The Cohomology of B_G

by

R. James Milgram*

In the theory of classifying spaces one of the most important -- as well as one of the most recalcitrant -- has been B_G, the classifying space for homotopy equivalences of the sphere, see e.g. [3], [6], [8]. However, compared to some other spaces, such as B_{PL}, B_{TOP} ([5]) B_G is much more accessible since G is homeomorphic to the union of the +1 and -1 components, Q_1 and Q_{-1} of $Q = \lim_{n\to\infty} (\Omega^n S^n)$ [6] and this latter space has been successfully studied, for example in [2].

In this note we shall show how to use the known information about Q to obtain information about G and B_G. In fact we have

<u>Theorem</u> A) $H^*(B_G, Z_2) \cong P(w_1 \ldots w_i \ldots) \otimes E(\ldots e_I \ldots)$ <u>where</u> $P(\ldots)$ <u>is a polynomial algebra, and</u> E <u>an exterior algebra. More exactly</u>

(i) I <u>runs over all sequences of integers</u> $0 \leq i_1 \leq i_2 \leq \ldots \leq i_n$ $(n \geq 2)$ <u>with</u> $i_1 = 0$ <u>implying</u> $n = 2$ <u>and</u> $i_2 > 0$.

(ii) $\dim w_i = i$, $\dim e_I = 1 + i_1 + 2i_2 + \ldots + 2^{n-1} i_n$.

Theorem A follows by Hopf algebra and spectral sequence arguments once we have determined the Pontrajagin product structure in $H_*(G,Z_2)$ which is given by

<u>Theorem</u> B) $H_*(SG,Z_2) \cong E(f_1,f_2,\ldots,f_i\ldots) \otimes P(f_{o1}\ldots f_I\ldots)$ <u>where</u> I <u>runs over admissible sequences as described in</u> A (i) <u>but</u> $\dim (f_i) = i$ <u>and</u>
$\dim f_I = i_1 + 2i_2 +\ldots+ 2^{n-2}i_n = \dim (e_I) - 1.$

In [section 8 of 4] we will discuss the analogue of B in the case of $H_*(G,Z_p)$ for p an odd prime. Hence, in the remainder of this note we will only use Z_2 coefficients.

1) The main difficulty in using results on Q to give information on G is that the structure of Q has been revealed by studying constructions based on <u>loop sums</u> $(*)$ while G has an entirely different multiplication, that of composition $(\cdot)$ -- and what is needed is information about Q in terms of composition products (regarding $\Omega^n S^n$ as the set of base point preserving maps $S^n \longrightarrow S^n$). In fact what is required is

<u>Theorem</u> 1.1. <u>The following diagram homotopy commutes</u>

$$\begin{array}{ccccc}
(Q\times Q)\times(Q\times Q) & \xrightarrow{\Delta\times\Delta} & Q^{(4)}\times Q^{(4)} & \xrightarrow{\text{Shuff}} & (Q\times Q)^{(4)} \\
\downarrow{\scriptstyle (*)\times(*)} & & & & \downarrow{\scriptstyle (\cdot)^4} \\
Q\times Q & \xrightarrow{\quad\cdot\quad} & Q & \xleftarrow{\;(*)^4\;} & Q^4
\end{array}$$

(The proof is elementary.)

Passing to homology we have

Corollary 1.2. *Let* a,b,c,d *be homology classes in* $H_*(Q)$, *then*

$$(a*b)\circ(c*d) = \sum_{i,j,k,s} (a_i'\circ c_j')_*(b_k'\circ c_j'')_*(a_i''\circ d_s')_*(b_k''\circ d_s'')$$

where $\Delta a = \Sigma a_i' \otimes a_i''$ *etc*.

In particular, if $b = d = J$ (the class of a point in Q_1) we have

Corollary 1.3. *If* $a,c \in Q_0$ *then*

$$(a*J)(c*J) = a*c*J + a\circ c*J + \Sigma \bar{a}_i'\circ \bar{c}_j'*\bar{a}_i''*\bar{c}_j''*J$$

where $\Delta a = a \otimes (\square) + (\square) \otimes a + \Sigma \bar{a}_i' \otimes \bar{a}_i''$ *etc. and* $\square$ *is the class of a point in* Q_0.

2) The results of §1 reduce the calculation of the Pontrajagin products in SG to studying the composition product in Q_0. Here the idea is to use the construction of Kudo-Araki [1] or Dyer-Lashof [2] to systematically build new classes from previously given ones and -- if we know the composition product on the given classes -- use some general position arguments essentially contained in the proof of 1.2 of [2] to calculate the composition on the new classes.

As the first step in this program we have

Proposition 2.1: The following diagram is equivariantly homotopy commutative

$$\begin{array}{ccccc}
(W_{\Sigma_2} \times Q\times Q) \times (W_{\Sigma_2} \times Q\times Q) & \xrightarrow{\text{Shuff}} & W_{\Sigma_2\oplus\Sigma_2}(Q^4) & \longrightarrow & W_{\Sigma_4}\times Q^4 \\
\downarrow{\scriptstyle \theta_2\times\theta_2} & & & & \downarrow{\scriptstyle \theta_4} \\
Q \times Q & & \xrightarrow{\quad (o) \quad} & & Q
\end{array}$$

where the θ_i are the maps in 1.1 of [2], K is any equivariant map covering the inclusion $\Sigma_2\oplus\Sigma_2 \subset \Sigma_4$ and

$$\lambda(x,y,z,w) = (x\cdot z,\ y\cdot z,\ x\cdot w,\ y\cdot w).$$

(The proof is a general position argument based on the proof of theorem 1.2 of [2].)

Lemma 2.2: Let $Q_i(J)$ be the i^{th} Kudo-Araki operation on J (the class of a point in Q_1), then $Q_i(J) \cdot Q_i(J) = Q_i(J) * Q_i(J)$.

(The proof is an explicit calculation of an element in $H_*(\Sigma_4, Z_2)$ based on 2.1.)

These $Q_i(J)$ are important because $Q_i(J)*-(2J) \in H^i(Q_o)$ are the classes e_i which, together with the classes obtained by iterating certain Kudo-Araki operations on them, form a set of generators for $H_*(Q_0)$ as a (loop sum) Pontrajagin ring. Thus to obtain the desired information about $e_i \circ e_i$ we must evaluate $(-J) \circ Q_i(J)$.

Lemma 2.3: Let $\chi_*: H_*(Q) \longrightarrow H_*(Q)$ be the canonical antiautomorphism (induced from the topological map $\chi: Q \to Q$ where $\chi(f)(t) = f(1-t)$). Then

(i) $(-J) \circ Q_i(a) = Q_i(-J \circ a)$

(ii) $Q_i(-J) = \chi_*(Q_i(J))$.

This now implies

Corollary 2.4: $(e_i * J) \circ (e_i * J) = 0$.

Proof: $e_i * J = Q_i(J) * -J$ and

$$(Q_i(J) * -J) \circ (Q_i(J) * -J) = \sum_r Q_r(J) \circ Q_r(J) * \chi_* Q_{i-r}(J)) * \chi_*(Q_{i-r}(J)) *$$
$$= Q_0[\sum_r Q_r(J) * \chi_* Q_{i-r}(J)] * J$$
$$= 0$$

since the term in square brackets is zero for the antiautomorphism.

Next, applying 2.1 inductively we have

Lemma 2.5: Let $I = (i_1, \ldots, i_k)$ be of length k and $K = (j_1 \ldots j_s)$ be of length s then

$$Q_I(J) \circ Q_K(J)$$

can be written as a sum

$$\sum_r Q_{S(r)}(J)$$

where each $S(r)$ has length $k + s$.

<u>Remark</u> 2.6: It is possible to evaluate the sum occurring in 2.5 explicitly. However, we do not need the explicit result to prove our main theorems.

<u>Remark</u> 2.7: The results 2.1, 2.2, 2.3 completely determine the composition product in $H_*(Q)$. Moreover, there are analogous results mod p which determine the mod p composition product in $H_*(Q;Z_p)$.

3. The proof of theorem B is now a direct, though tedious, calculation. Basically we take the sub-Hopf algebra $A \subset H_*(Q_1)$ generated by the classes $Q_I(J)* -(2^j-1)J$ where I has length $j > 1$, and shows it in a polynomial algebra. For this we use 2.5 and an ordering among the monomials $Q_I(J)$, together with the results of section 1.

Next, 2.4 and 2.5 show $H_*(Q_1)$ contains an exterior algebra E with generators $e_1 * J$. Moreover, E is disjoint from the set

$$H_*(Q_1) \cdot \bar{A}.$$

This shows, by a counting argument, that

$$H_*(Q_1) \cong E \otimes A$$

and the result follows.

University of Illinois at Chicago Circle

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