On some applications of the cobar construction

by

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The purpose of this note is to discuss several applications of the cobar construction. First we define a diagonal map in the cobar construction, so that $H(\Omega X)=HF(C_*(X))$ as Hopf algebras, strengthening the original result of Adams. The homotopies involved may be used to define operations (non-stable) to study if a space is a second suspension or not. Once one has a diagonal map in the cobar construction, one may take the cobar construction of the cobar construction. At this point, one has the chains of the double loop space. But the second iterate fails to be (co) associative. However, one can introduce (co) associating homotopies in the same spirit that Stasheff has solved the problem for the dual situation. Namely, if X is an H-space that is not associative but has an $A_{-\infty}$ form, one can go ahead and form its classifying space. Thus with the same type of analysis, one can iterate the cobar construction n-times.

Fuller details of the iteration of the cobar construction will appear in a joint paper with R.J. Milgram, to whom I am indebted for valuable conversations in this research.

Before defining the diagonal map in the cobar construction, I wish to give motivation by discussing the dual situation, the bar construction.

1. The bar construction and strongly homotopy multiplicative maps.

Let X be an associative H-space with unit e. The Dold-Lashof classifying space $B_\infty(X)$ is a filtered space

$$(e)=B_o(X)\subset B_1(X)\subset B_2(X)\subset \dots \subset B_n(X)\subset \dots \subset B_\infty(X)$$

where points in $B_n(X)$ can be written as

$$[x_o|t_1|\dots|t_n] \text{ where } (x_o,x_1,\dots,x_{n-1})\in X^n \text{ and } (t_1,\dots,t_n)\in I^n$$

with identifications

$$[x_o|t_1|\dots|t_n] = \begin{cases} [x_i|t_{i+1}|\dots|t_n] & \text{if } t_i=0 \quad (i<n) \\ e & \text{if } t_n=0 \\ [x_o|t_1|\dots|t_i|x_{i-1}\cdot x_i|t_{i+1}|\dots|t_n] & \text{if } t_i=1 \quad (i<n) \\ [x_o|t_1|\dots|t_{n-1}] & \text{if } t_n=1 \end{cases}$$

Now let K be a commutative ring with unit and let A be a connected DGA algebra over K. Let $\varepsilon:\bar{A}\longrightarrow K$ be the augmentation. Let $\bar{A}=\operatorname{Ker}\varepsilon$. Let $s\bar{A}$ be the graded module formed by $\bar{A}$ by raising degrees by 1. Then $\bar{B}_n(A)$ is defined to be $(s\bar{A})^n$, the tensor product of $s\bar{A}$ with itself n-times.

The (normalized) bar construction $\bar{B}(A)$ is the graded K-module with component $\bar{B}_n(A)$ in degree n. $(\bar{B}(A))_o$ is K. Elements of $\bar{B}_n(A)$ are written as linear combinations of elements

$$[a_1|\dots|a_n] = [a_1]\otimes\dots\otimes[a_n]$$

Elements of $\overline{B}_o(A)$ are written as scalar multiples of [ ] = the unit element of $\overline{B}(A)$. $\overline{B}(A)$ has a differential $d=d_E+d_I$ where

$$d_E([a_1|\dots|a_n]) = \sum_{i=1}^{n-1} (-1)^{u(i)}[a_1|\dots|a_i a_{i+1}|\dots|a_n]$$

$$d_I([a_1|\dots|a_n]) = \sum_{i=1}^{n} (-1)^{u(i-1)}[a_1|\dots|da_i|\dots|a_n]$$

where $u(i) = i+\sum_{K=1}^{i} \deg a_K$. $\overline{B}(A)$ is a DGA coalgebra with coproduct

$$\Delta:\overline{B}(A) \longrightarrow \overline{B}(A)\otimes\overline{B}(A) \quad \text{defined by}$$

$$\Delta[a_1|\dots|a_n] = \sum_{i=o}^{n} [a_1|\dots|a_i]\otimes[a_{i+1}|\dots|a_n]$$

If one assumes that X is a countable CW H-space with cellular multiplication, with cubical cells $\{e_1,e_2,\dots,e_n,\dots\}$, and if one takes $C_*(X)$ to be the cubical CW-chain complex of X, and one gives $B_n(X)$ a CW structure in a natural way with cells $[e_1|\dots|e_n]$, then checking the CW boundary gives immediately that

$$C_*(B_n(X)) \approx \overline{B}_n(C_*(X)), \quad \text{and hence}$$

$$H(B_\infty(X)) \approx H(\overline{B}(C_*(X))$$

Now we consider the problem of defining a multiplication in the classifying space. More generally, if X and Y are two associative

H-spaces and if $f:X \to Y$ is multiplicative, then one has induced in a natural way

$$B_\infty(X) \to B_\infty(Y).$$

However, requiring that f be multiplicative is too strong a condition. Sugawara [5] has defined a weaker condition of f so that there be an induced map of classifying spaces. Namely, that f be strongly homotopy multiplicative. That is, that for each positive integer i there exists

$$M_i:(X\times (I\times X)^n) \to Y \text{ so that}$$

$$M_o=f \text{ and}$$

$$M_i(x_o,t_1,x_1,\ldots,t_n,x_n) = \begin{cases} M_{i-1}(x_o,t_1,\ldots,t_{j-1},x_{j-1}x_j,t_{j+1},\ldots,t_n,x_n) & \text{if } t_j=0 \\ M_{j-1}(x_o,\ldots,t_{j-1},x_{j-1})\cdot M_{i-j}(x_j,t_{j+1},\ldots,t_n,x_n) & \text{if } t_j=1 \end{cases}$$

M_1 gives f homotopy multiplicative.

As one application, if $F \to E \to X$ is a principal quasifibering and X is a countable CW H-space, using a modified version of Sugawara's almost covering homotopy extension property [6], one may construct a strongly homotopy multiplicative map $\{M_i\}_{i=o}^{\infty}$ from $\Omega X \to F$ which induces

$$B_\infty(\Omega X) \longrightarrow B_\infty(F)$$

But $X \sim B_\infty(\Omega X)$ hence one has $X \rightarrow B_F$. One may prove a generalization of the Steenrod classification theorem. Let X be a countable CW space and F an associative CW-H-space. There is a one to one correspondence between homotopy classes of maps from X to $B_\infty(F)$ and equivalence classes of principal F-bundles over X.

In the above, given a map $X \rightarrow B_F$, one does not take an "induced fibering ". The construction is more complicated then that. One should mention that principal F-bundle and equivalence are in the sense of Dold and Lashof [3] if translations in F are monomorphisms, and one needs a slight modification in the definition of principal F-bundle otherwise.

The essential ideas of the proof of the above are contained in the author's PH.D dissertation in the classifying space.

Let X be an associative H-space with multiplication $m: X \times X \rightarrow X$. Suppose m forms a strongly homotopy multiplicative map. Then we have induced

$$B_\infty(X \times X) \rightarrow B_\infty(X \times X).$$

But

$$B_\infty(X) \times B_\infty(X) \sim B_\infty(X \times X)$$

Hence we have

$$B_\infty(X) \times B_\infty(X) \rightarrow B_\infty(X).$$

Now that there exist a strongly homotopy multiplicative map is a geometric condition on the space. Such a map may or may not exist.

Clark has given the algebraic analogue of this condition [ ]. Namely, let B and C be associative DGA algebras over a commutative ring with unit K. A strongly homotopy multiplicative map from B to C is an infinite sequence of K-module homomorphisms $\{h_1, h_2, \ldots, h_n, \ldots\}$ where each

$$h_m : B \otimes \ldots (m) \ldots \otimes B \longrightarrow C$$

of degree m-1 satisfies

$$dh_m(b_1 \otimes \ldots \otimes b_m) + h_m d(b_1 \otimes \ldots \otimes b_m) =$$

$$\sum_{i=1}^{m-1} h_{m-1}(-1)^{u(i)}(b_1 \otimes \ldots \otimes b_i \circ b_{i+1} \otimes \ldots \otimes b_m)$$

$$- \sum_{i=1}^{m-1} h_i((-1)^{u(i)}(b_1 \otimes \ldots \otimes b_i)) \cdot h_{m-i}(b_{i+1} \otimes \ldots \otimes b_m)$$

where

$$u(i) = i + \deg(b_1 \otimes \ldots \otimes b_i)$$

Such a strongly homotopy multiplicative map induces a morphism of DGA-coalgebras

$$\overline{B}(h) : \overline{B}(B) \longrightarrow \overline{B}(C).$$

Hence if A are the chains of an H-space X, if there exists a strongly homotopy multiplicative map $A\otimes A \longrightarrow A$ then we have induced

$$\overline{B}(A\otimes A) \longrightarrow \overline{B}(A)$$

and hence

$$\overline{B}(A)\otimes\overline{B}(A)\sim\overline{B}(A\otimes A) \longrightarrow \overline{B}(A).$$

2. <u>The cobar construction and strongly homotopy comultiplicative maps</u>.

<u>The cobar construction</u>. (Adams [1]). Recall that if C is a simply connected DGA coalgebra over K, a fixed commutative ring with unit, i.e., C is connected and $C_1=0$, then the cobar $\overline{F}(C)$ is the direct product of the D^n for all $n\geq 0$, where D^n is the n-fold tensor product of the desuspension of $\overline{C}=\mathrm{Ker}\ (\varepsilon)$, and where $\varepsilon: C \longrightarrow K$ is the augmentation. (Normally one takes the direct sum, but the free product will be more convenient). We will use infinite sum notation instead of the product notation. A typical element is therefore an infinite linear combination of elements of the form $x=[c_1|\ldots|c_n]$, where x has bidegree $(-n,m)$, and $m=\sum_{i=1}^{n}$ degree (c_i). The differential in $\overline{F}(C)$ is defined on elements of bidegree $(-1,*)$ by

$$d[c]=[-dc]+\sum_i (-1)^{\deg c_i'}[c_i'|c_i'']$$

where

$$\Delta(c)=c\otimes 1 + 1\otimes c + \sum_i c_i'\otimes c_i'', \quad \Delta:C \longrightarrow C\otimes C$$

being the diagonal mapping of C . The differential is extended to all of $\overline{F}(C)$ by the requirement that $\overline{F}(C)$ be a DGA-algebra.

The acyclic cobar construction is $F(C) = C\otimes\overline{F}(C)$ with the contracting homotopy $s:F(C) \longrightarrow F(C)$ defined by

$$s(c\otimes [c_1|\dots|c_n]) = \varepsilon(c)\cdot c_1\otimes [c_2|\dots|c_n]$$

and differential $d:F(C) \longrightarrow F(C)$ defined so that

$$ds(x) + sd(x) = x - \varepsilon(x)\otimes [\ ],$$

where $[\ ]$ is the unit element of $\overline{F}(C)$, and $\varepsilon:F(C) \longrightarrow K$ is the augmentation induced by the augmentations of C and $\overline{F}(C)$.

F(C) is a differential C-comodule with coaction

$$\Delta_{F(C)}:F(C) \longrightarrow C\otimes F(C)$$

given by

$$\Delta_{F(C)}(c\otimes z) = \Delta(c)\otimes z$$

and is a differential $\overline{F}(C)$-module with action

$$F(C)\otimes\overline{F}(C) \longrightarrow F(C)$$

given by

$$(c\otimes[c_1|\ldots|c_n])\cdot([b_1|\ldots|b_m]) = c\otimes[c_1|\ldots|c_n|b_1|\ldots|b_m]$$

We wish to intoduce some notation at this point to make the writing of some formulas easier. Let

$$C^k = C\otimes\ldots(k)\ldots\otimes C,$$

be the tensor product of C with itself k times.

Then for $1\leq i\leq k$, define $P_i:C^k \to C^k$ by

$$P_i(c_1\otimes\ldots\otimes c_k) = (-1)^{\sum_{j=1}^{i}\deg c_j}(c_1\otimes\ldots\otimes c_k)$$

and also define

$$d_k^-:C^k \longrightarrow C^k$$

by

$$d_k^-(c_1\otimes\ldots\otimes c_k)= \sum_{i=1}^{k}(-1^i P_{i-1}(c_1\otimes\ldots\otimes c_{i-1}\otimes dc_i\otimes c_{i+1}\otimes\ldots\otimes c_k)$$

Also, define $\Delta_i^k:C^k \longrightarrow C^{k+1}$ by

$$\Delta_i^k(c_1\otimes\ldots\otimes c_k)=c_1\otimes\ldots\otimes c_{i-1}\otimes\Delta(c_i)\otimes c_{i+1}\otimes\ldots\otimes c_k$$

Define

$$i_k : C^k \to \bar{F}(C) \quad \text{by} \quad i_k(c_1 \otimes \ldots \otimes c_k) = [c_1 | \ldots | c_k]$$

Since $\bar{F}(C)$ is defined on $\bar{C}$, if any c_i has degree 0 then $i_k(c_1 \otimes \ldots \otimes c_k) = 0$.

The following formula may be verified by induction:

$$d i_n = i_n \bar{d}_n + i_{n+1} \sum_{i=1}^{n} (-1)^{i+1} P_i \Delta_i^n$$

<u>Strongly homotopy comultiplicative maps</u>. (Drachman [ ]). Suppose C and D are DGA coalgebras over K and $h_1 : C \to D$ is a homomorphism of DGA-modules (but not necessarily a homomorphism of coalgebras). Then to say h_1 is the initial mapping of the strongly homotopy comultiplicative (SHCM) mapping $\{h_1, h_2, \ldots, h_n, \ldots\}$ will mean that for each integer $n \geq 1$, h_n is a K-module homomorphism of degree $n-1$

$$h_n : C \longrightarrow (D)^n$$

such that

$$\bar{d}_n h_n + h_n d = \sum_{i=1}^{n-1} (h_i \otimes h_{n-i}) P_1 \Delta + \sum_{i=1}^{n-1} (-1)^i P_i \Delta_i^{n-1} h_{n-1}$$

In particular, for $n=2$ we have

$$\bar{d}_2 h_2 + h_2 d = (h_1 \otimes h_1) P_1 \Delta - P_1 \Delta h_1$$

which says, except that the signs are different, that the following diagram is homotopy commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C\otimes C \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1\otimes h_1} \\ D & \xrightarrow{\Delta} & D\otimes D \end{array}$$

The motivation for the above definition is the following:

<u>Theorem</u>. If C and D are simply connected coalgebras over K and $h=\{h_1,\ldots,h_n,\ldots\}$ is a SHCM mapping from C to D, then h induces a morphism of DGA algebras

$$\overline{F}(h):\overline{F}(C) \longrightarrow \overline{F}(D)$$

<u>Proof</u>. First define $\overline{F}(h)$ for elements having one bar by

$$F(h)[c]=i_1h_1(c) + i_2h_2(c)+\ldots+i_nh_n(c)+\ldots$$

and then extend $\overline{F}(h)$ to all of $\overline{F}(C)$ by the requirement that $\overline{F}(h)$ be multiplicative.

<u>A diagonal map for the cobar construction</u>. Suppose C and D are DGA comodules. The usual way to define a morphism between $\overline{F}(C) \longrightarrow \overline{F}(D)$ is to have a comultiplicative map $h:C \longrightarrow D$ inducing

$$h_*: F(C) \longrightarrow F(D)$$

given by

$$[c_1|\ldots|c_n] \longrightarrow [h(c_1)|\ldots|h(c_n)].$$

To say $h:C \longrightarrow D$ is comultiplicative means the following dia diagram is commutative.

$$\begin{array}{ccc} C & \xrightarrow{\quad h \quad} & D \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow{\quad h\otimes h \quad} & D \otimes D \end{array}$$

However, in the case that C is the chain complex of a space X, and $D=C\otimes C$, the diagonal map $\Delta: C \longrightarrow C\otimes C$ is not comultiplicative, hence $\Delta_*:\overline{F}(C) \longrightarrow \overline{F}(C\times C) \approx \overline{F}(C)\otimes\overline{F}(C)$ is not defined this way. One can, however, form a SHCM map $h=\{h_1,\ldots,h_n,\ldots\}$ from C to $C\otimes C$ where $h_1=\Delta: C \rightarrow C\otimes C$ and in this case we have

$$\Delta:\overline{F}(C) \xrightarrow{\overline{F}(H)} F(C\otimes C) \xrightarrow{\quad\approx\quad} \overline{F}(C)\otimes\overline{F}(C),$$

a morphism of DGA-algebras.

We shall return to these maps shortly.

3. The Geometric Cobar Construction.

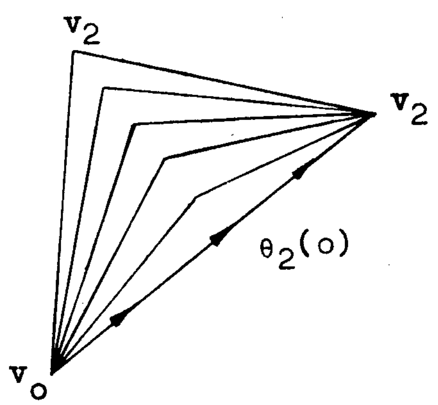
Just as the bar construction has a geometric realization, namely the classifying space which is a filtered space whose filtration corresponds to the Moore filtration of the bar construction, there is a geometric realization of the cobar construction. To see this, let X be a simply connected special countable CW complex. That is, X has one zero-cell e, no one-cells, and we assume that the cells are simplicial. We use the maps $\{\theta_n\}$ in Adams' original paper [ ]. Let $L_{i,j}(\sigma^n)$ be the paths in the n-simplex σ^n which start at the $i^{\underline{th}}$ vertex and end at the $j^{\underline{th}}$ vertex. Adams constructs maps

$$\theta_n : I^{n-1} \longrightarrow L_{o,n}(\sigma^n)$$

where $\theta_q = \omega : [0,1] \longrightarrow \sigma^1$ is given by

$$\omega(x) = (1-x, x).$$

We show a picture of $\theta_2 : I^1 \longrightarrow L_{o,2}(\sigma^2)$.



$\theta_2(o)$ is the path from v_o to v_2 travelling directly from v_o to v_2 along the edge. $\theta_2(1)$ is the path from v_o to v_2 first going from v_o to v_1 along the edge, and then going from v_1 to v_2 along the edge. $\theta_2(t)$ may be taken as two segments in the manner indicated.

In general, θ_n may be chosen so that

$$\theta_n(t_1,\ldots,y_{n-1}) = \begin{cases} L(f_i)\theta_i(t_1,\ldots,t_{i-1})\times L(\ell_i)\theta_{n-i}(t_{i+1},\ldots,t_{n-1}) \\ \qquad\qquad \text{if } t_i=1 \\ L(d_i)\theta_{n-1}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{n-1}) \\ \qquad\qquad \text{if } t_i=0 \end{cases}$$

In the above $d_i:\sigma^{n-1} \longrightarrow \sigma^n$, $f_i:\sigma^i \longrightarrow \sigma^n$, and $\ell_i:\sigma^{n-i} \longrightarrow \sigma^n$ are the injections of σ^{n-1} as the $i^{\underline{th}}$ face, σ^i and σ^{n-1} as the first and last faces, inducing

$$L_{o,n-1}(\sigma^{n-1}) \xrightarrow{L(d_i)} L_{o,n}(\sigma^n)$$

$$L_{o,i}(\sigma^i) \xrightarrow{L(f_i)} L_{o,i}(\sigma^n)$$

$$L_{o,n-i}(\sigma^{n-i}) \xrightarrow{L(\ell_i)} L_{i,n}(\sigma^n)$$

Since $L(f_i)\theta_i(t_1,\ldots,t_{i-1})$ lies in $L_{o,i}(\sigma^n)$ and $L(\ell_i)\theta_{n-i}(t_{i+1},\ldots,t_{n-1})$ lies in $L_{i,n}(\sigma^n)$, their product makes sense.

The above formula defines θ_n inductively on ∂I^{n-1}, and then one fills in as before. Then θ_n is 1-1 onto its image.

Now given a cell e_n of X with attaching map $f:\Delta^n \to X$, one constructs a cubical cell $[e_n]$ of one less dimension with attaching map

$$I^{n-1} \xrightarrow{\theta_n} L_{(o,n)}(\sigma^n) \xrightarrow{L(f)} \Omega(X).$$

Notice that since the o^{th} and n^{th} vertices are attached to the base point e by f, we land in the loop space instead of merely the path space of X. Let $[e_n]$ be the image of this attaching map in $\Omega(X)$.

Let $\overline{F}^{-n}(X)$ be the CW complex consisting of those points which are in the product on n or more cells $[e_{i_1}]\ldots[e_{i_k}]$ $(K \geq n)$. Then we have $\overline{F}^{o}(X)=\overline{F}(X)$ a filtered space

$$\ldots \subset \overline{F}^{-n}(X) \subset \overline{F}^{-n+1}(X) \subset \ldots \subset \overline{F}^{1}(X) \subset \overline{F}^{o}(X) \subset \Omega(X)$$

$\overline{F}(X)$ is H-homotopy equivalent to ΩX. In fact, is strongly homotopy multiplicatively equivalent to ΩX. (Their classifying spaces are equivalent.)

4. The Diagonal Map and Steenrod Squares in the Loop Space.

As we mentioned, to introduce a multiplication in the bar construction introduces an extra condition on the algebra. To introduce a coalgebra structure in the cobar construction, we wanted $\Delta:C \to C\otimes C$ to form the initial map of a strongly homotopy comultiplicative map. Fortunately, this is no extra condition when C is the chain group of a space. The existence of $\{h_2,\ldots,h_n,\ldots\}$ (with $h_1=\Delta$) can be demonstrated using an acyclic carriers argument. $h_n:C \to (C\otimes C)^n$ is constructed so σ lies in $(\sigma\times\sigma)^n$, and any two such sequences $\{h_1,h_2,\ldots,h_n,\ldots\}$, $\{h_1',h_2',\ldots,h_n',\ldots\}$ can be shown to be homotopic in an appropriate sense, so that their induced maps $\overline{F}(C) \to \overline{F}(C\otimes C)$ are homotopic in the usual sense.

It is possible to give explicit formulas for one choice of the h_n, which we do here. For the sake of notation, we will no longer refer to these maps as h_n's, but as $\{{}^1\Delta_n\}_{n=1}^{\infty}$.

To further simplify matters, we shall work mod (2) to avoid keeping track of signs. ${}^1\Delta_1:C \to C\otimes C$ is the standard Alexander-Whitney diagonal map. That is, if $\sigma=\langle v_o,\ldots,v_n\rangle$ is an n-simplex which we write

$$\sigma=\langle o,\ldots,n\rangle,$$

$${}^1\Delta_1\sigma=\sum_{i=o}^{n}\langle o,\ldots,i\rangle\otimes\langle i,\ldots,n\rangle$$

To simplify further, we merely write

$${}^1\Delta_1=(1,n).$$

This means

$${}^1\Delta_1\langle o,\ldots,n\rangle = \sum_{o\leq i_1\leq n}^{n} \langle o,\ldots,i_1\rangle\otimes\langle i_1,\ldots,n\rangle$$

$$^{1}\Delta_2 : C \rightarrow C^4$$

is given by

$$^{1}\Delta_2 = (1,24,3,n).$$

This means

$$^{1}\Delta_2 \langle o,\dots,n\rangle =$$

$$\sum_{o \le i_1 \le i_2 < i_3 \le i_4 \le n} \langle o,\dots,i_1\rangle \otimes \langle i_1,\dots,i_2,i_3,\dots,i_4\rangle \otimes \langle i_2,\dots i_3\rangle \otimes \langle i_4,\dots,n\rangle$$

Before giving the general formula, let us give $^{1}\Delta_3$ and $^{1}\Delta_4$ explicitly

$$^{1}\Delta_3 = (1,24,3,57,6,n) + (1,246,3,7,5,n)$$

$$\begin{aligned} ^{1}\Delta_4 = &(1,24,3,57,6,810,9,n) + (1,24,3,579,6,10,8,n) \\ &+(1,246,3,79,5,10,8,n) + (1,246,3,7,5,810,9,n) \\ &+(1,2468,3,9,5,10,7,n) \end{aligned}$$

We note that the second term of $^{1}\Delta_3$ gives us a term that is not a cup-i.

To give the general formula, we define the sequence $\{k_1,\dots,k_{n-1}\}$ of positive integers to be admissible if

1) $\sum_{i=1}^{n-1} k = 2n-2$ and

2) for each i, $2 \leq i \leq n$, the inequality

$$\sum_{j=1}^{i-1} k_j \neq 2i-3 \quad \text{holds.}$$

For such an admissible sequence $k=\{k_1,\ldots,k_{n-1}\}$, we define $u_i(k)=((2\sum_{j=1}^{i-1} k_j)-i+2,\ldots,2\sum_{j=1}^{i} k_j-i)$. That is, $u_i(k)$ has listed every other integer between $(2\sum_{j=1}^{i-1} k_j)-i+2$ and $2\sum_{j=1}^{i} k_j-i$.

(Summation over the empty set being understood to be 0.)

Then $s_i(k)$ is defined to be the minimum of the set $\{1,2,3,\ldots,3_n-3\}$ - $\{$all entries listed in $u_1(k),s_1(k),\ldots$

$$\ldots,u_{i-1}(k),s_{i-1}(k),u_i(k)\}$$

Then finally we define

$$^1\Delta_n = \sum_{k=(k_1,\ldots,k_{n-1}) \text{ is admissible}} (1,u_1(k)+1,s_1(k)+1,u_2(k)+1,s_2(k)+1,\ldots,u_{n-1}(k)+1,s_{n-1}(k)+1,n)$$

For example, in $^1\Delta_4$ appears the term

$$(1,24,3,57,6,810,9,n)$$

which means that this term applied to $\sigma = \langle o,\dots,n\rangle$ gives

$$\sum_{\substack{0\le i_1\le i_2<i_3\le i_4\le i_5<i_6\\ \le i_7\le i_8<i_9\le i_{10}\le n}} \begin{aligned}&\langle o,\dots,i_1\rangle\otimes\langle i,\dots,i_2,i_3,\dots,i_4\rangle\otimes\langle i_2,\dots,i_3\rangle\\ &\otimes\langle i_4,\dots,i_5,i_6,\dots,i_7\rangle\otimes\langle i_5,\dots,i_6\rangle\\ &\otimes\langle i_7,\dots,i_8,i_9,\dots,i_{10}\rangle\\ &\otimes\langle i_8,\dots,i_9\rangle\otimes\langle i_{10},\dots,n\rangle\end{aligned}$$

Hence we know the diagonal map in the loop space explicity. That is, returning to the original diagram,

$$\begin{array}{ccc} \overline{F}(C) & \xrightarrow{\ {}^1\Delta\ } & \overline{F}(C)\otimes\overline{F}(C)\\ & \searrow & \uparrow\\ & & \overline{F}(C\otimes C)\end{array}$$

$${}^1\Delta[c_1] = \sum_{n=1}^{\infty} i_n\,{}^1\Delta_n(c) \quad \text{and hence}$$

$${}^1\Delta[c_1|\dots|c_k] = \sum_{s=k}^{\infty} i_s \sum_{i_1+\dots+i_k=s} {}^1\Delta_{i_1}(c_1)\otimes\dots\otimes{}^1\Delta_{i_k}(c_k)$$

where in the above, is puts s-bars around its term. That is,

$$i_s(c_1\otimes c_1^1\otimes\dots\otimes c_s\otimes c_s^1) = [c_1\otimes c_1'|\dots|c_s\otimes c_s']$$

Such a general formula for the maps $\{{}^1\Delta_n\}$ will be useful in looking at certain (unstable) operations, but for the problem at hand we have done too much. Since ρ is zero except if in each term $c_i \otimes c_i'$, c_i or c_i' has degree, in each ${}^1\Delta_n(\sigma)$ only one term remains. That is, we may write the much smaller expression.

$$ {}^1\Delta[c] = \rho \sum_{n=1}^{\infty} i_n {}^1\delta_n(c) $$

where

$$ {}^1\delta_1 = \Delta = (1,n) $$

$$ {}^1\delta_2 = (1,24,3,n) $$

$$ {}^1\delta_3 = (1,246,3,7,5,n) $$

$$ {}^1\delta_4 = (1,2468,3,9,5,10,7,n) $$

$$ \vdots $$

$$ {}^1\delta_n = (1,2468\ldots 2\cdot n,3,2\cdot n+1,5,\cdots,n) $$

We also note that if σ is a k-simplex and n>k then ${}^1\Delta_n$ vanishes on σ. Hence if $c=\sigma$ is a k-simplex,

$$ {}^1\Delta[c] = \rho \sum_{n=1}^{k} i_n {}^1\delta_n(c) $$

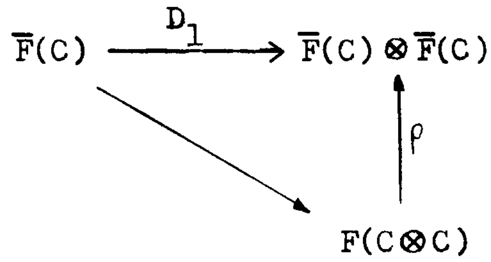
<u>The Steenrod Squares in the Loop Space</u>.

Now that we have a diagonal map in the cobar construction, we can introduce the cup-i products. For instance, cup-1 is defined by letting

$D_1 : \overline{F}(C) \longrightarrow \overline{F}(C) \otimes \overline{F}(C)$ to be a homotopy between Δ and $T\Delta$, and in general

$D_{i+1}=\overline{F}(C) \longrightarrow \overline{F}(C)\otimes\overline{F}(C)$ to be a homotopy between D_i and TD_i.

For example, in defining cup-1, one looks at the diagram



where $D_1 i_s = \rho \sum_{k=1}^{\infty} i_k D^1_{s,k}$ where $D^1_{s,k}: C^s \longrightarrow (C\otimes C)^k$ is of degree $k-s+1$.

One can give simplicial formulas for the $D^1_{s,k}$ and in general the $D^i_{s,k}$ just as in the case of the cup product.

A. Zacharion [1] has written down explicity cup-i products in the case that $\Delta: C \longrightarrow C\otimes C$ is co-commutative, which of course does not apply to our case.

The maps $\{{}^1\Delta_n\}$ may be used to introduce certain (unstable) operations to detect if a space can be a second suspension. By iterating the cobar construction in a suitable manner, one gets higher order maps which can be used to detect if a space can be an $n^{\underline{th}}$ suspension.

Iterating the Cobar Construction.

As we mentioned, starting with a simplicial chain complex, taking the cobar construction gives us a cubical chain complex with a diagonal map described above. Since we have (co-) associativity, we can iterate the cobar construction once and we have explicitly the differential in the 2^{nd} loop space. But then we loose co-associativity. Hence extra homotopies are needed to define the differential, just as in the analogous case of an H-space which is not associative but has a certain collection of associating homotopies. In the n^{th} iteration of the cobar construction, there will be a differential

$$^{n}d : \overline{F}^{n}(C) \longrightarrow \overline{F}^{n}(C)$$

determined by an "n-matrix" of functions and also a diagonal map

$$^{n}\Delta : \overline{F}^{n}(C) \longrightarrow \overline{F}^{n}(C) \otimes \overline{F}^{n}(C)$$

determined by an n-matrix of functions.

The matrix for ^{n}d will have entries

$$^{n}d_{i_1,\ldots,i_n} : C \longrightarrow C^{i_1 \ldots i_n}$$

for each sequence $\{i_1,\ldots,i_n\}$ of positive integers.

Similarly, the matrix

$^{n}\Delta$ will have entries

$$^{n}\Delta_{i_1,\ldots,i_n} : C \longrightarrow C^{2i_1 \ldots i_n}$$

I will not go into further details about the iteration of the cobar construction at this time.

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