ORIENTABILITY AND EXTENSIONS
OF COHOMOLOGY THEORIES

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1. Introduction. Dold [1] has established a Thom isomorphism theorem for a generalized cohomology theory h provided that the sphere bundle in question is h-orientable. In the case of ordinary cohomology this condition of orientability can be eliminated by introducing cohomology with local coefficients. Our purpose is to carry out an analogous construction for any cohomology theory. Similar results have been obtained by MacAlpine [2] and Berstein (unpublished) using different constructions.

2. Cohomology theories. Let $\mathcal{P}^*$ be the category of finite CW-pairs, B a space, and $\mathcal{P}^*(B)$ the category of triples (X,A,f) with $(X,A) \in \mathcal{P}^*$ and $f : X \longrightarrow B$. A *cohomology theory* h on $\mathcal{P}^*(B)$ consists of contravariant functors $h^n : \mathcal{P}^*(B) \longrightarrow \mathcal{A}$ ($\mathcal{A}$ = category of abelian groups) and natural transformations $d^n : h^n \circ T \longrightarrow h^{n+1}$ (where $T(X,A,f) = (A,\varphi,f_{|A})$) such that

(2.1) *For* $(X \times I, A \times I, F) \in \mathcal{P}^*(B)$,

$$i_0^* : h^n(X \times I, A \times I, F) \longrightarrow h^n(X,A,f_0)$$

is an isomorphism, where $i_0(x) = (x,0)$, $x \in X$, *and* $f_0 = Fi_0$.

(2.2) _For_ $(X,A,f) \in \mathcal{P}^*(B)$, _the sequence_

$$\cdots \xrightarrow{d^{n-1}} h^n(X,A,f) \xrightarrow{j^*} h^n(X,f) \xrightarrow{i^*} h^n(A,f_{|A}) \xrightarrow{d^n} \cdots$$

is exact.

(2.3) _If_ $X = A_1 \cup A_2$ _where_ A_1, A_2 _are subcomplexes of_ X _and_ $i : (A_1, A_1 \cap A_2) \longrightarrow (X, A_2)$ _is inclusion then_

$$i^* : h^n(X,A_2,f) \longrightarrow h^n(A_1,A_1 \cap A_2, f_{|A_1})$$

is an isomorphism.

EXAMPLE 1. If B is a point, h is a generalized cohomology theory as in [5].

EXAMPLE 2. Let π be an abelian group, $A(\pi)$ its automorphism group and L the universal bundle of coefficients over $K(A(\pi),1)$ with fibre π. Define $H^n : \mathcal{P}^*(K(A(\pi),1)) \longrightarrow \mathfrak{a}$ by $(X,A,f) \longrightarrow H^n(X,A;f^{-1}(L))$, the n-th singular cohomology group of (X,A) with coefficients in the induced bundle $f^{-1}(L)$.

It follows from (2.1) that

$$(2.4) \qquad i_1^* : h^n(X \times I,\ A \times I,\ F) \longrightarrow h^n(X,A,f_1)$$

is also an isomorphism, where $i_1(x) = (x,1)$, $x \in X$, and $f_1 = Fi_1$. Define

$$(2.5) \qquad F_{\#} : h^n(X,A,f_1) \longrightarrow h^n(X,A,f_0)$$

by $F_{\#} = i_0^* i_1^{*-1}$. Thus, $f_0 \simeq f_1$ implies that $h^n(X,A,f_0)$ is isomorphic to $h^n(X,A,f_1)$.

3. Spectra. A sectioned fibration $\mathcal{E} = (E,B,p,\Lambda)$ consists of a fibration (E,B,p) and a cross-section $\Lambda : B \longrightarrow E$. The loop space of $\mathcal{E}$ is $\Omega(\mathcal{E},\Lambda) = (\Omega(E,\Lambda),B,\Omega(p),\Omega(\Lambda))$ where

$$(3.1) \qquad \Omega(E,\Lambda) = \{\sigma : I \longrightarrow E \mid \sigma(I) \subset p^{-1}(b), \text{ some } b \in B,\ \sigma(0) = \sigma(1) = \Lambda(b)\}$$

For $(X,A,f) \in \mathcal{P}^*(B)$ we have the space of liftings

$$(3.2) \qquad \mathcal{L}(X,A,f;\mathcal{E}) = \{g : X \longrightarrow E \mid pg = f \text{ and } g_{|A} = \Lambda f_{|a}\}.$$

A B-spectrum $E = \{\mathcal{E}_k; \epsilon_k\}$ is a collection of sectioned fibrations $\mathcal{E}_k$ and maps $\epsilon_k : \mathcal{E}_k \longrightarrow \Omega(\mathcal{E}_{k+1};\Lambda_{k+1})$, $-\infty < k < \infty$.

Now, a cohomology theory $h(\ ;E)$ on $\mathcal{P}^*(B)$ may be constructed from E as follows. For $(X,A,f) \in \mathcal{P}^*(B)$ form the point spectrum $\mathcal{L}(X,A,f;E)$ whose k-th space is $\mathcal{L}(X,A,f;\mathcal{E}_k)$ and whose connecting maps are

$$\mathcal{S}(X,A,f;\mathcal{E}_k) \xrightarrow{\mathcal{S}(\epsilon_k)} \mathcal{S}(X,A,f;\Omega(\mathcal{E}_{k+1})) = \Omega(\mathcal{S}(X,A,f;\mathcal{E}_{k+1})).$$

Then let

$$(3.3) \qquad h^n(X,A,f;E) = \pi_{-n}(\mathcal{S}(X,A,f;E)).$$

4. Pairings. Suppose that cohomology theories h_i on $\mathcal{P}^*(B^i)$, $i = 1, 2, 3$, and $\mu : B^1 \times B^2 \longrightarrow B^3$ are given. A μ-pairing $\eta : h_1 \otimes h_2 \longrightarrow h_3$ consists of homomorphisms

$$(4.1) \qquad h_1^s(X,A_1,f_1) \otimes h_2^t(X,A_2,f_2) \longrightarrow h_3^{s+t}(X,A_1 \cup A_2, \mu(f_1 \times f_2))$$

$(X,A_1,f_1) \in \mathcal{P}(B^1)$, $(X,A_2,f_2) \in \mathcal{P}(B^2)$, having the following properties. (As usual, denote $\eta(u \otimes v)$ by $u \cup v$.)

(4.2) If g is such that $g : (X,A_1,f_1) \longrightarrow (X,A_1',f_1')$ and $g : (X,A_2,f_2) \longrightarrow (X,A_2',f_2')$ are maps then

$$g^*(u \cup v) = g^*(u) \cup g^*(v) .$$

For the next two properties let (X,A) and $f_j : X \longrightarrow B^j$, $j = 1, 2$, be given and let $i : A \longrightarrow X$ be inclusion.

(4.3) <u>If</u> $u \in h_1^{s-1}(A,f_1)$ <u>and</u> $v \in h_2^t(X,f_2)$ <u>then</u> $d(u \cup i^*(v)) = d(u) \cup v$.

(4.4) <u>If</u> $u \in h_1^s(X,f_1)$ <u>and</u> $v \in h_2^{t-1}(A,f_2)$ <u>then</u> $d(i^*(u) \cup v) = (-1)^s u \cup d(v)$.

These are the analogue of Steenrod's axioms [4] for the pairing of two ordinary cohomology theories to a third.

5. <u>The basic construction</u>. Let $(E(\mathfrak{G}_n),B(\mathfrak{G}_n),r)$ denote the universal $\mathfrak{G}_n$-bundle constructed by Milnor [3], let $B(\mathfrak{G}) = \bigcup_n B(\mathfrak{G}_n)$ and pick a base-point $b_0 \in B(\mathfrak{G})$. Let $\mu : B(\mathfrak{G}) \times B(\mathfrak{G}) \longrightarrow B(\mathfrak{G})$ endow $B(\mathfrak{G})$ with the usual H-space structure, and map (b_0,b_0) to b_0. Identify ρ^* with $\rho^*(b_0) \subset \rho^*(B(\mathfrak{G}))$. For a point spectrum $F = \{F_k; \epsilon_k\}$, let $\bar{h}(\ ;F)$ be the associated cohomology theory on ρ^* and for a pairing $\eta : (F,G) \longrightarrow D$ of point spectra let

$$\eta_* : \bar{h}(\ ;F) \otimes \bar{h}(\ ;G) \longrightarrow \bar{h}(\ ;D)$$

be the induced pairing as described by Whitehead [5].

I. From F we construct a cohomology theory $H(\ ;F)$ on $\mathcal{P}^*(B(\mathfrak{G}))$ which is an extension of $\bar{h}(\ ;F)$.

II. From $\eta : (F,G) \longrightarrow D$ we construct a μ-pairing

$$\eta_{\#} : H(\ ;F) \otimes H(\ ;G) \longrightarrow H(\ ;D)$$

which is an extension of η_*.

The construction I proceeds as follows. Regard S^n as the one-point compactification of R^n with y_n the point at infinity. Let $\mathfrak{G}_n$ act on S^n as the extension of the usual action on R^n. Now let $\mathfrak{G}_n$ act on $F \wedge S^n$ by $T[x,y] = [x,Ty]$, $T \in \mathfrak{G}_n$, and let $\mathfrak{F}_k^n = (F_k^n, B(\mathfrak{G}_n), p_k^n, \Delta_k^n)$ where

$$F_k^n = E(\mathfrak{G}_n) \times (F_{k-n} \wedge S^n)/\mathfrak{G}_n ,$$

$p_k^n[e,x,y] = [e]$ and $\Delta_k^n[e] = [e, x_{k-n}, y_n]$, x_{k-n} the base-point of F_{k-n}.

Define $\epsilon_k^n : F_k^n \longrightarrow \Omega(F_{k+1}^n ; \Delta_{k+1}^n)$ by

$$\epsilon_k^n[e,x,y](t) = [e, \epsilon_{k-n}(x)(t), y], \qquad 0 \leq t \leq 1.$$

We now have a $B(\mathfrak{G}_n)$-spectrum $F^n = \{\mathfrak{F}_k^n ; \epsilon_k^n\}$ and hence a cohomology theory $h(\ ;F^n)$ on $\mathcal{P}^*(B(\mathfrak{G}_n))$. To define $H(\ ;F)$ on $\mathcal{P}^*(B(\mathfrak{G}))$ we take a direct limit of the $h(\ ;F^n)$.

6. Thom classes. Let $\alpha = (A,X,p)$ be an $(n-1)$-sphere bundle with structural group $\mathcal{G}_n$ and $\Sigma(\alpha) = (\Sigma(A),X,\Sigma(p))$ its fibrewise suspension. Here $\Sigma(A) = A \times I/(\sim)$ where $(a,t) \sim (a',t)$ if $p(y) = p(y')$ and $t = 0$ or 1. Define $\zeta_1 : X \longrightarrow \Sigma(A)$ by $\zeta_1(x) = [p^{-1}(x),1]$ and let $X^+ = \zeta_1(X)$.

Let F be a ring spectrum and $f : X \longrightarrow B(\mathcal{G})$. A Thom class for (α,f) relative to F is an element

$$u \in H^n(\Sigma(A),X^+,f\Sigma(p);F)$$

whose restriction to each fibre is the n-fold suspension of $1 \in H^0(pt.,F)$.

(6.1) THEOREM. *If f is a classifying map for α then (α,f) has a Thom class.*

Now, if (α,f) has a Thom class u relative to F and D is an F-module, define, for $g : X \longrightarrow B(\mathcal{G})$,

$$(6.2) \quad \Phi_u : H^k(X,g;D) \longrightarrow H^{n+k}(\Sigma(A),X^+,\mu(f \times g)\ \Sigma(p);D)$$

by $\Phi_u(v) = u \cup \Sigma(p)^*(v)$.

(6.3) THEOREM. (Thom isomorphism theorem) *Φ_u is an isomorphism.*

REFERENCES

1. A. Dold, Relations between ordinary and extraordinary cohomology theories, Colloquium on Algebraic Topology; Aarhus Universitet (1962), pp. 2-9.

2. E. A. M. MacAlpine, Duality in a twisted homology theory, Bull. Amer. Math. Soc., 72 (1966), pp. 1051-1054.

3. J. W. Milnor, Construction of universal bundles II, Ann. of Math., 63 (1956), pp. 430-436.

4. N. E. Steenrod, Cohomology invariants of mappings, Ann. of Math., 50 (1949), pp. 954-988.

5. G. W. Whitehead, Generalized homology theories, Trans. Amer. Math. Soc., 102 (1962), pp. 227-283.

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