

MOD 2 COHOMOLOGY OF  $D2^n$  AND ITS EXTENSIONS BY  $Z_2$ 

by

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1. IDEA

Suppose that  $\bar{G}$  is a group of order  $2^n$  for which you want to find  $H^*(\bar{G}) = H^*(\bar{G}; Z_2)$  as an algebra over the Steenrod algebra  $\mathcal{A}$ . In principle this may be done as follows. Pick some invariant  $Z_2 \subseteq \bar{G}$  and compute  $H^*(\bar{G})$  from the spectral sequence for the extension  $1 \longrightarrow Z_2 \longrightarrow \bar{G} \longrightarrow G \longrightarrow 1$  (assuming inductively that you already know  $H^*(G)$ ). This involves two problems

- (A) Find  $E_\infty$  (i.e., compute the differentials),
- (B) Extension problem (including cup-products and action of  $Sq^i$ ).

Of course one cannot in general solve these problems. It was suggested to me by L. Kristensen that -- at any rate in favorable cases -- both problems can be attacked successfully using cochain operations in the sense of [K]. The following is a very preliminary report on that idea. It presents five infinite families of 2-primary groups for which the computations can be carried out without too much trouble. The method is certainly not limited to these five families. On the other hand, it would be unfair not to mention some serious drawbacks.

- (C) One must know  $\bar{G}$  pretty well.
- (D) The results come out in a highly non-functorial way.
- (E) Some examples that I have computed (but not included here)

indicate that the computational work may become tremendous, even for "nice, small groups."

## 2. COCHAIN OPERATIONS

Let us briefly recall some notions and results from [K].

$C^*(-; A)$  is the cochain functor on the category of CSS complexes  $C^i(X; Z_2 \oplus Z_2 \oplus \dots \oplus Z_2)$  is identified with  $C^i(X) \oplus \dots \oplus C^i(X)$ , where  $C^*(-) = C^*(X; Z_2)$ .  $Z^*(X)$  denotes the cocycles.

$\Theta^{(n)}$  is the set of natural transformations

$\theta: C^*(-; Z_2 \oplus Z_2 \oplus \dots \oplus Z_2) \longrightarrow C^*(-)$  (n summands  $Z_2$ ) satisfying

$$\theta(0) = 0,$$

$$\theta(C^i(X) \oplus C^i(X) \oplus \dots \oplus C^i(X)) \subseteq C^{i+k}(X)$$

for some fixed integer  $k$ , called the degree of  $\theta$ . Notice that  $\theta$  is not required to be additive.

$Q^{1,1}$  is the set of natural transformations

$\psi: C^*(X) \times C^*(X) \longrightarrow C^*(X)$ , satisfying

$$\psi(0, y) = \psi(x, 0) = 0$$

$$\psi(C^i(X) \times C^j(X)) \subseteq C^{i+j+k}(X)$$

for some fixed integer  $k$ , called the degree of  $\psi$ .

On  $\Theta^{(n)}$ , resp.  $Q^{1,1}$ , there is a differential  $\Delta$ , resp.  $\nabla$ , defined by the formula

$$(\Delta\theta)(x_1, \dots, x_n) = \delta\theta(x_1, \dots, x_n) + \theta(\delta x_1, \dots, \delta x_n),$$

$$\text{resp. } (\nabla\psi)(x, y) = \delta\psi(x, y) + \psi(\delta x, y) + \psi(x, \delta y).$$

And with  $Z\Theta^{(n)} = \ker \Delta$ ,  $ZQ^{1,1} = \ker \nabla$  one has exact sequences

$$\begin{aligned} \mathcal{O}^{(n)} &\xrightarrow{\Delta} Z\mathcal{O}^{(n)} \xrightarrow{\varepsilon} \mathcal{A} \oplus \mathcal{A} \oplus \dots \oplus \mathcal{A} \longrightarrow 0, \\ \mathcal{Q}^{1,1} &\xrightarrow{\nabla} Z\mathcal{Q}^{1,1} \xrightarrow{\varepsilon} \mathcal{A} \otimes \mathcal{A} \longrightarrow 0. \end{aligned}$$

For the definition of  $\varepsilon$  see [K]. Here we shall just need the following: If  $x \in Z^*(X)$  and  $\hat{x}$  denotes its class in  $H^*(X)$  then for any  $\theta$  in  $Z\mathcal{O} = Z\mathcal{O}^{(1)}$  one has  $(\varepsilon\theta)(\hat{x}) = (\theta x)^\wedge$ .

From the exact sequences one can get

Additivity defects: For any  $\theta \in Z\mathcal{O}$  there is an element  $d(\theta; -, -, \dots, -)$  in  $\mathcal{O}^{(n)}$  such that (with arbitrary cochains  $x_j$  in  $C^1(X)$ )

$$\delta d(\theta; x_1, \dots, x_n) + d(\theta; \delta x_1, \dots, \delta x_n) = \sum \theta(x_j) + \theta(\sum x_j),$$

$$d(\theta; x_1, \dots, x_n) = 0 \text{ if all but at most one } x_j \text{ is zero.}$$

Relational defects: If  $a \in Z\mathcal{O}$  and  $\varepsilon a = 0$  then there is an element  $\theta_a$  in  $\mathcal{O}$  such that (with  $x$  an arbitrary cochain)

$$\delta \theta_a(x) + \theta_a(\delta x) = a(x).$$

Cartan formula: Let  $a, a'_i, a''_i \in Z\mathcal{O}$  and assume that the diagonal  $\Psi$  in  $\mathcal{O}$  has  $\Psi(\varepsilon a) = \sum \varepsilon a'_i \otimes \varepsilon a''_i$ . Then there exists  $T_a \in \mathcal{Q}^{1,1}$  with

$$\delta T_a(x, y) + T_a(\delta x, y) + T_a(x, \delta y) =$$

$$a(xy) + \sum a'_i(x) \cdot a''_i(y) + d(a; \delta x \cdot y, x \cdot y, x \cdot \delta y) + \deg(x) d(a; x \cdot \delta y, x \cdot y)$$

(for arbitrary cochains  $x, y$  on  $X$ ).

We also need the Steenrod-cup- $i$ -products. To cochains  $x, y$  there is the cochain  $x \cup_i y$  of degree  $\deg(x) + \deg(y) - i$ .  $\cup_i$  is bilinear and satisfies

$$\delta(x \cup_i y) + \delta x \cup_i y + x \cup_i \delta y = x \cup_{i-1} y + y \cup_{i-1} x,$$

$$x \cup_0 y = xy, \quad x \cup_{-i} y = 0 \quad \text{for } i > 0,$$

$$(xy) \cup_1 z = x(y \cup_1 z) + (x \cup_1 z)y \quad (\text{Hirsch's formula}).$$

The formula

$$sq^i(x) = x \cup_{n-i} x + x \cup_{n-i+1} \delta x, \quad \text{where } n = \deg(x)$$

defines elements  $sq^i$  in  $Z\mathcal{O}$  having  $\epsilon(sq^i) = Sq^i$ . We introduce the abbreviations

$$d_i = d(sq^i; -, \dots, -) \in \mathcal{O}^{(n)},$$

$$T_i = T_{sq^i} \in Q^{1,1},$$

$$\theta_{i,j}^{p,q} = \theta_a, \quad \text{where } a = sq^i sq^j \dots + sq^p sq^q \dots, \quad \text{and}$$

$$Sq^i Sq^j \dots + Sq^p Sq^q \dots \quad \text{is supposed to be zero in } \mathcal{A}.$$

### 3. TRIVIAL LEMMAS

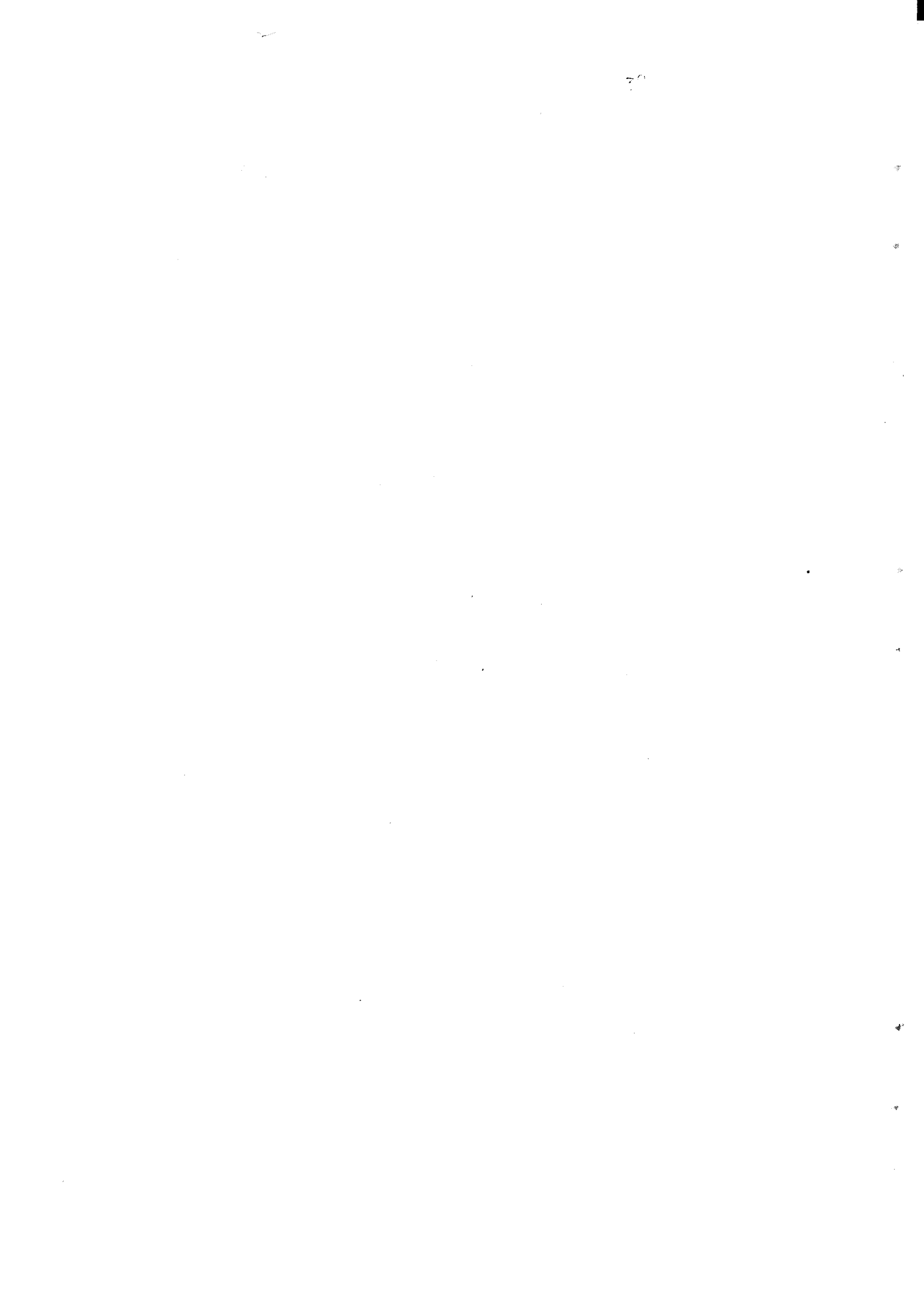
Let  $(E_r, d_r)$  be the spectral sequence for the extension

$$(S) \quad 1 \longrightarrow Z_2 \xrightarrow{i} \bar{G} \xrightarrow{p} G \longrightarrow 1$$

and let  $c \in H^2(G) = H^2(G; Z_2)$  be the characteristic class of (S) (see [ML]). One has

$$E_2^{**} = H^*(Z_2) \otimes H^*(G) = Z_2[t] \otimes H^*(G).$$

Lemma 1.  $c = d_2 t.$



$$\omega = \sum a_j(\tau) \bar{\beta}_j + \bar{\sigma}$$

is a q-decent cocycle on  $\bar{G}$ .

Remark.  $\bar{\quad}$  denotes  $p\#q\#$ .  $\omega$  is a cocycle on  $\bar{G}$ . q-decency means that  $\omega$  is a cocycle because of relations that hold already in  $C^*(H)$ . It would be more correct to say that the set  $((a_j, \beta_j)_{j, \sigma})$  is a q-decent representative of  $\omega$ , but it would also be more complicated.

A q-decent rearrangement of the above expression for  $\omega$  consists in repeated applications of the following operations.

- (4.1) Replace the term  $a(\tau)\bar{\beta}$  by  $c(\tau)\bar{\beta} + \theta(\bar{\gamma})\bar{\beta}$ , provided  $\Delta\theta = a + c$  ( $a, c \in Z\mathcal{D}$ ,  $\theta \in \mathcal{D}$ ),
- (4.2) Replace the term  $a(\tau)\bar{\beta}$  by  $a(\tau)\bar{\eta} + a(\bar{\gamma})\bar{\rho}$ , provided  $\delta\rho = \beta + \eta$  ( $\beta, \eta \in Z^*(H)$ ,  $\rho \in C^*(H)$ ),
- (4.3) Replace the term  $sq^k a(\tau)\bar{\beta}$  by  $a(\tau)a(\bar{\gamma})\bar{\beta}$ , provided  $\deg(a) = k - 2$  or by 0, provided  $\deg(a) < k - 2$  ( $a \in Z\mathcal{D}$ ,  $\beta \in Z^*(H)$ ).

A q-decent computation of  $sq^i \omega$  consists in the following

- (4.4) Write down the expression  $\sum_{j,k} sq^k a_j(\tau) sq^{i-k}(\bar{\beta}_j) + sq^i \bar{\sigma} + d_i(a_1(\bar{\gamma})\bar{\beta}_1, \dots, a_n(\bar{\gamma})\bar{\beta}_n, \delta\bar{\sigma}) + \sum_j T_i(a_j(\bar{\gamma}), \bar{\beta}_j)$ ,
- (4.5) Rearrange this expression by means of (4.1-3).

We shall write  $sq^i \omega \approx \sum c_j(\tau)\bar{\eta}_j + \bar{\rho}$  if the right hand side is the result of some q-decent computation of  $sq^i \omega$ .

A q-semidecent rearrangement of  $\sum a_j(\tau)\bar{\beta}_j$  consists in repeated applications of the following operations

- (4.6) Replace  $a(\tau)\bar{\beta}$  by  $c(\tau)\bar{\beta}$ , provided  $\epsilon a = \epsilon c$  ( $a, c \in Z\mathcal{D}$ ,  $\beta \in Z^*(H)$ ),

(4.7) Replace  $a(\tau)\bar{\beta}$  by  $a(\tau)\bar{\eta}$ , provided  $\hat{\beta} = \hat{\eta}$

( $\beta, \eta \in Z^*(H)$ ,  $a \in Z\theta$ ),

(4.8) Replace  $sq^k a(\tau)\bar{\beta}$  by  $a(\tau)a(\bar{\gamma})\bar{\beta}$ , provided  $\deg(a) = k - 2$  or by 0, provided  $\deg(a) < k - 2$  ( $a \in Z\theta, \beta \in Z^*(H)$ ).

A q-semidecent computation of  $sq^i w$  consists in the following

(4.9) Write down the expression  $\sum_{j,k} sq^k a_j(\tau) sq^{i-k}(\bar{\beta}_j)$ ,

(4.10) Rearrange this expression by means of (4.6-8).

We write  $sq^i w \approx \sum c_j(\tau)\bar{\eta}_j$  if the right hand side is the result of some q-semidecent computation of  $sq^i w$ .

It is obvious that any q-decent rearrangement of a q-decent cocycle  $w$  leads to a q-decent cocycle which is cohomologous to  $w$ ; in view of the defect-formulas one then gets

Lemma 4. Let  $w$  be a q-decent cocycle. Any q-decent computation of  $sq^i w$  leads to a q-decent cocycle cohomologous to  $sq^i w$ .

Addition of q-decent cocycles is of course defined. Also if  $w$  is a q-decent cocycle and  $\xi \in Z^*(H)$  then  $w\xi = \sum a_j(\tau)\bar{\beta}_j\xi + \bar{\sigma}\xi$  is a q-decent cocycle. It is now easy to prove

Lemma 5. Let  $w$  and  $w_k = \sum_j c_{kj}(\tau)\bar{\eta}_j + \bar{\rho}_k$  be q-decent cocycles.

If

$$sq^i w \approx \sum_{j,k} c_{kj}(\tau)\bar{\eta}_j\xi_k$$

then

$$sq^i w = \sum w_k \xi_k \text{ mod } \delta C^*(\bar{G}) + p^{\#} q^{\#} Z^*(H),$$

(and, hence

$$Sq^i \hat{\omega} \equiv \sum_k \hat{\omega}_k \hat{\xi}_k \pmod{p^*q^*H^*(H)}.$$

Lemma 5 of course is trivial; it is also useful; it allows us to compute  $Sq^i \hat{\omega} \pmod{p^*q^*H^*(H)}$  without bothering about additivity defects, Cartan-formula defects, non-commutativity of  $Z^*(H)$  and the like.

Remark. This section could easily be made more precise, e.g. by introducing the graded  $Z_2$ -module  $E(q)$  with  $E^n(q) = \Sigma(Z_2^{n-k-1} \otimes Z^k(H)) \oplus C^n(H)$  and introducing equivalence relations  $\approx$  and  $\cong$  in  $E(q)$ .

### 5. $H^*(D2^n)$

$D2^n$  = the dihedral group of order  $2^n$  has generators  $a_n, b_n$ , and relations  $a_n^{2^{n-1}} = b_n^2 = 1, b_n a_n b_n^{-1} = a_n^{-1}$ . Clearly  $D2 = Z_2$  (generator  $b_1$ ) and  $D4 = Z_2 \oplus Z_2$  (generators  $a_2, b_2$ ). The formulas

$$\begin{aligned} i_n b_1 &= a_n^{2^{n-2}}, & p_n a_n &= a_{n-1}, & p_n b_n &= b_{n-1}, & \tilde{p}_2 a_2 &= b_1, \\ \tilde{p}_2 b_2 &= 1, \end{aligned}$$

define a projection  $\tilde{p}_2: D4 \longrightarrow Z_2$  and a group extension

$$(S_n) \quad 1 \longrightarrow Z_2 \xrightarrow{i_n} D2^n \xrightarrow{p_n} D2^{n-1} \longrightarrow 1.$$

If  $\tau_0 \in Z^1(Z_2)$  is the standard cocycle we put

$$\begin{aligned} \xi_n &= p_n^{\#} p_{n-1}^{\#} \dots p_3^{\#} \tilde{p}_2^{\#} \tau_0 \in Z^1(D2^n), & \eta_n &= p_n^{\#} p_{n-1}^{\#} \dots p_3^{\#} p_2^{\#} \tau_0 \in Z^1(Z_2), \\ x_n &= \hat{\xi}_n, & y_n &= \hat{\eta}_n \in H^1(D2^n). \end{aligned}$$

Also let  $\left( \binom{(n)}{E_r}, \binom{(n)}{d_r} \right)$  be the spectral sequence for  $(S_n)$  and put

$$u_{n-1} = \binom{(n)}{d_2} t \in H^2(D2^{n-1}).$$

Theorem. For all  $m > 2$  one has

$$(a_m) \quad H^*(D2^m) = Z_2[x_m, y_m, u_m]/(x_m^2 + x_m y_m),$$

$$(b_m) \quad Sq^1 u_m = u_m y_m.$$

Proof: Let  $c_m$  and  $A_{n-1}$  be the statements

$$(c_m) \quad Sq^1 u_m \equiv u_m y_m \pmod{p_m^* H^*(D2^{m-1})},$$

$$(A_{n-1}) \quad (a_m) \text{ and } (c_m) \text{ hold for } 3 \leq m \leq n-1, (b_m) \text{ holds for } 3 \leq m \leq n-2.$$

We shall prove  $(A_n)$  by induction on  $n$ . The induction starts by verifying  $(A_3) = (a_3) \wedge (c_3)$ ; this is done precisely as step 2 and step 3 in the following inductive step, so we leave it to the reader (one has to know that  $u_2 = x_2^2 + x_2 y_2$ ; but that is easily gotten from lemma 1). Hence take  $n > 3$  and assume by induction that  $(A_{n-1})$  is true.

Step 1. From [W] it is easy to see that  $\dim H^2(D2^n) = 3$ . Also in  $\left( \binom{(n)}{E_r}, \binom{(n)}{d_r} \right)$  the elements  $t \otimes x_{n-1}$ ,  $t \otimes y_{n-1}$ ,  $t \otimes (x_{n-1} + y_{n-1})$ , and  $u_{n-1}$  cannot survive. Hence  $t^2$  must survive. But this implies that

$$\begin{aligned} \binom{(n)}{d_3} t^2 &= Sq^1 u_{n-1} \text{ belongs to } \binom{(n)}{d_2} t \cdot H^1(D2^{n-1}) = \\ &= u_{n-1} H^1(D2^{n-1}). \end{aligned}$$

From  $(c_{n-1})$  it is then easy to derive  $(b_{n-1})$ .

Step 2. We now know  $Sq^1 u_{n-1}$ , and lemma 3 gives

$$E_\infty = Z_2[t^2] \otimes Z_2[x_{n-1}, y_{n-1}] / (x_{n-1}^2 + x_{n-1}y_{n-1}).$$

Hence

$$(a'_n) \quad H^*(D2^n) = Z_2[x_n, y_n, w_n] / (x_n^2 + x_n y_n) \quad \text{for any } w_n \in H^2(D2^n) \\ \text{with } i_n^* w_n = t^2.$$

Lemma 2 implies that  $i_n^* u_n = t^2$ , so  $(a'_n) \implies (a_n)$ .

Step 3. From  $(n)_{d_2} t = u_{n-1}$ , and  $Sq^1 u_{n-1} = u_{n-1} y_{n-1}$  there are

$$\gamma \in Z^2(D2^{n-1}), \quad \delta \in C^2(D2^{n-1}), \quad \tau \in C^1(D2^n),$$

such that  $\hat{\gamma} = u_{n-1}$  and

$$\delta \tau = \bar{\gamma}, \quad (i_n^{\#} \tau)^\wedge = t, \quad sq^1 \gamma = \gamma \eta_{n-1} + \delta \delta.$$

Then

$$\omega = sq^1 \tau + \tau \eta_n + \bar{\delta}$$

is a decent (i.e. identity-decent) cocycle on  $D2^n$  whose class  $\hat{\omega}$  is a possible choice for the above  $w_n$ . A semidecent computation yields

$$sq^1 \omega \approx sq^1 \tau \cdot \eta_n + \tau \cdot \eta_n^2,$$

so by lemma 5

$$(c'_n) \quad Sq^1 \hat{\omega} \equiv \hat{\omega} y_n \pmod{p_n^* H^*(D2^{n-1})}.$$

A comparison of  $(a_n)$  and  $(a'_n)$  gives a relation of the form

$$\hat{w} = u_n + \lambda x_n^2 + \mu y_n^2,$$

in view of which it is easy to get  $(c_n)$  from  $(c'_n)$ .

## 6. $H^*(Q2^n)$

$Q2^n$  = generalized quaternion group of order  $2^n$  has generators  $\bar{a}_n, \bar{b}_n$ , and relations  $\bar{a}_n^{2^{n-1}} = 1$ ,  $\bar{b}_n^2 = \bar{a}_n^{2^{n-2}}$ ,  $\bar{b}_n \bar{a}_n \bar{b}_n^{-1} = \bar{a}_n^{-1}$ . There is the extension

$$(\bar{S}_{n+1}) \quad 1 \longrightarrow Z_2 \xrightarrow{\bar{i}_{n+1}} Q2^{n+1} \xrightarrow{\bar{p}_{n+1}} D2^n \longrightarrow 1,$$

and the classes  $\bar{x}_n, \bar{y}_n \in H^1(Q2^{n+1})$ ,  $\bar{u}_n \in H^2(Q2^{n+1})$ .

Theorem.  $\exists w_n \in H^4(Q2^n)$  such that

$$H^*(Q2^n) = Z_2[\bar{x}_{n-1}, \bar{y}_{n-1}, w_n] / (\bar{x}_{n-1}^2 + \bar{x}_{n-1} \bar{y}_{n-1} + \delta_{3,n} \bar{y}_{n-1}^2, \bar{y}_{n-1}^3)$$

$$\text{Sq}^i w_n = 0 \quad \text{for } i = 1, 2, 3.$$

Proof: Let us leave the case  $n = 3$  to the reader, just noticing that by lemma 1 one has  $d_2 t = x_2^2 + x_2 y_2 + y_2^2$  in the spectral sequence for  $(\bar{S}_3)$ .

We must find  $d_2 t$  in the s.s. for  $(\bar{S}_n)$  where now  $n > 3$ . Lemma 2 gives  $i_{n-1}^* d_2 t \neq 0$ , and hence  $d_2 t = u_{n-1} + ax_{n-1}^2 + by_{n-1}^2$ , where the coefficients  $a$  and  $b$  are still to be determined. To do so we borrow from [C-E] the fact that  $\dim H^4(Q2^n) = 1$ . Since  $t^4$  must survive (by lemma 3) we must kill off all of  $E_2^{0,4} = H^4(D2^{n-1})$ . Now write down a basis for  $E_2^{0,4}$  and write down a basis for the boundaries that are available to do the killing. It then follows that  $a = 0$ ,  $b = 1$ , so that  $d_2 t = u_{n-1} + y_{n-1}^2$ . Lemma 3 then gives us

$$E_\infty = Z_2[t^4] \otimes Z_2[x_{n-1}, y_{n-1}] / (x_{n-1}^2 + x_{n-1}y_{n-1}, y_{n-1}^3), \text{ so}$$

$$H^*(Q2^n) = Z_2[\bar{x}_{n-1}, \bar{y}_{n-1}, w_n] / (\bar{x}_{n-1}^2 + \bar{x}_{n-1}\bar{y}_{n-1}, \bar{y}_{n-1}^3)$$

for any  $w_n \in H^4(Q2^n)$  with  $\bar{i}_n^* w_n = t^4$ .

Since  $Sq^2 Sq^1(u_{n-1} + y_{n-1}^2) = Sq^1(u_{n-1} + y_{n-1}^2)(u_{n-1} + y_{n-1}^2)$  there is  $\gamma \in Z^2(D2^{n-1})$ ,  $\sigma \in C^4(D2^{n-1})$ ,  $\tau \in C^1(Q2^n)$  with

$$\hat{\gamma} = u_{n-1} + y_{n-1}^2 \text{ and}$$

$$\delta\tau = \bar{\gamma}, (\bar{i}_n^{\#}\tau)^\wedge = t, sq^2 sq^1 \gamma = sq^1 \gamma \cdot \gamma + \delta\sigma.$$

Then

$$\hat{w} = sq^2 sq^1 \tau + sq^1 \tau \cdot \bar{\gamma} + \bar{\sigma}$$

is a decent cocycle on  $Q2^n$  with  $\hat{w}$  a possible choice for the above  $w_n$ . Semidecent computations immediately give

$$sq^1 \hat{w} \approx 0,$$

$$sq^2 \hat{w} \approx sq^2 sq^1 \tau \cdot \bar{\gamma} + sq^1 \tau \cdot \bar{\gamma}^2,$$

$$sq^3 \hat{w} \approx sq^2 sq^1 \tau \cdot \bar{\gamma}^2 + sq^1 \tau \cdot \bar{\gamma}^3.$$

Since  $\bar{p}_n^* H^*(D2^{n-1}) = 0$  in dimensions  $> 3$ , and since we have lemma 5 we get the desired result, namely  $Sq^i \hat{w} = 0$  for  $i = 1, 2, 3$ .

## 7. EXTENSIONS OF $D2^n$ BY $Z_2$

An extension  $1 \longrightarrow Z_2 \xrightarrow{i} G \xrightarrow{p} D2^n \longrightarrow 1$  is classified by its characteristic class  $c \in H^2(D2^n)$ . The group  $G$  will be denoted  $G(c)$ . It is easy to see that we get six different groups, namely,

$$\begin{aligned}
G(0) &= \mathbb{Z}_2 \times D2^n, \\
G(u_n) &= D2^{n+1}, \\
G(u_n + y_n^2) &= Q2^{n+1}, \\
G(x_n^2) &= G(x_n^2 + y_n^2), \\
G(y_n^2), \\
G(u_n + x_n^2) &= G(u_n + x_n^2 + y_n^2).
\end{aligned}$$

We shall compute  $H^*(G(c))$  for the three last mentioned cases.  $\bar{\phantom{x}}$  will continue to denote pullback through  $G(c) \xrightarrow{p} D2^n \xrightarrow{q} H$  (for any  $q$ ).

### 8. $H^*(G(x_n^2))$

Lemma 3 does not apply, but it is easy to get

$$\begin{aligned}
E_\infty &= \mathbb{Z}_2[t^2] \otimes \mathbb{Z}_2[x_n, y_n, u_n] / (x_n^2, x_n y_n) \\
&\quad + \sum t^{2i+1} \otimes (x_n + y_n) / (x_n^2 + x_n y_n).
\end{aligned}$$

Hence, if  $v, z$  are classes on  $G(x_n^2)$  representing  $t^2$  and  $t \otimes (x_n + y_n)$  in  $E_\infty$ , then the ring  $H^*(G(x_n^2))$  is generated by  $\bar{x}_n, \bar{y}_n, \bar{u}_n, v$  and  $z$ .

Assume  $n > 3$ . We shall first use  $p_n$ -decent cocycles; let

$\gamma = \xi_{n-1}^2 \in \mathbb{Z}^2(D2^{n-1})$ ; there is  $\rho \in C^1(D2^{n-1})$  with

$\delta\rho = \gamma + \xi_{n-1}\eta_{n-1}$ . Choose  $\tau \in C^1(G(x_n^2))$  with  $\delta\tau = \bar{\gamma}$ ,  $(i\#\tau)^\wedge = t$ .

Then

$$\underline{v} = sq^1\tau + \theta_{11}(\bar{\xi}_{n-1}), \quad \underline{z} = \tau(\bar{\xi}_{n-1} + \bar{\eta}_{n-1}) + \bar{\xi}_{n-1}\bar{\rho},$$

are  $p_n$ -decent cocycles on  $G(x_n^2)$ , and their classes  $v$ , and  $z$  are possible choices for the above  $v, z$ . Easy  $p_n$ -semidecent computations + lemma 5 give

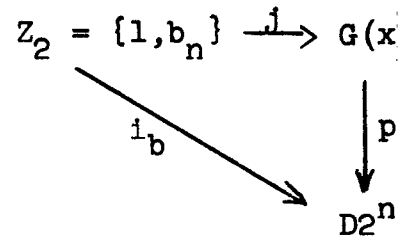
$$Sq^1 z \equiv (v+z)(\bar{x}_n + \bar{y}_n), \quad Sq^2 z \equiv v(\bar{x}_n + \bar{y}_n)^2$$

modulo  $p^*p_n^*H^*(D2^{n-1})$ . Since  $p_n^*u_{n-1} = 0$  (by definition of  $u_{n-1}$ ) and  $p^*p_n^*(x_{n-1}^2) = 0$ , this gives coefficients  $a, b$  such that

$$(8.1) \quad Sq^1 z = (v+z)(\bar{x}_n + \bar{y}_n) + a\bar{y}_n^3,$$

$$(8.2) \quad Sq^2 z = v(\bar{x}_n + \bar{y}_n)^2 + b\bar{y}_n^4.$$

To find  $a$  and  $b$  notice that the inclusion  $i_b: Z_2 = \{1, b_n\} \subseteq D2^n$  has  $i_b^\# \eta_n = \tau_0$ ,  $i_b^\# \xi_n = 0$ ; from the last one of these, it is easy to see that  $i_b$  factors through  $p$  like this



Then  $j^\# \bar{\xi}_{n-1} = 0$ , so  $j^\# \tau$  is a cocycle; let  $(j^\# \tau)^\wedge = vt$ . Now it follows that  $j^*v = j^*z = vt^2$ . Then apply  $j^*$  to (8.1) and (8.2) to get  $a = b = 0$ , i.e.,

$$(8.3) \quad Sq^1 z = (v + z)(\bar{x}_n + \bar{y}_n), \quad Sq^2 z = v(\bar{x}_n + \bar{y}_n)^2.$$

A  $p_n$ -semidecent rearrangement of

$$\underline{z}\bar{\xi}_{n-1} = \tau(\bar{\xi}_{n-1} + \bar{\eta}_{n-1})\bar{\xi}_{n-1} + \bar{\xi}_{n-1}\bar{\rho}\bar{\xi}_{n-1}$$

+ lemma 5 gives  $z\bar{x}_n \in p^*p_n^*H^*(D2^{n-1})$ . Hence as above there is a

coefficient  $a$  with  $z\bar{x}_n = a\bar{y}_n^3$ . Apply  $j^*$  to get  $a = 0$ , i.e.,

$$(8.4) \quad z\bar{x}_n = 0.$$

To find  $Sq^1 v$  notice that  $v = sq^1 \tau + \theta_{11}(\bar{\tau}_0)$  is actually  $\tilde{p}_2 p_3 \dots p_n$ -decent. Lemma 5 then easily gives  $Sq^1 v \in p^* p_n^* \dots p_3^* \tilde{p}_2^* H^*(Z_2)$ , but this group is 0, so

$$(8.5) \quad Sq^1 v = 0.$$

In  $H^*(G(x_n^2))$  we have now established the following relations

$$\bar{x}_n^2 = 0, \quad \bar{x}_n \bar{y}_n = 0, \quad z^2 = v(\bar{x}_n + \bar{y}_n)^2 (= v\bar{y}_n^2), \quad z\bar{x}_n = 0.$$

From the appearance of  $E_\infty$  it is not hard to see that there are no relations independent of the above. Hence

Theorem. Let  $n > 3$ .  $\exists v, z \in H^2(G(x_n^2))$  such that

$$H^*(G(x_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, \bar{u}_n, v, z] / (\bar{x}_n^2, \bar{x}_n \bar{y}_n, \bar{y}_n^2, z^2 + v\bar{y}_n^2, z\bar{x}_n) \\ Sq^1 z = (v + z)(\bar{x}_n + \bar{y}_n), \quad Sq^1 v = 0.$$

Remark. For  $n = 3$  there is no  $p_n$ -decent cocycle representing  $z$  (since in  $D^4$  there is no cochain  $\rho$  with  $\delta\rho = \epsilon_2^2 + \epsilon_2\eta_2$ ).

However,  $z$  does have a decent cocycle-representative  $\tau(\bar{\epsilon}_3 + \bar{\eta}_3) + \bar{\epsilon}_3\bar{\rho}$  with  $\rho \in C^1(D^8)$  and  $\delta\rho = \epsilon_3^2 + \epsilon_3\eta_3$ . A little bit of work then shows that the theorem is still true for  $n = 3$ , except that  $Sq^1 z = (v + z)(\bar{x}_3 + \bar{y}_3) + \bar{u}_3\bar{y}_3$ .

9.  $H^*(G(y_n^2))$

Here  $E_\infty = Z_2[t^2] \otimes Z_2[x_n, y_n, u_n] / (x_n^2 + x_n y_n, y_n^2)$ , so

$H^*(G(y_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, \bar{u}_n, v]/(\bar{x}_n^2 + \bar{x}_n \bar{y}_n, \bar{y}_n^2)$  for any  $v \in H^2$  with  $i^*v = t^2$ .

We take  $\gamma = \bar{\tau}_0^2$ , ( $= p^{\#} p_n^{\#} \dots p_3^{\#} p_2^{\#} \tau_0^2$ ), and choose  $\tau \in C^1(G(y_n^2))$  with  $\delta\tau = \bar{\tau}_0^2$ ,  $(i^{\#}\tau)^{\wedge} = t$ . Then  $\underline{v} = \text{sq}^1\tau + \theta_{11}(\bar{\tau}_0)$  is a  $p_2 p_3 \dots p_n$ -decent cocycle representing one particular choice of the above  $v$ . From lemma 5 and a  $p_2 p_3 \dots p_n$ -semidecent computation one gets  $\text{Sq}^1 v \in p^* p_n^* \dots p_2^* H^*(Z_2) = p^*(Z_2[y_n])$ ; but  $p^* y_n^2 = 0$ , so  $\text{Sq}^1 v = 0$ . Hence

Theorem.  $\exists v \in H^2(G(y_n^2))$  such that

$$H^*(G(y_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, \bar{u}_n, v]/(\bar{x}_n^2 + \bar{x}_n \bar{y}_n, \bar{y}_n^2),$$

$$\text{Sq}^1 v = 0.$$

10.  $H^*(G(u_n + x_n^2))$

Put  $c = u_n + x_n^2$  ( $= d_2 t$ ),  $z = x_n + y_n$  (and  $\zeta = \xi_n + \eta_n$ ). Then

$$(10.1) \quad \text{Sq}^1 c = c y_n + x_n^3, \quad \text{Sq}^2 \text{Sq}^1 c = \text{Sq}^1 c (c + z^2).$$

Also  $(0:c) = 0$  and  $(c:\text{Sq}^1 c) = (c, z)$ , so lemma 3 gives

$E_{\infty} = Z_2[t^4] \otimes Z_2[x_n, z]/(x_n z, x_n^3) + \Sigma t^{4i+2} \otimes z Z_2[z]$ . Hence, if  $w, v$  are classes on  $G(u_n + x_n^2)$  representing  $t^4$  and  $t^2 \otimes z$  in  $E_{\infty}$ , then  $\bar{x}_n, \bar{z}, w$ , and  $v$  generate  $H^*(G(u_n + x_n^2))$  as a ring.

It is not hard to see that there is a complete set of relations of the form

$$(10.2) \quad \bar{x}_n \bar{z} = 0, \quad \bar{x}_n^3 = 0 \quad (\text{from the basis}),$$

$v^2$  is a combination of  $w, v$ , and powers of  $\bar{x}_n$  and  $\bar{z}$ ,

$v \bar{x}_n$  is a polynomial in  $\bar{x}_n$  and  $\bar{z}$ .

We now choose  $\gamma \in Z^2(D2^n)$ ,  $\tau \in C^1(G(u_n + x_n^2))$ ,  $\rho, \sigma \in C^*(D2^n)$  such that

$$(10.3) \quad \begin{aligned} \hat{\gamma} &= c, \quad \delta\tau = \bar{\gamma}, \quad (i^{\#}\tau)^{\wedge} = t, \quad \delta\sigma = sq^1\gamma \cdot \zeta + \gamma\zeta^2, \\ \delta\rho &= sq^2sq^1\gamma + sq^1\gamma(\gamma + \zeta^2). \end{aligned}$$

There is the inclusion  $i_b:Z_2 = \{1, b_n\} \subseteq D2^n$ . It has  $i_b^{\#}\xi_n = 0$ ,  $i_b^{\#}\eta_n = i_b^{\#}\zeta = \tau_0$ . Also it lifts through  $p_{n+1}:D2^{n+1} \longrightarrow D2^n$ , so  $i_b^*u_n = 0$ ; it follows that  $i_b^{\#}\gamma$  is a coboundary; replace  $\gamma$  by  $\gamma + p_n^{\#}p_{n-1}^{\#}\dots p_2^{\#}i_b^{\#}\gamma$  (and adjust the choice of  $\tau$ ,  $\rho$ , and  $\sigma$  if needed). We then have

$$(10.4) \quad i_b^{\#}\xi_n = 0, \quad i_b^{\#}\eta_n = i_b^{\#}\zeta = \tau_0, \quad i_b^{\#}\gamma = 0.$$

Then  $i_b^{\#}\rho$  and  $i_b^{\#}\sigma$  are cocycles, so adjusting them by adding a power of  $\zeta$  we may assume that

$$(10.5) \quad (i_b^{\#}\rho)^{\wedge} = 0, \quad (i_b^{\#}\sigma)^{\wedge} = 0.$$

From (10.3) it is seen that

$$\begin{aligned} \omega &= sq^2sq^1\tau + sq^1\tau(\bar{\gamma} + \bar{\zeta}^2) + \bar{\rho}, \\ \underline{v} &= sq^1\tau \cdot \bar{\zeta} + \tau\bar{\zeta}^2 + \bar{\sigma} \end{aligned}$$

are decent cocycles, whose classes serve as the above  $w, v$ . Semidecent computations take the form

$$sq^1\omega \cong sq^1\tau \cdot sq^1\bar{\gamma} + sq^1\tau \cdot sq^1(\bar{\gamma} + \bar{\zeta}^2) \cong 0.$$

$$sq^2\omega \cong sq^2sq^1\tau \cdot (\bar{\gamma} + \bar{\zeta}^2) + sq^1\tau \cdot (\bar{\gamma} + \bar{\zeta}^2)^2 \cong \omega(\bar{\gamma} + \bar{\zeta}^2).$$

$$sq^3\omega \cong sq^1\tau \cdot sq^1\bar{\gamma}(\bar{\gamma} + \bar{\zeta}^2) + sq^2sq^1\tau \cdot sq^1(\bar{\gamma} + \bar{\zeta}^2) \cong \omega sq^1(\bar{\gamma} + \bar{\zeta}^2).$$

$$sq^1 \underline{v} \approx sq^1 \tau \cdot \bar{\zeta}^2 + sq^1 \tau \cdot \bar{\zeta}^2 \approx 0.$$

$$\begin{aligned} sq^2 \underline{v} &\approx sq^2 sq^1 \tau \cdot \bar{\zeta} + \tau(\bar{\gamma} \bar{\zeta}^2 + \bar{\zeta}^4) \\ &\approx \omega \bar{\zeta} + sq^1 \tau(\bar{\gamma} + \bar{\zeta}^2) \bar{\zeta} + \tau(\bar{\gamma} \bar{\zeta}^2 + \bar{\zeta}^4) \\ &\approx \omega \bar{\zeta} + \underline{v}(\bar{\gamma} + \bar{\zeta}^2). \end{aligned}$$

$$\begin{aligned} sq^3 \underline{v} &\approx sq^1 \tau \cdot sq^1 \bar{\gamma} \bar{\zeta} + sq^2 sq^1 \tau \cdot \bar{\zeta}^2 + sq^1 \tau \cdot \bar{\zeta}^4 \\ &\approx \omega \bar{\zeta}^2 + sq^1 \tau [sq^1 \bar{\gamma} \bar{\zeta} + \bar{\zeta}^4 + (\bar{\gamma} + \bar{\zeta}^2) \bar{\zeta}^2] \\ &\approx \omega \bar{\zeta}^2. \end{aligned}$$

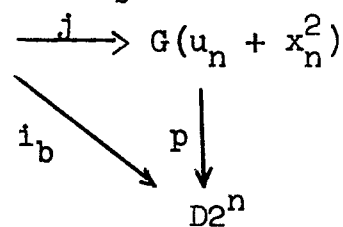
$$\underline{v} \bar{\xi}_n = sq^1 \tau \cdot \bar{\zeta} \bar{\xi}_n + \tau \bar{\zeta}^2 \bar{\xi}_n \approx 0.$$

In dimensions above 3 one has  $p^*H^*(D2^n) = Z_2[\bar{z}]$ , so lemma 5 gives us coefficients  $a_i, b_i$ , and  $a$  such that

$$(10.6) \quad \begin{aligned} Sq^1 w &= a_1 \bar{z}^5, Sq^2 w = w \bar{z}^2 + a_2 \bar{z}^6, Sq^3 w = a_3 \bar{z}^7, \\ Sq^1 v &= b_1 \bar{z}^4, Sq^2 v = w \bar{z} + v \bar{z}^2 + b_2 \bar{z}^5, Sq^3 v = v^2 = w \bar{z}^2 + b_3 \bar{z}^6, \end{aligned}$$

$$(10.7) \quad v \bar{x}_n = a \bar{z}^4.$$

To find these coefficients first notice that  $i_b$  factors through  $p: G(u_n + x_n^2) \longrightarrow D2^n$  like this



(this follows from  $i_b^\# \gamma = 0$ ). From (10.3-5) it is easy to see that  $j^*z = t$  and  $j^*w = 0, j^*v = 0$ . Putting that much information into (10.6) it follows that  $a_i = b_i = 0$ . Also  $j^* \bar{x}_n = 0$  follows from (10.7).

Altogether we have proved

Theorem.  $\exists w \in H^4(G(u_n + x_n^2)), v \in H^3(G(u_n + x_n^2))$  such that  
 $H^*(G(u_n + x_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, w, v] / (\bar{x}_n^2 + \bar{x}_n \bar{y}_n, \bar{x}_n^3, v \bar{x}_n, v^2 + w(\bar{x}_n + \bar{y}_n)^2),$   
 $Sq^1 w = 0, Sq^2 w = w(\bar{x}_n + \bar{y}_n)^2, Sq^3 w = 0,$   
 $Sq^1 v = 0, Sq^2 v = w(\bar{x}_n + \bar{y}_n) + v \bar{y}_n^2.$

## 11. REFERENCES

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