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Differential Topology

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Differential topology may be defined as the study of those properties of differentiable manifolds which are invariant under diffeomorphism (differentiable homeomorphism). Typical problems falling under this heading are the following:

(1) Given two differentiable manifolds, under what conditions are they diffeomorphic?

(2) Given a differentiable manifold, is it the boundary of some differentiable manifold-with-boundary?

(3) Given a differentiable manifold, is it parallelizable?

All of these problems concern more than the topology of the manifold, yet they do not belong to differential geometry, which usually assumes additional structure (e.g., a connection or a metric).

The most powerful tools in this subject have been derived from the methods of algebraic topology. In particular, the theory of characteristic classes is crucial, whereby one passes from the manifold M to its tangent bundle, and thence to a cohomology class in M which depends on this bundle.

These notes are intended as an introduction to the subject; we will go as far as possible without bringing in algebraic topology. Our two main goals are Whitney's theorem that a differentiable n-manifold can be imbedded as a closed subset of the euclidean space R^{2n+1} (see §1.32); and Thom's theorem that the non-orientable cobordism group ^{n} is isomorphic to a certain stable homotopy group (see §3.15).

Chapter I is mainly concerned with approximation theorems. First the basic definitions are given and the inverse function theorem is exploited. (§1.1 - 1.12). Next two local approximation theorems are proved, showing that a given map can be approximated by one of maximal rank. (§1.13 - 1.21). Finally locally finite coverings are used to derive the corresponding global theorems: namely Whitney's imbedding theorem and Thom's transversality lemma (§1.35).

Chapter II is an introduction to the theory of vector space bundles, with emphasis on the tangent bundle of a manifold. Chapter III makes use of the preceding material in order to study the cobordism groups ^{n}.

Chapter I Imbeddings and Immersions of Manifolds.

Notation. If x is in the euclidean space R^n, the coordinates of x are denoted by $(x^1,\ldots,x^n)$. Let $\|x\| = \max |x^i|$; let $C^n(r)$ denote the set of x such that $\|x\| < r$; and $C^n(x_o,r)$ the set of x such that $\|x-x_o\| < r$. The closure of a cube C is denoted by $\overline{C}$.

A real valued function $f(x^1,\ldots,x^n)$ is differentiable if the partials of f of all orders exist and are continuous (i.e., "differentiable" means C^∞). A map $f\colon U \longrightarrow R^p$ (where U is an open set, in R^n) is differentiable if each of the coordinate functions $f^1,\ldots,f^p$ is differentiable. Df denotes the Jacobian matrix of f; one verifies that $D(gf) = Dg \cdot Df$. The notation $\partial(f^1,\ldots,f^p)/\partial(x^1,\ldots,x^n)$ is also used. If $n = p$, $|Df|$ denotes the determinant.

1.1. Definition. An n-manifold M^n is a Hausdorff space with a countable basis which is locally homeomorphic to R^n.

A differentiable structure $\mathcal{D}$ on a manifold M^n is a collection of real-valued functions, each defined on an open subset of M, such that:

1) For every point p of M there is a neighborhood U of p and a homeomorphism h of U onto an open subset of R^n such that a function f, defined on the open subset W of U, is in $\mathcal{D}$ if and only if fh^{-1} is differentiable.

2) If U_i are open sets contained in the domains of f and $U = \cup U_i$, then $f|U \in \mathcal{D}$ if and only if $f|U_i$ is in $\mathcal{D}$, for each i.

A *differentiable manifold* M^n is a manifold provided with a differentiable structure $\mathcal{D}$; the elements of $\mathcal{D}$ are called the *differentiable functions* on M. Any open set U and homeomorphism h which satisfy the requirements of 1) above are called a *coordinate system* on M. *Notation*. A coordinate system is sometimes denoted by the coordinate functions: $h(p) = (u^1(p),\ldots,u^n(p))$.

1.2. *Alternate definition*. Let a collection (U_i,h_i) be given, where h_i is a homeomorphism of the open subset U_i of M^n onto an open subset of R^n, such that

a) the U_i cover M

b) $h_j\, h_i^{-1}$ is a differentiable map on $h_i\ (U_i \cap U_j)$, for all i,j.

Define a *coordinate system* as an open set U and homeomorphism h of U onto an open subset of R^n such that $h_i h^{-1}$ and $h\, h_i^{-1}$ are differentiable on $h(U \cap U_i)$ and $h_i(U \cap U_i)$ respectively , for each i. Define a *differentiable structure* on M as the collection of all such coordinate systems. A function f, defined on the open set V, is *differentiable* if $f\, h^{-1}$ is differentiable on $h(U \cap V)$, for all coordinate systems (U, h).

One shows readily that these two definitions are entirely equivalent.

1.3. Definition. Let M_1, M_2 be differentiable manifolds. If U is an open subset of M_1, $f: U \longrightarrow M_2$ is differentiable if for every differentiable function g on M_2, gf is differentiable on M_1.

If $A \subset M_1$, a function $f: A \longrightarrow M_2$ is differentiable if it can be extended to a differentiable function defined on a neighborhood U of A.

$f: M_1 \longrightarrow M_2$ is a diffeomorphism if f and f^{-1} are defined and differentiable.

(A coordinate system (U,h) on M^n is then an open set U in M and a diffeomorphism h of U onto an open set in R^n.)

If $A \subset M$, we have just defined the notion of differentiable function for subsets of A. Suppose that A is locally diffeomorphic to R^k: this collection is easily shown to be a differentiable structure on A. In this case, A is said to be a differentiable submanifold of M.

The following lemma is familiar from elementary calculus.

1.4. Lemma. Let $f: C^n(r) \longrightarrow R^n$ satisfy the condition

$\left|\dfrac{\partial f^i}{\partial x^j}\right| \leq b$, for all i,j. Then $\|f(x)-f(y)\| \leq bn\ \|x-y\|$, for all $x,y \in \bar{C}^n$.

1.5. Theorem (inverse function theorem). Let U be an open subset of R^n, let $f\colon U \longrightarrow R^n$ be differentiable, and let Df be non-singular at x_o. Then f is a diffeomorphism of some neighborhood of x_o onto some neighborhood of $f(x_o)$.

Proof: We may assume $x_o = f(x_o) = 0$, and that $Df(x_o)$ is the identity matrix.

Let $g(x) = f(x) - x$, so that $Dg(0)$ is the zero matrix. Choose $r > 0$ so that $x \in U$ and $Df(x)$ is non-singular and $|\partial g^i/\partial x_j| \leq 1/2n$, for all x with $\|x\| < r$.
Assertion. If $y \in C(r/2)$, there is exactly one $x \in C(r)$ such that $f(x) = y$:

(*) By the previous lemma, $\|g(x)-g(x_o)\| \leq 1/2\|x-x_o\|$ on $C(r)$. Let us define $x_o = 0$, $x_1 = y$, $x_{n+1} = y - g(x_n)$. This is defined, since $x_n - x_{n-1} = g(x_{n-2}) - g(x_{n-1})$, so that $\|x_n-x_{n-1}\| \leq 1/2\ \|x_{n-2}-x_{n-1}\|$; and thus $\|x_n\| \leq 2\|y\|$ for each n. Hence the sequence x_n converges to a point x with $\|x\| \leq 2\|y\|$, so that $x \in C(r)$. Then $x = y - g(x)$, so that $f(x) = y$. This proves the existence of x. To show uniqueness, note that if $f(x) = f(x_1) = y$, then $g(x_1) - g(x) = x - x_1$, contradicting (*).

Hence $f^{-1}: C(r/2) \longrightarrow C(r)$ exists. Note that $\|f(x)-f(x_1)\| \geq \|x-x_1\| - \|g(x)-g(x_1)\| \geq 1/2\ \|x-x_1\|$, so that $\|y-y_1\| \geq 1/2\ \|f^{-1}(y)-f^{-1}(y_1)\|$. Hence f^{-1} is continuous; the image of $C(r/2)$ under f^{-1} is open because it equals $C(r) \cap f^{-1}(C(r/2))$, the intersection of two open sets. *

To show that f^{-1} is differentiable, note that $f(x) = f(x_1) + Df(x_1)\cdot(x-x_1) + h(x,x_1)$, where $(x-x_1)$ is written as a column matrix and the dot stands for matrix multiplication. Here $h(x,x_1)/\|x-x_1\| \longrightarrow 0$ as $x \longrightarrow x_1$. Let A be the inverse matrix of $Df(x_1)$. Then

$$A\cdot(f(x)-f(x_1)) = (x-x_1) + A\cdot h(x,x_1), \quad \text{or}$$

$$A\cdot(y-y_1) + A\cdot h_1(y,y_1) = f^{-1}(y) - f^{-1}(y_1),$$

where $h_1(y,y_1) = -h(f^{-1}(y), f^{-1}(y_1))$. Now

$$\frac{h_1(y,y_1)}{\|y-y_1\|} = -\frac{h(x,x_1)}{\|x-x_1\|}\ \frac{\|x-x_1\|}{\|y-y_1\|}\ .$$

Since $\|x-x_1\|/\|y-y_1\| \leq 2$, $h_1(y,y_1)/\|y-y_1\| \longrightarrow 0$ as $y \longrightarrow y_1$. Hence $D(f^{-1}) = A = (D(f))^{-1}$.

This means that $D(f^{-1})$ is obtained as the composition of the following maps:

$$C(r/2) \xrightarrow[f^{-1}]{} C(r) \xrightarrow[Df]{} GL(n) \xrightarrow[\text{matrix inversion}]{} GL(n);$$

where $GL(n)$ denotes the set of non-singular $n \times n$ matrices, considered as a subspace of n^2-dimensional euclidean space.

Since f^{-1} is continuous and Df and matrix inversion are C^∞, $D(f^{-1})$ is continuous, i.e., f^{-1} is C^1. In general, if f^{-1} is C^k, then by this argument $D(f^{-1})$ is also, i.e., f^{-1} is of class C^{k+1}. This completes the proof.

1.6. Lemma. Let U be an open subset of R^n, let $f\colon U \longrightarrow R^p$ $(n \leq p)$, $f(0) = 0$, and let $Df(0)$ have rank n. Then there exists a diffeomorphism g of one neighborhood of the origin in R^p onto another so that $g(0) = 0$ and $gf(x^1,\ldots,x^n) = (x^1,\ldots,x^n,0,\ldots,0)$, in some neighborhood of the origin.

Proof: Since $\partial(f^1,\ldots,f^p)/\partial(x^1,\ldots,x^n)$ has rank n, we may assume that $\partial(f^1,\ldots,f^n)/\partial(x^1,\ldots,x^n)$ is the submatrix which is non-singular. Define $F\colon U \times R^{p-n} \longrightarrow R^p$ by the equation

$$F(x^1,\ldots,x^p) = f(x^1,\ldots,x^n) + (0,\ldots,0,x^{n+1},\ldots,x^p).$$

F is an extension of f, since $F(x^1,\ldots,x^n,0,\ldots,0) = f(x^1,\ldots,x^n)$.

DF is non-singular at the origin, since its determinant everywhere equals $|\partial(f^1,\ldots,f^n)/\partial(x^1,\ldots,x^n)|$. Hence F has a local inverse g, so that g maps one neighborhood of the origin in R^p onto another, and

$$gF(x^1,\ldots,x^p) = (x^1,\ldots,x^p).$$

Hence

$$gf(x^1,\ldots,x^n) = (x^1,\ldots,x^n,0,\ldots,0).$$

1.7. Corollary. Let A^k be a differentiable submanifold of M^n. Given $x \in A$, there is a coordinate system (U,h) on M about x, such that $h(U \cap A) = h(U) \cap R^k$ (where R^k is considered as the subspace $R^k \times 0$ of $R^k \times R^{n-k} = R^n$).

Proof: Let (U_1,h_1) be a coordinate system on M about x; by hypothesis, there is a differentiable map f of a neighborhood V of x in M into R^k such that $f|V \cap A = f_1$ is a diffeomorphism whose range is an open set W in R^k. We may assume $U_1 = V$, and $h_1(x) = f(x) = 0$.

Now $f\, h_1^{-1}\, h_1\, f_1^{-1}$ is the identity on W, so that its Jacobian, which equals $D(fh_1^{-1})\cdot D(h_1\, f_1^{-1})$ is non-singular. Hence $D(h_1\, f_1^{-1})$ has rank k, so that by the previous lemma, there is a diffeomorphism g of some neighborhood $V_1 \subset h_1(U_1)$ of 0 onto another such that $g(0) = 0$ and $g\, h_1\, f_1^{-1}\, (x^1,\ldots,x^k) = (x^1,\ldots,x^k,0,\ldots,0)$.

Then $U = h_1^{-1}(V_1)$ and $h = g\, h_1$ will satisfy the requirements of the lemma.

1.8. Lemma. Let U be an open subset of R^n, let $f\colon U \longrightarrow R^p$, $f(0) = 0$, $(n \geq p)$, and let $Df(0)$ have rank p. Then there is a diffeomorphism h of some neighborhood of the origin in R^n onto another such that $h(0) = 0$ and $fh(x^1,\ldots,x^n) = (x^1,\ldots,x^p)$.

Proof: We may assume $\frac{\partial(f^1,\dots,f^p)}{\partial(x^1,\dots,x^p)}$ is non-singular at 0, since $Df(0)$ has rank p. Define $F\colon U \longrightarrow R^n$ by the equation $F(x^1,\dots,x^n) = (f^1(x),\dots,f^p(x), x^{p+1},\dots,x^n)$. Then $DF(0)$ is non-singular; let h be the local inverse of F. Let g project R^n onto the subspace R^p; $f = gF$. Then $fh(x^1,\dots,x^n) = gF\,h(x^1,\dots,x^n) = g(x^1,\dots,x^n) = (x^1,\dots,x^p)$.

1.9. Exercise. Let U be an open subset of R^n, $f\colon U \longrightarrow R^p$, $f(0) = 0$; and let $Df(x)$ have rank k for all x in U. Then there are local diffeomorphisms h and g of R^n and R^p respectively such that

$$g\,f\,h\,(x^1,\dots,x^n) = (x^1,\dots,x^k,0,\dots,0).$$

1.10. Definition. If $f\colon M_1 \longrightarrow M_2$, the rank of f at x is the rank of $D(h_2 f\, h_1^{-1})$ at $h_1(x)$, where (U_1, h_1) and (U_2, h_2) are coordinate systems about x and $f(x)$, respectively. The differentiable map $f\colon M^n \longrightarrow M^p$ is an immersion if rank $f = n$ everywhere $(n \leq p)$. It is an imbedding if it is also a homeomorphism into.

If $f\colon M^n \longrightarrow M^p$, then $y \in M^p$ is a regular value of f if rank $f = p$ on the entire set $f^{-1}(y)$. Otherwise, y is a critical value. (If $y \notin f(M^n)$, y is thus a regular value of f.)

1.11. Exercise. If A is a differentiable submanifold of M, the inclusion $A \longrightarrow M$ is an imbedding and conversely if $f: M_1 \longrightarrow M$ is an imbedding then $f(M_1)$ is a differentiable submanifold.

1.12. Exercise. If y is a regular value of $f: M^n \longrightarrow M^p$, then $f^{-1}(y)$ is a differentiable submanifold of M^n of dimension $n - p$ (or is empty).

1.13. Definition. A subset A of R^n has measure zero if it may be covered by a countable collection of cubes $C(x,r)$ having arbitrarily small total volume. In such a case, $R^n - A$ is everywhere dense (i.e., it intersects every open set).

1.14. Lemma. Let U be an open subset of R^n; let $f: U \longrightarrow R^n$ be differentiable. If $A \subset U$ has measure 0, so does $f(A)$.

Proof: Let C be any cube with $\overline{C} \subset U$. Let b denote the maximum of $|\partial f^i/\partial x^j|$ on $\overline{C}$ for all i,j. By 1.4, $\|f(x)-f(y)\| \leq b\,n\,\|x-y\|$ for $x,y \in \overline{C}$.

Now $A \cap C$ has measure zero; let us cover $A \cap C$ by cubes $C(x_i,r_i)$ contained in C, such that $\sum_{i=1}^{\infty} r_i{}^n < \varepsilon$. Then $f(C(x_i,r_i)) \subset C(f(x_i), bn\,r_i)$, so that $f(A \cap C)$ is covered by cubes of total volume $b^n n^n \Sigma\, r_i{}^n < b^n n^n \varepsilon$. Hence $f(A \cap C)$ has measure zero.

Since A can be covered by countably many such cubes C, $f(A)$ has measure zero.

1.15. Corollary. If $f: U \longrightarrow R^p$ is differentiable, where U is an open subset of R^n and $n < p$, then $f(U)$ has measure 0.

Proof: Project $U \times R^{p-n}$ onto U and apply f. Since $U \times 0$ has measure 0 in R^p, so does $f(U)$.

1.16. Definition. If $A \subset M$, A has measure 0 if $h(A \cap U)$ has measure 0 for every coordinate system (U, h).

1.17. Corollary. If $f: M^n \longrightarrow M^p$ is differentiable and $n < p$, then $f(M^n)$ has measure zero.

1.18. Definition. Let $M(p, n)$ denote the space of $p \times n$ matrices, with the differentiable structure of the euclidean space R^{pn}. Let $M(p,n; k)$ denote the subspace consisting of matrices of rank k. Thus $M(p,n;n)$ is an open subset of $M(p,n)$ if $p \geq n$; the determinental criterion for rank proves this. More generally, we have:

1.19. Lemma. $M(p,n;k)$ is a differentiable submanifold of $M(p,n)$ of dimension $k(p+n-k)$, where $k \leq \min(p,n)$.

<u>Proof</u>: Let $E_o \in M(p,n;k)$; we may assume that E_o is of the form $\begin{pmatrix} A_o & B_o \\ C_o & D_o \end{pmatrix}$, where A_o is a non-singular $k \times k$ matrix. There is an $\varepsilon > 0$ such that if all the entries of $A - A_o$ are less than ε, A must also be non-singular. Let U consist of all matrices in $M(p,n)$ of the form $E = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, with all the entries of $A - A_o$ less than ε.

Then E is in $M(p,n;k)$ if and only if $D = CA^{-1}B$: For the matrix

$$\begin{pmatrix} I_k & 0 \\ X & I_{p-k} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ XA + C & XB + D \end{pmatrix}$$

has the same rank as E. If $X = -CA^{-1}$, this matrix is $\begin{pmatrix} A & B \\ 0 & -CA^{-1}B+D \end{pmatrix}$. If $D = CA^{-1}B$, this matrix has rank k. The converse also holds, for if any element of $-CA^{-1}B+D$ is different from zero, this matrix has rank $> k$.

Let W be the open set in euclidean space of dimension $(pn-(p-k)(n-k)) = k(p+n-k)$ consisting of matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, with all the entries of $A - A_o$ less than ε. The map $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \to \begin{pmatrix} A & B \\ C & CA^{-1}B \end{pmatrix}$ is then a diffeomorphism of W onto the neighborhood $U \cap M(p,n;k)$ of E_o.

1.20. Theorem. Let U be an open set in R^n, and let $f: U \longrightarrow R^p$ be differentiable, where $p \geq 2n$. Given $\varepsilon > 0$, there is a $p \times n$ matrix $A = (a^i_j)$ with each $|a^i_j| < \varepsilon$, such that $g(x) = f(x) + A \cdot x$ is an immersion. (x written as a column matrix.)

Proof: $Dg(x) = Df(x) + A$; we would like to choose A in such a way that $Dg(x)$ has rank n for all x. I.e., A should be of the form $Q - Df$, where Q has rank n.

We define $F_k: M(p,n;k) \times U \longrightarrow M(p,n)$ by the equation

$$F_k(Q,x) = Q - Df(x).$$

Now F_k is a differentiable map, and the domain of F_k has dimension $k(p+n-k) + n$. As long as $k < n$, this expression is monotonic in k (its partial with respect to k is $p + n-2k$). Hence the domain of F_k has dimension not greater than $(n-1)(p+n-(n-1)) + n = (2n-p) + pn - 1$ for $k < n$. Since $p \geq 2n$, this dimension is strictly less than $pn = \dim M(p,n)$.

Hence the image of F_k has measure zero in $M(p,n)$, so that there is an element A of $M(p,n)$, arbitrarily close to the zero matrix, which is not in the image of F_k for $k = 0,\ldots,n-1$. Then $A + Df(x) = Dg(x)$ has rank n, for each x.

1.21. Theorem. Let U be an open subset of R^n; and let $f\colon U \longrightarrow R^p$ be differentiable. Given $\varepsilon > 0$, there are matrices A $(p \times n)$ and B $(p \times 1)$ with entries less than ε in absolute value, such that

$$g(x) = f(x) + A\cdot x + B$$

has the origin as a regular value.

Remark. The following much more delicate result has been proved by A. Sard: The set of critical values of any differentiable map has measure zero.

Proof of 1.21. Note that the theorem is trivial if $p > n$, since then $f(U)$ has measure zero, and we may choose $A = 0$ and B small in such a way that 0 is not in the image of g.

Assume $p \leqq n$. We wish $Dg(x_o) = Df(x_o) + A$ to have rank p, where x_o ranges over all points such that

$$g(x_o) = 0 = f(x_o) + A\cdot x_o + B.$$

Hence A is of the form $Q - Df(x)$, and B is of the form $-f(x) - A\cdot x$, where Q is to have rank p.

We define $F_k\colon M(p,n;k) \times U \longrightarrow M(p,n) \times R^p$ by the equation

$$F_k(Q,x) = (Q\text{-}Df(x), -f(x) - (Q\text{-}Df(x))\cdot x).$$

Then F_k is differentiable. If $k < p$, the dimension of its domain is not greater than $(p-1)(p+n-(p-1)) + n = p + pn - 1$. Hence the image of F_k, $k = 0,\ldots,p-1$ has measure zero; so that there is a point (A,B) arbitrarily close to the origin which is not in any such image set. This completes the proof.

1.22. Definition. A covering of X is locally-finite if every point has a neighborhood which intersects only finitely many elements of the covering. A refinement of a covering of X is a second covering each element of which is contained in an element of the first covering. A Hausdorff space is paracompact if every open covering has a locally-finite open refinement.

If X is paracompact, and U_α is an open covering, there is a locally-finite open covering V_α with $V_\alpha \subset U_\alpha$ for each α. For let W_β be a locally-finite refinement of U_α; choose $\alpha(\beta)$ so that $W_\beta \subset U_{\alpha(\beta)}$ for each β. Set $V_{\alpha_0} = \bigcup_{\alpha(\beta)=\alpha_0} W_\beta$. Given a neighborhood intersecting only finitely many W_β, it intersects only finitely many V_α as well.

1.23. Theorem. If X is locally compact and Hausdorff, having a countable basis, X is paracompact.

Proof: Let $U_1, U_2, \ldots$ be a basis for X with $\bar{U}_i$ compact for each i. There exists a sequence $A_1, A_2, \ldots$ of compact sets whose union is X, such that $A_i \subset \operatorname{Int} A_{i+1}$:

Set $A_1 = \bar{U}_1$. Given A_i compact, let k be the smallest integer such that A_i is contained in $U_1 \cup \ldots \cup U_k$; let A_{i+1} equal the closure of this set union $\bar{U}_{i+1}$.

Let O be an open covering of X. Cover the compact set $A_{i+1} - \text{Int } A_i$ by a finite number of open sets $V_1, \ldots, V_n$ where each V_i is contained in some element of O, and in the open set $\text{Int } A_{i+2} - A_{i-1}$. Let P_i denote the collection $\{V_1, \ldots, V_n\}$, and let $P = P_0 \cup P_1 \cup \ldots$. P refines O, and since any compact closed neighborhood C is contained in some A_i, C can intersect only finitely many elements of P.

1.24. Exercise. Prove: A paracompact space is normal. (First prove that it is regular.)

1.25. Theorem. Let M^n be a differentiable manifold, $\{U_\alpha\}$ an open covering of M. There is a collection (V_j, h_j) of coordinate systems on M such that

1) $\{V_j\}$ is a locally-finite refinement of $\{U_\alpha\}$.
2) $h_j(V_j) = C^n(3)$
3) If $W_j = h_j^{-1}(C(1))$, then $\{W_j\}$ covers M.

$C^n(1)$ *

Proof: The proof proceeds along lines similar to the previous one. The only difference is that one chooses the V_j to satisfy 2), and makes sure that the sets $h_j^{-1}(C(1))$ also cover $A_{i+1} - \text{Int } A_i$.

1.26. We wish to construct a C^∞ function $\varphi(x_1,\dots,x_n)$ such that $\varphi = 1$ on $\overline{C}(1)$, $0 < \varphi < 1$ on $C(2) - \overline{C}(1)$, $\varphi = 0$ on $R^n - C(2)$.

This function may be defined by the equation $\varphi(x_1,\dots,x_n) = \Pi_1^n \psi(x_i)$, where

$$\psi(x) = \frac{\lambda(2+x)\cdot\lambda(2-x)}{\lambda(2+x)\cdot\lambda(2-x) + \lambda(x-1) + \lambda(-x-1)}$$

and

$$\begin{aligned} \lambda(x) &= e^{-1/x} && \text{if} \quad x > 0 \\ &= 0 && \text{if} \quad x \leq 0. \end{aligned}$$

Note that the denominator in the expression for ψ is always positive, and that $\psi(x) = 1$ for $|x| \leq 1$

$$\begin{aligned} 0 < \psi(x) &< 1 \quad \text{if} \quad 1 < |x| < 2 \\ \psi(x) &= 0 \quad \text{if} \quad |x| \geq 2. \end{aligned}$$

1.27. Definition. Let $f,g\colon X \to Y$, where Y is metrizable, and let $\delta(x)$ be a positive continuous function defined on X. Then g is a δ-<u>approximation to</u> f if $d(f(x), g(x)) < \delta(x)$ for all x. [If one takes the δ-approximations to f to be a neighborhood of f in the function space $F(X,Y)$, this imposes a topology on the function space, independent of the metric on Y (X,Y paracompact).]

1.28. <u>Theorem.</u> Given a differentiable map $f\colon M^n \to R^p$ where $p \geq 2n$, and a continuous positive function δ on M^n, there exists an immersion $g\colon M^n \to R^p$

which is a δ-approximation to f. If rank $f = n$ on the closed set N, we may choose $g|N=f|N$.

Proof: Rank $f = n$ on a neighborhood U of N. Cover M^n by U and $M^n - N$. Let (V_i,h_i) be a refinement of this covering, constructed as in 1.25. As before, $h_i(W_i) = C(1)$ and $h_i(V_i) = C(3)$. Let $h_i(U_i) = C(2)$. Let the V_i be so indexed with positive and negative integers that those V_i with non-positive indices are the ones contained in U. Let $\varepsilon_i = \min$ of $\delta(x)$ on the compact set $\overline{U}_i$.

Set $f_o = f$. Given $f_{k-1}: M^n \longrightarrow R^p$, having rank n on $N_{k-1} = \cup_{j<k} \overline{W}_j$, consider $f_{k-1} h_k^{-1}: C(3) \longrightarrow R^p$. Let A be a $p \times n$ matrix; let $F_A: C(3) \longrightarrow R^p$ be defined by the equation

$$F_A(x) = f_{k-1} h_k^{-1}(x) + \varphi(x) A\cdot(x),$$

where (x) is written (as usual) as a column matrix $(n \times 1)$; A is yet to be chosen; and $\varphi(x)$ is the function defined in 1.26.

First, we want $F_A(x)$ to have rank n on the set $K = h_k(N_{k-1} \cap \overline{U}_k)$; we are given that $f_{k-1} h_k^{-1}$ has rank n on K. Now

$$D(F_A(x)) = D(f_{k-1} h_k^{-1}(x)) + A\cdot(x)\cdot D\varphi(x) + \varphi(x)A.$$

($D\varphi$ is a $1 \times n$ matrix.) The map of $K \times M(p,n)$ into

$M(p,n)$ which carries (x,A) into $D(F_A(x))$ is continuous. It carries $K \times (0)$ into the open subset $M(p,n;n)$ of $M(p,n)$. Hence if A is sufficiently small, this map will carry $K \times A$ into $M(p,n;n)$; our first requirement is that A be this small.

Secondly, we require A to be small enough that $\|A\cdot(x)\| < \varepsilon_k/2^k$ for all $x \in C(3)$.

Finally, by 1.20, A may be chosen arbitrarily small so that $f_{k-1}\, h_k^{-1}(x) + A\cdot(x)$ has rank n on $C(2)$. Let A be chosen to satisfy this requirement.

We then define $f_k\colon M^n \longrightarrow R^p$ by the equation:

$$\begin{aligned} f_k(y) &= f_{k-1}(y) + \varphi(h_k(y))\, A\cdot(h_k(y)) && \text{for } y \in V_k \\ &= f_{k-1}(y) && \text{for } y \in M-\overline{U}_k . \end{aligned}$$

These definitions agree on the overlapping domains, so that f_k is differentiable. By the first condition on A, it has rank n on N_{k-1}; by the third condition it has rank n on $\overline{W}_k$. By the second condition, f_k is a $\delta/2^k$ approximation to f_{k-1}.

We define $g(x) = \lim\limits_{k\to\infty} f_k(x)$. Since the covering V_i is locally finite, all the f_k agree on a given compact set for k sufficiently large; it follows that g is differentiable and has rank n everywhere. It is also a δ-approximation to f.

1.29. Lemma. If $p > 2n$, any immersion $f: M^n \longrightarrow R^p$ can be δ-approximated by a 1-1 immersion g. If f is 1-1 in a neighborhood U of the closed set N, we may choose $g|N=f|N$.

Proof: Choose a covering $\{U_\alpha\}$ of M such that $f|U_\alpha$ is an imbedding (possible by 1.6). Let (V_i, h_i) be the locally finite refinement constructed in 1.25; let $\varphi(x)$ be the function constructed in 1.26. Let $\varphi_i(y) = \varphi(h_i(y))$ for $y \in V_i$; $= 0$ for other y. Then φ_i is differentiable. As before, we assume (V_i, h_i) refines the covering $(U, M\text{-}N)$ and that those V_i with non-positive indices are the ones contained in U. Let $f_o = f$. Given the immersion $f_{k-1}: M^n \longrightarrow R^p$, we define f_k by the equation

$$f_k(y) = f_{k-1}(y) + \varphi_k(y)b_k, \text{ where } b_k \text{ is}$$

a point of R^p yet to be chosen. By the argument of the previous theorem, if b_k is chosen sufficiently small, f_k will have rank n everywhere. The first requirement is that b_k be this small; the second requirement is that b_k be small enough that f_k be a $\delta/2^k$ approximation to f_{k-1}.

Finally, let N^{2n} be the open subset of $M^n \times M^n$ consisting of pairs (y, y_o), with $\varphi_k(y) \neq \varphi_k(y_o)$. Consider the differentiable map of N^{2n} into R^p which carries (y, y_o) into $-[f_{k-1}(y) - f_{k-1}(y_o)]/[\varphi_k(y)-\varphi_k(y_o)]$. Since $2n < p$, the image of N^{2n} has measure 0, so that b_k may be chosen arbitrarily small and not in this image.

It follows that $f_k(y) - f_k(y_o) = 0$ if and only if $\varphi_k(y) - \varphi_k(y_o) = 0$ and $f_{k-1}(y) - f_{k-1}(y_o) = 0$ $(k > 0)$.

Define $g(y) = \lim_{k \to \infty} f_k(y)$. If $g(y) = g(y_o)$ and $y \neq y_o$, it would follow that $f_{k-1}(y) = f_{k-1}(y_o)$ and $\varphi_k(y) = \varphi_k(y_o)$ for all $k > 0$. The former condition implies that $f(y) = f(y_o)$, so that y and y_o cannot belong to any one set U_i. Because of the latter condition, this means that neither is in any set U_i for $i > 0$. Hence, they lie in U, contradicting the fact that f is 1-1 on U.

1.30. Definition. Let $f\colon M^n \to R^p$. The limit set $L(f)$ is the set of $y \in R^p$ such that $y = \lim f(x_n)$ for some sequence $\{x_1, x_2, \dots\}$ which has no limit point on M^n.

Exercise. Show the following:

1) $f(M)$ is a closed subset of R^p if and only if $L(f) \subset f(M)$

2) f is a topological imbedding if and only if f is 1-1 and $L(f) \cap f(M)$ is vacuous.

1.31. Lemma. There exists a differentiable map $f\colon M^n \to R$ with $L(f)$ empty.

Proof: Let (V_i, h_i) and φ be chosen as in 1.25 and 1.26 with i ranging over positive integers; let $\varphi_i(y) = \varphi h_i(y)$ if $y \in V_i$; $= 0$ otherwise.

Define $f(y) = \Sigma_j(j\,\varphi_j(y))$. This sum is finite, since V_i is a locally finite covering. If $\{x_i\}$ is a set of points of M having no limit point, only finitely many lie in any compact subset of M. Given m, there is an integer i such that x_i is not in $\overline{W}_1 \cup \ldots \cup \overline{W}_m$. Hence $x_i \in \overline{W}_j$ for some $j > m$, whence $f(x_i) > m$. Thus the sequence $f(x_m)$ cannot converge.

1.32. Corollary. Every M^n can be differentiably imbedded in R^{2n+1} as a closed subset.

Proof: Let $f: M^n \longrightarrow R \subset R^{2n+1}$ differentiably, with $L(f) = 0$. Set $\delta(x) \equiv 1$, and let g be a 1-1 immersion which is a δ-approximation to f. Then $L(g)$ is empty, so that g is a homeomorphism.

1.33. Definition. Let $f: M^n \longrightarrow N^p$ differentiably. Let N_1^{p-q} be a differentiable submanifold of N. Let $f(x) \in N_1$. Let $(u^1,\ldots,u^n)$ be a coordinate system about x; and let $(v^1,\ldots,v^p)$ be a coordinate system about $f(x)$ such that on N_1, $v^1 = \ldots = v^q = 0$ (see 1.6). Consider the condition that

$$\frac{\partial v^i}{\partial u^j} \qquad \begin{array}{l} i = 1,\ldots,q \\ j = 1,\ldots,n \end{array}$$

have rank q at x. This is the transverse regularity condition for f and N_1 at x. [Exercise: Show that this condition is independent of coordinate system.] Note that the set of points on which the transverse regularity condition is satisfied is an open subset of $f^{-1}(N_1)$. f is said to be transverse regular on N_1 if the condition is satisfied for each x in $f^{-1}(N_1)$.

1.34. Lemma. If $f: M^n \longrightarrow N^p$ is transverse regular on N_1^{p-q} then $f^{-1}(N_1)$ is a differentiable submanifold of dimension $n - q$ (or is empty).

Proof: Let π project R^p onto its first q components; $\pi: R^p \longrightarrow R^q$. If $(V,h) = (v^1,\ldots,v^p)$ is the coordinate system hypothetesized in 1.33, then $N_1 \cap V = h^{-1}\pi^{-1}(0)$ (here 0 denotes the origin in R^q); and $f^{-1}(N_1 \cap V) = (\pi hf)^{-1}(0)$. Since πhf has rank q at $x \in f^{-1}(N_1 \cap V)$, the origin is a regular value of πhf. Hence $(\pi hf)^{-1}(0)$ is a differentiable submanifold of M of dim $n-q$ (see 1.12).

1.35. Theorem. Let $f: M^n \longrightarrow N^p$ be differentiable; let N_1^{p-q} be a closed differentiable submanifold of N. Let A be a closed subset of M such that the transverse regularity condition for f and N_1 holds at each x in $A \cap f^{-1}(N_1)$. Let δ be a positive continuous function on M. There exists a differentiable map $g: M^n \longrightarrow N^p$ such that

(1) g is a δ-approximation to f,
(2) g is transverse regular on N_1, and
(3) $g|A = f|A$.

Proof: There is a neighborhood U of A in M such that f satisfies the transverse regularity condition on $U \cap f^{-1}(N_1)$. Cover N by $N - N_1 = Y_o$ and coordinate system (Y_i, k_i) for $i > 0$; with coordinate functions $(v^1, \ldots, v^q)$ such that $v^1 = \ldots = v^q = 0$ on N_1. Now the open sets $f^{-1}(Y_i)$ cover M, as do the open sets U, $M - A$. Let (V_j, h_j) be a refinement of both coverings, constructed as in 1.25. Recall that $h_j(V_j) = C(3)$, $h_j(U_j) = C(2)$, $h_j(W_j) = C(1)$, and the W_j cover M. The V_j are to be indexed with positive and negative integers so that those V_j which are contained in U are the ones with non-positive indices.

Let φ be as in 1.26, and define $\varphi_i(x) = \varphi(h_i(x))$ for $x \in V_i$ and $\varphi_i(x) = 0$ elsewhere. For each j choose $i(j) \geq 0$ so that $f(V_j)$ is contained in $Y_{i(j)}$.

Set $f_o = f$.

Suppose f_{k-1} is defined and satisfies the transverse regularity condition for N_1 at each point of the intersection of $f_{k-1}^{-1}(N_1)$ with $\bigcup_{j<k} \overline{W}_j$. Furthermore suppose that $f_{k-1}(\overline{U}_j) \subset Y_{i(j)}$ for each j. Setting $i = i(k)$, it follows in particular that $f_{k-1}(\overline{U}_k) \subset Y_i$.

Consider $\pi\, k_i\, f_{k-1}\, h_k^{-1} : C(2) \longrightarrow R^q$; by 1.21. Sards there is an arbitrarily small affine function $L(x) = A \cdot (x) + B$ such that when added to the previous function, the resulting map has the origin as a regular value. Consider R^q as the first q coordinates in R^p, and define

$f_k(x) = k_1^{-1}(k_1 f_{k-1}(x) + L(h_k(x))\ \varphi_k(x))$ for x in a neighborhood of $\overline{U}_k$

$k_1^{-1}(k_1 f_{k-1}(x) - \varphi_k(x)\ b_k)$

$= f_{k-1}(x)$ for x in $M - U_k$.

Here L is yet to be chosen. Of course, we must choose L small enough that $k_1 f_{k-1}(x) + \overset{(x)}{L}\varphi_k(x)$ lies in $C(1)$ for $x \in \overline{U}_k$, in order that k_1^{-1} may be applied to it. This is the first requirement on L. Secondly, we choose L small enough that f_k is a $\delta/2^k$ approximation to f_{k-1}. Thirdly choose L small enough so that $f_k(\overline{U}_j)$ is contained in $Y_{i(j)}$ for each j. This is possible since only a finite number of the sets $\overline{U}_j$ can intersect $\overline{U}_k$.

$k_i(y_i)$

*

Now f_k by definition satisfies the transverse regularity condition for N_1 at each point of $f_k^{-1}(N_1) \cap \overline{W}_k$. We want to choose L small enough that the condition is satisfied at each point of the intersection of $f_k^{-1}(N_1)$ with $\cup_{j<k} \overline{W}_j$. It is sufficient to consider the intersection of this set with $\overline{U}_k$; let this intersection be denoted by K. Consider the function which maps the pair (x,L) $(x \in K)$ into $(f_k(x)\ ,\ D(\pi k_1\ f_k\ h_k^{-1})(h_k(x))$ in $N \times M(q,n)$. This function is continuous and carries $K \times (0)$ into the set' f_{k-1} $[(N-N_1) \times M(q,n)] \cup [N_1 \times M(q,n;q)]$, which is open in $N \times M(q,n)$. Hence for L sufficiently small, (K,L) is

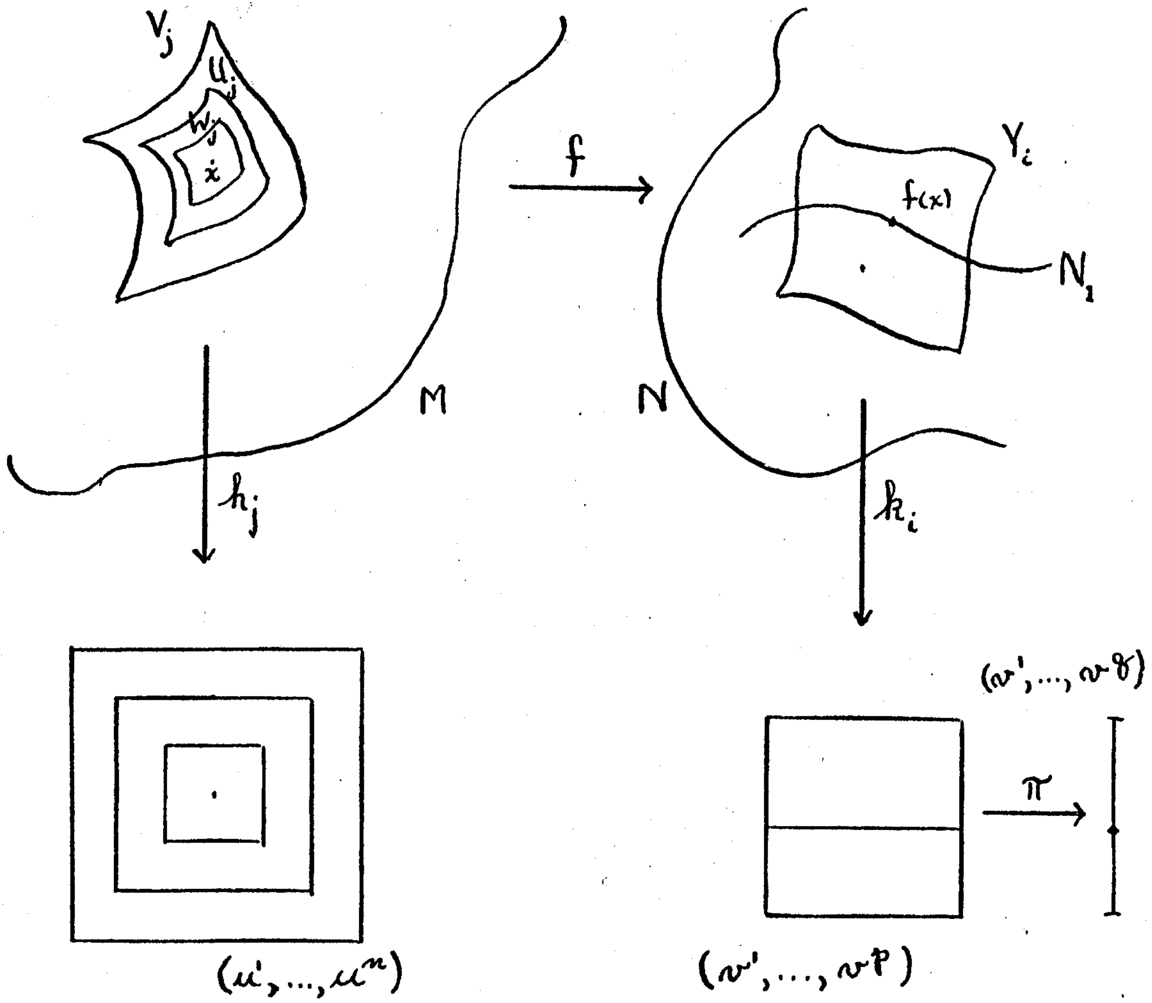
*

$= N \times M(q,n) - N_1 \times (M(q,n;q))$

closed

carried into this set, so that f_k satisfies the transverse regularity condition for N_1 at each point of $f_k^{-1}(N_1) \cap (\cup_{j \leq k} \overline{W}_j)$.

We define $g(x) \doteq \lim_{k \to \infty} f_k(x)$, as usual.



Chapter II

Vector Space Bundles

2.1 Definition. An n-dimensional real vector space bundle ξ is a triple (π, a, s). π is a continuous map of E onto B (E, B topological spaces; B Hausdorff). Let $F_b = \pi^{-1}(b)$; it is called a fibre. s maps $R \times E \longrightarrow E$ and carries $R \times F_b$ into F_b; a is defined on $\bigcup_b (F_b \times F_b) \subset E \times E$ and carries $F_b \times F_b$ into F_b.

The following must be satisfied:

(1) F_b is an n-dimensional real vector space with s and a as scalar product and vector addition, respectively.

(2) (Local triviality) For each b in B, there is a neighborhood U of b and a homeomorphism $\varphi : U \times R^n \longrightarrow \pi^{-1}(U)$ such that φ is a vector space isomorphism of $b' \times R^n$ onto $F_{b'}$, for each b' in U.

If in (2) the neighborhood U may be taken as all of B, the bundle is said to be the trivial bundle.

If ξ, η are n-dimensional and p-dimensional vector space bundles, respectively, we define the product bundle $\xi \times \eta$ as follows:

$$E(\xi \times \eta) = E(\xi) \times E(\eta)$$

$$B(\xi \times \eta) = B(\xi) \times B(\eta)$$

$$(\pi \times \lambda)(x, y) = (\pi(x), \lambda(y))$$

where π, λ are the projections in ξ, η respectively and $F_b(\xi \times \eta)$ has the usual product structure for vector spaces.

If U is a subset of $B(\xi)$, then $\xi|U$ denotes the bundle $\pi: \pi^{-1}(U) \longrightarrow U$. It is called the <u>restriction</u> of the bundle to U.

2.2 <u>Definition</u>. Let M^n be a differentiable manifold and let x_o be in M. A <u>tangent vector at x_o</u> is an operation X which assigns to each differentiable function f defined in a neighborhood of x_o, a real number. The following conditions must be satisfied:

1) If g is a restriction of f, $X(g) = X(f)$.

2) $X(cf + dg) = cX(f) + dX(g)$ (c,d real numbers)

3) $X(f \cdot g) = X(f) \cdot g(x_o) + f(x_o) \cdot X(g)$, where the dot means ordinary real multiplication.

Then $X(1) = X(1 \cdot 1) = X(1) + X(1)$, by 3). Hence $X(1) = 0$; and $X(c)$ also $= 0$, by 2).

If one thinks of a tangent vector as being the velocity vector of a curve lying in the manifold, then $X(f)$ is merely the derivative of f with respect to the parameter of the curve. This is made more precise below.

2.3 <u>Lemma</u>. Let $(u^1,\ldots,u^n)$ be a coordinate system about x. Let X be a tangent vector at x. Then X may be written uniquely as a linear combination of the operators $\frac{\partial}{\partial u^i}$:

$$X = \Sigma\, \alpha^i \frac{\partial}{\partial u^i}$$

Proof: We assume $u(x)$ is the origin. Given any $f(u^1,\ldots,u^n)$ define

$$g_1(u^1,\ldots,u^n) = \frac{f(u^1,\ldots,u^n) - f(0,u^2,\ldots,u^n)}{u^1} \quad \text{if } u^1 \neq 0$$

$$= \partial f/\partial u^1\,(0,u^2,\ldots,u^n) \quad \text{if } u^1 = 0 .$$

To see that g_1 is differentiable, note that

$$g_1(s,u^2,\ldots,u^n) = \int_0^1 \frac{\partial f}{\partial u^1}(s\,t,u^2,\ldots,u^n)\,dt .$$

(Then $f(u^1,\ldots,u^n) = u^1\, g_1(u^1,\ldots,u^n) + f(0,u^2,\ldots,u^n)$.)

Similarly, $f(0,u^2,\ldots,u^n) = u^2\, g_2(u^2,\ldots,u^n) + f(0,0,u^3,\ldots,u^n)$, where $g_2(0) = \partial f/\partial u^2(0)$. Finally, we have

$$f(u^1,\ldots,u^n) = \Sigma\, u^i\, g_i + f(0) , \quad \text{where} \quad g_i(0) = \frac{\partial f}{\partial u^i}(0) .$$

Thus $X(f) = \Sigma\, X(u^i)\, g_i(0) + 0\cdot X\,(g_i)$

$$= \Sigma\, \alpha^i \frac{\partial f}{\partial u^i}(0) , \quad \text{where} \quad \alpha^i = X(u^i) .$$

Remark. If $(v^1,\ldots,v^n)$ is another coordinate system about x, and $X = \Sigma\, \beta^j \dfrac{\partial}{\partial v^j}$, then $\alpha^i = X(u^i) = \Sigma_j\, \beta^j \dfrac{\partial u^i}{\partial v^j}$. The α^i are called the components of the vector X with respect to the coordinate system $(u^1,\ldots,u^n)$.

2.4 Alternate definition. A tangent vector at x is an assignment to every coordinate system $(u^1,\dots,u^n)$ about x of an element $(\alpha^1,\dots,\alpha^n)$ of R^n, with the requirement that if (β^j) is assigned to the system $(v^1,\dots,v^n)$, then $\alpha^i = \Sigma_j \frac{\partial u^i}{\partial v^j} \beta_j$. The derivation operator X is then defined as $\Sigma\, \alpha^i \frac{\partial}{\partial u^i}$. One checks readily that

a) $X(f)$ is independent of the coordinate system used, and

b) $X(f)$ satisfies requirements 1), 2), and 3) for a tangent vector.

2.5. Definition. For each x in M, the tangents at x form an n-dimensional vector space (the operations $\partial/\partial u^i$ form a basis, by 2.3). Let the totality of these be denoted $E(\tau)$; define $\pi\colon E(\tau) \longrightarrow M$ as mapping the tangent vector X at x_o into x_o. The local product structure is given by $\varphi_u\colon U \times R^n \longrightarrow E$, where $(U,h) = (u^1,\dots,u^n)$ is a coordinate system on M, and φ is defined as follows:

$$\varphi(x_o, a^1,\dots,a^n) = \text{the tangent vector } X = \Sigma\, a^i \frac{\partial}{\partial u^i} \text{ at } x_o .$$

Since φ is to be a homeomorphism, this structure imposes a topology on E; since $\varphi_v^{-1}\varphi_u$ is a homeomorphism on $(U \cap V) \times R^n$

this topology is unambiguously determined. One checks immediately that φ gives us a vector space isomorphism for each fibre.

Indeed, $\varphi_V^{-1}\varphi_u$ is a C^∞ map on $(U \cap V) \times R^n$, so that E is a differentiable manifold of dimension 2n (using definition 1.2 of a differentiable manifold). The map π is differentiable of rank n .

This bundle τ is called the tangent bundle of M .

2.6 Definition. If $f: M_1 \longrightarrow M_2$, there is an induced map $df: E(\tau_1) \longrightarrow E(\tau_2)$ defined as follows : $df(X) = Y$, where $Y(g) = X(gf)$. If X is a vector at x_o, Y is a vector at $f(x_o)$. This is clearly linear on each fibre; it is called the derivative map.

If (U,h) and (V,k) are coordinate systems about x_o, $f(x_o)$ respectively, and (α^i), (β^j) are the respective components of X and Y with respect to these coordinate systems, then $(\beta^j) = D(kfh^{-1})\cdot(\alpha^i)$ where the vector components are written as column matrices, as usual.

2.7 Definition. Let ξ,η be two n-dimensional vector space bundles.

A bundle map $f: \xi \longrightarrow \eta$ is a continuous map of $E(\xi)$ into $E(\eta)$ which carries each fibre isomorphically onto a fibre. The induced map $f_B: B(\xi) \longrightarrow B(\eta)$ is automatically continuous.

If $B(\xi) = B(\eta)$ and the induced map is the identity, f is said to be an equivalence. Note that if f is an equivalence, it is a homeomorphism: Locally f is just a map $U \times R^n \longrightarrow V \times R^n$. The projection of f^{-1} into the factor U is continuous, because f_B^{-1} is the identity. But f may be given by a non-singular matrix function of $x \in U$; f^{-1} is the inverse of this matrix, so that the projection of f^{-1} into the factor R^n is continuous. Hence f^{-1} is continuous.

If there is an equivalence of ξ onto η, we write $\xi \simeq \eta$.

2.8 Lemma. Given a bundle η with projection map $\lambda: E(\eta) \longrightarrow B(\eta)$, and a map $f: B_1 \longrightarrow B(\eta)$, there is a bundle $\pi: E_1 \longrightarrow B_1$ and a bundle map $g: E_1 \longrightarrow E(\eta)$ such that $\lambda g = f\pi$. Furthermore, E_1 is unique up to an equivalence.

[E_1 is called induced bundle and is often denoted by $f^*\eta$.]

Proof: Let E_1 be that subset of $B_1 \times E(\eta)$ consisting of points (b,e) such that $f(b) = \lambda(e)$. Define $\pi(b,e) = b$; $g(b,e) = e$. To show that E_1 is a vector space bundle, let $\varphi: V \times R^n \longrightarrow E(\eta)$ be a product neighborhood in $E(\eta)$, and let $f(U) \subset V$. Then define $\varphi_1: U \times R^n \longrightarrow E_1$ by $\varphi_1(b,x) = (b,\varphi(f(b),x))$. This is continuous and 1 - 1; its image equals $\pi^{-1}(U)$. Its inverse carries (b,e) into $(b,p\,\varphi^{-1}(e))$ (where p projects $V \times R^n$ onto R^n), so that it is continuous. The map g is an isomorphism on each fibre.

Now suppose $g': E' \longrightarrow E(\eta)$ is a bundle map, where $\pi': E' \longrightarrow B_1$ is a bundle and $\lambda g' = f\pi'$. We map $E' \longrightarrow E_1$ by mapping $e' \longrightarrow (\pi'(e'), g'(e'))$ in E_1. Because g' is an isomorphism on each fibre, so is this map; and it induces the identity on the base space. Hence it is an equivalence.

2.9 Definition. Let ξ, η be two bundles over B. The Whitney sum $\xi \oplus \eta$ is a bundle defined as follows: Consider the product bundle $E(\xi) \times E(\eta) \longrightarrow B \times B$; let d be the diagonal map $B \longrightarrow B \times B$. The induced bundle $d^*(\xi \times \eta)$ is defined as the Whitney sum $\xi \oplus \eta$.

Note that the fibre over b in $\xi \oplus \eta$ is merely $F_b(\xi) \times F_b(\eta)$, so that $\dim(\xi \oplus \eta) = \dim \xi + \dim \eta$.

Note also the commutativity and associativity of $\oplus$. I.e., $\xi \oplus \eta \simeq \eta \oplus \xi$ and $(\xi \oplus \eta) \oplus \zeta \simeq \xi \oplus (\eta \oplus \zeta)$. The proof is left as an exercise.

2.10 <u>Definition</u>. If ξ, η are bundles over B, then $g: E(\xi) \longrightarrow E(\eta)$ is a <u>homomorphism</u> if

1) it maps each fibre linearly into a fibre, and
2) the induced map on B is the identity.

Note that an equivalence is both a bundle map and a homomorphism. An <u>imbedding</u> of bundles is a 1 - 1 homomorphism.

2.11 <u>Theorem</u>. If $f: E(\xi) \longrightarrow E(\eta)$ maps each fibre linearly into a fibre, then f may be factored into a homomorphism followed by a bundle map.

<u>Proof</u>: Let π_1, π_2 be the projections in ξ, η, respectively.

Let $f_B: B(\xi) \longrightarrow B(\eta)$ be the map induced by f. Let $E_1 = f_B^* \eta$ be the bundle induced by f_B; let g be the bundle map $E_1 \longrightarrow E(\eta)$ and π the projection $E_1 \longrightarrow B(\xi)$.

Define $h: E(\xi) \longrightarrow B(\xi) \times E(\eta)$ by the equation $h(e) = (\pi_1(e), f(e))$. The image of h actually lies in that subset of $B(\xi) \times E(\eta)$ which is E_1; then h is a homomorphism. From the definition, $f = gh$.

$$\begin{tikzcd}
E(\xi) \arrow[r, "h"] \arrow[d, "\pi_1"] & E_1 \arrow[r, "g"] \arrow[d, "\pi"] & E(\eta) \arrow[d, "\pi_2"] \\
B(\xi) \arrow[r, "1"'] & B(\xi) \arrow[r, "f_B"'] & B(\eta)
\end{tikzcd}$$

note

$f_B(\pi_1(e)) = \pi_2(f(e))$

2.12 Lemma. Let ξ, η be bundles over B of dimensions n, p, respectively; let $g: \xi \longrightarrow \eta$ be a homomorphism. If g is onto, then kernel g is a bundle. If g is $1 - 1$, then cokernel g (i.e., the quotient, $\eta/\text{image } g$) is a bundle.

Proof: Suppose g is $1 - 1$ (i.e., has rank n when restricted to each fibre). In $E(\eta)$, we define $e \sim e'$ if $e - e'$ exists and is in the image of g. We identify the elements of these equivalence classes; the resulting identification space is defined to be $E(\eta/g(\xi))$. It is a bundle over B with projection naturally defined and each fibre is a vector space of dim $p - n$. We need only to show the existence of a local product structure.

Let U be an open set in B, with $\xi|U$ equivalent to $U \times R^n$ and $\eta|U$ equivalent to $U \times R^p$. Let g_o denote the homomorphism of $U \times R^n \longrightarrow U \times R^p$ induced by g. Now $(\eta/g(\xi))|U$ is equivalent to the quotient $U \times R^p/g_o(U \times R^n)$, so that it suffices to show that this latter quotient is locally a product.

g_o is given by a matrix $M(b) \in M(p,n)$ which depends continuously on the point $b \in U$. Given b_o, we may assume that in a neighborhood U_o of b_o, the first n rows are independent. We define $h: U_o \times R^n \times R^{p-n} \longrightarrow U_o \times R^p$ as the linear function on R^p whose matrix (non-singular) is

$$\left(\begin{array}{c|c} M(b) & \begin{array}{c} 0 \\ \hline I_{p-n} \end{array} \end{array}\right)$$

The image of $U_o \times R^n \times 0$ under h is just $g_o(U_o \times R^n)$; since h is an equivalence, it induces an equivalence of

$$U_o \times R^{p-n} \simeq \frac{U_o \times R^n \times R^{p-n}}{U_o \times R^n \times 0} \quad \text{onto} \quad \frac{U_o \times R^p}{g_o(U_o \times R^n)} .$$

not zero

Secondly, suppose g is onto (i.e., it has rank p on each fibre). $E(g^{-1}(0))$ is defined as that subset of $E(\xi)$ consisting of points e with $g(e) = 0$. Again, we need to show the existence of a local product structure. Let U, g_o, and $M(b)$ be as above. Given b_o, we may assume that the first p columns of $M(b)$ are independent in the neighborhood U_o of b_o . We define $h\colon U_o \times R^n \longrightarrow U_o \times R^p \times R^{n-p}$ by the matrix function

$$\left(\begin{array}{c|c} \multicolumn{2}{c}{M(b)} \\ \hline 0 & I_{n-p} \end{array}\right)$$

Now h followed by the natural projection of $U_o \times R^p \times R^{n-p}$ onto $U_o \times R^p$ equals $g_o | U_o$. Hence h^{-1} maps $U_o \times 0 \times R^{n-p}$ onto $g_o^{-1}(U_o \times 0)$; since h is an equivalence, the restriction of h^{-1} to $U_o \times 0 \times R^{n-p}$ is also.

Remark. If g is onto, $\xi/g^{-1}(0)$ is a bundle, being the quotient of the inclusion homomorphism $g^{-1}(0) \to \xi$. If g is 1 - 1, $g(\xi)$ is a bundle, being the kernel of the projection homomorphism $\eta \to \eta/g(\xi)$.

2.13 Definition. If φ is a non-negative function on B, the carrier of φ is the closure of the set of x with $\varphi(x) > 0$. A partition of unity is a collection φ_α of continuous non-negative functions on B, such that the sets C_α = carrier φ_α form a locally-finite covering of B, and $\Sigma\, \varphi_\alpha(x) = 1$ (this is a finite sum for each x).

2.14 Lemma. Let B be a normal space; U_α a locally-finite open covering of B. Then there is a partition of unity φ_α with carrier $\varphi_\alpha \subset U_\alpha$ for each α.

Proof: First, we show that there is an open covering V_α of B with $\bar{V}_\alpha \subset U_\alpha$ for each α. Assume the U_α indexed by a set of ordinals (well ordering theorem). Let V_α be defined for all $\alpha < \beta$ and assume that the sets V_α along with the sets U_α for $\alpha \geq \beta$ cover B. Consider the set $A(\beta) = B - \cup_{\alpha < \beta} V_\alpha - \cup_{\alpha > \beta} U_\alpha$. Then $A(\beta) \subset U_\beta$. Let V_β be an open set containing the closed set $A(\beta)$, with $\bar{V}_\beta \subset U_\beta$ (normality). This completes the construction of the V_α.

Now let g_α be a function which is positive on $\bar{V}_\alpha$ and 0 outside U_α (normality again). Define $\varphi_{\alpha_o}(x) = g_{\alpha_o}(x)/\Sigma\, g_\alpha(x)$. Since U_α is locally-finite, the sum in the denominator is finite and positive, so φ_α is well-defined.

Remark. If B is a differentiable manifold, φ_α may be chosen to be differentiable: Cover B with coordinate systems (V_i, h_i) as in 1.25 refining the covering $U_\alpha, B - \bar{V}_\alpha$. Let $\varphi_i(y) = \varphi(h_i(y))$ for $y \in V_i$, and $= 0$ otherwise (φ as in 1.26). Let $g_\alpha(y) = \Sigma\, \varphi_i(y)$, where the sum extends over all i such that $V_i \subset U_\alpha$.

2.15 Lemma. Let B be paracompact and let $0 \longrightarrow \xi \xrightarrow{i} \eta \xrightarrow{\varphi} \zeta \longrightarrow 0$ be an exact sequence of homomorphisms of bundles. Then there is equivalence $f\colon \eta \longrightarrow \xi \oplus \zeta$, with fi the natural inclusion and φf^{-1} the natural projection.

Proof. Let $\dim \xi = n$; $\dim \zeta = p$.

We first construct a Riemannian metric on η (i.e., a continuous inner product in $E(\eta)$). Let U_α be a locally-finite covering of B with $\eta | U_\alpha$ trivial; let g_α be the corresponding projection of $\eta | U_\alpha$ onto R^{n+p}. Let φ_α be a partition of unity with carrier $\varphi_\alpha \subset U_\alpha$.

If e, e' are in $E(\eta)$ and $\pi(e) = \pi(e')$, define $e \cdot e' = \Sigma_\alpha \varphi_\alpha(\pi(e))\, g_\alpha(e) \cdot g_\alpha(e')$, where the dot on the right hand side is the ordinary scalar product in R^{n+p}. This is a finite sum; it satisfies the axioms for a scalar product.

The way we use the Riemannian metric is to break η up into $iE(\xi)$ and its orthogonal complement. Let ξ' be the image of ξ in η and let $E(\zeta')$ be defined as that subset of $E(\eta)$ consisting of elements which are orthogonal to $i(E(\xi))$. In order to show that ζ' has a local product structure, consider the homomorphism

$$h: \eta \longrightarrow \xi'$$

which sends each vector into its orthogonal projection in ξ'. [Verification that h is continuous. Over any coordinate neighborhood U we can choose a basis $a_1, \ldots, a_n$ for the fibre of ξ'. Then the function h carries $v \in E(\eta)$ into $\Sigma\, t_j a_j \in E(\xi') \subset E(\eta)$, where $t_j = \Sigma\, B_{jk}(v \cdot a_k)$ and where (B_{jk}) denotes the inverse matrix to $(a_j \cdot a_k)$.] Since h is onto, its kernel ζ' is again a vector space bundle.

Now the bundle $i(\xi) = \xi'$ is equivalent to ξ. It remains to show that ζ' is equivalent to ζ and that η is equivalent to $\xi' \oplus \zeta'$. The former follows immediately from the fact that $\varphi|\zeta'$ is a homomorphism; from rank considerations it must be 1 - 1 and onto as well. The latter follows by noting that $E(\xi' \oplus \zeta')$ is defined as the subset of $E(\xi') \times E(\zeta')$ consisting of points (e_1, e_2) such that $\pi(e_1) = \pi(e_2)$. Consider the map f of $E(\xi' \oplus \zeta')$ into $E(\eta)$ obtained by taking (e_1, e_2) into their sum in $E(\eta)$ (this sum exists because e_1 and e_2 lie in the same fibre). This is clearly a homomorphism; from rank considerations, it must be 1 - 1 and onto.

2.16 Definition. Let M_1, M_2 be differentiable manifolds.

Let f be an immersion $M_1 \longrightarrow M_2$. The normal bundle ν_f is defined as follows:

Let τ_1, τ_2 be the tangent bundles of M_1, M_2 respectively. By 2.11, the map $df: E(\tau_1) \longrightarrow E(\tau_2)$ may be factored into a homomorphism h of $E(\tau_1)$ into $E(f^*\tau_2)$ followed by a bundle map g. Now h is a 1 - 1 homomorphism because f is an immersion; hence by 2.12, $f^*\tau_2$/image $g\overset{h}{\wedge}$ is a bundle over M_1. It is called the normal bundle ν_f. *

Then $0 \longrightarrow \tau_1 \longrightarrow f^*\tau_2 \longrightarrow \nu_f \longrightarrow 0$ is an exact sequence of homomorphisms, so that by 2.15, $f^*\tau_2$ is equivalent to $\tau_1 \oplus \nu_f$. Indeed, given a Riemannian metric on $f^*\tau_2$, ν_f is equivalent to the orthogonal complement of the image of τ_1.

Let us consider the case $M_2 = R^{n+p}$, where $\dim M_1 = n$. Then τ_2 is the trivial bundle, so that $f^* \tau_2$ is as well. (Proof: If $f\colon B \longrightarrow B(\eta)$ and η is trivial, so is $f^* \eta$. We have

$$\begin{tikzcd}
 & B \times R^n \arrow[d, "\pi"] \\
f\colon B_1 \arrow[r] & B
\end{tikzcd}$$

$E(f^* \eta)$ is defined as that subset of $B_1 \times (B \times R^n)$ consisting of points (b_1, b, x) such that $f(b_1) = \pi(b,x)$; i.e., of all points $(b_1, f(b_1), x)$. If we map this into (b_1, x), we obtain an equivalence of $f^* \eta$ with the bundle $B_1 \times R^n \longrightarrow B_1$.)

Thus $\tau_1 \oplus \nu_f$ is equivalent to a trivial bundle. In what follows, we investigate the following question: Given ξ, does there exist an η with $\xi \oplus \eta$ trivial? Using 1.28, this is always the case for ξ the tangent bundle of an n-manifold, and indeed η may be chosen also to have dimension n. A more general answer appears in 2.19.

2.17 Definition. Let $f\colon M_1 \longrightarrow M_2$; let $\dim M_1 = n$, $\dim M_2 = p$. If f has rank p at every point of M_1, it is said to be regular. If f is regular, the homomorphism $h\colon \tau_1 \longrightarrow f^* \tau_2$ given by 2.11 is an onto map. By 2.12, the kernel of h is a bundle α_f. It is called the bundle along the fibre.

Note that $f^{-1}(y)$ is a submanifold of M_1 of dim n-p (by 1.12 or 1.34). The inclusion i_y of $f^{-1}(y)$ into M_1 induces an inclusion di_y of its tangent bundle into τ_1. The kernel of h consists precisely of the vectors which are in the image of some di_y, i.e., the vectors tangent to the submanifolds $f^{-1}(y)$ are the ones carried into 0 by h.

One has the exact sequence $0 \longrightarrow \alpha_f \longrightarrow \tau_1 \xrightarrow{g} f^* \tau_2 \longrightarrow 0$, so that by 2.15, τ_1 is equivalent to $\alpha_f \oplus f^* \tau_2$.

2.18 Definition. A bundle ξ is of finite type if B is normal and may be covered by a finite number of neighborhoods $U_1, \ldots, U_k$ such that $\xi | U_i$ is trivial for each i.

2.19 Lemma. ξ is of finite type if B is compact, or paracompact finite dimensional.

The former statement is clear; let us consider the latter. By definition, the dimension of B is not greater than n if every open covering has an open refinement such that (*) no point of B is contained in more than n + 1 elements of the refinement. It is a standard theorem of topology than an n-manifold has dimension n in this sense.

Cover B by open sets U, with $\xi | U$ trivial; let $\{V_\alpha\}$ be an open refinement of this covering satisfying (*).

By 1.22, we may assume that $\{V_\alpha\}$ is locally-finite as well. Let φ_α be a partition of unity with carrier $\varphi_\alpha \subset V_\alpha$ for each α (2.14).

Let A_i be the set of unordered $i + 1$ tuples of distinct elements of the index set of $\{\varphi_\alpha\}$. Given a in A_i, where $a = \{\alpha_o, \dots, \alpha_i\}$, let W_{ia} be the set of all x such that $\varphi_\alpha(x) < \min[\varphi_{\alpha_o}(x), \dots, \varphi_{\alpha_i}(x)]$ for all $\alpha \neq \alpha_o, \dots, \alpha_i$. Each set W_{ia} is open, and W_{ia} and W_{ib} are disjoint if $a \neq b$. Also W_{ia} is contained in the intersection of the carriers of $\varphi_{\alpha_o}, \dots, \varphi_{\alpha_i}$, and hence in some set V_α. If we set X_i equal to the union of all sets W_{ia}, for fixed i, the result is that $\xi | X_i$ is trivial. (For $\xi | W_{ia}$ is trivial, and the W_{ia} are disjoint.)

Finally, the sets $X_o, \dots, X_n$ cover B. Given x in B, x is contained in at most $n + 1$ of the sets V_α, so that at most $n+1$ of the functions φ_α are positive at x. Since some φ_α is positive at x, x is contained in one of the sets W_{ia} for $0 \leq i \leq n$.

[The intuitive idea of the proof is as follows: Consider an n-dimensional simplicial complex, with φ_α the barycentric coordinate of x with respect to the vertex α. The sets W_{oa} will be disjoint neighborhoods of the vertices, the sets W_{1a} disjoint neighborhoods of the open 1 - simplices, and so on.]

2.20 Theorem. If ξ is of finite type, there is a bundle η such that $\xi \oplus \eta$ is trivial.

Proof: We proceed by showing that ξ may be imbedded in a trivial bundle $B \times R^m$, so that we have the exact sequence $0 \longrightarrow \xi \xrightarrow{i} B \times R^m \longrightarrow B \times R^m/i(\xi) \longrightarrow 0$ by 2.12. The theorem then follows from 2.15. (Paracompactness is not needed since the trivial bundle clearly has a Riemannian metric.)

Cover B by finitely many neighborhoods $U_1, \ldots, U_k$ with $\xi | U_i$ trivial for each i. Let $\varphi_1, \ldots, \varphi_k$ be a partition of unity with carrier $\varphi_i \subset U_i$ for each i (2.14). Let f_i denote the equivalence of $E(\xi | U_i)$ onto $U_i \times R^n$; let $f_i^1, \ldots, f_i^n$ denote the coordinate functions of its projection into R^n.

We define $h: E(\xi) \longrightarrow B \times R^{nk}$ as follows:

$$h(e) = (\pi(e) \;,\; \varphi_i(\pi(e)) \cdot f_i^j(e)) \qquad \begin{array}{l} i = 1, \ldots, k \\ j = 1, \ldots, n \end{array}$$

(no summation is indicated). This is well-defined, since $\varphi_i(\pi(e)) = 0$ unless $e \in E(\xi | U_i)$. It is clearly a homomorphism, since each f_i^j is linear on $E(\xi | U_i)$. To show that it is 1 - 1, let $e \neq 0$. Then for some i, $\varphi_i(\pi(e)) > 0$. Since f_i is an equivalence, $f_i^j(e) \neq 0$ for some j. Hence $h(e) \neq (\pi(e), 0)$, as desired.

2.21 Definition. The bundle ξ is s-equivalent to η if there are trivial bundles o^p, o^n such that $\xi \oplus o^p \simeq \eta \oplus o^n$.

Here $o^p = B \times R^p$.. Symmetry and reflexivity are clear. To show transitivity, assume $\xi \oplus o^p \simeq \eta \oplus o^q$ and $\eta \oplus o^r \simeq \zeta \oplus o^s$. Then $\xi \oplus o^p \oplus o^r \simeq \zeta \oplus o^s \oplus o^q$.

Note that s-equivalence differs from equivalence. E.g., consider the two-sphere S^2 in R^3. Then $\tau^2 \oplus \nu^1 = o^3$. The normal bundle ν^1 is easily seen to be trivial; but it is a classical theorem of topology that τ^2 is not (it does not admit a non-zero cross-section). Hence τ^2 is s-trivial, but not trivial.

2.22 Theorem. The set of s-equivalence classes of vector space bundles of finite type over B forms an abelian group under $\oplus$.

Proof. To avoid logical difficulties, we consider only subbundles of $B \times R^m$, for all m. This suffices, since any bundle ξ of finite type may be imbedded in some $B \times R^m$, by 2.20.

The class of trivial bundles o^p is the identity element. The existence of inverses is the substance of 2.20.

2.23 Corollary. Given two immersions of the differentiable manifold M in euclidean space, their normal bundles are s-equivalent.

2.24 Definition. M^n is a π-manifold if M may be immersed in some R^{n+p} so that its normal bundle is trivial.

This is equivalent to the requirement that τ^n be s-trivial: Let τ^n be s-trivial. If we take some immersion of M into R^{n+p}, then $\tau^n \oplus \nu^p$ is trivial by 2.16, so that ν^p is s-trivial, i.e. $\nu^p \oplus o^q = o^{p+q}$ for some q. Consider the composite immersion $M \longrightarrow R^{n+p} \subset R^{n+p+q}$. The normal bundle of M in R^{n+p+q} is just $\nu^p \oplus o^q$, which is trivial.

Conversely, if ν^p is trivial for some immersion, then τ^n is s-trivial because $\tau^n \oplus \nu^p$ is trivial.

2.25 Definition. Let $G_{p\,n}$ denote the set of all n-dimensional vector subspaces of R^{n+p} (i.e., all n-dim hyperplanes through the origin). It is called the Grassman manifold of n-planes in $n+p$ space.

Its topology is obtained as follows: Consider $M(n,n+p;n)$; we identify two elements of this set if the hyperplanes spanned by their row vectors are the same. $G_{p\,n}$ is in 1 - 1 correspondence with this identification space, and is given the identification topology. Let ρ be the projection of $M(n,n+p;n) \longrightarrow G_{p\,n}$.

Now $\rho(A) = \rho(B)$ if and only if $A = CB$ for some non-singular $n \times n$ matric C : The hyperplane $\rho(A)$ consists of all points $(x',\ldots,x^{n+p}) \in R^{n+p}$ which equal $(c',\ldots,c^n)\cdot A$ for some choice of constants c^i. If $\rho(A) = \rho(B)$, then

$$(1,0,\ldots,0)\cdot A = (c_1^1,\ldots,c_1^n)\cdot B$$

$$(0,1,\ldots,0)\cdot A = (c_2^1,\ldots,c_2^n)\cdot B \text{ ,etc., for some choice of } c_i^j .$$

Then $IA = CB$, where C has rank n because A does. The converse is clear.

(a) $G_{p\,n}$ is locally euclidean. Let $A \in M(n,n+p;n)$; after permuting the columns, we may assume $A = (P,Q)$ where P is $n \times n$ and non-singular. Let U be the set of all such A; it is an open set in $M(n,n+p;n)$, being the inverse image of the non-zero reals under the continuous map $(P,Q) \longrightarrow \det P$. If $\rho(P,Q) = \rho(R,S)$, where P is non-singular, then $(P,Q) = (CR,CS)$ for some non-singular C. Hence R is necessarily non-singular; it follows that $\rho^{-1}(\rho(U)) = U$, so that $\rho(U)$ is open in $G_{p\,n}$ (by definition of the identification topology).

We show $\rho(U)$ homeomorphic with $R^{p\,n}$. Define $\varphi : U \longrightarrow R^{p\,n}$ by $\varphi(P,Q) = P^{-1}Q$. If $\rho(P,Q) = \rho(R,S)$ then $(P,Q) = (CR,CS)$, so that $P^{-1}Q = (CR)^{-1}(CS) = R^{-1}S$. Hence

φ induces a continuous map $\varphi_o : \rho(U) \longrightarrow R^{p\,n}$. Define $\psi : R^{p\,n} \longrightarrow \rho(U)$ by $\psi(Q) = \rho(I,Q)$ where Q is an $n \times p$ matrix. One checks immediately that ψ and φ_o are inverses of each other.

$$\begin{tikzcd}
M(n,n+p;n) \arrow[d, "\rho"] & \supset & U \arrow[d, "\rho"] \arrow[drr, "\varphi"] & & \\
G_{p\,n} & \supset & \rho(U) \arrow[rr, shift left, "\varphi_o"] & & R^{p\,n} \arrow[ll, shift left, "\psi"]
\end{tikzcd}$$

(b) To show that $G_{p\,n}$ is Hausdorff, we show that ψ maps every compact set into a closed set (this will clearly suffice). Let K be a compact subset of $R^{p\,n}$; we show $\varphi^{-1}(K)$ is closed in $M(n,n+p;n)$. $\varphi^{-1}(K)$ consists of all matrices (P,Q) with P non-singular and $P^{-1}Q \in K$. Let $(P,Q) \in M(n,n+p;n)$ be the limit of the sequence (P_i,Q_i) of elements of $\varphi^{-1}(K)$. Since K is compact some subsequence of the sequence $\varphi(P_i,Q_i) = P_i^{-1} Q_i$ converges to a point R of K . Then the corresponding subsequence of the sequence Q_i converges to PR , so that $(P,Q) = P(I,R)$. Since (P,Q) has rank n it follows that P is non-singular, so that $(P,Q) \in \varphi^{-1}(K)$, as desired.

[illegible]
X X

X

Hence $G_{p\,n}$ is a manifold of dimension $p\,n$.

(c) $G_{p\,n}$ is a differentiable manifold and ρ is a differentiable map. A function f on the open set V in $G_{p\,n}$ belongs to the differentiable structure $\mathcal{D}$ if $f\rho$ is differentiable. To show that this satisfies the conditions for a differentiable structure, we show that $(\rho(U),\varphi_o)$, as defined in (a), is a coordinate system. Let f be defined on $V \subset \rho(U)$. Given $Q \in R^{p\,n}$, $f\,\varphi_o^{-1}(Q) = f\rho\,(I,Q)$ so that $f\,\varphi_o^{-1}$ is differentiable if $f\rho$ is. Conversely, given $(P,Q) \in V$, $f\rho\,(P,Q) = f\,\varphi_o^{-1}\,\varphi_o\,\rho\,(P,Q) = f\,\varphi_o^{-1}(P^{-1}Q)$, so that $f\rho$ is differentiable if $f\,\varphi_o^{-1}$ is.

(d) $G_{p\,n}$ is compact. Let L be the subset of $M(n,n+p;n)$ consisting of matrices whose rows are orthonormal vectors. L is a closed and bounded subset of $R^{n(n+p)}$. Since $\rho(L) = G_{p\,n}$ (the Gram-Schmidt orthogonalization process proves this), $G_{p\,n}$ is compact.

(e) $G_{p\,n}$ is diffeomorphic to $G_{n\,p}$. Geometrically, the homeomorphism h is defined as carrying each hyperplane into its orthogonal complement. It is clearly 1 - 1; to show it differentiable we use the coordinate system $(\rho(U),\varphi_o)$ defined in (a). Let g map U into $M(p,n+p;p)$ by carrying (P,Q) into $(-(P^{-1}Q)^{\tau}, I_p)$; it is differentiable (τ denotes transpose). The row space of (P,Q) is the same as that of $(I_n, P^{-1}Q)$, while the row vectors of this matrix are orthogonal to those of $(-(P^{-1}Q)^{\tau}, I_p)$ (multiply the one by the transpose of the other). Hence g induces $h|\rho(U)$, so that the latter is differentiable.

2.26 Definition. Let $E(\gamma_p^n)$ be defined as that subset of $G_{p\,n} \times R^{n+p}$ consisting of pairs (H,x) where x is a vector lying in the hyperplane H. It is called the universal bundle (for reasons we shall see). The projection π maps (H,x) into H; the fibre is thus an n-dimensional subspace of R^{n+p}.

γ_p^n is an n-dimensional vector space bundle over $G_{p\,n}$. We need to show the existence of a local product structure. Let $(\rho(U), \varphi_o)$ be a coordinate neighborhood on $G_{p\,n}$, as in (a) above. We define $h: \rho(U) \times R^n \longrightarrow \pi^{-1}\rho(U)$ as carrying $(H,(x^1,\ldots,x^n))$ into $(x^1,\ldots,x^n) \circ (I_n, Q)$ where $Q = \varphi_o(H)$. This is a vector in the hyperplane H; h is clearly an isomorphism on each fibre. Its inverse is continuous, since it sends $(H,(y^1,\ldots,y^{n+p}))$ in $G_{p\,n} \times R^{n+p}$ into $(H,(y^1,\ldots,y^n))$ in $\rho(U) \times R^n$.

2.27 Definition. ξ is a differentiable vector space bundle if $E(\xi)$ and $B(\xi)$ are differentiable manifolds, and if the homeomorphisms

$$U \times R^n \longrightarrow \pi^{-1}(U)$$

which specify the local product structure can be chosen as diffeomorphisms.

It follows that $\pi : E \longrightarrow B$ is differentiable of maximum rank. Note that B can be differentiably imbedded in E by mapping b into the 0-vector of F_b . The normal bundle of this imbedding is just ξ .

Examples of differentiable bundles include the tangent bundle of a manifold, the normal bundle of an immersed manifold, and the universal bundle γ_p^n above. In the latter case, $E(\gamma_p^n)$ is imbedded differentiably in $G_{p\ n} \times R^{n+p}$.

2.28 Theorem. Let ξ^n be an n-dimensional vector space bundle. The following conditions are equivalent:

(a) ξ is of finite type.

(b) There is a bundle η^p such that $\xi^n \oplus \eta^p$ is trivial.

(c) There is a bundle map $\xi^n \longrightarrow \gamma_p^n$ for some p . (Thus the terminology "universal bundle" for γ_p^n .)

Proof: We have already shown that (a) implies (b) (2.20); the bundle η^p there constructed has dimension $n(k - 1)$, where k is the number of elements in the covering $U_1, \ldots, U_k$ of $B(\xi) = B$ such that $\xi | U_i$ is trivial.

(b) implies (c): Condition (b) means that ξ^n may be imbedded in the trivial bundle $B(\xi) \times R^{n+p}$; let f be this imbedding. We wish to define g and g_B in the following diagram:

$$\begin{tikzcd} E(\xi) \arrow[r, "g"] \arrow[d, "\pi"'] & E(\gamma_p^n) \arrow[d] \\ B(\xi) \arrow[r, "g_B"'] & G_{p\,n} \end{tikzcd}$$

Since f is a 1-1 homomorphism, $f(F_b)$ is the cartesian product of b and an n-dim hyperplane H^n in R^{n+p} ; let $g_B(b)$ equal this hyperplane H^n . If $e \in F_b$, then $f(e) = (b,x)$, where x is a vector in the hyperplane H^n ; let $g(e) = (H^n,x)$ in $G_{p\,n} \times R^{n+p}$. Then g(e) actually lies in the subset of $G_{p\,n} \times R^{n+p}$ which constitutes $E(\gamma_p^n)$. From rank considerations, g is automatically an isomorphism on each fibre.

It remains to show that g is continuous. Locally, g just looks like a map $U \times R^n \longrightarrow G_{p\,n} \times R^{n+p}$. We factor it into a continuous map $h: U \times R^n \longrightarrow M(n,n+p;n) \times R^{n+p}$ followed by the projection $\rho \times 1$ into $G_{pn} \times R^{n+p}$. Locally, f looks like a map $U \times R^n \longrightarrow B \times R^{n+p}$. Let $e_1,\dots,e_n$ be a basis for R^n ; we define h(b,x) as $(A, p_2\, f(b,x))$. Here p_2 projects $B \times R^{n+p}$ onto its second factor and A is the matrix having $p_2 f(b,e_1),\ldots,p_2 f(b,e_n)$ as its rows. Then h is continuous, and $(\rho \times 1)h$ equals g.

(Note: The converse assertion, (c) implies (b), can be proved by the same argument.)

(c) implies (a): Being compact, G_{pn} is covered by finitely many neighborhoods U_i with $\gamma_p^n | U_i$ trivial. (In fact $(n+p)!/n!p!$ neighborhoods will suffice.) If f is a bundle map $\xi^n \longrightarrow \gamma_p^n$ then the sets $f_B^{-1}(U_i) = V_i$ cover B, and $\xi | V_i$ is equivalent to the bundle induced by $f_B \colon V_i \longrightarrow G_{pn}$ (the uniqueness part of 2.8). Then $\xi | V_i$ is trivial (since it is induced from a trivial bundle).

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Chapter III

The Cobordism Theory of Thom

3.1 *Definition.* An *n-manifold-with-boundary* Q is a Hausdorff space with a countable basis which is locally homeomorphic with H^n (the subset of R^n such that $x^1 \geq 0$). The *boundary* ∂Q is that subset of Q corresponding to R^{n-1} under the local homeomorphism (R^{n-1} being the subset of R^n with $x^1 = 0$). ∂Q is well-defined, since the image of an open set in R^n under a homeomorphism of it into R^n must be open (Brouwer theorem on invariance of domain). It is clear that ∂Q is an (n-1)-manifold.

A *differentiable structure* $\mathcal{D}$ on Q is a collection of real-valued functions f defined on open subsets of Q such that

1) every point of Q has an open neighborhood U and a homeomorphism h of U into an open subset of H^n, such that f is in $\mathcal{D}$ if and only if fh^{-1} is differentiable. (f is defined on an open subset of U; fh^{-1} differentiable means that it may be extended to a neighborhood of h(U) *in R^n* so as to be differentiable).

2) If U_i are open sets contained in the domain of f and $U = \cup U_i$, then $f|U \in \mathcal{D}$ if and only if $f|U_i \in \mathcal{D}$ for each i .

As before, (U,h) is called a <u>coordinate system</u> on Q, and one can define differentiable structures alternatively by means of coordinate systems.

We impose an additional condition on $\mathcal{D}$ in 3.2.

3.2 <u>Definition</u>. Let M_1, M_2 be compact differentiable n-manifolds. They are said to lie in the same <u>cobordism class</u> $(M_1 \sim M_2)$ if there is a compact differentiable $n+1$ manifold-with-boundary Q such that ∂Q is diffeomorphic with the disjoint union of M_1 and M_2 (denoted by $M_1 + M_2$).

Symmetry and reflexivity of this relation are clear. To show transitivity, we impose the additional condition on $\mathcal{D}$ that there is a neighborhood U of ∂Q in Q which is diffeomorphic with $\partial Q \times [0,1)$, the diffeomorphism being the identity on $\partial Q \times 0$. This is redundant, but we assume it to avoid proving it. Transitivity follows:

Let $M_1 + M_2$ be diffeomorphic with ∂Q_1 and $M_2 + M_3$ diffeomorphic with ∂Q_2; let h_1, h_2 be the diffeomorphisms. We form a new space Q_3 from $Q_1 \cup Q_2$ by identifying each point of $h_1(M_2)$ with its image under $h_2 h_1^{-1}$. There is then a homeomorphism of $M_2 \times (-1,1)$ into this space which equals h_1 when restricted to $M_2 \times 0$, and is a diffeomorphism of $M_2 \times [0,(-1)^i)$ into Q_i for $i = 1,2$. (It is derived from the postulated "product neighborhoods" $\partial Q_i \times [0,1)$.) If this is taken to be a coordinate system on Q_3 , Q_3 becomes a differentiable manifold-with-boundary, and $M_1 + M_3$ is diffeomorphic with ∂Q_3 . Q_1 and Q_2 are diffeomorphic with subsets of Q_3 .

3.3 Definition. As usual, there are logical difficulties involved in considering these cobordism classes. One way of avoiding them is to consider only manifolds-with-boundary imbedded in some euclidean space R^p: If Q_1 is a differentiable manifold-with-boundary and $Q_2 = \partial Q_1 \times [0,1)$, then the space Q_3 constructed in the preceding paragraph is a differentiable manifold, so that it may be imbedded in some euclidean space. Hence Q_1 may so be imbedded.

With these restrictions, the set of cobordism classes of n-manifolds forms an abelian group (denoted by $\mathcal{N}^n$) under the operation + (disjoint union). If $M_1 \sim M_1'$ and $M_2 \sim M_2'$, this means that $M_i + M_i'$ is diffeomorphic with ∂Q_i . Then $(M_1 + M_2) + (M_1' + M_2')$ is diffeomorphic with $\partial(Q_1 \cup Q_2)$, so that $M_1 + M_2 \sim M_1' + M_2'$ and the operation + is well-defined on cobordism classes. The zero element is the vacuous manifold or the n-sphere (or ∂Q , where Q is any compact differentiable (n+1)-manifold-with-boundary). The remaining axioms are clear. Note that $M + M$ is diffeomorphic with $\partial(M \times [0,1])$, so that every element is of order 2.

The groups $\mathfrak{N}^n$ are called the (non-orientable) cobordism groups. Let $\mathfrak{N}$ denote the direct sum $\mathfrak{N}^0 \oplus \mathfrak{N}^1 \oplus \mathfrak{N}^2 \oplus \ldots$. There is a bilinear symmetric pairing of $\mathfrak{N}^i$, $\mathfrak{N}^j$ into $\mathfrak{N}^{i+j}$, i.e. a homomorphism of $\mathfrak{N}^i \otimes \mathfrak{N}^j$ into $\mathfrak{N}^{i+j}$ induced by the operation of cartesian product.

First, $(M_1 + M_2) \times M_3 = (M_1 \times M_3) + (M_2 \times M_3)$ by definition of cartesian product. Second, if $M_1 \sim 0$, i.e. $M_1 = \partial Q$, then $M_1 \times M_2$ is diffeomorphic with $\partial(Q \times M_2)$, so that $M_1 \times M_2 \sim 0$.

Since $M_1 \times M_2 \sim M_2 \times M_1$, and since $M_1 \times p \sim M_1$ (where p is a point-manifold), this pairing makes $\mathfrak{N}$ into a (graded) commutative ring with unit. Indeed, it is a graded algebra over the field Z_2 .

3.4 Remark. The general result of Thom is the following

Theorem. $\mathfrak{N}$ is a polynomial algebra over Z_2 with one generator in each positive dimension except those of the form $2^m - 1$. If n is even, projective n-space is a generator.

This theorem means that there are compact manifolds M^2, M^4, M^5,... such that every compact manifold is in the cobordism class of a disjoint union of products of these manifolds, and that there are no relations among the generators (except commutativity and associativity of products).

Thom's procedure is to show that $\mathcal{N}^n$ is isomorphic with the $(n+k)^{\underline{th}}$ homotopy group of a certain space T_k ; and then to compute these homotopy groups. We shall consider only the first of these two problems in the present notes.

3.5 <u>Definition</u>. Let h be an imbedding of the differentiable manifold M^n in R^{n+k} ; consider the normal bundle of this imbedding. Using the standard Riemannian metric for the tangent bundle to R^{n+k} , this normal bundle is equivalent to the orthogonal complement of the image in the tangent bundle of R^{n+k} of the tangent bundle of M^n (2.16); this complement we denote by ν^k . Define e as the canonical map of $E(\nu^k)$ into R^{n+k} which maps the vector v normal to M^n at x into its end point. (Described differently, one maps the tangent bundle to R^{n+k} into itself canonically by mapping the vector v , based at x , into the point $x + v$ of R^{n+k} . This map is differentiable; its restriction to $E(\nu^k)$ is the map e .)

Consider M^n as the zero vectors of $E(\nu^k)$. Then we have the

3.6 <u>Theorem</u>. There is a neighborhood of M^n in $E(\nu^k)$ which is mapped diffeomorphically onto a neighborhood of M^n in R^{n+k} .

Proof: Note that e is differentiable, and that it has rank $n+k$ at points of $M^n \subset E(v^k)$. (This is easily checked by computing the derivative matrix of e with respect to a local coordinate system.) Hence e has rank $n+k$ in some neighborhood of M^n in $E(v^k)$, so that it is a local homeomorphism at points of M^n: it maps a neighborhood of each $x \in M^n$ homeomorphically onto a neighborhood of $f(x)$. We then appeal to the topological lemma:

If $f: X \longrightarrow Y$ is a local homeomorphism and the restriction of f to the closed subset A is a homeomorphism, then f is a homeomorphism on some neighborhood V of A. (X, Y are Hausdorff spaces with countable bases; X is locally compact.) This lemma is proved as follows:

(1) If A is compact, the lemma holds. For otherwise, there would be points x, y arbitrarily close to A such that $f(x) = f(y)$. Since A has a compact neighborhood, we may choose sequences x_n, y_n converging to x, y, respectively, in A such that $x_n \neq y_n$ and $f(x_n) = f(y_n)$. Hence $f(x) = f(y)$ so that $x = y$, f being a homeomorphism on A. But then f is not a local homeomorphism at x.

(2) Let A_o be a compact subset of A. Then there is a neighborhood U_o of A_o such that $\bar{U}_o$ is compact and f is a homeomorphism on $\bar{U}_o \cup A$: It will suffice for f to be $1-1$,

since f is a local homeomorphism. By (1), let V_o be a neighborhood of A_o so that $f|\bar{V}_o$ is 1-1. If no neighborhood of A_o in V_o satisfies the requirements for U_o, there is a sequence of points x_n of $X - A$ converging to $x \in A_o$ with $f(x_n) \in f(A)$. Choose $y_n \in A$ with $f(x_n) = f(y_n)$. Since f is continuous, $f(y_n)$ converges to $f(x)$; since f is a homeomorphism on A, y_n converges to x. Since $x_n \neq y_n$, this contradicts the fact that f is a local homeomorphism at x.

(3) Express A as the union of an ascending sequence of compact sets $A_1 \subset A_2 \subset \ldots$. Let V_1 be a neighborhood of A_1 such that $\bar{V}_1$ is compact and f is a homeomorphism on $\bar{V}_1 \cup A$ (by (2)). Given V_i a neighborhood of A_i satisfying these conditions, consider the set $\bar{V}_i \cup A_{i+1}$. It is a compact subset of $\bar{V}_i \cup A$, and f is a homeomorphism on $\bar{V}_i \cup A$. Hence by (2) there is a neighborhood V_{i+1} of $\bar{V}_i \cup A_{i+1}$ with $\bar{V}_{i+1}$ compact, such that f is a homeomorphism on $\bar{V}_{i+1} \cup A$. We proceed by induction. f is 1 - 1 on $V = \cup V_{i+1}$, so that it is a homeomorphism on V (being a local homeomorphism -onto).

3.7 <u>Corollary</u>. Any differentiable submanifold of R^{n+k} is a differentiable neighborhood retract.