ON THE HOMOTOPY GROUPS OF THE EXCEPTIONAL LIE GROUPS

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Let G_2, F_4, E_6, E_7, E_8 be the compact connected, simply connected forms of these exceptional groups.

The purpose of this note is to describe how to compute the homotopy groups of these exceptional Lie groups.

In fact,

$\pi_i(G_2)$ and $\pi_i(F_4)$ are calculated in [6],

${}^2\pi_i(E_6)$, ${}^2\pi_i(E_7)$ and ${}^2\pi_i(E_8)$ are calculated in [3],

${}^p\pi_i(E_6)$, ${}^p\pi_i(E_7)$ and ${}^p\pi_i(E_8)$ (for odd prime p) will be calculated by making use of the results in [7].

§0. The regularity

Let G be a compact, connected, simply connected, simple Lie group. The well known Hopf theorem states that

(1) $H^*(G;Q) \cong H^*(X(G);Q)$,

where $X(G) = S^{n_1} \times \ldots \times S^{n_\ell}$, with n_i = odd, ℓ = rank G and Σn_i = dim. G.

Recall [8] that a prim p is called regular if there exists a map $f:X(G) \to G$ such that

$$f^*:H^*(G;Z_p) \cong H^*(X(G);Z_p).$$

When one computes the homotopy groups of a Lie group G, one of the most useful theorems is the following ([4] and [8])

<u>Theorem</u> (Kumpel-Serre)

A prime p is regular if and only if $p \geq N(G) = \frac{\dim.\ G}{\mathrm{rank}\ G} - 1$.

The immediate corollary is

(2) $^{p}\pi_i(G) \cong {^{p}\pi_i}(X(G))$ for $p \geq N(G)$

So one can know $^{p}\pi_i(G)$ from the known results on sphere.

TABLE I

G	dim. G	N(G)	p-torsion	$(n_1, \ldots, n_e)$
G_2	14	6	2	(3,11)
F_4	52	12	2,3	(3,11,15,23)
E_6	78	12	2,3	(3,9,11,15,17,23)
E_7	133	18	2,3	(3,11,15,19,23,27,35)
E_8	248	30	2,3,5	(3,15,23,27,35,39,47,59)

To compute the p-component of $\pi_i(G)$ for a prime $p < N(G)$, we use the following two methods, namely:

(A) Using the homotopy exact sequence of the bundle,

(B) killing homotopy methods due to Cartan-Serre-Whitehead.

§1. The cases where G has p-torsions.

(I) $G = G_2$ and F_4 for $p = 2$.

To compute $^{2}\pi(G_2)$ we use the homotopy sequence of the bundle $G_2/SU(3) = S^6$. The characteristic class of this bundle is the generator of $\pi_5(SU(3)) \cong Z$.

The 2-components of $\pi_i(F_4)$ are calculated by making use of the exact sequence of the homogeneous space F_4/G_2. Here one has

$H^*(F_4/G_2;Z_2) \cong \Lambda(x_{15},x_{23})$, where $Sq^8 x_{15} = x_{23}$. (The result $\pi_{14}(F_4) \cong Z_2$ is important in this calculation.)

(II) $G = E_6, E_7$ and E_8 for $p = 2$.

The 2-primary components of $\pi_i(G)$ for $G = E_6$, E_7 and E_8 are calculated up to $i = 22$, 25 and 28 respectively in [3] by making use of the killing method.

(III) $G = E_6$, E_7 and E_8 for $p = 3$, $G = E_8$ for $p = 5$.

In these cases one can also calculate the p-components of $\pi_i(G)$ by the killing homotopy method to some extent.

§2. The cases where G has no p-torsions.

We have that

(2.1) $H^*(G;Z_p) = \Lambda(x_{n_1}, x_{n_2}, \dots, x_{n_\ell})$

(For the values of p and $(n_1, \dots, n_\ell)$ see Table I.)

One of the main results of [7] is the following

<u>Theorem</u> In (2.1) $\mathcal{P}^1 x_s = x_t$ if and only if $t - s = 2(p - 1)$ and $s \neq 2p + 1$.

(1) $G = G_2$ for $p = 3$.

Consider the bundle $G_2/S^3 = V_{7,2}$ and choose a map $f: S^{11} \to V_{7,2}$ such that it induces the isomorphism

$$H^*(V_{7,2};Z_3) \cong H^*(S^{11};Z_3).$$

Then we have the induced bundle f^*G_2;

$$\begin{array}{ccccccc} S^3 & \rightarrow & f^*G_2 & \rightarrow & G_2 & \leftarrow & S^3 \\ & & \downarrow & & \downarrow & & \\ & & S^{11} & \xrightarrow{f} & V_{7,2} & & \end{array}$$

and it gives the isomorphisms of 3-primary components:

$${}^3\pi_i(f^*G_2) \cong {}^3\pi_i(G_2),$$

where the characteristic class of f^*G_2 is α_2, a generator of ${}^3\pi_{10}(S^3) \cong Z_3$.

(II) $G = E_7$ for $p = 5$ and $G = E_7$ and E_8 for $p = 7$.

The above theorem enables us to calculate ${}^p\pi_i(G)$ to some extent by the killing homotopy method.

(III) $G = G_2$, F_4 and E_6 for any $p \geq 5$ and $G = E_7$ and E_8 for any $p \geq 11$.

The following are the results of [7], although they were developed after the Conference.

Let $B_n(p)$ be S^{2n+1}-bundle over $S^{2n+2p-1}$ with the characteristic class $\alpha_1(2n+1)$, a generator of ${}^p\pi_{2n+2(p-1)}(S^{2n+1})$.

Let X and Y be simply connected, finite CW-complexes. Let p be a prime. X is called p-equivalent to Y if and only if there exists a map $f:X \rightarrow Y$ such that

$$f^*:H^*(Y;Z_p) \cong H^*(X;Z_p).$$

Then we have

<u>Theorem</u>

(i) G_2 is 5-equivalent to $B_1(5)$

(ii) F_4 is 5-equivalent to $B_1(5) \times B_7(5)$

F_4 is 7-equivalent to $B_1(7) \times B_5(7)$

F_4 is 11-equivalent to $B_1(11) \times S^{11} \times S^{15}$.

(iii) E_6 is 5-equivalent to $B_1(5) \times B_4(5) \times B_7(5)$

E_6 is 7-equivalent to $B_1(7) \times B_5(7) \times S^9 \times S^{17}$.

E_6 is 11-equivalent to $B_1(11) \times S^9 \times S^{11} \times S^{15} \times S^{17}$.

(iv) E_7 is 11-equivalent to $B_1(11) \times B_7(11) \times S^{11} \times S^{19} \times S^{27}$.

E_7 is 13-equivalent to $B_1(13) \times B_5(13) \times S^{15} \times S^{19} \times S^{23}$.

E_7 is 17-equivalent to $B_1(17) \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27}$.

(v) E_8 is 11-equivalent to $B_1(11) \times B_7(11) \times B_{13}(11) \times B_{19}(11)$.

E_8 is 13-equivalent to $B_1(13) \times B_7(13) \times B_{11}(13) \times B_{17}(13)$.

E_8 is 17-equivalent to $B_1(17) \times B_7(17) \times B_{13}(17) \times S^{23} \times S^{39}$.

E_8 is 19-equivalent to $B_1(19) \times B_{11}(19) \times S^{15} \times S^{27} \times S^{35} \times S^{47}$.

E_8 is 23-equivalent to $B_1(23) \times B_7(23) \times S^{23} \times S^{27} \times S^{35} \times S^{39}$.

E_8 is 29-equivalent to $B_1(29) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S$

Thus the p-components of $\pi_i(G)$ can be read off from those of $B_n(p)$ and S^m.

§3. An application

Consider the Hurewicz map

$\pi_*(G)/\text{tors} \to PH_*(G)/\text{tors}$,

where PH_* is a module of primitive elements in the coalgebra.

Smith [9] proposed the following

Problem. To find the least integer N(t) such that N(t)·x is a spherical class for an element $x \in PH_t(G)$.

The above results, of course, can be applied to this problem. Note that N(3) = 1.

$G = G_2$ $\quad N(11) = 2^3 \cdot 3 \cdot 5$

$G = F_4$ $\quad N(11) = 2^3 \cdot 5,\ N(15) = 2^3 \cdot 3 \cdot 7,\ N(23) = 2^6 \cdot 3^a \cdot 5 \cdot 7 \cdot 11$

$G = E_6$ $\quad N(9) = 2,\ N(11) = 2^2 \cdot 5,\ N(13) = 2^b \cdot 3 \cdot 7$

$\quad N(17) = 2^c \cdot 3 \cdot 5,\ N(23) = 2^d 3^e \cdot 5 \cdot 7 \cdot 11$

$G = E_7$ $\quad N(11) = 2 \cdot 5$, etc.

This problem is closely related to the following

Problem Let G and X(G) be as in §1. To find a mapping degree $d(G): X(G) \to G$ for G the exceptional groups.

For instance: $d(G_2) = 120$. I guess that d(G) is a function of rank G, dim. G and the order of Weyl group of G.

§4. Appendix. The table of the homotopy groups of exceptional groups. (These facts are found in [3], [6] and [7].)

$\pi_1(G)$

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	11	12	13	14	15	16	17	18	19
G_2	$Z+Z_2$	0	0	$Z_{168}+Z_2$	Z_2	$Z_6+Z_2+Z_2$	Z_8+Z_2	Z_{240}	Z_6
F_4	$Z+Z_2$	0	0	Z_2	Z	Z_2+Z_2	Z_2	$Z_{720}+Z_3$	Z_2
E_6	Z	Z_{12}	0	0	Z	0	$Z+Z_2$	$Z_{720}+Z_6$	Z_3
E_7	Z	Z_2	Z_2	0	Z	Z_2	Z_2	Z_{12} or Z_{36}	$Z+Z_{[illegible]}$
E_8	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0

	20	21	22	23	24	25	26	27	2[illegible]
G_2	Z_2	0	$Z_{5544}+Z_2$	$Z_2+Z_2+Z_2$ or Z_4+Z_2					
F_4	0	Z_3+Z_3	Z_{27} or Z_9	$Z+Z_2+Z_2$ or $Z+Z_4$					
E_6	Z_{1512}	Z_3+Z_3	$Z_{27}+Z_3$ or Z_9+Z_3						
E_7	Z_2	Z_6	Z_{108} or Z_{36}	$Z+Z_2+Z_2$	$Z_2+Z_2+Z_2$	Z_6+Z_2			
E_8	0	Z_2	0	$Z+Z_2$	Z_2+Z_2	Z_6	0	Z	Z_3

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