HIGHER ORDER LINKING NUMBERS

by

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§ 1. Introduction

One of the oldest and most useful invariants of algebraic topology is the linking number of two cycles of appropriate dimensions in Euclidean n-space; in fact, for the case of two oriented closed curves in 3-space, the linking number concept goes back at least to Gauss, who gave a well known integral formula for its computation.

It is an easy exercise to show that the linking number of two cycles in Euclidean n-space is equivalent to a certain cup product in the complementary space. For example, suppose that S^p and S^q are disjoint spheres in Euclidean n-space, R^n, where $n = p + q + 1$, and $1 \leqq p < q \leqq n-2$. By the Alexander duality theorem, the complementary space, $R^n - (S^p \cup S^q)$, has infinite cyclic integral cohomology groups in dimensions p, q, and $p + q$. Therefore the cup product of the generating cohomology classes in dimensions p and q, will be a certain multiple of the generating cohomology class in dimension $p + q$. It can be shown that this multiple is equal to the linking number of S^p and S^q (up to a plus or minus sign).

Several years ago the author introduced the triple product and certain other higher order cohomology operations which can be defined in terms of cup products of suitable cochains (see [8]).

It is natural to expect that these higher order cohomology operations when applied to the complementary space of a collection of spheres in Euchidean n-space should give information about the linking properties of the spheres. It is the purpose of this paper to show the essential correctness of this conjecture. We shall illustrate some of the possibilities by means of various examples, and prove some relations between the different invariants that arise. It is our hope that these higher order operations will be useful in future work involving the concept of linking.

We can illustrate the basic idea by means of a simple example, the so-called "Borromean rings" (see Fig. 2). As is well known, any two of these three circles in 3-space are unlinked; yet it is impossible to pull the three of them apart. Since all linking numbers are zero, all cup products in the complementary space vanish. Thus the triple product of the three generating 1-dimensional cohomology classes is defined, and it turns out to be non-zero. This is one way of giving a rigorous mathematical proof that the three rings can not be pulled apart.

It should be pointed out that the cohomology invariants considered in this paper can be defined using any commutative ring for coefficients. It is not necessary to use the integers mod p as in the case of higher order operations based on the Steenrod operations. If one is interested in using the field of real numbers as coefficients, then by de Rham's theorem one can use differential forms to make the computations.

§ 2. Definitions, Terminology, Notations, etc.

Throughout this paper for the sake of simplicity we will use singular homology and cohomology with integer coefficients unless some other coefficient group is explicitly mentioned. The obvious generalization to other coefficient rings is left to the reader.

The triple product, as introduced by the author in [10], will be used extensively. Note that this definition differs from that of Kraines [7] by a sign; see page 433 of [7]. In addition to the properties listed in [10], we will use the following two further properties of the triple product.

The first property is a sort of Jacobi identity. Assume that u, v, and w are cohomology classes of dimensions p, q, and r respectively, and that the cup products $u \smile v$, $v \smile w$, and $w \smile u$ all vanish. Then the triple products $\langle u,v,w\rangle$, $\langle v,w,u\rangle$, and $\langle w,u,v\rangle$ are all defined, and satisfy the following relation:

$$(2.1) \qquad (-1)^{pr}\langle u,v,w\rangle + (-1)^{qp}\langle v,w,u\rangle + (-1)^{rq}\langle w,u,v\rangle = 0 .$$

This relation is to be understood modulo the smallest indeterminacy such that all three triple products are well defined; however, we will usually apply this relation when the indeterminacy is 0. For the proof of this relation, see Kraines [7], theorem 10.

For the next property, assume that u, v, and w are cohomology classes of dimensions p, q, and r, and that $u \smile v = 0$, $v \smile w = 0$. It then follows that $v \smile u = 0$ and $w \smile v = 0$ also, hence the triple products $\langle u,v,w\rangle$ and $\langle w,v,u\rangle$ are both defined, and have the same indeterminacy. We assert that

(2.2) $\langle u,v,w\rangle + (-1)^s\langle w,v,u\rangle = 0,$

where $s = pq + qr + rp$. For the proof, see Kraines, [7], theorem 8.

Suppose the hypotheses of (2.1) hold; then one can form six triple products with the cohomology classes u,v, and w, corresponding to the six possible permutations. For the sake of simplicity, let us assume that the indeterminacy is zero. It then follows from relations (2.1) and (2.2) that these six different triple products span a subgroup of the cohomology group $H^{p+q+r-1}$ which has rank at most 2; later on we will see examples where this rank is actually 2.

When using the Steenrod functional cup product, we will follow the definitions and notation laid down in [10]. In addition to the properties of the functional cup product stated in [10], we will also need the following anti-commutative law. Assume that u and v are cohomology classes of dimensions p and q respectively which satisfy the following conditions: $u \smile v = 0$ and $f^*(u) = 0$. Then the following relation holds:

(2.3) $L_f(u,v) = (-1)^{pq}R_f(v,u)$.

The proof is left to the reader; the most obvious procedure is to use the Steenrod "cup-1" products.

We will need to make use of the Alexander Duality theorem in the following form. Let X be a finite CW-complex which is a subset of the n-sphere, S^n. Then there exists a natural isomorphism

$$\gamma : H^q(X) \longrightarrow H_{n-q-1}(S^n - X),$$

where it is understood that in dimension 0, reduced homology and cohomology groups are used. Naturality means that if Y is also a finite CW-complex and $Y \subset X$, then the following diagram is commutative:

(2.4)
$$\begin{array}{ccc} H^q(X) & \longrightarrow & H^q(Y) \\ \downarrow \gamma & & \downarrow \gamma \\ H_{n-q-1}(S^n - X) & \longrightarrow & H_{n-q-1}(S^n - Y) \end{array} .$$

Here the horizontal arrows denote homomorphisms induced by inclusion maps.

§ 3. The Case of three disjoint spheres embedded in S^n.

In this section, we shall be concerned with the following situation: S_1, S_2, and S_3 are disjoint oriented spheres embedded in S^n, dimension $S_i = p_i$ where $1 \leq p_i \leq n-2$, the linking member of any two of the spheres is zero, and

(3.1)
$$p_1 + p_2 + p_3 = 2n-3 .$$

Usually we will assume that each S_i is embedded piece-wise linearly in S^n, although this is not essential for some of our statements. Let X denote the complementary space and

$$w_i \in H^{q_i}(X), \quad q_i = n - p_i - 1, \quad i = 1,2,3$$

the cohomology class which is the Alexander dual of the fundamental homology class of the sphere S_i (appropriately oriented). By the

Alexander Duality theorem, $H^{n-1}(X)$ is a free abelian group of rank two, while in dimensions between 0 and n-1 the cohomology of X is freely generated by the classes w_1, w_2, and w_3. Since any two of the spheres S_i and S_j have linking number 0, it follows that the cup products $w_i \smile w_j$ are all 0. Therefore the triple product $\langle w_i, w_j, w_k \rangle$ is defindd for any permutation (i,j,k) of the integers (1,2,3); it follows from equation (3.1) that

(3.2) $$q_i + q_2 + q_3 = n ;$$

therefore $\langle w_i, w_j, w_k \rangle \in H^{n-1}(X)$ (the indeterminacy is zero, since <u>all</u> cup products in X vanish).

Corresponding to the six different permutations of the integers (1,2,3), we obtain six triple products, which are related by the Jacobi identity (2.1) and the relation (2.2). Although the group $H^{n-1}(X)$ is free abelian of rank two, there is no canonical way to choose a basis, and thus at first sight there seems to be no easy way to specify the six elements of $H^{n-1}(X)$ determined by these triple products. We will shortly prove that they are essentially determined (up to sign) by a single integer.

In order to explain how this comes about, it is necessary to introduce certain other subspaces of S^n which are rather naturally associated with the piecewise-linearly embedded spheres S_1, S_2, and S_3. Let U_i be an open regular neighborhood of S_i in S^n, $i = 1,2,3$. It is assumed that these regular neighborhoods are chosen sufficiently small so that their closures are disjoint. Let B_i

denote the boundary of U_i ; then B_i is an (n-1)-dimensional manifold which is embedded piecewise-linearly in S^n. Let M denote the complement of $U = U_1 \cup U_2 \cup U_3$ in S^n ; then M is an n-dimensional manifold whose boundary is $B = B_1 \cup B_2 \cup B_3$. Also, M is a deformation retract of X, and for the sake of convenience, we will identify the cohomology groups of M and X by means of this obvious natural isomorphism. We will assume that S^n has been given a definite orientation. This induces an orientation for the manifold M, and we will orient B_1, B_2, and B_3 so that their orientations are consistent with that of M. These choices of orientation single out generating cohomology classes

$$\mu \in H^n(M,B) ,$$
$$\mu_i \in H^{n-1}(B_i), \quad i = 1,2,3 .$$

Now consider the following portion of the cohomology sequence of the pair (M,B) :

$$(3.3) \qquad H^{n-1}(M) \xrightarrow{g^*} H^{n-1}(B) \xrightarrow{\delta} H^n(M,B) \longrightarrow 0 .$$

All the groups in this sequence are free abelian groups and their ranks are, reading from left to right, 2, 3 and 1 respectively. It follows from the exactness of this sequence that g^* must be a monomorphism (This fact could also be proved by using the functorial nature of Alexander Duality). Therefore any element of $H^{n-1}(M)$ is completely specified by describing its image under the homomorphism g^* ; and to describe the image under g^*, it is convenient to use the fact that $\{\mu_1,\mu_2,\mu_3\}$ is a natural choice of basis for the group $H^{n-1}(B)$.

THEOREM 3.1. Let (i,j,k) <u>be any permutation of the integers</u> $(1,2,3)$. <u>Then there exists an integer</u> m_{ik} <u>such that</u>

$$g^*\langle w_i,w_j,w_k\rangle = m_{ik}(\mu_i-\mu_k) .$$

COROLLARY. <u>The six integers</u> m_{ik} <u>thus obtained are all equal in absolute value; to be precise,</u>

$$(-1)^{q_iq_k}m_{ik} = (-1)^{q_jq_i}m_{ji} = (-1)^{q_kq_j}m_{kj} ,$$

$$m_{ik} = (-1)^{q_iq_j + q_jq_k + q_kq_i}m_{ki} .$$

The corollary follows from the theorem by use of the identities (2.1) and (2.2).

Proof of theorem. Consider the portion of the cohomology sequence of the pair (M,B) written out in the paragraph above. In view of the way we chose the orientations μ and μ_i, it follows that

$$\delta(\mu_i) = \mu$$

for $i = 1,2,3$, Therefore if $x \in H^{n-1}(M)$ and $g^*(x) = a_1\mu_1 + a_2\mu_2 + a_3\mu_3$, it follows from exactness that $a_1 + a_2 + a_3 = 0$. Hence in order to prove theorem 3.1, it sufficies to prove that if we express $g^*\langle w_i,w_j,w_k\rangle$ as a linear combination of μ_i, μ_j, and μ_k, the coefficient of μ_j is 0. Alternatively, if $g_j^* : H^{n-1}(M) \longrightarrow H^{n-1}(B_j)$ denotes the homomorphism induced by the inclusion map, then it suffices to prove that

$$g_j^*\langle w_i,w_j,w_k\rangle = 0 .$$

We will prove that $g_j^* w_i = g_j^* w_k = 0$; it will then follow from lemma 5 of [10] that $g_j^* \langle w_i, w_j, w_k \rangle = 0$ as required.

The fact that $g_j^*(w_i) = 0$ if $i \neq j$ is an immediate consequence of the functorial nature of Alexander Duality as explained in § 2 and the assumption that the linking number of the spheres S_i and S_j is zero. For, the Alexander dual of the homomorphism $g_j^* : H^{q_i}(M) \longrightarrow H^{q_i}(B_j)$ is the homomorphism

$$H_{p_i}(U_1 \cup U_2 \cup U_3) \longrightarrow H_{p_i}(S^n - B_j)$$

and we have natural direct sum decompositions

$$H_{p_i}(U_1 \cup U_2 \cup U_3) = H_{p_i}(U_1) \oplus H_{p_i}(U_2) \oplus H_{p_i}(U_3),$$

$$H_{p_i}(S^n - B_j) = H_{p_i}(U_j) \oplus H_{p_i}(S^n - \overline{U}_j).$$

Now w_i is the Alexander dual of a generator of $H_{p_i}(U_i)$, $U_i \subset S^n - U_j$, and the degree of the homorphism

$$H_{p_i}(U_i) \longrightarrow H_{p_i}(S^n - \overline{U}_j)$$

is equal to the linking number of S_i and S_j. Q.E.D.

For some purposes, it might be more convenient to replace the exact sequence (3.1) above by the following sequence, which is a portion of the cohomology sequence of the pair (S^n, M):

(3.4) $$0 \longrightarrow H^{n-1}(M) \xrightarrow{\delta} H^n(S^n, M) \longrightarrow H^n(S^n) \longrightarrow 0 .$$

By the excision property, $H^n(S^n, M) \approx H^n(\overline{U}, B)$, and the latter group is naturally isomorphic to the direct sum of the infinite cyclic groups

$H^n(\overline{U}_i,B_i)$ for $i = 1,2,3$. The short exact sequences (3.1) and (3.2) are isomorphic; this follows from the commutativity of the following diagram:

$$\begin{array}{ccccc} & & H^{n-1}(B) & & \\ & \overset{g^*}{\nearrow} & & \overset{\delta'}{\searrow} & \\ H^{n-1}(M) & & & & H^n(\overline{U},B) \\ & \underset{\delta}{\searrow} & & \underset{g_o^*}{\nearrow} & \\ & & H^n(S^n,M) & & \end{array}$$

In this diagram, δ' and g_o^* are isomorphisms. Thus one can specify $\langle w_i,w_j,w_k\rangle$ by describing its image under δ rather than under g^*.

The integers m_{ik} of theorem 3.1 which determine all the triple products $\langle w_i,w_j,w_k\rangle$ are intimately related to another rather natural invariant which we will now describe. Let X_1 denote the complement of $S_2 \cup S_3$, X_2 the complement of $S_1 \cup S_3$, and X_3 the complement of $S_1 \cup S_2$ in S^n. For $i = 1,2,3$ we can consider the embeddings

$$f_i : S_i \longrightarrow X_i \quad .$$

As above, let (i,j,k) be any permutation of the integers $(1,2,3)$. With a slight abuse of notation, let us denote by

$$w_j \in H^{q_j}(X_i), \ w_k \in H^{q_k}(X_i)$$

the cohomology class which is the Alexander Dual of the fundamental homology classes on the spheres S_j and S_k respectively. Although this choice of notation conflicts with that introduced earlier in this section, no confusion will result, since cohomology classes which are denoted by the same symbol coincide under the homomorphism induced by the inclusion $X \subset X_i$.

Since the linking number of S_j and S_k is 0,

$$w_j \cup w_k = 0 .$$

The fact that the linking number of S_i and S_j is 0 implies that $f_i^*(w_j) = 0$; similarly $f_i^*(w_k) = 0$. Therefore the functional cup products $R_{f_i}(w_j,w_k)$ and $L_{f_i}(w_j,w_k)$ are both defined, and by lemma 3 of [10],

$$R_{f_i}(w_j,w_k) = L_{f_i}(w_j,w_k) .$$

Note that the indeterminacy is 0. From (3.1) it follows that

$$q_j + q_k - 1 = p_i .$$

Therefore $R_{f_i}(w_j,w_k)$ is an element of the infinite cyclic group $H^{p_i}(S_i)$, and hence it is a certain integral multiple of a generator of $H^{p_i}(S_i)$. We will prove that, <u>up to sign</u>, <u>the multiple which occurs is independent of the choice of the permutation</u> (i,j,k), <u>and is equal to the integer which determines the triple product</u> $\langle w_i,w_j,w_k\rangle$ in theorem 3.1 (for a more precise statement, see equations (3.7) below).

Before giving the proof, we will point out one corollary of this theorem. Since the functional cup product $R_{f_i}(w_j,w_k)$ is invariant under homotopies of the embedding map f_i, it follows that the triple product $\langle w_i,w_j,w_k\rangle$ is also invariant under such homotopies. Thus this triple product is not only an invariant of the homotopy type of the complementary space; it is also an invariant of certain homotopies of the embedding maps.

As a first step toward proving this result we will show that we may consider instead of the functional cup product $R_{f_i}(w_j, w_k)$ a certain other functional cup product which is easier to relate to the triple product $\langle w_i, w_j, w_k \rangle$.

$$\begin{array}{ccc}
H^*(S_i) & \xleftarrow{\quad f_i^* \quad} & H^*(X_i) \\
\uparrow \varphi_3 & & \downarrow \varphi_4 \\
H^*(\bar{U}_i) & \longleftarrow & H^*(M \cup U_i) \\
\uparrow \varphi_1 & & \downarrow \varphi_2 \\
H^*(B_i) & \xleftarrow{\quad g_i^* \quad} & H^*(M)
\end{array}$$

Fig. 1

For this purpose, consider the commutative diagram shown in figure 1. All homomorphisms in this diagram are induced by inclusion maps. The homomorphisms labelled φ_1 and φ_2 are monomorphisms, for which the image is a direct summand, while the homomorphisms labelled φ_3 and φ_4 are isomorphisms.

From this diagram, it follows readily that if $R_{f_i}(w_j, w_k)$ is a certain multiple of a generator of $H^{p_i}(S_i)$, then $R_{g_i}(w_j, w_k)$ is the same multiple of the corresponding generator of $H^{p_i}(B_i)$. (Note that we have identified w_j with $\varphi_2 \varphi_4(w_j)$ and w_k with $\varphi_2 \varphi_4(w_k)$). Thus we are led to study the functional cup products $R_{g_i}(w_j, w_k)$, where $g_i : B_i \longrightarrow M$ denotes the inclusion map.

In order to keep track of the various signs involved in the proof of our assertion, it is necessary to be more careful about choosing orientations from here on. This necessitates a bit of a digression. Since U_i is a regular neighborhood of S_i, it follows that S_i is a deformation retract of $\overline{U}_i$. The choice of orientation for the sphere S_i is equivalent to choosing a generator for the infinite cyclic group $H^{p_i}(S_i)$; let u_i denote the corresponding generator of $H^{p_i}(\overline{U}_i)$. By Alexander duality, $H^{q_i}(S^n-U_i)$ is infinite cyclic; we will use the chosen orientations to pick out a generator v_i of $H^{q_i}(S^n-U_i)$ as follows. Let

$$\varphi_i : H^{p_i}(\overline{U}_i) \longrightarrow H^{p_i}(B_i),$$
$$\psi_i : H^{q_i}(S^n-U_i) \longrightarrow H^{q_i}(B_i),$$

denote homomorphisms induced by inclusion maps. Since B_i is a sub-manifold of S^n of codimension 1, it follows that φ_i and ψ_i are monomorphisms, and their images generate the cohomology of B_i in dimensions less than n-1 (see [9], lemmas 12 and 13). If $p_i = q_i$, then $H^{p_i}(B_i)$ is the direct sum of the two images. If v_i is a generator of $H^{q_i}(S^n-U_i)$, then the cup product $(\varphi_i u_i) \smile (\psi_i v_i)$ is a generator of $H^{n-1}(B_i)$; this follows from the Poincaré duality theorem. <u>We will assume that</u> v_i <u>is chosen so that</u>

$$(\varphi_i u_i) \smile (\psi_i v_i) = \mu_i, \quad i = 1,2,3.$$

Let $\eta_i : H^{q_i}(S^n-U_i) \longrightarrow H^{q_i}(M)$, $i = 1,2,3$, denote homomorphisms induced by inclusion maps; then we denote by w_i the cohomology class $\eta_i(v_i)$. What we have done is to specify an explicit way of choosing the cohomology classes w_i mentioned earlier. Note that the following diagram is commutative:

$$\begin{array}{ccc} H^{q_i}(S^n - U_i) & \xrightarrow{\eta_i} & H^{q_i}(M) \\ & \psi_i \searrow \quad \swarrow g_i^* & \\ & H^{q_i}(B_i) & \end{array}$$

It follows that

$$g_i^*(w_i) = \psi_i(v_i), \quad i = 1,2,3.$$

We can now continue with the proof. Since $g_i^*(w_j) = 0$ if $i \neq j$, (see the proof of theorem 3.1), we can apply lemma 4 of [10] to obtain the following results:

$$(3.5) \qquad g_i^*\langle w_i, w_j, w_k\rangle = -(g_i^*\overline{w}_i) \smile L_{g_i}(w_j, w_k) \ ,$$

$$(3.6) \qquad g_k^*\langle w_i, w_j, w_k\rangle = R_{g_k}(w_i, w_j) \smile (g_k^* w_k) \ .$$

(Note that both of these formulas are correct with zero indeterminacy, although if $p_i = q_i$, $L_{g_i}(w_j, w_k)$ has non-zero indeterminacy). For the sake of completeness we recall that in the course of proving theorem 3.1, we proved that

$$(3.7) \qquad g_j^*\langle w_i, w_j, w_k\rangle = 0.$$

We now apply these formulas for the case where (i,j,k) is a <u>cyclic</u> permutation of $(1,2,3)$. There exist integers m_1, m_2, and m_3 such that

$$R_{g_1}(w_2, w_3) = L_{g_1}(w_2, w_3) = m_1 \cdot \varphi_1(u_1) \ ,$$

$$R_{g_2}(w_3, w_1) = L_{g_2}(w_3 w_1) = m_2 \cdot \varphi_2(u_2),$$

$$R_{g_3}(w_1, w_2) = L_{g_3}(w_1, w_2) = m_3 \cdot \varphi_3(u_3).$$

Using these results, and the fact that $(\varphi_i u_i) \smile (\psi_i v_i) = \mu_i$, we obtain the results depicted in the following table for the value of $g_{\ell *}\langle w_i, w_j, w_k\rangle$:

	g_1^*	g_2^*	g_3^*
$\langle w_1, w_2, w_3\rangle$	$-(-1)^{(p_1+1)q_1} m_1\mu_1$	0	$m_3\mu_3$
$\langle w_2, w_3, w_1\rangle$	$m_1\mu_1$	$-(-1)^{(p_2+1)q_2} m_2\mu_2$	0
$\langle w_3, w_1, w_2\rangle$	0	$m_2\mu_2$	$-(-1)^{(p_3+1)q_3} m_3\mu_3$

Using the fact that the sum of the coefficients in any row is 0 (already exploited in the proof of theorem 3.1), we find that

$$(3.8)\qquad \begin{cases} (-1)^{(p_1+1)q_1} m_1 = m_3 \,, \\ (-1)^{(p_2+1)q_2} m_2 = m_1 \,, \\ (-1)^{(p_3+1)q_3} m_3 = m_2 \quad . \end{cases}$$

This is the precise statement of the assertion in italics above. It is a sort of cohomological analog of the main theorem of a paper of Haefliger and Steer, [3]. On the one hand, the result of Haefliger and Steer is much more general in that they do not impose the condition $p_1 + p_2 + p_3 = 2n - 3$. On the other hand, Haefliger and Steer require that $p_i < n - 2$, while we only require that $p_i \leqq n - 2$ for all i.

It should be pointed out that many of the considerations of this section apply if we drop the hypothesis that the linking number of any two of the spheres is zero. For example, if we assume that S_1 and S_3 have a non-zero linking number ℓ, while the linking numbers of S_2 with either S_1 or S_3 are zero, we can still define the triple products $\langle w_1, w_2, w_3 \rangle$ and $\langle w_3, w_2, w_1 \rangle$. In addition, we can consider the functional cup products $L_{g_1}(w_2, w_3)$ and $R_{g_3}(w_1, w_2)$, and use lemmas 4 and 5 of [10] to extablish a relation between these concepts. The main difference is that there will be non-zero indeterminacy in this case, and the triple products and functional cup products take their values in certain quotient groups which have elements of finite order. The details are left to the reader. It should be noted that the assumptions we have made in this paragraph imply that p_2 = dimension $S_2 = n - 2$, and $p_1 + p_3 = n - 1$.

Another way to handle this same problem would be to use integers mod ℓ for coefficients for all homology groups, where ℓ = linking number of S_1 and S_3. Similarly, if $p_1 = p_2 = n - 2$, $p_3 = 1$, ℓ_1 = linking number of S_1 and S_3, ℓ_2 = linking number of S_2 and S_3, and S_2 and S_3 have linking number 0, then most of the foregoing considerations could be carried through if we used as coefficient ring for cohomology the integers mod d, where d is the greatest common divisor of ℓ_1 and ℓ_2. Finally, in the case where $p_1 = p_2 = p_3 = 1$, $n = 3$, and all three linking numbers are non-zero, it would be appropriate to use as coefficients the integers mod d, where d is the greatest common divisor of the three linking numbers.

§ 4. Use of Duality and Intersection Theory to Study Explicit Examples. In order to study explicit examples of the phenomena considered in the preceding section, it is sometimes convenient to shift from cohomology and cup products to homology and intersection theory by means of the duality theorems for manifolds. In the preceding section we were concerned with the cohomology groups $H^r(M)$ of the n-dimensional manifold with boundary M. By the duality theorems,

$$H^r(M) \approx H_{n-r}(M,B),$$

and the cup product pairing

$$H^r(M) \otimes H^s(M) \longrightarrow H^{r+s}(M)$$

is equivalent to the intersection theoretic pairing

$$H_{n-r}(M,B) \otimes H_{n-s}(M,B) \longrightarrow H_{n-(r+s)}(M,B).$$

By the excision property,

$$H_i(M,B) \approx H_i(S^n,\bar{U}),$$

and

$$H_i(S^n,\bar{U}) \approx H_i(S^n,S_1 \cup S_2 \cup S_3)$$

since each sphere S_j is a deformation retract of $\bar{U}_j$. Thus instead of computing triple products in $H^*(M)$, we can compute their analog in the intersection theory of $H_*(S^n,S_1 \cup S_2 \cup S_3)$. The duals of the cohomology classed w_1, w_2, and w_3 are homology classes which are represented by singular discs D_1, D_2, and D_3 respectively (of dimensions p_1+1, p_2+1, and p_3+1) such that the boundary of D_i is the sphere S_i. In order to apply intersection theory, it is necessary to choose these discs in general position with respect to each other.

An interesting special case is the case $n = 3$, $p_1 = p_2 = p_3 = 1$, the case of three circles in Euclidean 3-space (compactified by adding the point at infinity). Following the standard practice in knot theory, one can prescribe the position of the three circles in Euclidean 3-space by means of a regular projection onto a plane (see [1] , chap. I, section 3). Two examples are given in figures 2 and 3. The first figure

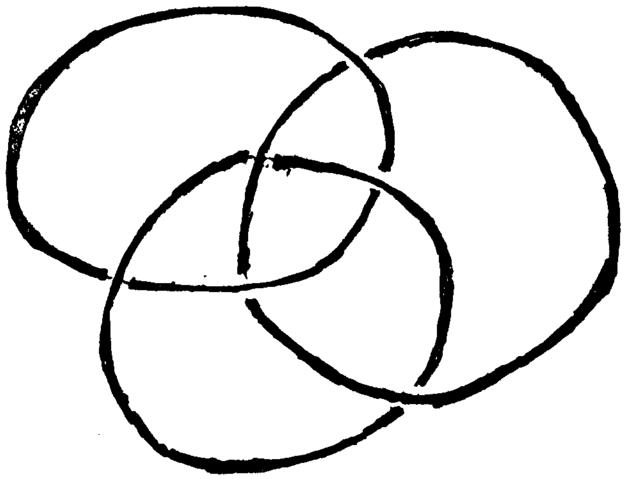


Fig. 2

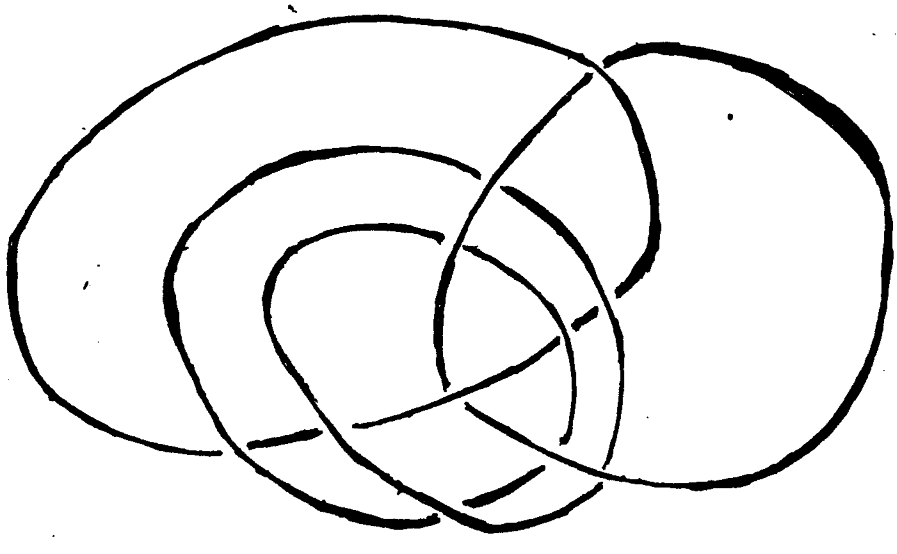


Fig. 3

depicts the well-known "Borromean Rings", and the second figure is a rather obvious modification of the first. For convenience, one can think of the circles as actually lying in the plane of diagram, except near the double points, where the two branches lie slightly above and slightly below the plane of the diagram. To obtain a singular disc which one of the circles bounds, it is convenient to take an infinite half-cylinder whose base is the circle in question and whose generators are half lines perpendicular to the plane of the diagram. From the topological point of view, such a half cylinder is a cone whose vertex is the point at infinity.

Using this idea, the reader can make explicit computations for the case of fig. 2 and fig. 3. It turns out that the integers m_{ik} of theorem 3.1 and all ± 1 for the Borromean rings, and ± 2 for the linkage shown in fig. 3. In this connection, it should be pointed out that the appropriate dual of the exact sequence (3.3) is the following:

$$0 \to H_1(S^n, S_1 \cup S_2 \cup S_3) \to H_0(S_1 \cup S_2 \cup S_3) \to H_0(S^n) \to 0 .$$

Next, we will give an explicit example of such a computation for arbitrary dimensions p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = 2n - 3$ and $1 \leq p_i \leq n - 2$. As in the preceding §, let

$$q_i = n - p_i - 1 ;$$

Then the integers q_1, q_2, q_3 satisfy the conditions $q_i \geqq 1$, and

$$q_i + q_2 + q_3 = n .$$

Thus (q_1, q_2, q_3) is an arbitrary sequence of non-negative integers, and n denotes their sum. We will consider S^n as the 1-point compactification of Euclidean n-space R^n, and

$$R^n = R^{q_1} \times R^{q_2} \times R^{q_3} .$$

Thus any point of R^n is represented as a triple (x,y,z), where $x \in R^{q_1}$, $y \in R^{q_2}$, and $z \in R^{q_3}$. The spheres S_1, S_2, and S_3 will be defined by the following equations:

$$S_1 : x = 0, \quad |y|^2 + |z|^2/4 = 1 ,$$

$$S_2 : y = 0, \quad |z|^2 + |x|^2/4 = 1 ,$$

$$S_3 : z = 0, \quad |x|^2 + |y|^2/4 = 1 .$$

The reader should clarify the meaning of these equations by drawing a diagram for the case $q_1 = q_2 = q_3 = 1$, $n = 3$. In that case, we again obtain the Borromean rings, hence we have here a true generalization of the Borromean rings.

It is not difficult to see that any two of these spheres have linking number 0; in fact, one can even prove the stronger statement that any two of them can be separated by a hypersphere in R^n. Consider, for example, the hypersphere in R^n whose equation is

$$\frac{|x|^2}{3^2} + \frac{|y|^2}{\left(\frac{1}{2}\right)^2} + \frac{|z|^2}{(3/2)^2} = 1 .$$

It is readily proved that S_1 is "outside" this hypersphere, and S_2 is "inside" it.

In order to avoid difficulties with signs, we will compute the dual of a triple product with mod 2 coefficients. Let $w_1 \in H_{p_1+1}(S^n, S_1 \cup S_2 \cup S_3, Z_2)$ be the homology class represented by the mod 2 relative cycle

$w_1' = \{(x,y,z) \mid x = 0, \ |y|^2 + |z|^2/4 \leq 1\}$, $w_2 \in H_{p_2+1}(S^n, S_1 \cup S_2 \cup S_3)$

the homology class of the relative cycle

$w_2' = \{(x,y,z) \mid y = 0 , \ |z|^2 + |x|^2/4 \leq 1\}$ and $w_3 \in H_{p_3+1}(S^n, S_1 \cup S_2 \cup S_3)$

the homology class of the cycle

$w_3' = \{(x,y,z) \mid z = 0, \ |x|^2 + |y|^2/4 \leq 1\}$.

These cycles are in general position. Their intersections are

$$w_1' \circ w_2' = \{(x,y,z) \mid x = y = 0, \ |z| \leq 1\} ,$$

$$w_2' \circ w_3' = \{(x,y,z) \mid y = z = 0, \ |x| \leq 1\} .$$

Therefore

$w_1' \circ w_2' = \partial(a)$, $w_2' \circ w_3' = \partial(b)$ where a and b are the mod 2 relative chains defined by the equations

$$a = \{(x,y,z) \mid |z|^2 + x_1^2/4 \leq 1, \ x_1 \geq 0, \ x_2 = \ldots = x_{q_1} = 0, \ y = 0\}.$$

$$b = \{(x,y,z) \mid |x|^2 + y_1^2/4 \leq 1, \ y_1 \geq 0, \ y_2 = \ldots = y_{q_2} = 0, \ z = 0\}.$$

It is now easy to compute the intersections

$$w_1' \circ b = \{(x,y,z) \mid x = z = 0 \ \ 0 \leq y_1 \leq 1, \ y_2 = \ldots = y_{q_2} = 0\} ,$$

$$a \circ w_3' = \{(x,y,z) \mid y = z = 0, \ 0 \leq x_1 \leq 1 \ \ x_2 = \ldots = x_{q_1} = 0\}$$

and the sum

$$w_1' \circ b + a \circ w_3'$$

is a relative 1-cycle (mod 2) which represents a non-zero element of the group $H_1(S^n, S_1 \cup S_2 \cup S_3, Z_2)$, since it is a path joining two distinct components of the subspace $S_1 \cup S_2 \cup S_3$; in fact, the end points of this path lie on S_1 and S_3. This sum obviously represents the dual of a certain triple product in $S^n - (S_1 \cup S_2 \cup S_3)$. The fact that it is non-zero proves that the spheres S_1, S_2, and S_3 can not be pulled apart.

§ 5. Application of a theorem of Hudson to the construction of Examples. It is the purpose of this section to show how a theorem of Hudson can be applied to construct non-trivial examples of the phenomena discussed in the preceding two sections.

Assume, as before, that $p_1 + p_2 + p_3 = 2n - 3$, and $1 \leqq p_i \leqq n - 2$. In addition, we will now assume that the notation is chosen so that $p_1 \geqq p_2 \geqq p_3$, and we make the additional assumption that $p_3 \leqq n - 3$ (it is readily seen that this additional assumption only excludes the case $p_1 = p_2 = p_3 = 1$). Let S_1, S_2, and S_3 be spheres of dimensions p_1, p_2, and p_3 respectively, and assume that S_1 and S_2 are embedded piecewise-linearly in S^n. Given any continuous map $f : S_3 \longrightarrow S^n - (S_1 \cup S_2)$, it is easy to check that the Embedding Theorem of J.F.P. Hudson [5] implies that f is homotopic to a PL-embedding.

We will now apply this result to the case where S_1 and S_2 are embedded in S^n in such a way that they can be separated by a hypersphere, and both S_1 and S_2 are unknotted. Then it is well known that the complementary space $S^n - (S_1 \cup S_2)$ has the (n-2)-type of $S^{q_1} \vee S^{q_2}$. Let α_1 and α_2 represent the homotopy classes of the composed maps

$$S^{q_1} \xrightarrow{i_1} S^{q_1} \vee S^{q_2} \xrightarrow{h} [S^n - (S_1 \cup S_2)]$$
$$S^{q_2} \xrightarrow{i_2} S^{q_1} \vee S^{q_2} \xrightarrow{h} [S^n - (S_1 \cup S_2)]$$

where h is an (n-2)-homotopy equivalence and i_1 and i_2 are inclusion maps. If we choose f to belong to the homotopy class of $m \cdot [\alpha_1 . \alpha_2]$, then it follows from Theorem IV of [10] that the functional cup product

$$R_f(w_1, w_2)$$

is $\pm m$ times a generator of $H^{p_3}(S_3)$. We can now apply equations (3.5) and (3.6) to determine the triple products $\langle w_i, w_j, w_k \rangle$ in the complementary space $S^n - (S_1 \cup S_2 \cup S_3)$, where S_3 is embedded piecewise linearly in $S^n - (S_1 \cup S_2)$ by a map of the homotopy class of f. Since we can choose the integer m arbitrarily, this shows that all possible values for the triple product $\langle w_i, w_j, w_k \rangle$ can be realized.

At this point it is perhaps appropriate to point out that the unknotting theorem of Hudson shows the role played by the condition $p_1 + p_2 + p_3 = 2n - 3$ on the dimension of the spheres S_1, S_2, S_3 embedded in S^n. In order to state the result precisely, it is convenient to introduce the following terminology: We shall say that the pair of spheres, S_i and S_j is embedded <u>trivially</u> in S^n if there exists a piecewise linear hypersphere $S^{n-1} \subset S^n$ such that S_i and S_j are in different components of $S^n - S^{n-1}$. Similarly, we shall say that the triple of spheres, S_1, S_2, S_3, is embedded trivially in S^n if there exists a disjoint pair of hyperspheres S_1^{n-1} and S_2^{n-1} embedded piecewise linearly in S^n such that S_1, S_2, and S_3 are contained in different components of

$S^n - (S_1^{n-1} \cup S_2^{n-1})$.

PROPOSITION 5.1. Let S_1, S_2, and S_3 be disjoint spheres which are embedded piecewise linearly in S^n: Assume that the dimensions p_1, p_2, and p_3 satisfy the following inequalities:

$$1 \leq p_i \leq n - 2 , \quad i = 1,2,3,$$
$$p_1 + p_2 + p_3 < 2n - 3.$$

Then if each pair of these spheres is embedded trivially, the triple S_1, S_2 S_3 is also embedded trivially.

Roughly speaking, this theorem says that if $p_1 + p_2 + p_3 < 2n-3$, then we can not have the phenomenon illustrated by the Borromean rings.

Proof: Assume that the notation is chosen so that $p_1 \geq p_2 \geq p_3$; note that this implies $p_3 < n - 2$, hence we can apply the unknotting theorem [4] to the inclusion map $S_3 \longrightarrow S^n - (S_1 \cup S_2)$. Since the pair (S_1, S_2) is embedded trivially, the manifold $S^n - (S_1 \cup S_2)$ is homeomorphic to the connected sum of the manifolds $S^n - S_1$ and $S^n - S_2$. From this it follows easily that $S^n - (S_1 \cup S_2)$ is of the same (n-2)-type as the wedge, or one-point union,

$$(S^n - S_1) \vee (S^n - S_2) .$$

The inclusion map $[S^n - (S_1 \cup S_2)] \longrightarrow [S^n - S_i]$, $i = 1,2$, corresponds to the retraction of the wedge onto the subspace $S^n - S_i$. Since the pair (S_i, S_3) is trivial for $i = 1$ or 2, it follows that the embedding $S_3 \longrightarrow S^n - S_i$ is homotopic to a constant map. Using these remarks above, we now wish to conclude that the embedding $S_3 \longrightarrow S^n - (S_1 \cup S_2)$ is also homotopic to a constant map. In order to reach this conclusion, it is necessary to make use of the following three observations:

(a) The space $S^n - S_i$ is (q_i-1)-connected, where $q_i = n - p_i - 1$.

(b) For $i < q_1 + q_2 - 1$, the homotopy group $\pi_i[(S^n-S_1)\vee(S^n-S_2)]$ is naturally isomorphic to the direct sum, $\pi_i(S^n-S_1) \oplus \pi_i(S^n-S_2)$.

(c) The hypothesis that $p_1 + p_2 + p_3 < 2n - 3$ implies that $p_3 < q_1 + q_2 - 1$.

We leave it to the reader to prove these observations in succession; from them, the desired conclusion follows.

Once we have proved that the embedding $S_3 \longrightarrow S^n - (S_1 \cup S_2)$ is homotopically trivial, it follows from the unknotting theorem of [4] that the given embedding of S_3 in $S^n - (S_1 \cup S_2)$ and a "trivial" embedding of S_3 in $S^n - (S_1 \cup S_2)$ are PL ambient isotopic. From this the desired result follows.

Q.E.D.

Remark: The same result holds for differentiable embeddings instead of PL embeddings. In that case, it is necessary to use a theorem of Haefliger [2] instead of the result of Hudson.

§ 6. Some Remarks on the Case of Two disjoint spheres embedded in S^n.

In this section we will consider rather briefly the case of two disjoint spheres, S_1 and S_2, embedded piecewise linearly in S^n; the dimension of S_i will be denoted by p_i as before, and we will assume that

(6.1) $2p_1 + p_2 = 2n - 3,$

and $1 \leq p_i \leq n - 2$ for $i = 1$ or 2. If we let $q_i = n - p_i - 1$, then it follows that

(6.2) $n = 2q_1 + q_2 .$

We will always assume that S_1 and S_2 have linking number 0 (the only case in which it could be non-zero would be the case where $p_1 = n - 2$ and $p_2 = 1$). Let

$$X = S^n - (S_1 \cup S_2)$$

and $w_i \in H^{q_i}(X)$ denote the cohomology class which is Alexander dual to the fundamental homology class on the sphere S_i. Since the linking number of S_1 and S_2 is 0, $w_1 \smile w_2 = 0$, and $w_1 \smile w_1 = 0$ automatically. Therefore we can consider the following triple products

$$\langle w_1, w_1, w_2\rangle,\ \langle w_1, w_2, w_1\rangle \text{ and } \langle w_2, w_1, w_1\rangle$$

all have values in the infinite cyclic group $H^{n-1}(X)$; the indeterminacy is 0. The Jacobi identity (2.1) gives the following relation:

(6.3) $(-1)^{q_1 q_2}\langle w_1, w_1, w_2\rangle + (-1)^{q_1}\langle w_1, w_2, w_1\rangle + (-1)^{q_1 q_2}\langle w_2, w_1, w_1\rangle = 0$

while the relation (2.2) gives the following two equations:

(6.4) $[1 + (-1)^{q_1}]\langle w_1, w_2, w_1\rangle = 0$

(6.5) $\langle w_1, w_1, w_2\rangle + (-1)^{q_1}\langle w_2, w_1, w_1\rangle = 0 .$

In case q_1 is even, then (6.4) implies that $\langle w_1, w_2, w_1\rangle = 0$ and then (6.3) or (6.5) implies $\langle w_1, w_1, w_2\rangle = -\langle w_2, w_1, w_1\rangle$. In case q_1 is odd, then (6.5) implies that $\langle w_1, w_1, w_2\rangle = \langle w_2, w_1, w_1\rangle$, and then (6.3) implies that $\langle w_1, w_2, w_1 \rangle = (-1)^{q_2} \cdot 2\langle w_1, w_1, w_2 \rangle$.

We can also consider the embedding $f : S_2 \longrightarrow S^n - S_1$; by the Alexander duality theorem, $S^n - S_1$ has the homology of a sphere of dimension q_1; let

$$w_1 \in H^{q_1}(S^n - S_1)$$

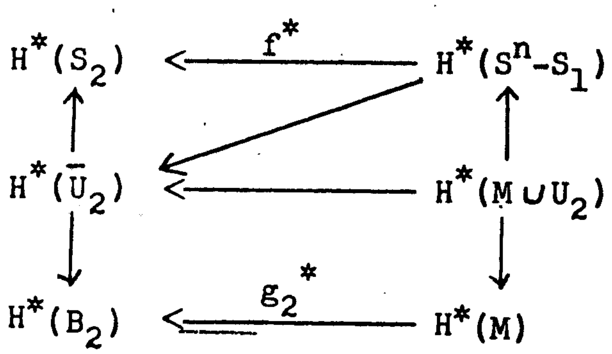
denote a generator. Since $w_1 \smile w_1 = 0$ and $f^*(w_1) = 0$, we can consider the functional cup product

$$R_f(w_1, w_1) = L_f(w_1, w_1) \in H^{p_2}(S_2) .$$

Of course if q_1 is odd, then these functional cup products are automatically zero, because of equation (2.3). However, if q_1 is even, they may be non-zero.

Exactly as in section 3, we can establish a relation between these functional cup products and the triple products described above. Since the method is so much like that in section 3, we will only give a brief indication of the results, leaving the details to the reader.

Let U_i be an open regular neighborhood of S_i, $i = 1,2$. We assume these regular neighborhoods chosen sufficiently small so that their closures are disjoint. Denote the boundary of U_i by B_i, and let $M = S^n - (U_1 \cup U_2)$. Then consideration of the following diagram



shows that consideration of the functional cup products $R_f(w_1,w_1)$ and $R_{g_2}(w_1,w_1)$ are essentially equivalent problems (here we are denoting by w_1 a cohomology class of $H^{q_1}(S^n-S_1)$ or of $H^{q_1}(M)$ without discriminating between the two). As in section 3, $g_2^*(w_2)$ is a generator of $H^{q_2}(B_2)$ and $g_2^*(w_1) = 0$. By applying lemmas 4 and 5 of [10], we obtain the following equations:

(6.6) $$g_2^*\langle w_1,w_1,w_2\rangle = R_{g_2}(w_1,w_1)\smile g_2^* w_2$$

(6.7) $$g_2^*\langle w_1,w_2,w_1\rangle = 0$$

(6.8) $$g_2^*\langle w_2,w_1,w_1\rangle = -(g_2^*\overline{w}_2)\smile L_{g_2}(w_1,w_1) .$$

To interpret these results observe that

$$g_2^* : H^{n-1}(M) \longrightarrow H^{n-1}(B_2)$$

is an isomorphism of infinite cyclic groups. This leads to the following result:

PROPOSITION 6.1. *If the integer* $q_1 = n - p_1 - 1$ *is odd*, then

$$\langle w_1,w_1,w_2\rangle = \langle w_1,w_2,w_1\rangle = \langle w_2,w_1,w_1\rangle = 0 .$$

Note that this result is definitely stronger than those deduced previously by use of the identities (2.1) and (2.2).

On the other hand, in case q_1 is even, equations (6.6)-(6.8) do not give us any new information about the three triple products. However equation (6.6) or (6.8) shows that the interesting triple products, $\langle w_1,w_1,w_2\rangle = -\langle w_2,w_1,w_1\rangle$ are completely determined by the functional cup products $R_{g_2}(w_1,w_1) = L_{g_2}(w_1,w_1)$ and vice-versa.

We leave it to the reader to show that one can actually construct examples where these functional cup products (and hence the corresponding triple products) are non-zero. Compare also a short paper by Zeeman [11].

In constructing such examples, the reader should recall that the functional cup product is essentially the Hopf invariant of the map $f : S_2 \longrightarrow S^n - S_1$.

A very interesting special case is the case where $p_1 = p_2$ and $q_1 = q_2$ is even. We must then have

$$p_1 = p_2 = 4k - 1$$
$$n = 6k$$
$$q_1 = q_2 = 2k$$

for some positive integer k. In this case not only can we consider the triple products $\langle w_1, w_1, w_2 \rangle = - \langle w_2, w_1, w_1 \rangle$; we can also consider $\langle w_2, w_2, w_1 \rangle = - \langle w_1, w_2, w_2 \rangle$. As far as the author knows, the only relation which is known to hold between these two pairs of triple products follows from a theorem of Kervaire ([6], lemma 5.1). It is an open question to determine whether there are any other relations between these triple products in the complementary space of the two spheres.

§ 7. <u>The case of</u> k <u>disjoint spheres embedded in</u> S^n, $k > 3$. Let $S_1, S_2, \ldots, S_k$ be disjoint spheres embedded piecewise linearly in S^n. We let p_i denote the dimension of S_i and assume that

$1 \leqq p_i \leqq n - 2$. As before, we denote the complementary space of the union of the spheres by X and let $w_i \in H^{q_i}(X)$ denote the cohomology class which is Alexander dual to the fundamental homology class on the sphere S_i; here $q_i = n - p_i - 1$. In order to generalize the ideas of the preceding section, we wish to impose conditions on the dimensions $p_1, p_2, \ldots, p_k$ such that the k-tuple product, if defined, will be a subset of the group $H^{n-1}(X)$ (which is free abelian of rank $k-1$). This requires that

$$\left(\sum_{i=1}^{k} q_i\right) - (k-2) = n - 1$$

which is equivalent to the condition

$$\sum_{i=1}^{k} p_i = (k-1)n - (2k-3) \ . \tag{7.1}$$

It is interesting to note that this condition is satisfied for $p_1 = p_2 = \cdots = p_k = 1$, $n = 3$, i.e. for the case of k circles in S^3.

In order to insure that the k-tuple product $\langle w_1, w_2, \ldots, w_k \rangle$ is defined, one would naturally want to assume that any $(k-1)$ of the spheres are embedded trivially in some sense. One way to attempt to construct examples is to assume that the spheres $S_1, S_2, \ldots, S_{k-1}$ are embedded in S^n trivially. Then the complementary space $S^n - (S_1 \cup S_2 \cup \ldots \cup S_{k-1})$ has the $(n-2)$-type of a wedge $S^{q_1} \vee S^{q_2} \vee \ldots \vee S^{q_{k-1}}$; let $\alpha_1, \alpha_2, \ldots \alpha_{k-1}$ denote the homotopy classes of the embedding of the "dual" spheres $S^{q_1}, S^{q_2}, \ldots, S^{q_{k-1}}$ in the complementary space; let f denote a mapping of S_k into the complementary space which belongs to the homotopy class of the iterated Whitehead product

$$[[\cdots[[\alpha_1,\alpha_2]\alpha_3]\cdots]\alpha_{k-1}] \; .$$

One then hopes that the map f is homotopic to an embedding, and that this embedding will have the desired properties. In certain special cases, this can actually be carried out. However, the general case looks rather complicated.

It is natural to ask whether or not the analogue of proposition 5.1 holds for k spheres embedded in S^n. Presumably, this analogue would read as follows: assume dimension $S_i = p_i$ for $i = 1,2,\ldots,k$, $1 \leq p_i \leq n-2$, and that

$$\sum_{i=1}^{k} p_i < (k-1)n - (2k-3);$$

then if any $(k-1)$ of the spheres is embedded trivially, all k of them are embedded trivially. The unknotting theorem of Hudson is not adequate to prove this statement. In fact, it does not work even in the simple case $k = 4$, $n = 10$, $p_1 = p_2 = p_3 = p_4 = 6$. Apparently, if the analog of theorem 5.1 is true, some other method of proof will have to be used.

One could also consider embeddings such that a k-tuple product with certain variables repeated is defined and non-zero, as in § 6. For example, one might consider embeddings of three spheres S_1, S_2, and S_3 in S^n such that $\langle w_1,w_1,w_2,w_3\rangle$ or $\langle w_1,w_2,w_1,w_3\rangle$ is defined and is a non-zero subset of $H^{n-1}(S^n-(S_1\cup S_2\cup S_3))$. We leave these various possibilities to the reader to investigate.

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