On A Family of Tertiary Operations (I)

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1. Over the past several years, a great deal has been discovered using secondary cohomology operations. However, relatively little has been done using higher cohomology operations. One of the reasons for this is the difficulty in determining enough information about the relations which determine the operations, which in many cases is almost insurmountable already for teriary operations. It is our purpose to report on a method which enables us to evaluate certain types of tertiary operations with only a small amount of specific information about their relations, and to give an example in which this method applies.

Let A be the mod 2 Steenrod algebra. In his thesis Peter May [6] states that there are elements $c_i \in H^{3,11\cdot 2^{i-1}}(A)$, $i \geq 1$ which together with those products $h_i h_j h_k$ which Adams [1; Th. 2.5.1] shows are linearly independent from a Z_2 basis for $H^3(A)$. We will show that the tertiary operations Ψ_i associated with c_i are defined and contain zero on certain universal Thom classes.

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In particular, let $BO[k]$ be the total space of the $(k - 1)$ - connected covering of BO, and let U_k be the Thom class of the universal bundle over $BO[k]$ induced from the universal bundle over BO. For $i = 4a + b$, $0 \leq b \leq 3$, let $\varphi(i) = 8a + 2^b$.

Theorem 1.1. *Let* $i > 1$. *If* $k \leq \varphi(i + 2)$, Ψ_i *is not defined on* U_k. *If* $k > \varphi(i + 2)$, Ψ_i *is defined on* U_k *and* $0 \in \Psi_i(U_k)$. *If* $i = 1$, Ψ_1 *is not defined on* U_k for $k \leq \varphi(2) = 4$ *and* $\Psi_1(U_k) = 0$ *mod 0 for* $k > 4$.

In proving this result, we also show the following: Let $\varphi_{i,i}$ be secondary operation based on the relation

$$Sq^{2^i}\, Sq^{2^i} + \sum_{j=0}^{i-1} Sq^{2^{i+1}-2^j}\, Sq^{2^j} = 0$$

i.e., the operation associated to $h_i^2 \in H^2(A)$.

Theorem 1.2. *If* $i \leq \varphi(i)$, $\varphi_{i,i}$ *is not defined on* U_k. If $K > \varphi(i)$, $\varphi_{i,i}$ *is defined on* U_k *and* $0 = \varphi_{i,i}(U_k)$ mod 0.

We note that the $\varphi_{i,i}$'s and the Ψ_i's are defined similarly on U_{BF}, i.e., the Thom class of the universal sphere bundle. Gitler and Stasheff [3] proved that $0 \notin \varphi_{1,1}$ there and recently Mahowald has shown $0 \notin \varphi_{i,i}$

for $i > 1$. It can also be shown that Ψ_1 is non zero, but the question for Ψ_i for $i > 1$ is more complicated since it is not known whether c_i is a permanent cycle or not in the Adems spectral sequence.

2. We will construct our tertiary operations in the sense of Maunder [5]. We require a chain complex

$$C_o \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3$$

which is admissible in his sense. Fix $i \geq 1$ and let

$$C_o = A \circ c \qquad \dim c = 0$$

$$C_1 = \oplus A \circ c_j \quad 0 \leq j \leq i+1 \qquad \dim c_j = 2^j$$

$$C_2 = \oplus A \circ c_{j,k} \quad 0 \leq j \leq k \leq i+1, \ k \neq j+1 \qquad \dim c_{j,k} = 2^i + 2^j$$

$$C_3 = A \circ b. \qquad \dim b = 11 \circ 2^{i-1}$$

Define $d_1(c_j) = Sq^{2^j} c$. To define d_2, we follow Adem [1] and use the Adem's method [2] to write the Adem relation

$$Sq^{2^j} Sq^{2^k} + \sum \lambda_r Sq^{2^j+2^k-r} Sq^r = 0, \ \lambda_r \in Z_2$$

as $Sq^{2^j} Sq^{2^k} + \sum_{r=0}^{k-1} \alpha_{j,k,r} Sq^{2^r} = 0$ (recall $j \leq k$ and $j \neq k-1$) where $\alpha_{j,k,r} \in A$, and we define

$$d_2(c_{j,k}) = Sq^{2^j} c_k + \sum_{r=0}^{k-1} \alpha_{j,k,r} c_r$$

As we will require it later, we note here that we can choose the $\alpha_{j,k,r}$'s in A_{k-1}, where A_{k-1} is the subalgebra of A generated by $\{Sq^1, \ldots, Sq^{2^k}\}$, since Sq^{2^j}, $Sq^{2^k} \in A_{k-1}$.

To define d_3, we construct an element $z \in C_{2,11\cdot 2^{i-1}}$ such that $d_2(z) = 0$ and which corresponds to May's element $c_i \in H^3(A)$. Since c_i is represented by Massey product $\langle h_i, h_{i-1}, h_{i+1}{}^2\rangle$ we start with $z' = Sq^{2^i} Sq^{2^{i-1}} c_{i+1,\, i+1}$ and add to z' other elements of C_2 to get z. Now $d_2(z') = \sum_{j=0}^{i+1} \gamma_j\, c_j$, where $\gamma_j \in A$ and in fact

$\gamma_{i+1} = Sq^{2^i}\, Sq^{2^{i-1}}\, Sq^{2^{i+1}}$. We will first find explicit elements $\beta_{j,i+1} \in A$ such that

$$d_2(z' + \sum_{j=0}^{i-1} \beta_{j,i+1})c_{j,i+1}) = 0 \cdot \sum_{j=0}^{1} \alpha_j\, c_j$$

and show that there are elements $\beta_{j,k} \in A$ such that $d_2(\sum \beta_{j,k}\, c_{j,k}) = \sum_{j=0}^{i} \alpha_j\, c_j$, where the first summation is taken over $0 \leq j \leq k \leq i$, $k \neq j + 1$.

To shorten notation, we write $Sq^{i_1}\, Sq^{i_2} \ldots Sq^{i_r}$ as $Sq^{i_1, \ldots, i_r}$.

Then $Sq^{2^i, 2^{i-1}, 2^{i+1}} = Sq^{2^i}(Sq^{2^{i-1}+2^{i+1}} + \sum_{j=0}^{i-2} Sq^{2^{i-1}+2^{i+1}-2^j,\, 2^j})$

$$= Sq^{2^i+2^{i+1},2^{i-1}} + \sum_{j=0}^{i-2} Sq^{2^i,2^{i-1}+2^{i+1}-2^j,2^j}$$

$$= Sq^{2^i+2^{i+1}} Sq^{2^{i-1}} + \sum_{j=0}^{i-2} [Sq^{2^i+2^{i+1},2^{i-1}-2^j}$$

$$+ \sum_{k=0}^{j} Sq^{2^i+2^{i+1}-2^k,2^{i-1}-2^j+2^k}]$$

Thus: Proposition 2.1 We can let $\beta_{i-1,i+1} = Sq^{2^i+2^{i+1}}$ and $\beta_{j,i+1} = Sq^{2^i+2^{i+1},2^{i-1}-2^j} + \sum_{k=0}^{j} Sq^{2^i+2^{i+1}-2^k,2^{i-1}-2^j+2^k}$, $0 \leq j \leq i-2$.

As we shall need it in our application, we note here:

Corollary 2.2: Each $\beta_{j,i+1}$ is a sum of admissible monomials, each of which has excess $\geq 2^{i+1} + 2^{i-2}$, $0 \leq j \leq i-2$, and $\beta_{i-1,i+1}$ has excess $= 2^i + 2^{i+1}$.

Even though we have done it for the $\beta_{j,i+1}$'s it would be a practical impossibility to determine all the $\beta_{j,k}$'s explicitly, mainly because of the complicated form of the $\alpha_{j,k,r}$'s. We will be able to prove they exist by proving:

Theorem 2.3 The Sequence

$$A \circ c \xleftarrow{d_1} \bigoplus_{j=0}^{i} A \circ c_j \xleftarrow{d_2} \oplus A \circ c_{j,k} \quad 0 \leq j \leq k \leq i,\ k \neq j+1$$

is exact.

This allows us to compute the ψ_i's. For, up to this point, all that we have constructed is a $z'' = \sum \beta_{j,i+1}c_{j,i+1}$ and we have yet to complete z'' to z. To do this, we need to know that there are $_{j,k}$'S ϵ A such that

$$d_2(\sum \beta_{j,k}c_{j,k}) = d_2(z'')$$

where the first term is summed over $0 \leq j \leq k \leq i, k \neq j + 1$. Since we have constructed z'' so that $d_2(z'') \in \bigoplus_{j=0}^{i} A \cdot c_i$, Theorem 2.3 gives us the result, and we define

$$z = \sum \beta_{j,k}\ c_{j,k} \qquad 0 \leq j \leq k \leq i + 1,\ k \neq j + 1$$

and define $d_3(b) = z$.

Before proving 2.3, we point out a corollary. Ad ms [1; p 79] notes that any secondary operation φ associated with a relation $\sum \alpha_j\ \beta_j = 0$, where $\beta_j = Sq^{2^j}$, is a linear combination of the "basic" operations $\varphi_{r,m}$, where $\varphi_{r,m}$ is the secondary operation associated with the relation $Sq^{2^r}\ Sq^{2^m} + \sum \alpha_{r,m,i}2^i = 0$ and consequently associated with the elements $c_{r,m} \in C_2$, above. However, the only thing which limits which $\varphi_{r,m}$'s is the degree of the relation. Now by his reasoning and by Theorem 2.3, we have

Corollary 2.4 *If* φ *is a secondary operation associated with a relation* $\sum \alpha_j Sq^{2^j} = 0$ *and* $j \leq i$, *then* φ *is an A-linear combination* (in the sense of Adams [1; p 79]) *of the Adams operations* $\varphi_{r,m}$ *where* $r \leq m \leq i$.

Outline of proof of 2.3. Let A_r be the subalgebra of A generated by $\{Sq^1, \ldots, Sq^{2^{r+1}}\}$, let $H^*(A_r) = \mathrm{Ext}_{A_r}(Z_2, Z_2)$, and let $h_j = \{[\xi_1^{2^j}]\} \in H^1(A_r)$, where $\xi_1 \in A_r^*$ is the projection of the Milnor basis element $\xi_1 \in A^*$. Then by a straightforward modification of Adam's proof for $H^*(A)$ [1, Th$^{\underline{m}}$ 2.5.1], we have

Theorem 2.5

(0) $H^0(A_r)$ *has as a base the unit element* 1.

(1) $H^1(A_r)$ *has as a base the elements* h_j, $0 \leq j \leq r + 1$

(2) $H^2(A_r)$ *has as a base the elements* $h_j h_k$ $0 \leq j \leq k \leq r + 1$, $k \neq j + 1$.

Thus similarly to Adams [1, pp 87-8], under the duality between Ext, Tor and the generators of a minimal resolution, we may construct the first three terms of a minimal resolution of Z_2 over A_r

$$0 \longleftarrow Z_2 \overset{\varepsilon}{\longleftarrow} D_o \overset{d_1}{\longleftarrow} D_1 \overset{d_2}{\longleftarrow} D_2$$

as follows. Let $D_o = A_r \circ c'$, $\dim c' = 0$, and $\varepsilon(c') = 1$; let $D_1 = \bigoplus_{j=0}^{r+1} A_r \circ c_j'$, $\dim c_j' = 2^j$, and $d_1(c_j') = Sq^{2^j} c_j'$ and let $D_2 = \bigoplus A_r\ c'_{j,k}$, $0 \leq j \leq k \leq r+1$, $k \neq j+1$, $\dim c'_{j,k} = 2^j + 2^k$, and $d_2(c'_{j,k}) = Sq^{2^j}\ c'_k + \sum_{r=0}^{k-1} \alpha_{j,k,r}\ c'_r$, where the $\alpha_{j,k,r}$'s are the same as in the un-primed case by their choice above. Therefore, we have the sequence

$$A_r\ c' \overset{d_1'}{\longleftarrow} \bigoplus_{j=0}^{r+1} A_r\ c_j' \overset{d_2'}{\longleftarrow} \oplus A_r\ c'_{j,k} \quad \begin{matrix} 0 \leq j \leq k \leq r+1 \\ k \neq j+1 \end{matrix}$$

is exact. However A is an A_r module and by Milnor and Moore [7, 4.4], A is a free A_r-module.

Therefore the sequence

$$A \oplus_{A_r} C_o' \overset{1\otimes d_1}{\longleftarrow} A \oplus_{A_r} C_1' \overset{1\otimes d_2'}{\longleftarrow} A \oplus_{A_r} C_2'$$

is exact. Since the C_j''s are free

$A \otimes_{A_r} C_o' \cong A \circ c_o$, $A \otimes_{A_r} C_1' \cong \bigoplus_{j=0}^{1+1} A \circ c_j$, and

$A \otimes_{A_r} C_2' \cong \oplus A \circ c_{j,k}$, $0 \leq j \leq k \leq r+1$, $k \neq j+1$,

and under these isomorphisms, $1\otimes d_j'$ corresponds to d_j. By letting $r = i - 1$, we are done.

At this point, we have a chain complex, $\mathcal{C}$,

$$C_o \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3$$

where $d_3(b) = z = \sum \beta_{j,k} c_{j,k}$ which we claim determines a tertiary operation Ψ_i; we have only to show that $\mathcal{C}$ is admissible in the sense of Maunder [5 ; 2.3]. To do this, by Maunder [5 ; 2.4.1] it is sufficient to show

$$\sum \beta_{j,k} \varphi_{j,k} = 0 \quad \text{mod } 0 \text{ indeterminancy}$$

where $\varphi_{j,k}$ are the operations associated with $Sq^{2^j} Sq^{2^k} + \sum_{r=0}^{k-1} \alpha_{j,k,r} Sq^{2^r} = 0$, i.e., are associated with the chain complex

$$C_o \longleftarrow C_1 \xleftarrow{d_2} A \cdot c_{j,k}$$

But $\sum \beta_{j,k} \varphi_{j,k}$ is a secondary operation φ associated with the chain complex,

$$C_o \xleftarrow{d_1} C_1 \xleftarrow{d_2} A \cdot z$$

which has zero indeterminacy since $d_2(z) = 0$, and differs from the zero operation by primary operation $\gamma \in A$ such that $\gamma \cdot c \in (\text{Coker } d_1)_{11 \cdot 2^{i-1}}$, by Adams [1, Th. 3.6.2]. But it is easy to check that d_1 is onto in dim $11 \cdot 2^{i-1}$, so $\gamma = 0$.

What we are really doing here, of course, is to determine:

Corollary 2.5: Ψ_i is based on the relation

$$\sum \beta_{j,k}\, \varphi_{j,k} = 0 \quad 0 \leq j \leq k \leq i + 1, \quad k \neq j + 1.$$

3. In [12], Stong has determined $H^*(BO[k]; Z_2)$. As we shall need some of his results, we quote them here. As above, for $n = 4a + b$, $0 \leq b \leq 3$, let $\varphi(n) = 8a + 2^b$. (Thus, the $(n+1)^{st}$ non-zero homotopy group of BO is in dimension $\varphi(n)$, and consequently, the $(n+1)^{st}$ different $BO[k]$ is $BO[\varphi(n)]$.)

Proposition 3.1 (Stong). In $H^*(BO[\varphi(n)]; Z_2)$, $W_{2^n} \neq 0$ and $W_i = 0$ for $i < 2^n$. If $p_n : BO[\varphi(n+1)] \longrightarrow BO[\varphi(n)]$ is the projection, then $p_n^* = 0 \bmod 2$ in dimensions $< 2^{n+1}$.

Corollary 3.2 Let $MO[k]$, be the Thom space of the universal bundle over $BO[k]$, let $U_n \in H^*(MO[\varphi(n)]; Z_2)$ be the Thom class, and let $\varphi_{j,k}$ be the mod 2 secondary operation associated with $Sq^{2^j} Sq^{2^k} + \sum_{r=0}^{k-1} \alpha_{j,k,r} Sq^{2^r} = 0$, $0 \leq j \leq k$, $k \neq j + 1$. Since $Sq^p(U_n) = U_n \circ W_p$ and $\varphi_{j,k}$ is defined on U_n only when $Sq^{2^p} U_n = 0$, $p \leq k$, we have

(I). $\varphi_{j,k}$ <u>is not defined on</u> U_n for $k \geq n$,

(II). $\varphi_{j,n-1}(U_n)$ <u>is defined</u>, <u>and</u>

(III). $0 \in \varphi_{j,k}(U_n)$, $k < n - 1$.

We can be a little more precise with II:

<u>Proposition 3.3</u> (I). $\varphi_{n-1,n-1}U_n = 0$ <u>mod</u> 0.

(II) <u>For</u> $n < 3$, <u>there is at least one</u> j, $0 \leq j \leq n - 3$, such that $0 \notin \varphi_{j,n-1}(U_n)$.

Note that these two propositions include 1.2.

Proof: Part II was essentially proven by Stong in [12]. By Adams [1], for $n > 3$ there are $\gamma_{i,j} = \gamma_{i,j,n} \in A$ such that $Sq^{2^n} = \sum \gamma_{i,j}\, \varphi_{i,j}$, $0 \leq i \leq j \leq n - 1$, $j \neq i + 1$.

Therefore,

$$0 \neq W_n \cdot U_n = Sq^{2^n} U_n = \sum_{j \leq n-1} \gamma_{i,j}\, \varphi_{i,j}\, U_n = 0 + \sum_{i \leq n-3} \varphi_{i,n-1}U_n,$$

by 3.2. We will outline the proof that $\varphi_{n-1,n-1}(U_n) = 0$ below.

We now construct the setting in which we prove Theorem 1.1. Let $E \xrightarrow{p} X$ be an m-plane bundle, $D \longrightarrow X$ and $S \longrightarrow X$ be the associated disk and sphere bundles to p. Let $A \subset X$, closed, and denote by D_A, S_A the restrictions of D, S to A, respectively. Assume that $\{D_A, S\}$ is an excisive couple in D, so that the inclusim map, i, induces an isomorphism

$$i^* : H^*(D_A \cup S, S) \longrightarrow H^*(D_A, S)$$

(see Spanier [9, p 188]). Let $U \in H^m(D, S)$ be the Thom class and $U_A \in H^m(D_A, S_A)$ be its restriction to (D_A, S_A), so that

$$H^r(X) \xrightarrow{p^*} H^r(D) \xrightarrow{\cup U} H^{r+m}(D,S)$$

and

$$H^r(A) \xrightarrow{p^*} H^r(D_A) \xrightarrow{\cup U_A} H^{r+m}(D_A, S_A)$$

are the Thom isomorphisms. Hereafter, we confuse notation and denote by $\cup U$ the isomorphism

$\cup U = (\cup U) \circ p^* : H^r(X) \longrightarrow H^{r+m}(D, S)$ and by $\cup U_A$ the isomorphism $\cup U_A = (i^*)^{-1}(\cup U_A) \circ p^* : H^r(A) \longrightarrow H^{r+m}(D_A \cup S, S)$. Further, there is a homomorphism $H^r(X, A) \longrightarrow H^{r+m}(D, D_A \cup S)$ given by the composit

$$H^r(X, A) \xrightarrow{p^*} H^r(D, D_A) \xrightarrow{\cup U} H^{r+m}(D, D_A \cup S)$$

which we denote by $\cup U_{X,A}$.

<u>Proposition 3.4</u> <u>The homomorphism</u> $\cup U_{X,A}$ <u>is an isomorphism.</u>

This result is well known and we outline the proof only since we will need it below. In the following diagram

(Diag. 1)

$$\begin{array}{ccccccccc}
\cdots\longleftarrow H^{m+r}(D_A\cup S,S) & \overset{\bar{j}^*}{\longleftarrow} & H^{m+r}(D,S) & \overset{\bar{k}^*}{\longleftarrow} & H^{m+r}(D,D_A\cup S) & \overset{\delta}{\longleftarrow} & H^{m+r-1}(D_A\cup S) & \longleftarrow\cdots \\
\uparrow \cup U_A & & \uparrow \cup U & & \uparrow \cup U_{X,A} & & \uparrow \cup U_A & \\
\cdots\longleftarrow H^{r}(A) & \overset{j^*}{\longleftarrow} & H^{r}(X) & \overset{k^*}{\longleftarrow} & H^{r}(X,A) & \overset{\delta}{\longleftarrow} & H^{r-1}(A) & \longleftarrow\cdots
\end{array}$$

where j, k, $\bar{j}$, $\bar{k}$ are the inclusions, the rows are exact, being the sequences associated with the triple $(D, D_A\cup S, S)$ and the pair (X, A), and each of the squares commute (see Spanier [10; 5.6.8 and 5.6.12]). Therefore, $\cup U_{X,A}$ is an isomorphism by the 5-lemma.

<u>Corollary 3.5</u> $H^*(D,D_A\cup S) = U \cdot H^*(X,A)$, <u>where</u> U <u>is the Thom class of</u> (D, S).

We now outline the proof of 3.3.1. Let $(Sq^{2^n-2^i})$ be the vector $(Sq^{2^{n-1}}, Sq^{2^n-2^{n-2}},\ldots,Sq^{2^n-1})$ and let (Sq^{2^i}) be the vector $(Sq^{2^{n-1}}, Sq^{2^{n-2}},\ldots, Sq^1)$. Then as noted above $\varphi_{n-1,n-1}$ is based on the relation

$$(Sq^{2^n-2^i})(Sq^{2^i}) = \sum Sq^{2^n-2^i}\, Sq^{2^i} = 0$$

and we wish is evaluate $\varphi_{n-1,n-1}(U_n)\subset U_n \cdot H^{2^n-1}(BO[\varphi(n)])$. Using the notation of Diagram I, let $X = BO[\varphi(n-1)]$ and $A = BO[\varphi(n)]$. Then, by Peterson and Stein [8]

$$\varphi_{n-1,n-1}(U_n) = (Sq^{2^n-i})_{\bar{j}}\,(Sq^{2^i}\,U_{n-1})$$

mod $\sum Sq^{2^n-2^i}(U_n \circ H^{2^i-1}(BO[\varphi(n)])) + \bar{j}^*(U_{n-1} \cdot H^{2^n-1}(BO[\varphi(n-1)])$, where $\Theta_{\bar{j}}$ is the functioned cohomology operation defined by Steenrod [10]. But the indeterminacy is 0. For

$\bar{j}^*(U_{n-1} \cdot H^{2^n-1}(BO[\varphi(n-1)])) = U_n \cdot j^* H^{2^n-1}(BO[\varphi(n-1)]) =$

$U_n \circ 0 = 0$ by 3.1, and

$Sq^{2^n-2^i}(U_n \cdot H^{2^{i-1}}(BO[\varphi(n)]) = U_n \circ Sq^{2^n-2^i} H^{2^i-1}(BO[\varphi(n)])$

by 3.1 $= U_n \circ 0$ for dimensional reasons. Thus, we have only to evaluate $(Sq^{2^n-2^i})_{\bar{j}}(Sq^{2^i} U_{n-1})$, i.e., we find an $x \in H^{2^n-1}(BO[\varphi(n)])$ and a

$z_i \in H^{2^i}(BO[\varphi(n)], BO[\varphi(n-1)])$ such that

$\bar{k}^*(U_{n-1} \cdot z_i) = Sq^{2^i} U_{n-1}$ and $\sum Sq^{2^n-2^i}(U_{n-1} \circ z_i) = S(U_n \circ x)$. We then have $\varphi_{n-1,n-1}(U_n) = U_n \circ x$. However $Sq^{2^i} U_{n-1} = 0$ for $i < n - 1$, by 3.2, so we let $z_i = 0$, $i < n - 1$ and pick $\alpha \in H^{2^{n-1}}(BO[\varphi(n)], BO[\varphi(n-1)])$ such $k^*(\alpha) = W_{2^{n-1}}$, so that $\bar{k}^*(U_n \circ \alpha) = U_n \circ W_{2^{n-1}}$. Then $Sq^{2^{n-1}}(U_n \circ \alpha) =$

$(Sq^{2^{n-1}} U_n) \circ \alpha + U_n \circ Sq^{2^{n-1}} \alpha + 0$

$= (U_n \circ W_{2^{n-1}}) \circ \alpha + U_n \circ \alpha^2 = U_n \circ \alpha^2 + U_n \circ \alpha^2 = 0.$

Thus, we must have $x = 0$.

We are now ready to prove 1.1. We again use Diagram I, this time with $A = BO[\varphi(n + 1)]$, $X = BO[\varphi(n)]$, and we will use a Peterson - Stein formula for tertiary operations, i.e., Maunder's Axiom 5 [4], and we proceed similarly to the above. Since Ψ_{n-2} is based on the relation $\sum \beta_{i,r}\, \varphi_{i,r} = 0$, $0 \leq i \leq r \leq n - 1$, $r \neq i + 1$, by 2.5 it is defined on U_m if $\varphi_{i,r}(U_m)$ is defined and contains 0, simultaneously, $0 \leq i \leq r \leq n - 1$. Therefore, for $n > 3$ by 3.2 Ψ_{n-2} is not defined on U_m for $m \leq \varphi(n)$ and is defined otherwise, by 3.1 and 3.2.

The case for Ψ_1 is exceptional and trivial. Since $H^t(BO[8]) = 0$ for $t < 8$, $\varphi_{i,r}(U_3) = 0$ for $r \leq 2$. Therefore, $\Psi_1(U_3) = 0$ is defined and $\subset U_3 \cdot H^9(BO[8]) = U_3 \cdot 0 = 0$. Now $\Psi_1(U_k) = 0$ for $k > 3$ follows by naturality and for dimensional reasons.

To evaluate $\Psi_{n-2}(U_{n+1})$, we find an

$$x \in H^{11\cdot 2^{n-3}-2}(BO[\varphi(n + 1)]) \text{ and}$$

$z_{i,r} \in H^{2^i+2^r-1}(BO[\varphi(n)],\ BO[\varphi(n + 1))$ such that

$\bar{k}^*(U_n \cdot z_{i,r}) \in \varphi_{i,r}(U_n)$ and

$\sum \beta_{i,r}(U_n \cdot z_{i,r}) = \delta x \quad 0 \leq i \leq r \leq n - 1, \quad r \neq i + 1.$

We then have

$$x \cdot U_{n+1} \in \Psi_{n-2}(U_{n+1}) + \operatorname{Im} \bar{j}^*$$

However, $\operatorname{Im} \bar{j}^* = U_{n+1} \cdot j^* H^{11\cdot 2^{n-3}-2}(BO[\varphi)]) = 0$, by

3.1, since $11 \cdot 2^{n-3}-2 < 2^{n+1}$.

By 3.2 and 3.3, we can choose $z_{i,r} = 0$ for $r \neq n-1$ or $i = n-1$. Further $\beta_{i,n-1}(U_n \cdot z_{i,n-1}) = U_n \cdot \beta_{i,n-1} z_{i,n-1}$ by the construction of $\beta_{i,n-1}$ in §1 and since $Sq^p U_n = 0$, $p < 2^n$; and $\beta_{i,n-1}(z_{i,n-1}) = 0$ by [11; II§5], since the excess $\beta_{i,n-1} > \dim z_{i,n-1}$, $0 < i < n-3$. Therefore $0 = \sum \beta_{i,r} \Psi_{i,r}$ so we must have $x = 0$.

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