\ \mu(r,\ s))$$

where μ is the product in $\mathcal{R}$ and $r \in R_i$, $s \in R_j$.

The universal $n^{\underline{th}}$ order product, corresponding to the n-tuple, $(R_{j_1},\ldots,\ R_{j_n})$, $R_{j_i} \in \mathcal{R}$, is denoted $\theta_n(j_1,\ldots,\ j_n)$ and is an element of $[E_n(j_1,\ldots,\ j_n),\ \Omega^{n-2}R(j_1,\ldots,\ j_n)]$ where Ω is the loop space, square brackets indicate homotopy classes, and $E_n(j_1,\ldots,\ j_n)$ is the universal example for $\theta_n(j_1,\ldots,\ j_n)$.

We first construct the universal examples $E_k(j_1,\ldots,\ j_n)$, $2 \leq k \leq n$. If $k = 2$ set $E_2(j_1,\ldots,\ j_n) = R_{j_1} \times\ldots\times R_{j_n}$.

$\theta_2(j_1, j_2): R_{j_1} \times R_{j_2} \to R(j_1, j_2)$ is the product μ in $\mathcal{R}$. Assume inductively that $\theta_r(j_1,\ldots, j_r)$ has been defined for $r < n$ and $E_k(j_1,\ldots, j_r)$ has been defined for $k < n$ and $r \geq k$. Assume furthermore that for $t \geq k + 1$ and $s \leq n - t$ there are canonical projections $q^k_{s,t}: E_k(j_1,\ldots, j_n) \to E_k(j_s,\ldots, j_{s+t})$ such that

$$q^k_{r-s+1,k-1}\, q^k_{s,t} = q^k_{r,k-1}$$

whenever $s \leq r < r + k - 1 \leq s + t$.

For $r \geq n$ we set $E_n(j_1,\ldots, j_r)$ equal to the fibre space induced from the path fibration by

$$\Phi : E_{n-1}(j_1,\ldots, j_r) \to \prod_{i=1}^{r-n+2} \Omega^{n-3}R(j_i,\ldots, j_{i+n-2})$$

where $\Phi = \prod_{i=1}^{r-n+2} \theta_{n-1}(j_i,\ldots, j_{i+n-2})\, q^{n-1}_{i,n-2}$.

We regard $E_n(j_1,\ldots, j_r) \subset E_{n-1}(j_1,\ldots, j_r) \times \prod_{i=1}^{r-n+2} P\Omega^{n-3}R(j_i,\ldots,j_{i+n-2})$. $q^n_{s,t}$ is then $q^{n-1}_{s,t}$ on the first factor and the obvious projection on the second factor. This completes the definition of $E_n(j_1,\ldots, j_r)$.

Let $h(j_i,\ldots, j_{i+k-1}): E_n(j_1,\ldots, j_n) \to P\Omega^{k-2}R(j_i,\ldots, j_{i+k-1})$ be the composite of $E_n(j_1,\ldots, j_n) \to E_{k+1}(j_1,\ldots, j_n)$ and the projection

$E_{k+1}(j_1,\ldots, j_n) \to P\,\Omega^{k-2}R(j_i,\ldots, j_{i+k-1})$. (More precisely the second projection should be included in the induction hypothesis). Also let $h(j_i):E_n(j_1,\ldots, j_n) \to R_{j_i}$ be the composite

$$E_n(j_1,\ldots, j_n) \to E_2(j_1,\ldots, j_n) \to R_{j_i}$$

where the second map is the obvious projection.

We generalize the definition given by Kraines [2] and define

$$\theta_n(j_1,\ldots, j_n) = \sum_{i=1}^{n-1} (-1)^{i+1} h(j_1,\ldots, j_i) \cdot h(j_{i+1},\ldots,j_n)$$

For this to be meaningful, we must define what we mean by multiplication and addition in the above expression.

Let $L^n \subset \dot{I}^n$ be $\{(t_1,\ldots, t_n) \mid t_i = 0 \text{ some } i\}$. We take as our model for $\Omega^n X$, the set of maps $f:(\dot{I}^{n+1}, L^{n+1}) \to (X, *)$; and for our model for $P\Omega^{n-1}X$, the set of maps $f:(I^n, L^n) \to (X, *)$. Each of these spaces is given the compact open topology.

Let $F:X \to \Omega^n Y$. The $i^{\underline{th}}$ face of F , $F^i:X \to P\Omega^{n-1}Y$, is defined by $F^i(x)(t_1,\ldots, t_n) = F(x)(t_1,\ldots, t_{i-1},1,t_i,\ldots, t_n)$. We write $F = \sum_{i=1}^{n+1} (-1)^{i+1} F^i$. Conversely given maps $F^1,\ldots, F^{n+1}$, $F^i:X \to P\Omega^{n-1}Y$, which are compatible (in the obvious sense) we define their sum

$F = \sum_{i=1}^{n+1} (-1)^{i+1} F^i$ to be the map which has F^i as its $i^{\underline{th}}$ face.

This is addition in the sense of the homotopy addition theorem.

Let $f:X \to P\Omega^{n-1}R_j$ and $g:X \to P\Omega^{m-1}R_k$ $(P\Omega^{-1}R_j = R_j)$ be given. Define $(f \cdot g):X \to P\Omega^{m+n-1}R(j,k)$ by

$$(f \cdot g)(x)(t_1,\dots,t_{m+n}) = \mu(f(x)(t_1,\dots,t_n),g(x)(t_{n+1},\dots,t_{m+n}))$$

where μ is the product in the ringed set $\mathcal{R}$.

Assume inductively that for $k < n$,

$$(\theta_k(j_1,\dots,j_k))^i = h(j_1,\dots,j_i) \cdot h(j_{i+1},\dots,j_k)$$

then the maps $h(j_1,\dots,j_i) \cdot h(j_{i+1},\dots,j_n)$ are compatible and the above definition of $\theta_n(j_1,\dots,j_n)$ makes sense and satisfies the inductive hypothesis.

<u>Definition 2</u>: Given $\varphi:X \to E_n(j_1,\dots,j_n)$, the $n^{\underline{th}}$ order Massey product $M_n(\varphi)$, is defined by

$$M_n(\varphi) = \varphi^*[\theta_n(j_1,\dots,j_n)] \in [X,\Omega^{n-2}R(j_1,\dots,j_n)]$$

We say φ is of type $(f_1,\dots,f_n)$ if $h_{j_i}\varphi = f_i$ for $i = 1,\dots,n$. The set of $n^{\underline{th}}$ order Massey products of type $(f_1,\dots, f_n)$ is defined by

$$\langle f_1,\dots,f_n\rangle = \{M_n(\varphi) \mid \varphi \text{ is of type } (f_1,\dots, f_n)\}.$$

The definition of the external product is a bit more complicated. First note that if $n > 2$,

$$[X_1 \times \cdots \times X_n, \Omega^{n-2}R(j_1,\ldots,j_n)] \text{ has } [\wedge(X_1,\ldots,X_n),\Omega^{n-2}R(j_1,\ldots,j_n)]$$

as a direct summand, where $\wedge$ is the smash product. Let q be the projection onto the above summand.

We say $\varphi : X_1 \times \cdots \times X_n \to E_n(j_1,\ldots,j_n)$ is special if $h(j_i,\ldots,j_{k+i-1})\varphi$ factors through $X_i \times \cdots \times X_{k+i-1}$ for $1 \le k \le n$ and $1 \le i \le n+1-k$.

<u>Definition 3</u>: Given $\varphi : X_1 \times \cdots \times X_n \to E_n(j_1,\ldots,j_n)$, the $n^{\underline{th}}$ order Toda product, $T_n(\varphi)$, is defined by

$$T_n(\varphi) = q(\varphi^*[\theta_n(j_1,\ldots,j_n)]) \in [\wedge(X_1,\ldots,X_n),\Omega^{n-2}R(j_1,\ldots,j_n)]$$

We say φ is of type $(f_1,\ldots,f_n)$ if φ covers

$f_1 \times \cdots \times f_n : X_1 \times \cdots \times X_n \to E_2(j_1,\ldots,j_n)$. The set of $n^{\underline{th}}$ order Toda products of type $(f_1,\ldots,f_n)$ is defined by

$$\{f_1,\ldots,f_n\} = \{T_n(\varphi) \mid \varphi \text{ is special of type } (f_1,\ldots,f_n)\}.$$

(Special maps are used to keep indeterminacy small.)

Thus the Massey products are the internal products and the Toda products are the external products associated to the universal products, $\theta_n(j_1,\ldots,j_n)$.

We note that $M_n(\varphi)$ and $T_n(\varphi)$ are unique elements while $\langle f_1, \ldots, f_n \rangle$ and $\{f_1, \ldots, f_n\}$ are (perhaps empty) subsets of the appropriate homotopy groups.

THE PROPERTIES OF THE HIGHER PRODUCTS

Corresponding to each property of $M_n(\varphi)$ $(T_n(\varphi))$ there is a corollary about $\langle f_1, \ldots, f_n \rangle$ $(\{f_1, \ldots, f_n\})$. We state only the theorems about $M_n(\varphi)$ and $T_n(\varphi)$ and leave the corollaries to the reader.

<u>Theorem 1</u> (Naturality)

a) Let $f: X \to Y$ and $\varphi: Y \to E_n(j_1, \ldots, j_n)$ then

$$f^*(M_n(\varphi)) = M_n(\varphi f) .$$

b) Let $f_i: X_i \to Y_i$, $1 \le i \le n$ and $\varphi: Y_1 \times \ldots \times Y_n \to E_n(j_1, \ldots, j_n)$ then $(\wedge(f_1, \ldots, f_n))^* T_n(\varphi) = T_n(\varphi(f_1 \times \ldots \times f_n))$.

<u>Definition 4</u>: Let $\mathcal{R}$ and $\mathcal{S}$ be ringed sets. $F: \mathcal{R} \to \mathcal{S}$ is said to be a morphism of ringed sets if $F = \{f(j): R(j) \to S(j),\ j \in J\}$ is such that the following diagram commutes for all $(i, j) \in J \times J$.

$$\begin{array}{ccc} R(i) \times R(j) & \xrightarrow{f(i) \times f(j)} & S(i) \times S(j) \\ \Big\downarrow \theta_2^R(i, j) & & \Big\downarrow \theta_2^S(i, j) \\ R(i, j) & \xrightarrow{f(i, j)} & S(i, j) \end{array}$$

<u>Theorem 2</u>: Let $F: \mathcal{R} \to \mathcal{S}$ be a morphism of ringed sets and let θ_n^R and θ_n^S be $n^{\underline{th}}$ order universal products corresponding to $\mathcal{R}$ and $\mathcal{S}$ respectively. Then there exists $F_n: E_n^R(j_1, \ldots, j_n) \to E_n^S(j_1, \ldots, j_n)$ which covers $f(j_1) \times \ldots \times f(j_n)$ and such that

$$\theta_n^S F_n = \Omega^{n-2}(f(j_1, \ldots, j_n))\ \theta_n^R \quad .$$

<u>Corollary</u>: Let $F: \mathcal{R} \to \mathcal{S}$ be a morphism of ringed sets

a) if $\varphi: X \to E_n^R(j_1, \ldots, j_n)$ then $(\Omega^{n-2} f(j_1, \ldots, j_n))_* M_n(\varphi) = M_n(F_n \varphi)$

b) if $\varphi: X_1 \times \ldots \times X_n \to E_n^R(j_1, \ldots, j_n)$ then

$(\Omega^{n-2} f(j_1, \ldots, j_n))_* T_n(\varphi) = T_n(F_n \varphi)$.

<u>Definition 5</u>: If $[R_i, R_i]$ is a group for each $R_i \in \mathcal{R}$, let $k: R_i \to R_i$ be $(Id + \ldots + Id)$ (k times) . We say $\mathcal{R}$ is linear if the following diagram commutes.

$$\begin{array}{ccc} R_i \times R_j & \longrightarrow & R_i \times R_j \\ \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle \theta_2} \\ R(i, j) & \xrightarrow{k} & R(i, j) \end{array}$$

where the top map is either $1 \times k$ or $k \times 1$.

<u>Theorem 3</u> (Linearity). If $\mathcal{R}$ is linear then for each t , $1 \le t \le n$, there is a map $k_t^n: E_n(j_1, \ldots, j_n) \to E_n(j_1, \ldots, j_n)$ which covers

$1 \times \cdots \times k \times \cdots \times 1 : R_{j_1} \times \cdots \times R_{j_t} \times \cdots \times R_{j_n} \to R_{j_1} \times \cdots \times R_{j_t} \times \cdots \times R_{j_n}$ and

such that $\theta_n k_t^n = k\theta_n$.

<u>Corollary</u>: a) Let $\varphi : X \to E_n(j_1, \ldots, j_n)$, then $M_n(k_t^n \varphi) = kM_n(\varphi)$.

b) Let $\varphi : X_1 \times \cdots \times X_n \to E_n(j_1, \ldots, j_n)$, then $T_n(k_t^n \varphi) = kT_n(\varphi)$.

<u>Theorem 4</u> (Higher Associativity)

a) Let $\varphi : X \to E_{n-1}(j_1, \ldots, j_n)$ be of type $(f_1, \ldots, f_n)$, then

$$f_1 \cdot M_{n-1}(q_{2,n-2}^{n-1}\varphi) = (-1)^{n+1} M_{n-1}(q_{1,n-2}^{n-1}\varphi) \cdot f_n$$

b) The projections of $f_1 \cdot T_{n-1}(q_{2,n-2}^{n-1}\varphi)$ and $(-1)^{n+1} T_{n-1}(q_{1,n-2}^{n-2}\varphi) \cdot f$ into $[\wedge(X_1, \ldots, X_n), \Omega^{n-3}R(j_1, \ldots, j_n)]$ are equal.

<u>Theorem 5</u>: a) Let $\varphi_{n-1} : X \to E_{n-1}(j_1, \ldots, j_n)$ be a map of type $(f_1, \ldots, f_n$ which can be lifted to $E_n(j_1, \ldots, j_n)$ then $\{M_n(\varphi) \mid \varphi \text{ lifts } \varphi_{n-1}\}$ is a coset of

$$f_1 \cdot [X, \Omega^{n-2}R(j_2, \ldots, j_n)] + [X, \Omega^{n-2}R(j_1, \ldots, j_{n-1})] \cdot f_n$$

b) Let $\varphi_{n-1} : X_1 \times \cdots \times X_n \to E_{n-1}(j_1, \ldots, j_n)$ be a special map of type $(f_1, \ldots, f_n)$ which can be lifted to $E_n(j_1, \ldots, j_n)$ **then**

$\{T_n(\varphi) \mid \varphi$ lifts $\varphi_{n-1}\}$ is a coset of

$$f_1 \cdot [\wedge(X_2,\ldots,X_n),\Omega^{n-2}R(j_2,\ldots,j_n)] + [\wedge(X_1,\ldots,X_{n-1}),\Omega^{n-2}R(j_1,\ldots,j_{n-1})]\cdot f_n$$

<u>Theorem 6</u>: Let $\varphi:\Sigma X \to E_n(j_1,\ldots,j_n)$, then $M_n(\varphi) = 0$.

<u>Corollary</u>: Let $\Omega_*:[X,\Omega^{n-2}R(j_1,\ldots,j_n)] \to [\Omega X,\Omega^{n-1}R(j_1,\ldots,j_n)]$ be the cohomology suspension. For all $\varphi:X \to E_n(j_1,\ldots,j_n)$, $\Omega_* M_n(\varphi) = 0$.

<u>Theorem 7</u>: In $\Sigma\mathcal{R}$ let $E_n(j_1,\ldots,j_n)$ and $E_n(\Sigma j_1,\ldots,\Sigma j_n)$ be the universal examples corresponding to $(R_{j_1},\ldots,R_{j_n})$ and $(\Sigma R_{j_1},\ldots,\Sigma R_{j_n})$ respectively. There is a function σ_n which associates a map, $\sigma_n(\varphi):\Sigma X_1 \times\cdots\times \Sigma X_n \to E_n(\Sigma j_1,\ldots,\Sigma j_n)$, of type $(\Sigma f_1,\ldots,\Sigma f_n)$, to each map $\varphi:X_1\times\cdots\times X_n \to E_n(j_1,\ldots,j_n)$ of type $(f_1,\ldots,f_n)$ such that $T_n(\sigma_n(\varphi)) = \Sigma^n T_n(\varphi)$. Thus 'stable' Toda products can be defined.

<u>Remark 1</u>: If $\mathcal{R}$ is not associative but is homotopy associative then we define the analogue of Stasheff's A_n-forms [5]. If $\mathcal{R}$ has A_n forms for $n \le k$, θ_n may be defined.

<u>Remark 2</u>: Under a more general definition of ringed set, the matrix Massey products of May [3] and the matrix Toda products of Cohen [1] are special cases of the above construction. In particular, one considers a set of topological spaces and products between distinguished pairs of spaces. The theorems stated above remain valid in this more general setting.

Remark 3: The commutator product in ΩX is not associative or homotopy associative; however, it does satisfy a Jacobi identity. In this case, there is an analagous construction which yields higher Samelson products and higher group commutators.

REFERENCES

[1] J. M. Cohen: The decomposition of stable homotopy, Ann. Math. (2), 87 (1968), 305-320.

[2] D. Kraines: Massey higher products, Trans. Amer. Math. Soc. 124 (1966) 431-449.

[3] J. P. May: Matrix Massey products (to appear).

[4] G. J. Porter: Higher products (to appear).

[5] J. D. Stasheff: On homotopy associativity of H-spaces, I, Trans. Amer. Math. Soc. 108 (1963), 275-292.

University of Pennsylvania
Philadelphia, Pennsylvania