Extended power operation in homotopy theory

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§1. Introduction.

Given an CW complex K, we denote by $K^p(K^{(p)})$ the direct product (the reduced join) of p-copies of K, where p denotes a prime. Then as the cyclic permutations of factors, Π, the cyclic group of order p, operates on $K^p(K^{(p)})$. Let W be an acyclic Π-free complex. In the present note, we will mainly concern with the complexes $W\times_\Pi K^p$ and $W\times_\Pi K^{(p)}/W\times_\Pi *$ and the reduced powers in such complexes.

In their elaborate paper [2], Dyer and Lashof have considered the case in which K is an associative H-space and there exists a map $\theta_p; W\times_\Pi K^p \longrightarrow K$ which is an extension of the product map, and they have determined the structure of homology of infinite loop spaces. By making use of their results, we can describe the actions of reduced powers in infinite loop spaces.

As the next application, we will introduce Toda's work [6] about the stable homotopy groups of spheres. The key point of his methods is the construction of a functor $K \longrightarrow ep^r(K) = W^r\times_\Pi K^{(p)}/W^r\times_\Pi *$ where W^r denotes the r-skeleton of W and the determination of homotopy type $ep^r(K)$ for reasonable K and r.

§2. Cohomology operations in $W\times_{\Pi}K^p$.

To determine the actions of the reduced powers in $H^*(W\times_{\Pi}K^p : Z_p)$, we describe the homology and cohomology structure of $W\times_{\Pi}K^p$ following to [2], [5]. Let W be the acyclic Π-free complex having a single Π-generator e_i for each dimension i. Let $\{x_i\}$, $i = 1,2,\ldots,$ be a Z_p-basis of $H_*(K;Z_p)$ and $\{u_i\}$, $i = 1,2,\ldots,$ be the dual basis of $H^*(K;Z_p)$. Then the following homology classes form a Z_p-basis of $H_*(W\times_{\Pi}K^p;Z_p)$

$$e_i \otimes {}_{\Pi}x_j^p, \quad i \geq 0, \quad j \geq 1,$$

$e_0 \otimes {}_{\Pi}(x_{j1} \otimes \cdots \otimes x_{jp})$, $j_s \neq j_t$ for some s,t and $(j_1,\ldots,j_p)$ runs through each representative of the classes obtained by cyclic permutations

and $H^*(W\times_{\Pi}K^p;Z_p)$ has the following Z_p-basis.

$$w_i \times Pu_j, \quad i \geq 0, \quad j \geq 1.$$

$$z_k, \quad k = 1,2,\ldots \quad \text{and} \quad z_k \in \ker d^*$$

where w_i is the dual of homology class e_i and P denotes the external reduced power of Steenrod and $d; W\times_{\Pi}K \longrightarrow W\times_{\Pi}K^p$ the diagonal map. It is well known that the reduced powers are defined by the following equality.

$$\nu(q)d^*P(u) = \sum(-1)^i w_{(q-2i)(p-1)} \times P^i u + \sum(-1)^i w_{(q-2i)(p-1)-1} \times \beta P^i u$$

where $\nu(q) = m.(-1)^{m \cdot \frac{q(q+1)}{2}}$, $m = \frac{p-1}{2}$, $q = \dim u$ and β denotes Bockstein operation.

Proposition 2.1.

$$P^n(P(u)) = \sum \binom{(q-2i)m}{n-pi} w_{2(n-pi)(p-1)} P(P^i u)$$

$$- \mu(q) \sum \binom{(q-2i)m-1}{n-pi-1} w_{2(n-pi)(p-1)-p} P(\beta P^i u) + z$$

where $z \in \ker(d^*)$ and $\mu(q)$ denotes $(m!)^{-1}(-1)^{m(q+1)}$.

Proposition 2.2. We can choose basic elements z_k, $k = 1,2,\ldots,$ in $\ker(d^*)$ such that $w_i \times Pu_j$ and z_k form a Z_p-basis of $H^*(W\times_\pi K^p;Z_p)$ and $w_i \times Pu_j$ is dual to $e_i \otimes {}_\pi x_j^p$ for all i,j.

These propositions are proved by direct calculations and the duality in chain level. Now to state the following theorem, we denote by P_*^n the dual of P^n, i.e., $\langle P_*^n x,u\rangle = \langle x,P^n u\rangle$ for all $u \in H^*(X:Z_p)$.

Theorem 2.3.

$$P_*^n(e_{c+2n(p-1)} \otimes_\pi x^p) = \sum \binom{[c/2]+qm}{n-pi} e_{c+2pi(p-1)} \otimes {}_\pi(P_*^i x)^p$$

$$+ \mu(q)\varepsilon(c+1) \sum \binom{[c+1/2]+qm-1}{n-pi-1} e_{c+p+2pi(p-1)} \otimes {}_\pi(P_*^i \Delta x)^p$$

where Δ is the homology Bockstein operation, $\mu(q) \not\equiv 0(p)$, $\varepsilon(t) = 0$ if t odd, $= 1$ if t is even and $[x]$ denotes the Gaussian symbol.

Now let K be an H_p^∞-space in a sense of [2], i.e., K is an H-space with an associative product $\varphi: K \times K \longrightarrow K$ and $\varphi^{p-1}: K^p \longrightarrow K$ has Π-equivariant extension $\theta_p: W \times K^p \longrightarrow K$, where Π operates trivially on K. For example, every infinite loop space is an H_p^∞-space for all p. Then Dyer-Lashof operation Q_i is defined by

$$Q_i(x) = (\theta_p/\Pi)_*(e_i \otimes_\Pi x^p).$$

Corollary 2.4.

$$P_*^n Q_{c+2n(p-1)} x = \sum \binom{[c/2]+qm}{n-pi} Q_{c+2pi(p-1)} P_*^i x$$
$$+ \mu(q)\, \varepsilon\, (c+1) \sum \binom{[c+1/2]+qm-1}{n-pi-1} Q_{c+p+2pi(p-1)} P_*^i \Delta x.$$

§3. Application to homotopy theory.

Let α_1 be the first element of order p in the stable homotopy groups of spheres and let γ be an arbitrary element of order p. We denote C_{α_1} the mapping cone of α_1. Then it is obvious that $\alpha_1 \cdot \gamma^p = 0$ if and only if $\gamma^p: S^{k+pn} \longrightarrow S^k$ can be extended over C_{α_1}.

To research such extendability, we will consider the following construction, introduced by Toda [6].

Let $f: X \longrightarrow Y$ be base pointed finite CW complexes and a base point preserving continuous map. We denote by $ep^r(X)$ the complex $(W^r \times_\Pi X^{(p)})/(W^r \times_\Pi x_0^{(p)})$, where W^r indicates the r-skeleton of W and $X^{(p)}$ is the reduced join of p-copies of X. Of course we can define a canonical map $ep^r(f); ep^r(X) \longrightarrow ep^r(Y)$. It is easily seen that ep^r is a homotopy functor and $ep^0(X) = X^{(p)}$, $ep^0(f) = f^{(p)} = f \wedge \ldots \wedge f$ and $ep^r(X) \subset ep^s(X)$, $ep^s(f)\big|_{ep^r(f)} = ep^r(f)$ if $r \leq x$.

Remark. Let ξ be a vector bundle over K and let T_ξ be the Thom complex. We denote by 0_r the trivial 0 dimensional vector bundle over W^r. Then the vector bundle $0_r \times \xi^p$ over $W^r \times K^p$ admits Π-action. Since $W^r \times K^p$ is Π-free, we have a vector bundle $p(\xi) = 0_r \times \xi/\Pi$ over $W^r \times_\Pi K^p$. Then it is easily shown that $T_{p(\xi)} = ep^r(T_\xi)$. As the special case, there exists a map $ep^r(MSO(n)) \longrightarrow MSO(pn)$ extending the product $MSO(n)^{(p)} \longrightarrow MSO(pn)$.

Now we describe homotopy properties of $ep^r(X)$ for $X = S^n$ and $M_p^{n+1} = S^n \cup_p e^{n+1}$, the co-Moore space.

Lemma 3.1. $ep^0(S^n) = S^{pn}$ is the retract of $ep^{p-1}(S^n)$ for all n.

Lemma 3.2. $ep^{p-1}(M_p^{n+1})$ has the subcomplex

$$S^{pn} \cup_{\alpha_1} e^{pn+2p-2} \cup_p e^{pn+2p-1},$$

such that the inclusion map induces the isomorphism on H_{pn}.

The first lemma is an easy consequence of the results on the stable homotopy groups of spheres and the second can be proved by making use of the formulae in §2.

Theorem 3.3. (Toda). If γ is of order p, then $\alpha_1 \cdot \gamma^p = 0$.

Proof. Since $\gamma; S^{m+t} \longrightarrow S^m$ is of order p, there is a map $\bar{\gamma}; M_p^{m+1+t} \longrightarrow S^m$ such that $\bar{\gamma}|_{S^{m+t}} = \gamma$. Apply the functor ep^{p-1} to $\bar{\gamma}$. By the above lemmas, we have the following diagram

$$S^{pm+pt} \cup_{\alpha_1} e^{pm+pt+2p-2} \cup_p e^{pm+pt+2p-1} \subset ep^{p-1}(M_p^{m+t+1})$$

$$\xrightarrow{ep^{p-1}(\bar{\gamma})} ep^{p-1}(S^m) \xrightarrow{r} S^{pm}$$

and the restriction on S^{pm+pt} of the composite map coincides with γ^p, and this proves the theorem.

Let β_1 be a generator of p component of $G_{2p(p-1)-2}$, then similar arguments show that

Theorem 3.4. (Toda). If $\alpha_1 \cdot \gamma = 0$, then $\beta_1 \cdot \gamma^p = 0$.

Corollary 3.5. $\beta_1^{p^2+1} = 0$.

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