Relative Stable Homotopy

by

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Our object is to develop spectral sequences which converge, under suitable conditions, to the set (which has under these conditions a natural abelian group structure) of fiber-wise homotopy classes of cross-sections of a fibration. The proper context for this problem is the homotopy theory of spaces <u>over</u> a fixed space B, i.e., $X \to B$, pointed spaces being generalized to spaces over B with a cross-section. Many of the constructions and theorems of ordinary homotopy theory generalize in this context; the culmination of our study is a spectral sequence of the Adams type which converges, under certain stability conditions, to the cross-sections (modulo a given prime) of the pull-back of the following diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle p} \\ K & \xrightarrow{f} & B \end{array}$$

and which has as its E_2-term $\mathrm{Ext}_{A(B)}(H^*(p), H^*(K))$. This E_2-term was first found during a study of Mahowald's computations ([Mah]) and the theory sketched below resulted from an attempt to situate this E_2-term in a complete spectral sequence.

§1 - Let B be a fixed topological space. Then a <u>B_p-space</u> is a space X with a projection $p_X : X \to B$; a <u>B_i-space</u> X is one with an injection $i_X : B \to X$; and a <u>B-space</u> X is a B_i-space and B_p-space with $p_X i_X = 1$. The corresponding

3 notions of mappings and homotopies are the obvious ones.

The cone, suspension, path-space and loop-space constructions can be generalized to B-spaces; for example, loop space of X = $\mathcal{L}X$ = $\{\omega \in X^I \mid p_X[\omega(t)] = b$, some $b \in B$; $\omega(0) = \omega(1) = i_X(b)\}$; suspension of X = $\mathcal{S}X$ = quotient of the union of $X \times I$ and B modulo the identifications $(x,0) \sim (x,1) \sim p_X(x)$

$$(i_X(b),t) \sim (i_X(b),t^1)$$

These are again B-spaces; these 4 functors are pair-wise adjoint, as usual.

Examples: (a) If B = *, a one-point space, then a B_p-space is a space, a B_i-space (or B-space) is a pointed space.

(b) If C is a pointed space, then $B \times C$ is a B-space.

(c) If $X = B \times C$, then its path-space $\mathcal{P}X = B \times \underline{P}C$, and $\mathcal{L}X = B \times \Omega C$, where $\underline{P},\Omega$ denote the ordinary path-space and loop-space functors.

(d) If $f : X \to B \times C$ is a B-map, then $f = (p_X,g)$ and $gi_X = *$.

Let $f : X \to Y$ be a map into a B-space. Then the B-induced fibration with classifying map f is

$$\mathcal{E}(f) = \{(x,\omega) \in X \times \mathcal{P}Y \mid f(x) = \omega(1)\} .$$

This is a B_p-space, and a B-space if f is a B-map.

Example-(d) If $Y = B \times C$ and $f = (p_X,g)$ is a B-map, then $\mathcal{E}(f)$ and $E(g)$ are homeomorphic; here $E(g)$ denotes the ordinary induced fibration with classifying map g.

There is an "operation" of $\mathcal{L}Y$ on $\mathcal{E}(f)$, $\mu : \mathcal{L}Y \times_B \mathcal{E}(f) \to \mathcal{E}(f)$. If $[A,C]_p$ denotes the set of B_p-homotopy classes ("free homotopy classes") of B_p-maps $A \to C$, then $[K,\mathcal{L}Y]_p$ has a natural group structure for any B_p-space K.

The notions of $\mathcal{H}$-spaces and $\mathcal{H}'$-spaces, analogues of H-spaces and H'-spaces, exist and have much the same properties as their counterparts in the ordinary theory.

Fundamental to establishing a stable homotopy theory of B-spaces is the following generalized Freudenthal theorem.

<u>Theorem</u>: <u>If dim</u> $X < 2n - 1$ <u>and the fiber of</u> PY <u>is</u> (n-1)-<u>connected, then</u>

$$\mathcal{S} : [X,Y]_p \to [\mathcal{S}X, \mathcal{S}Y]_p$$

<u>is a set-equivalence</u>

<u>Corollary</u>: <u>If</u> $F \to X \to B$ <u>is a fibration such that</u>: (1) <u>cross-sections exist</u>, (2) <u>dim</u> $B < 2n-1$, (3) <u>F is</u> (n-1)-<u>connected. Then the set of fiber-wise homotopy classes of cross-sections has a "natural" abelian group structure</u>.

§2 - The ordinary mapping sequences can also be generalized. If $f : X \to Y$ is a mapping into a B-space, its <u>B-fiber</u> F is the pull-back

$$\begin{array}{ccc} F & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \underset{i_Y}{\longrightarrow} & Y \end{array}$$

Thus, if f is a fibration, then $F = f^{-1}(i_Y(B))$.

Theorem: If $f : X \to Y$ is a B-map with B-fiber F, and f is a fibration, then for any B_p-space K, we have an exact sequence,

$$\ldots\ [K,\mathcal{L}X]_p \to [K,\mathcal{L}Y]_p \to [K,F]_p \to [K,X]_p \to [K,Y]_p.$$

Theorem: If $g : Y \to Z$ is a B-map and is also a homotopy-multiplicative map of $\mathcal{H}$-spaces, then we have an exact sequence,

$$\ldots\ [K,\mathcal{L}Y]_p \to [K,\mathcal{L}Z]_p \to [K,\mathcal{E}(g)]_p \to [K,Y]_p \to [K,Z]_p.$$

§3 - Using the results of §1 and §2, if we have a tower of induced fibrations over B with cross-sections, we can set up in the standard way an exact couple and so a spectral sequence converging (at any rate if the tower is finite) to the group of classes of cross-sections or liftings. More precisely,

Theorem: Let K be a B_p-space and $A_{on} \to A_{on-1} \to \cdots \to A_o \to B$ a tower of induced fibrations with cross-sections. Suppose each $A_{oi} \to A_{oi-1}$ is B-induced (see example (d), §1) with classifying map $A_{oi-1} \to \mathcal{L}^{i-1}A_i$. Assume, finally, either (a) $\dim K < 2k-1$ and all $A_{oi} \to A_{oi-1}$ have (ordinary) fibers which are (k-1)-connected, or (b) all A_{oi} are $\mathcal{L}$-spaces and maps between them are $\mathcal{L}$-maps.

Then there exists a spectral sequence "converging" to $\sum_r [K,\mathcal{L}^r A_{on}]_p$ and such that $E_1^{s,t} = [K\mathcal{L}^t A_s]_p$ $s \geq 0$, $t \geq 0$, $r = t - s$.

§4 - The spectral sequence of §3 can be applied to any decomposition of a fibration whose cross-section classes are to be enumerated (provided, of course, the assumptions of the theorem are satisfied). The Moore-Postnikov decomposition in the stable range (up to the (2k-1)-stage where the fiber is (k-1)-connected) is such a decomposition. Of particular interest is another one, described below, which generalizes the Adams decomposition of a space, [A].

Let ν be a fixed prime, H^* denote cohomology with coefficients Z_ν, A the Steenrod algebra mod ν, and A(X) the "Steenrod algebra" of X, ([Me] and [Ma-P]). Let $P_0 : E_0 \to B_0$ be a "universal fibration" with (k-1)-connected fiber F such that: (1) $P_0^* : H^*(B_0) \to H^*(E_0)$ is an epimorphism below dimension 2k, (2) $H^*(F)$ consists of transgressive elements below dimension 2k. Then we can construct a "relative Adams decomposition "of P_0 as follows: Pick an $A(B_0)$-set of generators for ker P_0^* in dimensions below 2k, and use these as k-invariants for constructing $P_1 : X_1 \to X_0$, $g_1 : E_0 \to X_1$. Then it can be shown, using results of [Me], that (1) and (2) are again satisfied by g_1. We can therefore repeat the construction to obtain $X_2 \to X_1$, $E_0 \to X_2$, etc. ...

This construction is the "correct" one and leads to a convergent spectral sequence because of the following theorem:

Theorem: <u>If we "restrict" such a relative Adams decomposition to a single point of B_0, we obtain an Adams decomposition of the fiber F.</u>

Combining all the above, we obtain:

Theorem: <u>Let $p : E \to B$ be a fibration satisfying the following conditions:</u>

Conditions: (1) p is induced from $P_0 : E_0 \to B_{01}$ with fiber F.

(2) p_0^* is an epimorphism in dimensions below $2k$

(3) F is $(k-1)$-connected and $H^*(F)$ consists of universally transgressive elements below dimension $2k$.

(4) $\dim B < 2k-1$.

Then there exists a "relative Adams spectral sequence" converging to $\sum_r [B \chi^r E]_p \Big/$ elements of finite order prime to ν, and such that

$$E_2^{s,t} = \mathrm{Ext}_{A(B_0)}^{s,t} (H^*(P_0), H^*(B)).$$

The differentials in this spectral sequence, generalizing the classical case [Mau], may be identified as "twisted" cohomology operations associated with a certain chain-complex of $A(B_0)$-modules, namely the $A(B_0)$-resolution of $H^*(p_0)$ realized by the relative Adams decomposition. Still more generally, the differentials in the spectral sequence of §3 can be identified as generalized operations associated with a "split" sequence of B-spaces (see [Sp] for the case $B = *$) from which the tower can be built (see [G] for the case $B = *$); in other words, the differentials are higher Toda brackets in the category of B-spaces.

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