Author: Prof Sergei Novikov, University of Maryland, MD

The **Novikov Conjecture** attempts to answer following question: Which expressions of the rational Pontryagin characteristic classes are homotopy invariant for the closed manifolds and how should we classify them?

Since the beginning of topology in the late 19th Century, topological invariants were defined using additional structures. Poincaré defined homology (Betti numbers) through combinatorics and the invention of simplicial complexes. Using smooth skew symmetric tensor fields (i.e. differential forms) Poincaré also defined cohomology (now, "de Rham cohomology"). Even the definition of the dimension of a manifold requires either a coordinate structure or a combinatorial form, after which it must be proved that this quantity is invariant under continuous homeomorphisms (i.e. is topologically invariant).

By definition, the characteristic classes of smooth manifolds are invariant under diffeomorphisms. They were discovered in the late 1930s by Stiefel, Whitney (classes w_k in the cohomology groups H^k of a manifold with modulo 2 coefficients), and by Pontryagin (classes p_k in the cohomology groups H^{4k} with integral coefficients). There are also Chern classes c_k for complex manifolds and bundles depending on complex structure.

The Pontryagin classes can be expressed through the Riemann curvature tensor of any Riemannian metric as differential forms whose integrals along the cycles yield a "rational" homological Pontryagin class: these integrals are integers which do not change under small variations of metric (i.e. are diffeomorphism invariant). They determine only the image of class p_k in the groups $H^{4k}(M, Q)$ with rational coefficients (i.e. "modulo torsion").

However, it turned out that the dimension of closed manifolds, homology groups (rings), Stiefel-Whitney classes and Pontryagin classes modulo 12 are each homotopy invariant. At the same time the very fact that the homotopy group of spheres $\pi_{n+3}(S^n)$ are finite for $n > 4$ (in fact they are equal to $Z/24Z)$) implies that rational Pontryagin classes are **not** homotopy invariant, even for the direct product of spheres.

Only one expression in the rational Pontryagin classes is homotopy invariant for simply connected manifolds of dimension $4k$. Consider the celebrated Hirzebruch polynomial $L_k(p_1, ..., p_k)$ (the L-genus). The integral along the whole $4k$-dimensional closed manifold

$$< L_k, [M^{4k}] >= \tau(M^{4k})$$

is equal to the "signature" τ of quadratic form naturally defined on the middle dimensional homology group by the intersection index or by the ring structure of cohomology–it is an obvious homotopy invariant. For example, $L_0 = 1, L_1 = (1/3)p_1$ and $L_2 = (1/45)(p_1^2 - 7p_2)$; let us define all of them. Take the formal "Hirzebruch series" $u/\tanh(u)$. Consider the product

$$\prod_{i=1}^{N} u_i/\tanh(tu_i) = \sum_{j=0}^{\infty} t^{2k} L_k + O(t^{2N})$$

where $N \to \infty$. By definition, the Pontryagin class p_k enters this expression as an elementary symmetric polynomial in the variables $u_1^2, ..., u_N^2$.

As clarified by the theory of formal groups in complex cobordism theory, the function $\tanh(u)$ appears here because its inverse function

$$g(v) = \tanh^{-1}(v) = 1/2[\log(1+v) - \log(1-v)]$$

is a "logarithm" of the formal group associated with signature, i.e. its derivative is a generating function for the signatures of complex projective spaces

$$dg(v)/dv = 1 + \sum_{i>0} \tau(CP^{2i})v^{2i} = \sum_{j\geq 0} v^{2j}$$

Such a formula is true for the Hirzebruch series of all "multiplicative characteristics" from the Pontryagin-Chern classes: other functions appear instead of tanh. There are several very important characteristics among them (Todd genus, $\widehat{A}$-genus, Elliptic genus, et cetera).

Such expressions are extremely important for topology and Riemannian geometry (cobordisms, homotopy groups of spheres, discovery of nontrivial differential structures on spheres and topological non-invariance of the torsion part of Pontryagin classes), algebraic geometry (Riemann-Roch theorems), and for partial differential equations (the index of elliptic operators) as it was revealed in the 1950s and early 1960s. These polynomials play a fundamental role in the proof of topological invariance of rational Pontryagin classes. Much later characteristic classes became important also for quantum field theory.

The topological invariance of rational Pontryagin classes was proved by Novikov in 1965/66. Let us formulate two corollaries of this result for the closed simply connected manifolds:

1. Homotopy equivalent manifolds can be non-homeomorphic ($n > 5$) (which disproves the Hurewicz conjecture going back to 1930s);

2. There exists only a finite number of differential structures on the manifold ($n > 4$).

The development of these ideas solved several fundamental problems of multidimensional topology. Unfortunately, some of these works remain unfinished.

The proof is based on an idea quite similar to that of the "Grothendieck etale topology" (all known proofs use it). The approach to the homeomorphism problems requires work with special toric nonsimply connected open and closed submanifolds and their coverings. In 1965 final proof was obtained applying the "surgery" technic developed by algebraic and differential topology in process of solving the classification problems. In 1990s new proof appeared (solving problem posed by Novikov in 1965). It is based on the nonstandard homological arguments avoiding any use of the surgery technic.

The first version of the Novikov conjecture appeared in 1965 as a byproduct of this work. It was proved step by step between 1965 and 1970 starting with the work of Novikov himself and finished by several famous topologists: It states that all integrals of the classes L_k along the cycles which are the intersections of cycles of codimension one are homotopy invariant.

The **Novikov Conjecture** as we know it today was finally formulated in 1970. Personal discussion of S.Novikov with A.Borel in Princeton (1967) was a very important step for the understanding what kind of form this conjecture might have: the homotopy types of symmetric manifolds with nonpositive curvature were added to the list of objects under consideration here.

The Novikov conjecture claims:

1.Consider every closed orientable manifold with fundamental group G and every cohomology class $y \in H^*(G, Q)$ defining the corresponding class in cohomology of every manifold with fundamental group G. Then the expressions

$$< y \bigcup L_k(p_1, ..., p_k), [M^n] >$$

integrated along the whole manifold are homotopy invariant. (They are called "the higher signatures").

2.There are no other homotopy invariant expressions.

Many mathematicians attempted to prove this conjecture, and a large branch of functional analysis ("The Fredholm representations") and of K-theory were developed for that. In many cases the Novikov conjecture is proved, such as for the discrete subgroups of Lie groups, hyperbolic groups in the sense of Gromov, as well as other classes of groups). However, the Novikov conjecture has not been proved for all groups. Such a proof has been a very active area of research during the past 40 years, since 1970.

A very interesting analog of this conjecture was found for the spin manifolds with positive scalar curvature, namely, the Gromov-Lawson-Rosenberg conjecture: the L-genus should be replaced by the A-genus, and similar expressions $< y \bigcup A(p_1, ..., p_k), [M] >$ play the role of obstructions to the existence of a metric with positive scalar curvature. However, such a family of obstructions may be incomplete in the nonsimply connected case.)

Recommended Reading

S.Ferry, A.Ranicki, J.Rosenberg: A History and the Survey of the Novikov Conjecture, London Math Society Notes, v 226,227, Proceedings of the Conference, Cambridge Univ Press, Cambridge, 1995

M.Kreck, W.Lueck: The Novikov Conjecture: Geometry and Algebra (Kindle Edition), Birkhauser, Basel, 2005. (A Review of this book can be found in the Bulletin of London Math Society 42 (2010), pp 181-190)

A.Bartels, W.Lueck, H.Reich. On the Farrell-Jones Conjecture and its applications. Journal of Topology, 1 (2008), pp 57-86