SECONDARY OPERATIONS ASSOCIATED WITH B(n)

by John R. Harper

Let A denote the mod 2 Steenrod Algebra. Let B(n) denote the left-ideal in A consisting of all operations which annihilate classes of dimension n or less [6]. Computation of the groups $\mathrm{Ext}_A^{s,t}(B(n), Z_2)$ gives information which can be used to study higher order cohomology operations. The situation for secondary operations is simplest. Such operations correspond to relations of the form $0 = \Sigma_i a_i b_i$ with $a_i \epsilon A$, $b_i \epsilon B(n)$. These operations are defined on all n-dimensional (mod 2) classes. The set of all such operations has the structure of a graded left A-module, [1], [3]. Generators for $\mathrm{Ext}_A^{1,t}(B(n), Z_2)$ as an abelian group are in one-to-one correspondence with a basis for the module of operations.

For S>1 the situation is more complicated. Higher order relations detected by the Ext groups need not correspond to higher order operations. Maunder's axioms [5] for higher order operations solve the problem of determining what additional information is needed for a higher order relation to give an operation.

In [2] detailed computations of $Ext_A^{s,t}(B(n), Z_2)$ are given for $S = 0$, all t and $S = 1$, $t \leq 3n+4$. This note describes a technique for introducing products in exact couples which is used there and which may be of more general interest.

Let A now be a connected, graded, Hopf algebra of finite type over a field K with $A_n = 0$ for $n<0$. Filter A with a (say) decreasing filtration F^p and let G^p denote the associated graded.

Applying $Ext_A(\ ,K)$ to the short exact sequences $0 \to F^{p+1} \to F^p \to G^p \to 0$ produces a collection of long exact sequences which can be fitted together to obtain an exact couple, $\langle D, E \rangle$. We have

$$E^{p,q,t} = Ext_A^{p+q,t}(G^p, K)$$
$$D^{p,q,t} = Ext_A^{p+q,t}(F^{p-1}, K)$$

with $p \geq q$, $p+q \geq 0$.

Theorem 1. If there exists an A-map $D: G^{p+q} \to G^p \otimes G^q$ then a product P can be introduced in the exact couple $\langle D, E \rangle$ such that the differentials are derivations,

$$P: Ext_A^{s,t}(G^p, K) \otimes Ext_A^{u,v}(G^q, K) \to Ext_A^{S+u,t+v}(G^{p+q}, K).$$

Sketch proof: First use the Hom-$\otimes$ interchange to obtain an external cohomology product p, [4]. Let X and Y be resolutions of G^p and G^q respectively.

$$\begin{array}{ccc} H^s(\mathrm{Hom}_A(X,K))\otimes H^u(\mathrm{Hom}_A(Y,K)) & & \\ & \searrow & \\ p \big\downarrow & & H^{s+u}(\mathrm{Hom}_A(X,K)\otimes \mathrm{Hom}_A(Y,K)) \\ & \swarrow \ \text{Hom-}\otimes & \\ H^{s+u}(\mathrm{Hom}_{A\otimes A}(X\otimes Y,K)), & & \end{array}$$

MacLane [4] shows that under the hypotheses on A, p is an isomorphism and commutes with connecting homomorphisms. Next, the diagonal $\Delta:A\to A\otimes A$ induces a change of rings $\Delta^{\#}$,

$$\Delta^{\#}:H^{s+u}(\mathrm{Hom}_{A\otimes A}(X\otimes Y,K))\to H^{s+u}(\mathrm{Hom}_A(X\otimes Y,K)).$$

Now $D:G^{p+q}\to G^p\otimes G^q$ induces a map D^* in Ext,

$$D^*:H^{s+u}(\mathrm{Hom}_A(X\otimes Y,K))\to \mathrm{Ext}_A^{s+u}(G^{p+q},K).$$

Define $P = D^*\Delta\, p$. P commutes with connecting homomorphisms because all the factors do. Since the differentials in the exact couple are connecting homomorphisms obtained from short exact sequences

$$d^r:\ 0\to F^{p-r+1}/F^p\to F^{p-r}/F^p\to G^{p-r}\to 0,$$

the commutativity of P and connecting homomorphisms implies the differentials are derivations.

In [2] theorem 1 is used in the situation where A is the mod 2 Steenrod Algebra, $F^p = B(p-1)$. Here we only have the existence of $D: G^{p+q}\to G^p\otimes G^q$ in grades $t\leq 3(p+q)+1$. This accounts for the restriction on t in the calculation. The algebraic results obtained are the following.

Theorem A. $\mathrm{Ext}_A^{0,t}(B(n),Z_2)\cong Z_2$ for pairs (n,t) such that either

(a) $t = 2^i$ and $0\leq n<t$

or (b) $t \equiv 2^i(2^{i+1})$, $t>2^i$ and $n = t-r$ for some r, $0<r<2^{i+1}$.

Otherwise the group is 0. The corresponding generator of $B(n)$ can be chosen as Sq^t.

Theorem B. For $t\leq 3n+4$, $\mathrm{Ext}_A^{1,t}(B(n),Z_2)\cong Z_2$ for pairs (n,t) satisfying all of the following:

(a) Given t determine all non-negative integers i, j such that $t = m+3\cdot 2^j$, $m \equiv 2^i (2^{i+1})$, m may be negative.

(b) $n = m+2^j-r \quad 0 \leq r < 2^{i+1}-1$

(c) $n \not\equiv 2^j (2^{j+1})$

Theorem A is established by a detailed investigation of B(n). Theorem B is obtained from Theorem A using Theorem 1. For values of $s>1$, the same inductive procedure will work but the results get quite complicated.

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