ON THE COHOMOLOGY OF BSF AND BSPL

by
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Let BSF and BSPL be the classifying spaces for oriented spherical fibrings and oriented PL-bundles respectively. Let p be an odd prime. The purpose of this talk is to state some of what is known about $H^*(BSF;Z_p)$ and $H^*(BSPL;Z_p)$ and to use these results to compute the low dimensional torsion in Ω_*^{PL}, the oriented PL-cobordism ring.

Let $r = 2p - 2$, let $q_i \in H^{ir}(BSF)$ be[1] the Wu class defined by $\mathcal{P}^i(U) = q_i \cdot U$, where U denotes the Thom class. The first theorem is due to Peterson and Toda.

THEOREM 1. There is a Hopf algebra, C, over $\mathcal{A}$, the Steenrod algebra, such that $H^*(BSF) \approx (Z_p[q_i] \otimes E(\beta q_i)) \otimes C$, as Hopf algebras over $\mathcal{A}$.

Since the proof of this theorem has appeared, we refer the reader to [3].

The full structure of C, as a Hopf algebra over $\mathcal{A}$, is known in dimensions $\leq p^2 r - 2$ by results of Stasheff [4]. Milgram and May will have further comments on C in later talks.

The following corollary is also proved in [3].

COROLLARY 2. MSF is of the same homotopy type as a wedge of Eilenberg-MacLane spectra.

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[1]All cohomology groups are assumed to have Z_p coefficients unless otherwise stated.

Let J_{PL} : BSPL $\longrightarrow$ BSF be the natural map. It is not difficult to show that $J_{PL}{}^{*}(\beta q_i) = 0$ if $i \leq p$.

THEOREM 3. $J^{*}_{PL}(\beta q_{p+1}) = \mu\beta\mathcal{P}^1\beta\, \hat{J}_{PL}(e_1)$, where $\mu \not\equiv 0(p)$ and e_1 is the first exotic class of Gitler and Stasheff [2][2]. Also, $J^{*}_{PL}(\beta q_i) \neq 0$ if $i \geq p + 1$.

The proof of theorem 3 requires a detailed knowledge of the homotopy structure of BSF and some results about β_1, the element of order p in the (pr - 2) - stem.

Sullivan [5] has shown that BSPL has the same mod p homotopy type as BSO $\times$ B Coker J, where B Coker J is a space that π_*(B Coker J) = p-torsion of Coker (J : π_*(BSO) $\longrightarrow \pi_*$(BSF)). Theorem 3 shows that this splitting of BSPL is not as nice as it might be because the higher q_i's have a factor in the B Coker J piece. However, in low dimensions (e.g. $\leq p^2 r$), H^*(B Coker J)$\approx$ C and it is not unreasonable to conjecture that this is true in general.

Let $\Theta : \mathcal{A} \longrightarrow H^*$(MSPL) be defined by $\Theta(a) = a(U)$. Let $Q_i \varepsilon \mathcal{A}$ be the Milnor elements. $\Theta(Q_0) = 0 = \Theta(Q_1)$ and one might conjecture that $\Theta(Q_i) = 0$ for all i. However, as a corollary of theorem 3, we have the following result.

COROLLARY 4. $\Theta(Q_2) \neq 0$.

Sullivan's splitting above respects the universal bundle and hence MSPL is of the same mod p homotopy type as MSO $\wedge$ M Coker J. Hence, H^*(MSPL)$\approx H^*$(MSO) $\otimes$ H^*(M Coker J). To compute π_*(MSPL)$\approx \Omega_*^{PL}$ (by Williamson [6]), we wish to

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[2]This theorem shows that the first lemma on p. 32 of [6] is incorrect, so the calculations there are incorrect. The answers given in [6] are correct however.

compute $H^*(MSPL)$ as a module over $\mathcal{A}$ and apply the Adams spectral sequence. How $H^*(MSO) = \Sigma' \mathcal{A}$, where $'\mathcal{A} = \mathcal{A}/\mathcal{A}E$, and $E = E(Q_0, Q_1, Q_2 \ldots)$, the exterior algebra on the elements Q_i. The results of [1] show that the $\mathcal{A}$-module structure of $'\mathcal{A} \otimes N$ depends only on the E-module structure of N and further that $\mathrm{Ext}_{\mathcal{A}}('\mathcal{A} \otimes N, Z_p) \approx \mathrm{Ext}_E(N, Z_p)$, Hence, we must compute $H^*(M \text{ Coker } J)$ as an E-module. Using theorem 3 and the results of Stasheff, we compute $\mathrm{Ext}_E(H^*(M \text{ Coker } J), Z_p)$. One also notes that all differentials in the Adams spectral sequence are zero in the range $t - s \leq p^2 r - 1$. To simplify the statement, we let $p = 3$ in the following theorem.

THEOREM 5. In dimensions < 35, the 3-torsion of Ω_*^{PL} is given by the following table:

Generators	Dim.	Order	Detected by
$M_\alpha \times M^{11}$	$11 + \dim M_\alpha$	3	$y_\alpha \cdot e_1$
$M_\alpha \times M_1^{23}$	$23 + \dim M_\alpha$	3	$y_\alpha \cdot \mathcal{P}^3 e_1$
$M_\alpha \times M_2^{23}$	$23 + \dim M_\alpha$	3	$y_\alpha \cdot e_1 \cdot \beta e_1$
M_1^{27}	27	9	$\mathcal{P}^4 e_1 - e_1 \cdot \mathcal{P}^1 \beta e_1$
M_2^{27}	27	3	$\mathcal{P}^4 e_1$
M^{34}	34	3	$e_1 \cdot \mathcal{P}^3 e_1$

Here M_α are elements in Ω_*^{SO} detected by y_α, where $y_\alpha U$ are a basis of $H^*(MSO)$ over $'\mathcal{A}$. The dimensions such M_α appear in are 0, 8, 12, 16, 20, 20 in the range under discussion.

COROLLARY 6. In dimensions ≤ 26, all elements of Ω_*^{PL} are detected by ordinary characteristic classes. There is an M_1^{27} of order a such that $3M_1^{27}$ is not detected by an ordinary characteristic class.

It is hoped that soon there will be a determination of $H^*(B \text{ Coker } J)$ in a form so that theorem 5 can be generalized to all dimensions.

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