The non-existence of spaces with finitely generated stable homotopy

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The purpose of this talk is to prove the following:

Theorem 1: If $\tilde{H}^n(X; Z_p) \neq 0$ for any n, then $\sum_{i=0}^{\infty} {}_p\pi_i^s(X)$ (the p-primary component of the total stable homotopy group of a space X) is not a finitely generated group.

This is actually the restriction of the following theorem to spaces of finite type.

Theorem 2: If $\tilde{H}_n(X; Z_p) \neq 0$ for any n, then ${}_p\pi_i^s(X) \neq 0$ for infinitely many n.

Theorem 2 is proved in [2] by a generalization of the method used here for Theorem 1.

Most of the work here is algebra. Recall that a Noetherian module over a ring R is an R-module M such that every submodule of M is finitely generated. A ring R is Noetherian if and only if it is Noetherian as a module over itself. (left-, right-, or two-sided. For this , assume everything is left.)

Noetherian is much too strong a condition in topology. Polynomial algebras over Z or Z_p are Noetherian if they have finitely many generators, but not otherwise. Yet, the nicest things arising in topology usually have infinitely many ring generators. So we look at the following:

<u>Definition.</u> An R-module M is finitely presented if and only if there exists an exact sequence

$$0 \to K \to F \to M \to 0$$

where F and K are finitely generated R-modules and F is free. (We use f.p. and f.g. throughout.)

Then we call an R-module M <u>coherent</u> if and only if M itself and every finitely generated submodule of M is finitely presented. (In particular M is finitely generated.) Then we say that a ring R is coherent if and only if R is coherent as an R-module.

The following facts, once stated are easy to prove. Some are exercises in Bourbaki [1], pp. 62-63.

<u>Proposition 1.</u> A ring R is coherent if and only if every f.p. R-module is coherent.

<u>Proposition 2.</u> If $0 \to M \to N \to P \to 0$ is an exact sequence of R-modules or if

$$\begin{array}{ccc} M & \longrightarrow & N \\ & \searrow \quad \swarrow & \\ & R & \end{array}$$

is an exact triangle of R-modules, then if any two are coherent, so is the third.

<u>Proposition 3.</u> If $f: M \to N$ where M and N are coherent R-modules, then ker f, im f, cok f are coherent.

<u>Proposition 4.</u> If $R = \varinjlim_{\alpha} R_\alpha$ where each R_α is a left coherent ring and R is a right flat R_α-module, then R is a left coherent ring.

Proposition 4 leads to the fact that the Steenrod algebra for a prime p, A^* is coherent (it is the direct limit of finite, hence coherent, sub-Hopf algebras; and by Milnor-Moore [3] is free, hence flat, over each of them).

We now establish the following about spectra (anybody's definition of spectra will do).

From now on $H^*(\ ) = H^*(\ ; Z_p)$, p some prime.

<u>Lemma.</u> If $\underline{Y}$ is a spectrum and $\sum_{i=-\infty}^{\infty} \pi_i(\underline{Y}) \simeq (Z_p)^r$ for r finite, then $H^*(\underline{Y})$ is a coherent A^*-module.

<u>Proof.</u> By induction on r. If $r = 0$ then $H^*(\underline{Y}) = 0$ hence it is trivially coherent. Assume the theorem true up to $r-1$. Then if $\sum \pi_i(\underline{Y}) \simeq (Z_p)^r$, choose a non-trivial cohomology class of lowest degree in $H^*(\underline{Y})$. Let $f: \underline{Y} \to \underline{K}_p$ represent this class (where $\underline{K}_p$ is the Eilenberg-MacLane spectrum of Z_p). Then $f_*: \pi_*(\underline{Y}) \to \pi_*(\underline{K}_p)$ is onto . Thus if we look at the fibre of f, $\underline{E}_f$, we have $\sum \pi_i(\underline{E}_f) \simeq (Z_p)^{r-1}$ hence $H^*(\underline{E}_f)$ is coherent.

But now

$$\begin{array}{ccc} H^*(\underline{E}_f) & \longrightarrow & H^*(\underline{K}_p) \\ & \nwarrow \quad \swarrow & \\ & H^*(\underline{Y}) & \end{array}$$

is exact. Since $H^*(\underline{E}_f)$ and $H^*(\underline{K}_p) \simeq A^*$ are coherent. $H^*(\underline{Y})$ is also.

Now we can prove Theorem 1. Assume we have a space X with $\sum_{i=0}^{\infty} {}_p\pi_i^s(X)$ finitely generated. Then let $Y = X \wedge M$, where M is a Moore space $S^t \cup_{p\iota} e^{t+1}$ of type (Z_p, t). Then we have that

$$\sum_{i=0}^{\infty} {}_p\pi_i^s(Y) \simeq (Z_p)^r$$

for some finite r. Thus $\tilde{H}^*(Y) \simeq H^*(\underline{S}Y)$ is a coherent A^* module by the lemma ($\underline{S}Y$ is the suspension spectrum of Y), since $\pi_*^s(Y) \simeq \pi_*(\underline{S}Y)$.

Let $\theta \in \tilde{H}^*(Y)$. Let $f: A^* \to \tilde{H}^*(Y)$ be given by $f(a) = a\theta$. Then by Proposition 3 ker f is coherent, hence finitely generated. If $1 \notin \ker f$ then ker f contains only finitely many of the indecomposable elements P^{p^n} of A^* whence $P^{p^n}\theta = f(P^{p^n}) \neq 0$ for all n sufficiently large. But this contradicts the well-known properties of the Steenrod algebra. Thus $1 \in \ker f$ so $0 = f(1) = \theta$. Thus $\tilde{H}^*(Y) = 0$. Since $Y = X \wedge M$, this says $\tilde{H}^*(X) = 0$ and the theorem is proved.

For Theorem 2, replace finitely generated by:

Definition. A graded module M is weakly finitely generated if and only if it has generators in only finitely many degrees.

Similarly define weakly finitely presented and weakly coherent. Then in [2] we prove that every w.f.g. free A^*-module is w.coherent. This leads to the fact that $H^*(\underline{K}(G); Z_p)$ is weakly coherent for the Eilenberg-MacLane spectrum of any group G. Then induction on the number of non-trivial ${}_p\pi_i^s(X)$ finishes up the argument exactly as here.

Bibliography

1. N. Bourbaki, Algèbre commutative, Chapters 1-2, Hermann, Paris, 1961.

2. J. M. Cohen, Coherent graded rings and the non-existence of spaces of finite stable homotopy type, Comm. Math. Helv. (to appear).

3. J. W. Milnor and J. C. Moore, On the structure of Hopf algebras, Ann. of Math. 81(1965), 211-264.

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