Scottish Topology Seminars

The topological rigidity of the torus

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June 25, 2009

Introduction

One of the main goals in topology is to classify manifolds up to some equivalence relation:

1. homotopy equivalence,
2. homeomorphism
3. PL-homeomorphism
4. diffeomorphism

diffeomorphism $\Rightarrow$ PL-homeomorphism $\Rightarrow$ homeomorphism $\Rightarrow$ homotopy equivalence.

Remark: In dimension 2, all the classifications are the same.

In higher dimensions:

PL-homeomorphism $\not\Rightarrow$ diffeomorphism
$\rightarrow$ exotic spheres,

homeomorphism $\not\Rightarrow$ PL-homeomorphism
$\rightarrow$ high-dimensional torus,

homotopy equivalence $\not\Rightarrow$ homeomorphism
$\rightarrow$ lens spaces (3-dimensional).

Question Does a weak classification imply a stronger classification for some manifolds ?

Introduction

Definition

A topological manifold M is said to be **topologically rigid** *if every homotopy equivalence*

$$f : N \to M$$

with N a topological manifold is homotopic to a homeomorphism.

A few examples:

1. **Poincaré Conjecture** (now proven in all dimensions):

   $\mathbb{S}^n$ is topologically rigid.

2. **Mostow rigidity theorem**: If $f : M \to N$ is a homotopy equivalence between two complete hyperbolic n-manifolds ($n \geq 3$) of finite volume, then f is homotopic to an isometry.

Definition

A manifold is said to be **asphericical** *if its universal cover is contractible.*

Example:

Cartan-Hadamard theorem: A complete Riemannian manifold with non-positive curvature is aspherical.

Borel Conjecture: A compact aspherical manifold is topologically rigid.

Aim of this talk:

Theorem (Hsiang-Wall, 1969)

$\mathbb{T}^n$ *is topologically rigid* $(n \geq 5)$.

Key ingredients:

1. Surgery theory (hence the dimensional restriction)
2. Splitting obstructions
3. Topological manifolds theory (Kirby-Siebenmann)

Remark: The Borel conjecture is formulated in topological terms.
$\longrightarrow$ we would expect a solution involving only topological objects.

BUT !! Topological surgery cannot be fully developed only in the topological world. We need to work in the PL-world.

$\rightarrow$ key ingredients: computation of $\mathcal{S}_{PL}(\mathbb{T}^n \times \mathbb{D}^k)$, and algebra.

Introduction

structure of the talk:

1. Quick introduction to surgery theory,
2. The PL-structure set $\mathcal{S}_{PL}(\mathbb{T}^n \times \mathbb{D}^k)$,
3. PL-smoothing theory,
4. topological rigidity of the torus.

We will be considering different type of manifolds: CAT-manifolds ($CAT = TOP, PL, DIFF$) and CAT-isomorphisms

1. topological manifolds and homeomorphisms
2. PL-manifolds and PL-homeomorphisms
3. smooth manifolds and diffeomorphisms

Surgery theory

Goals:

1. decide whether a Poincaré space is homotopy equivalent to a CAT-manifold,
2. decide whether a homotopy equivalence between two CAT-manifolds is homotopic to a CAT-isomorphism.

main tool: Surgery exact sequence.

Surgery theory

Idea (very vague):

1. start with a map not too far from a homotopy equivalence,
2. try to perform surgery to obtain a homotopy equivalence.

What is a **not too far from a homotopy equivalence** map ?

Definition

A **degree one normal map**

$$(f, b) : (M, \nu_M) \to (X, \eta)$$

is a degree one map $f : M \to X$ *with* M *a CAT-manifold and* X *a CW-complex, together with a stable pullback bundle* $b : \nu_M \to X$ *with* ν_M *the stable normal bundle of* M *and* η *a stable bundle over* X*.*

$$\begin{array}{ccc} \nu_M & \xrightarrow{b} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow[f]{} & X \end{array}$$

advantage: such maps always exist for Poincaré spaces !!

Disadvantage: That's indeed NOT a homotopy equivalence...

Surgery theory

What is a **surgery** ?

Definition

Let M^m be a manifold, and $\mathbb{S}^n \hookrightarrow M$ an embedding with trivial normal bundle. A **surgery** *on M is the procedure of constructing a new manifold*

$$M' = \overline{M - (\mathbb{S}^n \times \mathbb{D}^{m-n})} \cup_{\mathbb{S}^n \times \mathbb{S}^{m-n-1}} (\mathbb{D}^{n+1} \times \mathbb{S}^{m-n-1})$$

by cutting out the embedded $\mathbb{S}^n \times \mathbb{D}^{m-n}$ and replacing it by $\mathbb{D}^{n+1} \times \mathbb{S}^{m-n-1}$.

Effect of a surgery: If $2n + 1 \leq m$,

$$\pi_i(M) \simeq \pi_i(M') \text{ for } i < n,$$

$$\pi_n(M') \simeq \pi_n(M)/ < x >,$$

where x is the class in $\pi_n(M)$ represented by the embedding $\mathbb{S}^n \hookrightarrow M$.

Surgery theory

One can extend the notion of a surgery on a manifold to surgery on a normal map.

Definition

Given a degree one normal map $(f, b) : M^m \to X$, a **framed** *n-***embedding** *in f is a commutative square*

$$\begin{array}{ccc} \mathbb{S}^n \times \mathbb{D}^{m-n} & \xrightarrow{\bar{g}} & M \\ \downarrow & & \downarrow f \\ \mathbb{D}^{n+1} \times \mathbb{D}^{m-n} & \xrightarrow[\bar{h}]{} & X \end{array}$$

Proposition

Given a degree one normal map $(f, b) : M \to X$ and a framed n-embedding, one can perform surgery on f in order to obtain a new degree one normal map $(f', b') : M' \to X$.
Furthermore, the surgery kills the homotopy class of the embedding

$$[g : \mathbb{S}^n \to M] \in \ker(\pi_n(M) \xrightarrow{f_*} \pi_n(X)).$$

Surgery theory

Proposition

A map between m-dimensional Poincaré spaces is a homotopy equivalence if and only if it induces isomorphism on π_1 and H_i for $2i \leq m$.

Lemma

A degree one map between Poincaré spaces induces epimorphisms on homology groups

Lemma

A degree one map between Poincaré spaces induces an epimorphism on π_1.

Surgery theory

Strategy to prove that a CW-complex has the homotopy type of a manifold:

1. Start with a degree one normal map $f : M^m \to X$.
2. Perform surgery on f so it induces an isomorphism on π_1.
3. Perform surgery on f so it induces an isomorphism on $H_i,\ 2i \leq m$.

Is it always possible to achieve this program ?

1. is always possible.
2. is always possible.
3. is always possible for $2i < m$ **BUT** problem in the middle dimension !

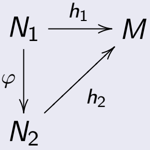
   $\to$ obstruction in $L_m(\pi_1(X))$.

The surgery exact sequence.

goal: computing the CAT-structure set $\mathcal{S}_{CAT}(M)$ of a topological manifold without boundary M^m, with $m \geq 5$.

Definition

The **CAT-structure set** $\mathcal{S}_{CAT}(M)$ *is the set of equivalence classes of homotopy equivalences* $h : N \to M$ *with* N *a CAT-manifold, under the following equivalence relation:*
$(h_1 : N_1 \to M) \simeq (h_2 : N_2 \to M)$ *if there exists a CAT-isomorphism* $\varphi : N_1 \to N_2$ *such that the following diagram*



commutes up to homotopy.

Proposition

M is topologically rigid if and only if $\mathcal{S}_{TOP}(M)$ consists of one element.

Surgery theory

We have the following exact sequence, for dim $M \geq 5$:

$$L_{m+1}(\pi_1(M)) \to \mathcal{S}_{CAT}(M) \to \mathcal{N}(M) \to L_m(\pi_1(M)).$$

where $\mathcal{N}(M)$ denotes the set of equivalence classes of degree one normal maps under some good (but technical!) equivalence relation.

Understanding the maps:

1. the obstruction map $\mathcal{N}(M) \to L_m(\pi_1(M))$,
2. the forgetful map $\mathcal{S}_{CAT}(M) \to \mathcal{N}(M)$,
3. the action of the L-group on the structure set $L_{m+1}(\pi_1(M)) \to \mathcal{S}_{CAT}(M)$.

Advantage: the objects involved in the surgery exact sequence have good algebraic properties.

1. the L-groups have been heavily studied and are know for a large class of groups (and completely known for $\mathbb{Z}^n$).
2. the normal set $\mathcal{N}(M)$ is isomorphic to $[M, G/CAT]$, where G/CAT is the homotopy fiber of a fibration $BG \rightarrow BCAT$. $\longrightarrow$ homotopy theoretic tools to compute it.

What is more difficult is to compute the maps !

the PL-structure set of the torus

We want to compute $\mathcal{S}_{PL}(\mathbb{T}^n \times \mathbb{D}^k), n + k \geq 5$. Here, we only consider the case $\mathcal{S}_{PL}(\mathbb{T}^n), n \geq 5$.

We have the following exact sequence:

$$L_{n+1}(\mathbb{Z}^n) \to \mathcal{S}_{PL}(\mathbb{T}^n) \to [\mathbb{T}^n, G/PL] \to L_n(\mathbb{Z}^n).$$

the PL-structure set of the torus

$$L_{n+1}(\mathbb{Z}^n) \to \mathcal{S}_{PL}(\mathbb{T}^n) \to [\mathbb{T}^n, G/PL] \to L_n(\mathbb{Z}^n).$$

Proposition

The surgery obstruction map $[\mathbb{T}^n, G/PL] \to L_n(\mathbb{Z}^n)$ is injective.

Corollary

The action of $L_{n+1}(\mathbb{Z}^n)$ on $\mathcal{S}_{PL}(\mathbb{T}^n)$ is transitive.

Hence it is enough to understand the action of $L_{n+1}(\mathbb{Z}^n)$ on $\mathcal{S}_{PL}(\mathbb{T}^n)$ to compute the PL-structure set of the torus.

Unfortunately, it is a bit technical !

Theorem

$$\mathcal{S}_{PL}(\mathbb{T}^n) \simeq H^3(\mathbb{T}^n, \mathbb{Z}_2).$$

key ingredients:

1. Rokhlin's theorem,
2. codimension one splitting obstructions.

the PL-structure set of the torus

Theorem (Rokhlin)

A compact 4-dimensional PL-manifold with vanishing w_1 and w_2 has a signature divisible by 16.

Remarks:
(i) For compact topological 4-dimension manifolds with vanishing w_1 and w_2, the signature is divisible by 8.
(ii) Freedman constructed the so-called E_8-manifold, a simply connected compact 4-dimensional topological manifold with signature 8. Hence, the E_8-manifold admits no smooth structure.
(iii) The $K3$ surface is a simply connected compact 4-dimensional smooth manifold with signature -16. Hence Rokhlin's theorem is optimal.

Codimension one splitting obstructions.

Definition

Let M be a PL-manifold, and $N \subset M$ be a codimension one two-sided submanifold. A map $f : M' \to M$ is said to be **splittable** *along N if there is a homotopic map $g : M' \to M$ transverse to N and such that the restriction $g_| : g^{-1}(N) \to N$ is a homotopy equivalence.*

Proposition

A homeomorphism is splittable along any submanifold.

Corollary

If a PL-manifold M is topologically rigid, a homotopy equivalence $h : N \to M$ is splittable along any codimension one two-sided submanifold.

There are obstructions to splitting a homotopy equivalence along a codimension one two-sided submanifold, living in the so-called Nil-groups $\widetilde{Nil}_0(\mathbb{Z}\pi_1(M))$.

How do these obstructions arise ? Start with a map $g : M' \longrightarrow M$ transverse to N, which yields a degree one map $g : |g^{-1}(N) \longrightarrow N$ and try to perform ambient surgery (called handle exchanges) to obtain a homotopy equivalence. Again, there will be a problem in the middle dimension...

Consider a PL-manifold M^m ($m \geq 5$) as a codimension one two-sided submanifold of $M \times \mathbb{S}^1$.

Theorem (Bass-Heller-Swan, Farrell-Hsiang)

There is the following decomposition

$$Wh(\pi_1(M) \times \mathbb{Z}) \simeq Wh(\pi_1(M)) \oplus \widetilde{K}_0(\mathbb{Z}\pi_1(M)) \oplus \widetilde{Nil}_0(\mathbb{Z}\pi_1(M))^2.$$

There are two obstructions to splitting a homotopy equivalence $f : N \to M \times \mathbb{S}^1$ *along* M:
- An obstruction living in $\widetilde{K}_0(\mathbb{Z}\pi_1(M))$,
- An obstruction $p(\tau) \in \widetilde{Nil}_0(\mathbb{Z}\pi_1(M))$, *where* τ *is the Whitehead torsion of* f, *and* $p : Wh(\pi_1(M) \times \mathbb{Z}) \to \widetilde{Nil}_0(\mathbb{Z}\pi_1(M))$ *is the projection given by the Bass-Heller-Swan formula.*

Thus, if the Whitehead group of $\pi_1(M) \times \mathbb{Z}$ vanishes, the homotopy equivalence f is splittable along M.

Theorem (Bass-Heller-Swan)

$$Wh(\mathbb{Z}^n) = 0.$$

Corollary

Any homotopy equivalence $f : M \to \mathbb{T}^n$ is splittable along any codimension one subtorus if $n \geq 6$

Idea to prove the theorem $\mathcal{S}_{PL}(\mathbb{T}^n) \simeq H^3(\mathbb{T}^n, \mathbb{Z}_2)$:

1. To any subset $J \subset \{1, \ldots, n\}$, one can define an element $\xi(J) \in L_{n+1}(\mathbb{Z}^n)$.
2. If $\xi(J)$ acts trivially on the PL-structure set, one can, using the splitting theorem, consider a homotopy equivalence between 4-manifolds.
3. Use Rokhlin's theorem to obtain a contradiction.

PL-smoothing theory

So far, we have been interested in the PL-structure set. However, we want to prove that $\mathbb{T}^n$ is topologically rigid, which is a problem formulated in the TOP-world.

$\rightarrow$ We first prove that any topological manifold homotopy equivalent to $\mathbb{T}^n$ admits a PL-structure, then use the data given by the previous computation of $\mathcal{S}_{PL}(\mathbb{T}^n)$.

PL-smoothing theory

Definition

A topological manifold M admits a **PL-structure** *if there exists a homeomorphism $f : M \to N$ with N a PL-manifold.*
A PL-manifold M admits a **smooth structure** *if there exists a PL-homeomorphism $f : M \to N$ with N a smooth manifold.*

Smoothing problem:

1. Given a topological manifold, does it admit a PL-structure ?
2. Given a PL-manifold, does it admit a smooth structure ?

tangent bundles and classifying spaces

We already know that there is a one-one correspondence between
- isomorphism classes of stable vector bundles over a smooth manifold M

and

- homotopy classes of maps $M \rightarrow BO$, with BO the classifying space of $O = lim_n O(n)$.

PL-smoothing theory

In the same way, we can define classifying spaces BPL and $BTOP$ for PL- and Top-manifolds. one-one correspondence between

- isomorphism classes of stable vector bundles over a CAT-manifold M

and

- homotopy classes of maps $M \longrightarrow BCAT$

PL-smoothing theory

There are natural maps $BO \to BPL \to BTOP$.

- Considering the composite map $M \xrightarrow{\xi} BPL \to BTOP$ is the same as forgetting the PL-structure on M and seeing ξ as a topological bundle over the topological manifold M.
- Considering the composite map $M \xrightarrow{\xi} BO \to BPL$ is the same as forgetting the DIFF-structure on M and seeing ξ as a PL-bundle over the PL-manifold M.

PL-smoothing theory

As for smooth manifolds, there is a notion of CAT- tangent bundle over a CAT manifold. This tangent bundle is classified by a map $M \xrightarrow{\tau_M} BCAT$.

- Considering the composite map $M \xrightarrow{\tau_M} BO \to BPL$ is the same as seeing M as a PL-manifold and taking its PL- tangent bundle.

- Considering the composite map $M \xrightarrow{\tau_M} BPL \to BTOP$ is the same as seeing M as a topological manifold and taking its topological tangent bundle.

naturality: For a CAT-isomorphism $f : M \to N$, the following triangle is commutative

$$\begin{array}{ccc} M & \xrightarrow{\tau_M} & BCAT \\ {\scriptstyle f}\downarrow & \nearrow_{\tau_N} & \\ N. & & \end{array}$$

PL-smoothing theory

Necessary condition for a solution to the smoothing problem:

There must exists a lift

$$
\begin{array}{ccc}
 & & BPL \\
 & \nearrow & \downarrow \\
M & \xrightarrow[\tau_M]{} & BTOP
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & BO \\
 & \nearrow & \downarrow \\
M & \xrightarrow[\tau_M]{} & BPL
\end{array}
$$

Is this a sufficient condition ?

PL-smoothing theorem

Theorem (Cairns-Hirsch)

A PL-manifold admits a smooth structure if and only if its tangent bundle $M \to BPL$ admits a lift to BO.

Theorem (Kirby-Siebenmann)

A topological manifold of dimension ≥ 5 admits a PL-structure if and only if its tangent bundle $M \xrightarrow{\tau_M} BTOP$ admits a lift to BPL.

key point: computation of $\mathcal{S}_{PL}(\mathbb{T}^n \times \mathbb{D}^k)$, $n + k \geq 5$, using surgery theory.

There is a natural (but technical) equivalence relation between PL-structures, called **concordance**.
The set of concordance classes of PL-structures on M is denoted $\mathcal{T}_{PL}(M)$.

Theorem (Kirby-Siebenmann)

If M is a topological compact manifold of dimension ≥ 5 which admits a PL-structure, then

$$\mathcal{T}_{PL}(M) \simeq [M, TOP/PL].$$

PL-smoothing theory

Since we have fibrations

$$
\begin{array}{ccc}
 & & TOP/PL \\
 & & \downarrow \\
 & \nearrow & BPL \\
 & & \downarrow \\
M & \xrightarrow[\tau_M]{} & BTOP
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & & PL/O \\
 & & \downarrow \\
 & \nearrow & BO \\
 & & \downarrow \\
M & \xrightarrow[\tau_M]{} & BPL
\end{array}
$$

we have to understand the homotopy fibers TOP/PL and PL/O

PL-smoothing theory

Theorem

PL/O is 6-connected.

Corollary

Every PL-manifold of dimension ≤ 5 is smoothable.

Proof.

obstructions $\omega_n \in H^{n+1}(M, \pi_n(PL/O))$.
In dimension ≥ 5, these obstructions vanish since dim $M \leq 5$.
In dimension ≤ 4, these obstructions vanish since PL/O is 6-connected.
Hence, by obstruction theory, a lift exists. Hence M admits a smooth structure by the Cairns-Hirsch theorem. □

PL-smoothing theory

Theorem (Kirby-Siebenmann)

$TOP/PL \simeq K(\mathbb{Z}_2, 3)$.

Corollary

By obstruction, there exists a class $\kappa \in H^4(BTOP, \mathbb{Z}_2)$ *such that* $M \to^{\tau_M} BTOP$ *admits to BPL if and only if* $\tau_M^* \kappa = 0 \in H^4(M, \mathbb{Z}_2)$.

Definition

$\kappa(M) = \tau_M^* \kappa$ *is called the* **Kirby-Siebenmann obstruction**.

PL-smoothing theory

Theorem

A topological manifold of dimension ≥ 5 admits a PL-structure if and only if its Kirby-Siebenmann obstruction vanishes.

In that case, the concordance classes of PL-structures are in bijection with

$$[M, TOP/PL] \simeq [M, K(\mathbb{Z}_2, 3)] \simeq H^3(M, \mathbb{Z}_2).$$

Topological rigidity of the torus.

Consider $f : M \to \mathbb{T}^n$ a homotopy equivalence, $n \geq 5$.

Problem: Does M admit a PL-structure ?

$\to$ obstructution in $H^4(M, \mathbb{Z}_2) \simeq H^4(\mathbb{T}^n, \mathbb{Z}_2)$.

PROBLEM !!!!! How to compute $\kappa(M)$? Of course, $\kappa(\mathbb{T}^n) = 0$,

BUT we don't have $\kappa(M) = f^*\kappa(\mathbb{T}^n)$, since κ lives in the cohomology of $BTOP$.

However, one can construct an intermediary characteristic class ω carrying enough data on $\kappa(M)$ to derive the existence of a PL-structure from its vanishing, and such that $\omega(M) = \omega(\mathbb{T}^n)$ for a manifold homotopy equivalent to $\mathbb{T}^n$.

topological rigidity

Proposition

A topological manifold homotopy equivalent to $\mathbb{T}^n$, $n \geq 5$ admits a PL-structure.

We can now use the previous data about $\mathcal{S}_{PL}(\mathbb{T}^n)$.We have

$$H^3(\mathbb{T}^n, \mathbb{Z}_2) \simeq \mathcal{T}_{PL}(\mathbb{T}^n) \rightarrow \mathcal{S}_{PL}(\mathbb{T}^n) \simeq H^3(\mathbb{T}^n, \mathbb{Z}_2).$$

Lemma

The map $\mathcal{T}_{PL}(\mathbb{T}^n) \rightarrow \mathcal{S}_{PL}(\mathbb{T}^n)$ is injective, hence bijective.

So given a homotopy equivalence $f : M \to \mathbb{T}^n$,

1. Give M a PL-structure,
2. Consider $f : M \to \mathbb{T}^n$ as an element of $\mathcal{S}_{PL}(\mathbb{T}^n)$,
3. Choose an homeomorphism $g : M' \to \mathbb{T}^n$ (with M' a PL-manifold) representing an element of $\mathcal{T}_{PL}(\mathbb{T}^n)$ mapped to the class of f.

Since f and g represent the same element in $\mathcal{S}_{PL}(\mathbb{T}^n)$, there exists a PL-homeomorphism $\varphi : M \to M'$ such that the following diagram commutes up to homotopy

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{T}^n \\ {\scriptstyle\varphi}\downarrow & \nearrow_{g} & \\ M'. & & \end{array}$$

In particular, f is homotopic to a homeomorphism, and $\mathbb{T}^n$ is topologically rigid.

What's next ?

The Borel Conjecture has been proven for a large class of groups: Bieberbach, non-positively curved, word hyperbolic ...
(Farrell-Hsiang, Farrell-Jones, Lueck-Bartels)
$\rightarrow$ uses some geometry of the groups involved (geometric group theory).

topological rigidity for non aspherical manifolds:
For example $\mathbb{S}^k \times \mathbb{S}^l$ $(k, l \geq 1,\ k + l \neq 3)$ is topologically rigid if and only if both k and l are odd (Kreck-Lueck).


That's all Folks!