MOD 2 COHOMOLOGY OF $D2^n$ AND ITS EXTENSIONS BY Z_2

by

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1. IDEA

Suppose that $\bar{G}$ is a group of order 2^n for which you want to find $H^*(\bar{G}) = H^*(\bar{G};Z_2)$ as an algebra over the Steenrod algebra $\mathfrak{A}$. In principle this may be done as follows. Pick some invariant $Z_2 \subseteq \bar{G}$ and compute $H^*(\bar{G})$ from the spectral sequence for the extension $1 \longrightarrow Z_2 \longrightarrow \bar{G} \longrightarrow G \longrightarrow 1$ (assuming inductively that you already know $H^*(G)$). This involves two problems

(A) Find E_∞ (i.e., compute the differentials),

(B) Extension problem (including cup-products and action of Sq^i).

Of course one cannot in general solve these problems. It was suggested to me by L. Kristensen that -- at any rate in favorable cases -- both problems can be attacked successfully using cochain operations in the sense of [K]. The following is a very preliminary report on that idea. It presents five infinite families of 2-primary groups for which the computations can be carried out without too much trouble. The method is certainly not limited to these five families. On the other hand, it would be unfair not to mention some serious drawbacks.

(C) One must know $\bar{G}$ pretty well.

(D) The results come out in a highly non-functorial way.

(E) Some examples that I have computed (but not included here) indicate that the computational work may become tremendous, even for "nice, small groups."

2. COCHAIN OPERATIONS

Let us briefly recall some notions and results from [K].

$C^*(-;A)$ is the cochain functor on the category of CSS complexes $C^i(X;Z_2\oplus Z_2\oplus\ldots\oplus Z_2)$ is identified with $C^i(X)\oplus\ldots\oplus C^i(X)$, where $C^*(-)=C^*(X;Z_2)$. $Z^*(X)$ denotes the cocycles.

$\mathcal{O}^{(n)}$ is the set of natural transformations

$\theta : C^*(-;Z_2\oplus Z_2\oplus\ldots\oplus Z_2)\longrightarrow C^*(-)$ (n summands Z_2) satisfying

$\theta(0)=0$,

$\theta(C^i(X)\oplus C^i(X)\oplus\ldots\oplus C^i(X))\subseteq C^{i+k}(X)$

for some fixed integer k, called the degree of θ. Notice that θ is not required to be additive.

$Q^{1,1}$ is the set of natural transformations

$\psi : C^*(X)\times C^*(X)\longrightarrow C^*(X)$, satisfying

$\psi(0,y)=\psi(x,0)=0$

$\psi\,(C^i(X)\times C^j(X))\subseteq C^{i+j+k}(X)$

for some fixed integer k, called the degree of ψ.

On $\mathcal{O}^{(n)}$, resp. $Q^{1,1}$, there is a differential Δ, resp. ∇, defined by the formula

$$(\Delta\theta)(x_1,\ldots,x_n)=\delta\theta(x_1,\ldots,x_n)+\theta(\delta x_1,\ldots,\delta x_n),$$

resp. $(\nabla\psi)(x,y)=\delta\psi(x,y)+\psi(\delta x,y)+\psi(x,\delta y)$.

And with $Z\mathcal{O}^{(n)}=\ker\Delta$, $ZQ^{1,1}=\ker\nabla$ one has exact sequences

$$\mathcal{B}^{(n)} \xrightarrow{\Delta} Z\mathcal{B}^{(n)} \xrightarrow{\varepsilon} \mathfrak{A} \oplus \mathfrak{A} \oplus \cdots \oplus \mathfrak{A} \longrightarrow 0,$$

$$Q^{1,1} \xrightarrow{\nabla} ZQ^{1,1} \xrightarrow{\varepsilon} \mathfrak{A} \otimes \mathfrak{A} \longrightarrow 0.$$

For the definition of ε see [K]. Here we shall just need the following: If $x \in Z^*(X)$ and $\hat{x}$ denotes its class in $H^*(X)$ then for any θ in $Z\mathcal{B} = Z\mathcal{B}^{(1)}$ one has $(\varepsilon\theta)(\hat{x}) = (\theta x)^\wedge$.

From the exact sequences one can get

Additivity defects: For any $\theta \in Z\mathcal{B}$ there is an element $d(\theta;-,-,\ldots,-)$ in $\mathcal{B}^{(n)}$ such that (with arbitrary cochains x_j in $C^i(X)$)

$$\delta d(\theta;x_1,\ldots,x_n) + d(\theta;\delta x_1,\ldots,\delta x_n) = \Sigma\theta(x_j) + \theta(\Sigma x_j),$$

$$d(\theta;x_1,\ldots,x_n) = 0 \text{ if all but at most one } x_j \text{ is zero.}$$

Relational defects: If $a \in Z\mathcal{B}$ and $\varepsilon a = 0$ then there is an element θ_a in $\mathcal{B}$ such that (with x an arbitrary cochain)

$$\delta\theta_a(x) + \theta_a(\delta x) = a(x).$$

Cartan formula: Let $a, a_i', a_i'' \in Z\mathcal{B}$ and assume that the diagonal Ψ in $\mathfrak{A}$ has $\Psi(\varepsilon a) = \Sigma\, \varepsilon a_i' \otimes \varepsilon a_i''$. Then there exists $T_a \in Q^{1,1}$ with

$$\delta T_a(x,y) + T_a(\delta x,y) + T_a(x,\delta y) =$$

$$a(xy) + \Sigma a_i'(x)\cdot a_i''(y) + d(a;\delta x\cdot y, x\cdot y, x\cdot\delta y) + \deg(x)d(a;x\cdot\delta y, x\cdot$$

(for arbitrary cochains x,y on X).

We also need the Steenrod-cup-i-products. To cochains x,y there is the cochain $x \cup_i y$ of degree $\deg(x) + \deg(y) - i$. $\cup_i$ is bilinear and satisfies

$$\delta(x \cup_i y) + \delta x \cup_i y + x \cup_i \delta y = x \cup_{i-1} y + y \cup_{i-1} x,$$

$$x \cup_o y = xy, \; x \cup_{-i} y = 0 \quad \text{for} \quad i > 0,$$

$$(xy) \cup_1 z = x(y \cup_1 z) + (x \cup_1 z)y \quad \textbf{(Hirsch's formula)}.$$

The formula

$$sq^i(x) = x \cup_{n-i} x + x \cup_{n-i+1} \delta x, \quad \text{where} \quad n = \deg(x)$$

defines elements sq^i in $Z\mathcal{O}$ **having** $\varepsilon(sq^i) = Sq^i$. We introduce the abbreviations

$$d_i = d(sq^i;-,-,\ldots,-) \in \mathcal{O}^{(n)},$$

$$T_i = T_{sq^i} \in Q^{1,1},$$

$\theta^{p,q\ldots}_{i,j\ldots} = \theta_a$, where $a = sq^i sq^j\ldots + sq^p sq^q\ldots$, and $Sq^i Sq^j\ldots + Sq^p Sq^q\ldots$ is supposed to be zero in $\mathcal{A}$.

3. TRIVIAL LEMMAS

Let (E_r, d_r) be the spectral sequence for the **extension**

(S) $$1 \longrightarrow Z_2 \xrightarrow{i} \bar{G} \xrightarrow{p} G \longrightarrow 1$$

and let $c \in H^2(G) = H^2(G;Z_2)$ be the characteristic class of (S) (see [ML]). One has

$$E_2^{**} = H^*(Z_2) \otimes H^*(G) = Z_2[t] \otimes H^*(G).$$

Lemma 1. $c = d_2 t$.

Lemma 2. If b_1 is the generator of Z_2 and $1 \longrightarrow Z_2 \xrightarrow{j} G \xrightarrow{q} H \longrightarrow 1$ is another extension then

$d_2 t \in \mathrm{im}(q^*)$ iff $j(b_1)$ lifts (through p) to a central involution (in $\bar{G}$),

$d_2 t \notin \ker(j^*)$ iff $j(b_1)$ lifts to an involution,

$d_2 t \in \ker(j^*)$ iff $j(b_1)$ lifts to an element of order 4.

Lemma 3. Suppose that $Sq^2 Sq^1 c \in (c, Sq^1 c)$ and that $(0{:}c) \cap (0{:}Sq^1 c) = 0$. Then

$$E_\infty = Z_2[t^4] \otimes H^*(G)/(c, Sq^1 c) + \Sigma\, t^{4i+1} \otimes (0{:}c)/(0{:}c)Sq^1 c + $$
$$+ \Sigma\, t^{4i+2} \otimes (c{:}Sq^1 c)/(c).$$

Remarks: 1. Lemma 3 abuses language as usual.

2. $(x, y, \ldots)$ denotes the ideal generated by $x, y \ldots$

3. $(a{:}b) = \{x; xb \in (a)\}$,

4. The assumption on $(0{:}c) \cap (0{:}Sq^1 c)$ is made in order to avoid Massey products as differentials; lemma 3 covers all the cases needed here.

4. COMPUTATIONS OR SOME TRIVIAL, USEFUL NONSENSE

Consider the extension (S) and a homomorphism $q{:}G \longrightarrow H$ (often q will be the identity). Suppose that there is a cocycle γ on H with $q^*\hat{\gamma} = c$ $(= d_2 t)$. One can then choose a cochain τ on $\bar{G}$ with $\delta\tau = \bar{\gamma}$, and $(i^{\#}\tau)^\wedge = t$. Let $a_j \in Z\mathcal{O}$, $\beta_j \in Z^*(H)$, $\sigma \in C^*(H)$. If

$$\delta\sigma = \Sigma a_j(\gamma)\beta_j$$

then we say that

$$\omega = \Sigma a_j(\tau)\bar{\beta}_j + \bar{\sigma}$$

is a q-<u>decent cocycle on</u> $\bar{G}$.

<u>Remark</u>. $\bar{\ }$ denotes $p^{\#}q^{\#}$. ω <u>is</u> a cocycle on $\bar{G}$. q-decency means that ω is a cocycle because of relations that hold already in $C^*(H)$. It would be more correct to say that the set $((a_j,\beta_j)_j,\sigma)$ is a q-decent representative of ω, but it would also be more complicated.

A q-<u>decent rearrangement</u> of the above expression for ω consists in repeated applications of the following operations.

(4.1) Replace the term $a(\tau)\bar{\beta}$ by $c(\tau)\bar{\beta} + \theta(\bar{\gamma})\bar{\beta}$, provided $\Delta\theta = a + c$ $(a,c \in Z\mathcal{O},\ \theta \in \mathcal{O})$,

(4.2) Replace the term $a(\tau)\bar{\beta}$ by $a(\tau)\bar{\eta} + a(\bar{\gamma})\bar{\rho}$, provided $\delta\rho = \beta + \eta$ $(\beta,\eta \in Z^*(H),\ \rho \in C^*(H))$,

(4.3) Replace the term $sq^k a(\tau)\bar{\beta}$ by $a(\tau)a(\bar{\gamma})\bar{\beta}$, provided $\deg(a) = k - 2$ or by 0, provided $\deg(a) < k - 2$ $(a \in Z\mathcal{O},\beta \in Z^*(H))$.

A q-<u>decent computation of</u> $sq^i\omega$ consists in the following

(4.4) Write down the expression $\sum_{j,k} sq^k a_j(\tau)\ sq^{i-k}(\bar{\beta}_j) +$

$$+ sq^i\bar{\sigma} + d_i(a_1(\bar{\gamma})\bar{\beta}_1,\ldots,a_n(\bar{\gamma})\bar{\beta}_n,\delta\bar{\sigma}) + \sum_j T_i(a_j(\bar{\gamma}),\bar{\beta}_j),$$

(4.5) Rearrange this expression by means of (4.1-3).

We shall write $sq^i\omega \approx \Sigma c_j(\tau)\bar{\eta}_j + \bar{\rho}$ if the right hand side is the result of some q-decent computation of $sq^i\omega$.

A q-<u>semidecent rearrangement of</u> $\Sigma a_j(\tau)\bar{\beta}_j$ consists in repeated applications of the following operations

(4.6) Replace $a(\tau)\bar{\beta}$ by $c(\tau)\bar{\beta}$, provided $\varepsilon a = \varepsilon c$ $(a,c \in Z\mathcal{O},\beta \in Z^*(H))$,

(4.7) Replace $a(\tau)\overline{\beta}$ by $a(\tau)\overline{\eta}$, provided $\hat{\beta} = \hat{\eta}$
$(\beta, \eta \in Z^*(H),\ a \in Z\Theta)$,

(4.8) Replace $sq^k a(\tau)\overline{\beta}$ by $a(\tau)a(\overline{\gamma})\overline{\beta}$, provided $\deg(a) = k - 2$
or by 0, provided $\deg(a) < k - 2$ $(a \in Z\Theta, \beta \in Z^*(H))$.

A q-<u>semidecent computation of</u> $sq^i\omega$ consists in the following

(4.9) Write down the expression $\sum_{j,k} sq^k a_j(\tau)\ sq^{i-k}(\overline{\beta}_j)$,

(4.10) Rearrange this expression by means of (4.6-8).

We write $sq^i\omega \approx \Sigma c_j(\tau)\overline{\eta}_j$ if the right hand side is the result of some q-semidecent computation of $sq^i\omega$.

It is obvious that any q-decent rearrangement of a q-decent cocycle ω leads to a q-decent cocycle which is cohomologous to ω; in view of the defect-formulas one then gets

<u>Lemma 4</u>. Let ω be a q-decent cocycle. Any q-decent computation of $sq^i\omega$ leads to a q-decent cocycle cohomologous to $sq^i\omega$.

Addition of q-decent cocycles is of course defined. Also if ω is a q-decent cocycle and $\xi \in Z^*(H)$ then $\omega\overline{\xi} = \Sigma\, a_j(\tau)\overline{\beta}_j\overline{\xi} + \overline{\sigma\xi}$ is a q-decent cocycle. It is now easy to prove

<u>Lemma 5</u>. Let ω and $\omega_k = \Sigma_j c_{kj}(\tau)\overline{\eta}_j + \overline{\rho}_k$ be q-decent cocycles. If

$$sq^i\omega \approx \sum_{j,k} c_{kj}(\tau)\overline{\eta}_j\overline{\xi}_k$$

then

$$sq^i\omega \equiv \Sigma\, \omega_k\, \overline{\xi}_k \bmod \delta C^*(\overline{G}) + p^{\#}q^{\#}Z^*(H),$$

(and, hence

$$Sq^1\hat{\omega} \equiv \Sigma\hat{\omega}_k\hat{\xi}_k \bmod p^*q^*H^*(H)).$$

Lemma 5 of course is trivial; it is also useful; it allows us to compute $Sq^1\hat{\omega}$ mod $p^*q^*H^*(H)$ without bothering about additivity defects, Cartan-formula defects, non-commutativity of $Z^*(H)$ and the like.

Remark. This section could easily be made more precise, e.g. by introducing the graded Z_2-module $E(q)$ with $E^n(q) =$ $= \Sigma(Z\mathcal{O}^{n-k-1} \otimes Z^k(H)) \oplus C^n(H)$ and introducing equivalence relations $\approx$ and $\approxeq$ in $E(q)$.

5. $H^*(D2^n)$

$D2^n$ = the dihedral group of order 2^n has generators a_n, b_n, and relations $a_n^{2^{n-1}} = b_n^2 = 1$, $b_na_nb_n^{-1} = a_n^{-1}$. Clearly $D2 = Z_2$ (generator b_1) and $D4 = Z_2 \oplus Z_2$ (generators a_2, b_2). The formulas

$$i_nb_1 = a_n^{2^{n-2}},\ p_na_n = a_{n-1},\ p_nb_n = b_{n-1},\ \tilde{p}_2a_2 = b_1,$$
$$\tilde{p}_2b_2 = 1,$$

define a projection $\tilde{p}_2 : D4 \longrightarrow Z_2$ and a group extension

$$(S_n) \qquad 1 \longrightarrow Z_2 \xrightarrow[i_n]{} D2^n \xrightarrow[p_n]{} D2^{n-1} \longrightarrow 1.$$

If $\tau_o \in Z^1(Z_2)$ is the standard cocycle we put

$$\xi_n = p_n^{\#}p_{n-1}^{\#}\cdots p_3^{\#}\tilde{p}_2^{\#}\tau_o \in Z^1(D2^n),\ \eta_n = p_n^{\#}p_{n-1}^{\#}\cdots p_3^{\#}p_2^{\#}\tau_o \in Z^1(Z_2),$$
$$x_n = \hat{\xi}_n,\ y_n = \hat{\eta}_n \in H^1(D2^n).$$

Also let $\left({}^{(n)}E_r, {}^{(n)}d_r\right)$ be the spectral sequence for (S_n) and put

$$u_{n-1} = {}^{(n)}d_2 t \in H^2(D2^{n-1}).$$

<u>Theorem</u>. For all $m > 2$ one has

(a_m) $\quad H^*(D2^m) = Z_2[x_m, y_m, u_m]/(x_m^2 + x_m y_m)$,

(b_m) $\quad Sq^1 u_m = u_m y_m$.

<u>Proof</u>: Let c_m and A_{n-1} be the statements

(c_m) $\quad Sq^1 u_m \equiv u_m y_m \bmod p_m^* H^*(D2^{m-1})$,

(A_{n-1}) $\quad (a_m)$ and (c_m) hold for $3 \leq m \leq n-1$, (b_m) holds for $3 \leq m \leq n-2$.

We shall prove (A_n) by induction on n. The induction starts by verifying $(A_3) = (a_3) \wedge (c_3)$; this is done precisely as step 2 and step 3 in the following inductive step, so we leave it to the reader (one has to know that $u_2 = x_2^2 + x_2 y_2$; but that is easily gotten from lemma 1). Hence take $n > 3$ and assume by induction that (A_{n-1}) is true.

<u>Step 1</u>. From [W] it is easy to see that $\dim H^2(D2^n) = 3$. Also in $\left({}^{(n)}E_r, {}^{(n)}d_r\right)$ the elements $t \otimes x_{n-1}$, $t \otimes y_{n-1}$, $t \otimes (x_{n-1}+y_{n-1})$, and u_{n-1} cannot survive. Hence t^2 must survive. But this implies that

$$\begin{aligned} {}^{(n)}d_3 t^2 = Sq^1 u_{n-1} \text{ belongs to } {}^{(n)}d_2 t \cdot H^1(D2^{n-1}) &= \\ = u_{n-1} H^1(D2^{n-1}). \end{aligned}$$

From (c_{n-1}) it is then easy to derive (b_{n-1}).

Step 2. We **now know** $Sq^1 u_{n-1}$, and lemma 3 gives

$$E_\infty = Z_2[t^2] \otimes Z_2[x_{n-1}, y_{n-1}]/(x_{n-1}^2 + x_{n-1}y_{n-1}).$$

Hence

(a_n') $\quad H^*(D2^n) = Z_2[x_n, y_n, w_n]/(x_n^2 + x_n y_n)$ for any $w_n \in H^2(D2^n)$ with $i_n^* w_n = t^2$.

Lemma 2 implies that $i_n^* u_n = t^2$, so $(a_n') \Longrightarrow (a_n)$.

Step 3. From ${}^{(n)}d_2 t = u_{n-1}$, and $Sq^1 u_{n-1} = u_{n-1} y_{n-1}$ there are

$$\gamma \in Z^2(D2^{n-1}), \quad \sigma \in C^2(D2^{n-1}), \quad \tau \in C^1(D2^n),$$

such that $\hat{\gamma} = u_{n-1}$ and

$$\delta\tau = \bar{\gamma}, \ (i_n^{\#}\tau)^\wedge = t, \ sq^1\gamma = \gamma\eta_{n-1} + \delta\sigma.$$

Then

$$\omega = sq^1\tau + \tau\eta_n + \bar{\sigma}$$

is a decent (i.e. identity-decent) cocycle on $D2^n$ whose class $\hat{\omega}$ is a possible choice for the above w_n. A semidecent computation yields

$$sq^1\omega \approx sq^1\tau \cdot \eta_n + \tau \cdot \eta_n^2,$$

so by lemma 5

(c_n') $\quad Sq^1\hat{\omega} \equiv \hat{\omega} y_n \mod p_n^* H^*(D2^{n-1}).$

A comparison of (a_n) and (a_n') gives a relation of the form

$$\hat{\omega} = u_n + \lambda x_n^2 + \mu y_n^2,$$

in view of which it is easy to get (c_n) from (c'_n).

6. $H^*(Q2^n)$

$Q2^n$ = generalized quaternion group of order 2^n has generators $\bar{a}_n, \bar{b}_n$, and relations $\bar{a}_n^{2^{n-1}} = 1$, $\bar{b}_n^2 = \bar{a}_n^{2^{n-2}}$, $\bar{b}_n \bar{a}_n \bar{b}_n^{-1} = \bar{a}_n^{-1}$. There is the extension

$$(\bar{S}_{n+1}) \qquad 1 \longrightarrow Z_2 \xrightarrow{\bar{i}_{n+1}} Q2^{n+1} \xrightarrow{\bar{p}_{n+1}} D2^n \longrightarrow 1,$$

and the classes $\bar{x}_n, \bar{y}_n \in H^1(Q2^{n+1})$, $\bar{u}_n \in H^2(Q2^{n+1})$.

__Theorem__. $\exists\, w_n \in H^4(Q2^n)$ such that

$$H^*(Q2^n) = Z_2[\bar{x}_{n-1}, \bar{y}_{n-1}, w_n]/(\bar{x}_{n-1}^2 + \bar{x}_{n-1}\bar{y}_{n-1} + \delta_{3,n}\bar{y}_{n-1}^2, \bar{y}_{n-1}^3)$$

$$Sq^i w_n = 0 \quad \text{for} \quad i = 1,2,3.$$

__Proof__: Let us leave the case $n = 3$ to the reader, just noticing that by lemma 1 one has $d_2 t = x_2^2 + x_2 y_2 + y_2^2$ in the spectral sequence for $(\bar{S}_3)$.

We must find $d_2 t$ in the s.s. for $(\bar{S}_n)$ where now $n > 3$. Lemma 2 gives $i_{n-1}^* d_2 t \neq 0$, and hence $d_2 t = u_{n-1} + a x_{n-1}^2 + b y_{n-1}^2$, where the coefficients a and b are still to be determined. To do so we borrow from [C-E] the fact that $\dim H^4(Q2^n) = 1$. Since t^4 must survive (by lemma 3) we must kill off all of $E_2^{0,4} = H^4(D2^{n-1})$. Now write down a basis for $E_2^{0,4}$ and write down a basis for the boundaries that are available to do the killing. It then follows that $a = 0$, $b = 1$, so that $d_2 t = u_{n-1} + y_{n-1}^2$. Lemma 3 then gives us

$$E_\infty = Z_2[t^4] \otimes Z_2[x_{n-1}, y_{n-1}]/(x_{n-1}^2 + x_{n-1}y_{n-1}, y_{n-1}^3), \quad \text{so}$$

$$H^*(Q2^n) = Z_2[\bar{x}_{n-1}, \bar{y}_{n-1}, w_n]/(\bar{x}_{n-1}^2 + \bar{x}_{n-1}\bar{y}_{n-1}, \bar{y}_{n-1}^3)$$

for any $w_n \in H^4(Q2^n)$ with $\bar{i}_n^* w_n = t^4$.

Since $Sq^2Sq^1(u_{n-1} + y_{n-1}^2) = Sq^1(u_{n-1} + y_{n-1}^2)(u_{n-1} + y_{n-1}^2)$ there is $\gamma \in Z^2(D2^{n-1})$, $\sigma \in C^4(D2^{n-1})$, $\tau \in C^1(Q2^n)$ with $\hat{\gamma} = u_{n-1} + y_{n-1}^2$ and

$$\delta\tau = \bar{\gamma}, \ (\bar{i}_n^{\#}\tau)^\wedge = t, \ sq^2sq^1\gamma = sq^1\gamma\cdot\gamma + \delta\sigma.$$

Then

$$\omega = sq^2sq^1\tau + sq^1\tau\cdot\bar{\gamma} + \bar{\sigma}$$

is a decent cocycle on $Q2^n$ with $\hat{\omega}$ a possible choice for the above w_n. Semidecent computations immediately give

$$sq^1\omega \approxeq 0,$$
$$sq^2\omega \approxeq sq^2sq^1\tau\cdot\bar{\gamma} + sq^1\tau\cdot\bar{\gamma}^2,$$
$$sq^3\omega \approxeq sq^2sq^1\tau\cdot\bar{\gamma}^2 + sq^1\tau\cdot\bar{\gamma}^3.$$

Since $\bar{p}_n^* H^*(D2^{n-1}) = 0$ in dimensions > 3, and since we have lemma 5 we get the desired result, namely $Sq^i\hat{\omega} = 0$ for $i = 1,2,3$.

7. EXTENSIONS OF $D2^n$ BY Z_2

An extension $1 \longrightarrow Z_2 \xrightarrow{i} G \xrightarrow{p} D2^n \longrightarrow 1$ is classified by its characteristic class $c \in H^2(D2^n)$. The group G will be denoted $G(c)$. It is easy to see that we get six different groups, namely,

$$G(0) = Z_2 \times D2^n,$$

$$G(u_n) = D2^{n+1},$$

$$G(u_n+y_n^2) = Q2^{n+1},$$

$$G(x_n^2) = G(x_n^2+y_n^2),$$

$$G(y_n^2),$$

$$G(u_n+x_n^2) = G(u_n+x_n^2+y_n^2).$$

We shall compute $H^*(G(c))$ for the three last mentioned cases. $\bar{}$ will continue to denote pullback through $G(c) \xrightarrow{p} D2^n \xrightarrow{q} H$ (for any q).

8. $H^*(G(x_n^2))$

Lemma 3 does not apply, but it is easy to get

$$E_\infty = Z_2[t^2] \otimes Z_2[x_n,y_n,u_n]/(x_n^2,x_ny_n)$$

$$+ \Sigma\, t^{2i+1} \otimes (x_n+y_n)/(x_n^2+x_ny_n).$$

Hence, if v,z are classes on $G(x_n^2)$ representing t^2 and $t \otimes (x_n+y_n)$ in E_∞, then the ring $H^*(G(x_n^2))$ is generated by $\bar{x}_n,\bar{y}_n,\bar{u}_n,v$ and z.

Assume $n \geq 3$. We shall first use p_n-decent cocycles; let $\gamma = \xi_{n-1}^2 \in Z^2(D2^{n-1})$; there is $\rho \in C^1(D2^{n-1})$ with $\delta\rho = \gamma + \xi_{n-1}\eta_{n-1}$. Choose $\tau \in C^1(G(x_n^2))$ with $\delta\tau = \bar{\gamma}$, $(i^{\#}\tau)^\wedge = t$. Then

$$\underline{v} = sq^1\tau + \theta_{11}(\bar{\xi}_{n-1}),\quad \underline{z} = \tau(\bar{\xi}_{n-1}+\bar{\eta}_{n-1}) + \xi_{n-1}\bar{\rho},$$

are p_n-decent cocycles on $G(x_n^2)$, and their classes v, and z are possible choices for the above v,z. Easy p_n-semidecent computations + lemma 5 give

$$Sq^1 z \equiv (v+z)(\bar{x}_n+\bar{y}_n), \; Sq^2 z \equiv v(\bar{x}_n+\bar{y}_n)^2$$

modulo $p^* p_n^* H^*(D2^{n-1})$. Since $p_n^* u_{n-1} = 0$ (by definition of u_{n-1}) and $p^* p_n^*(x_{n-1}^2) = 0$, this gives coefficients a,b such that

$$(8.1) \qquad Sq^1 z = (v+z)(\bar{x}_n+\bar{y}_n) + a\bar{y}_n^3,$$

$$(8.2) \qquad Sq^2 z = v(\bar{x}_n+\bar{y}_n)^2 + b\bar{y}_n^4.$$

To find a and b notice that the inclusion $i_b : Z_2 = \{1,b_n\} \subseteq D2^n$ has $i_b^\# \eta_n = \tau_o$, $i_b^\# \xi_n = 0$; from the last one of these, it is easy to see that i_b factors through p like this

$$\begin{array}{ccc} Z_2 = \{1,b_n\} & \xrightarrow{j} & G(x \\ & \searrow^{i_b} & \downarrow p \\ & & D2^n \end{array}$$

Then $j^\# \bar{\xi}_{n-1} = 0$, so $j^\# \tau$ is a cocycle; let $(j^\# \tau)^\wedge = \nu t$. Now it follows that $j^* v = j^* z = \nu t^2$. Then apply j^* to (8.1) and (8.2) to get $a = b = 0$, i.e.,

$$(8.3) \qquad Sq^1 z = (v + z)(\bar{x}_n + \bar{y}_n), \; Sq^2 z = v(\bar{x}_n + \bar{y}_n)^2.$$

A p_n-semidecent rearrangement of

$$\underline{z}\bar{\xi}_{n-1} = \tau(\bar{\xi}_{n-1} + \bar{\eta}_{n-1})\bar{\xi}_{n-1} + \bar{\xi}_{n-1}\bar{\rho}\,\bar{\xi}_{n-1}$$

+ lemma 5 gives $z\bar{x}_n \in p^* p_n^* H^*(D2^{n-1})$. Hence as above there is a coefficient a with $z\bar{x}_n = a\bar{y}_n^3$. Apply j^* to get $a = 0$, i.e.,

(8.4) $\quad z\bar{x}_n = 0.$

To find $Sq^1 v$ notice that $\underline{v} = sq^1\tau + \theta_{11}(\bar{\tau}_o)$ is actually $\tilde{p}_2 p_3 \dots p_n$-decent. Lemma 5 then easily gives $Sq^1 v \in p^* p_n^* \dots p_3^* \tilde{p}_2^* H^*(Z_2)$, but this group is 0, so

(8.5) $\quad Sq^1 v = 0.$

In $H^*(G(x_n^2))$ we have now established the following relations

$$\bar{x}_n^2 = 0,\ \bar{x}_n\bar{y}_n = 0,\ z^2 = v(\bar{x}_n + \bar{y}_n)^2\ (= v\bar{y}_n^2),\ z\bar{x}_n = 0.$$

From the appearance of E_∞ it is not hard to see that there are no relations independent of the above. Hence

<u>Theorem</u>. Let $n > 3$. $\exists\ v, z \in H^2(G(x_n^2))$ such that

$$H^*(G(x_n^2)) = Z_2[\bar{x}_n\ \bar{y}_n, \bar{u}_n, v, z]/(\bar{x}_n^2, \bar{x}_n, \bar{y}_n, z^2 + v\bar{y}_n^2, z\bar{x}_n)$$
$$Sq^1 z = (v + z)(\bar{x}_n + \bar{y}_n),\quad Sq^1 v = 0.$$

<u>Remark</u>. For $n = 3$ there is no p_n-decent cocycle representing z (since in D4 there is no cochain ρ with $\delta\rho = \xi_2^2 + \xi_2\eta_2$). However, z does have a decent cocycle-representative $\tau(\bar{\xi}_3 + \bar{\eta}_3) + \xi_3\bar{\rho}$ with $\rho \in C^1(D8)$ and $\delta\rho = \xi_3^2 + \xi_3\eta_3$. A little bit of work then shows that the theorem is still true for $n = 3$, <u>except</u> that $Sq^1 z = (v + z)(\bar{x}_3 + \bar{y}_3) + \bar{u}_3\bar{y}_3$.

9. $H^*(G(y_n^2))$

Here $E_\infty = Z_2[t^2] \otimes Z_2[x_n, y_n, u_n]/(x_n^2 + x_n y_n,\ y_n^2)$, so

$H^*(G(y_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, \bar{u}_n, v]/(\bar{x}_n^2 + \bar{x}_n\bar{y}_n, \bar{y}_n^2)$ for any $v \in H^2$ with $i^*v = t^2$.

We take $\gamma = \bar{\tau}_o^2$, $(= p^\# p_n^\# \ldots p_3^\# p_2^\# \tau_o^2)$, and choose $\tau \in C^1(G(y_n^2))$ with $\delta\tau = \bar{\tau}_o^2$, $(i^\#\tau)^\wedge = t$. Then $\underline{v} = sq^1\tau + \theta_{11}(\bar{\tau}_o)$ is a $p_2p_3\ldots p_n$-decent cocycle representing one particular choice of the above v. From lemma 5 and a $p_2p_3\ldots p_n$-semidecent computation one gets $Sq^1v \in p^*p_n^*\ldots p_2^*H^*(Z_2) = p^*(Z_2[y_n])$; but $p^*y_n^2 = 0$, so $Sq^1v = 0$. Hence

<u>Theorem</u>. $\exists$ $v \in H^2(G(y_n^2))$ such that

$$H^*(G(y_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, \bar{u}_n, v]/(\bar{x}_n^2 + \bar{x}_n\bar{y}_n, \bar{y}_n^2),$$

$$Sq^1v = 0.$$

10. $H^*(G(u_n + x_n^2))$

Put $c = u_n + x_n^2$ $(= d_2t)$, $z = x_n + y_n$ (and $\zeta = \xi_n + \eta_n$). Then

(10.1) $$Sq^1c = cy_n + x_n^3,\ Sq^2Sq^1c = Sq^1c(c + z^2).$$

Also $(0{:}c) = 0$ and $(c{:}Sq^1c) = (c,z)$, so lemma 3 gives $E_\infty = Z_2[t^4] \otimes Z_2[x_n,z]/(x_nz, x_n^3) + \Sigma\, t^{4i+2} \otimes zZ_2[z]$. Hence, if w,v are classes on $G(u_n + x_n^2)$ representing t^4 and $t^2 \otimes z$ in E_∞, then $\bar{x}_n$, $\bar{z}$, w, and v generate $H^*(G(u_n + x_n^2))$ as a ring. It is not hard to see that there is a complete set of relations of the form

(10.2) $\bar{x}_n\bar{z} = 0$, $\bar{x}_n^3 = 0$ (from the basis),

v^2 is a combination of w, v, and powers of $\bar{x}_n$ and $\bar{z}$,

$v\bar{x}_n$ is a polynomial in $\bar{x}_n$ and $\bar{z}$.

We now choose $\gamma \in Z^2(D2^n)$, $\tau \in C^1(G(u_n + x_n^2))$, $\rho,\sigma \in C^*(D2^n)$ such that

$$(10.3)\qquad \hat{\gamma} = c,\ \delta\tau = \bar{\gamma},\ (i^{\#}\tau)^\wedge = t,\ \delta\sigma = sq^1\gamma\cdot\zeta + \gamma\zeta^2,$$
$$\delta\rho = sq^2sq^1\gamma + sq^1\gamma(\gamma + \zeta^2).$$

There is the inclusion $i_b:Z_2 = \{1,b_n\} \subseteq D2^n$. It has $i_b^{\#}\xi_n = 0$, $i_b^{\#}\eta_n = i_b^{\#}\zeta = \tau_o$. Also it lifts through $p_{n+1}:D2^{n+1} \longrightarrow D2^n$, so $i_b^*u_n = 0$; it follows that $i_b^{\#}\gamma$ is a coboundary; replace γ by $\gamma + p_n^{\#}p_{n-1}^{\#}\cdots p_2^{\#}i_b^{\#}\gamma$ (and adjust the choice of τ, ρ, and σ if needed). We then have

$$(10.4)\qquad i_b^{\#}\xi_n = 0,\ i_b^{\#}\eta_n = i_b^{\#}\zeta = \tau_o,\ i_b^{\#}\gamma = 0.$$

Then $i_b^{\#}\rho$ and $i_b^{\#}\sigma$ are cocycles, so adjusting them by adding a power of ζ we may assume that

$$(10.5)\qquad (i_b^{\#}\rho)^\wedge = 0,\ (i_b^{\#}\sigma)^\wedge = 0.$$

From (10.3) it is seen that

$$\omega = sq^2sq^1\tau + sq^1\tau(\bar{\gamma} + \bar{\zeta}^2) + \bar{\rho},$$
$$\underline{v} = sq^1\tau\cdot\bar{\zeta} + \tau\bar{\zeta}^2 + \bar{\sigma}$$

are decent cocycles, whose classes serve as the above w,v. Semidecent computations take the form

$$sq^1\omega \approxeq sq^1\tau\cdot sq^1\bar{\gamma} + sq^1\tau\cdot sq^1(\bar{\gamma} + \bar{\zeta}^2) \approxeq 0.$$
$$sq^2\omega \approxeq sq^2sq^1\tau\cdot(\bar{\gamma} + \bar{\zeta}^2) + sq^1\tau\cdot(\bar{\gamma} + \bar{\zeta}^2)^2 \approxeq \omega(\bar{\gamma} + \bar{\zeta}^2).$$
$$sq^3\omega \approxeq sq^1\tau\cdot sq^1\bar{\gamma}(\bar{\gamma} + \bar{\zeta}^2) + sq^2sq^1\tau\cdot sq^1(\bar{\gamma} + \bar{\zeta}^2) \approxeq \omega sq^1(\bar{\gamma} + \bar{\zeta}^2).$$

$$\begin{aligned}
sq^1\underline{v} &\approx sq^1\tau\cdot\overline{\zeta}^2 + sq^1\tau\cdot\overline{\zeta}^2 \approx 0. \\
sq^2\underline{v} &\approx sq^2sq^1\tau.\overline{\zeta} + \tau(\overline{\gamma}\,\overline{\zeta}^2 + \overline{\zeta}^4) \\
&\approx \omega\overline{\zeta} + sq^1\tau(\overline{\gamma} + \overline{\zeta}^2)\overline{\zeta} + \tau(\overline{\gamma}\,\overline{\zeta}^2 + \overline{\zeta}^4) \\
&\approx \omega\overline{\zeta} + \underline{v}(\overline{\gamma} + \overline{\zeta}^2). \\
sq^3\underline{v} &\approx sq^1\tau\cdot sq^1\overline{\gamma}\,\overline{\zeta} + sq^2sq^1\tau\cdot\overline{\zeta}^2 + sq^1\tau\cdot\overline{\zeta}^4 \\
&\approx \omega\overline{\zeta}^2 + sq^1\tau[sq^1\overline{\gamma}\,\overline{\zeta} + \overline{\zeta}^4 + (\overline{\gamma} + \overline{\zeta}^2)\overline{\zeta}^2] \\
&\approx \omega\overline{\zeta}^2. \\
\underline{v}\,\overline{\xi}_n &= sq^1\tau\cdot\overline{\zeta}\overline{\xi}_n + \tau\,\overline{\zeta}^2\overline{\xi}_n \approx 0.
\end{aligned}$$

In dimensions above 3 one has $p^*H^*(D2^n) = Z_2[\overline{z}]$, so lemma 5 gives us coefficients a_i, b_i, and a such that

(10.6) $Sq^1w = a_1\overline{z}^5$, $Sq^2w = w\overline{z}^2 + a_2\overline{z}^6$, $Sq^3w = a_3\overline{z}^7$,

$Sq^1v = b_1\overline{z}^4$, $Sq^2v = w\overline{z} + v\overline{z}^2 + b_2\overline{z}^5$, $Sq^3v = v^2 = w\overline{z}^2 + b_3\overline{z}^6$,

(10.7) $v\overline{x}_n = a\overline{z}^4$.

To find these coefficients first notice that i_b factors through $p:G(u_n + x_n^2) \longrightarrow D2^n$ like this

$$\begin{array}{ccc}
Z_2 & \xrightarrow{\;j\;} & G(u_n + x_n^2) \\
 & \underset{i_b}{\searrow} & \downarrow p \\
 & & D2^n
\end{array}$$

(this follows from $i_b^{\#}\gamma = 0$). From (10.3-5) it is easy to see that $j^*z = t$ and $j^*w = 0$, $j^*v = 0$. Putting that much information into (10.6) it follows that $a_i = b_i = 0$. Also $j^*\overline{x}_n = 0$ follows from (10.7).

Altogether we have proved

<u>Theorem</u>. $\exists\ w \in H^4(G(u_n + x_n^2)),\ v \in H^3(G(u_n + x_n^2))$ such that

$$H^*(G(u_n + x_n^2)) = Z_2[\bar{x}_n, \bar{y}_n, w, v]/(\bar{x}_n^2 + \bar{x}_n\bar{y}_n, \bar{x}_n^3, v\bar{x}_n, v^2 + w(\bar{x}_n + \bar{y}_n)^2),$$

$$Sq^1 w = 0,\ Sq^2 w = w(\bar{x}_n + \bar{y}_n)^2,\ Sq^3 w = 0,$$

$$Sq^1 v = 0,\ Sq^2 v = w(\bar{x}_n + \bar{y}_n) + v\bar{y}_n^2.$$

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