NILPOTENCE = TORSION

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Nilpotent endomorphisms

Let $A$ be an associative ring with 1.

An endomorphism $\nu : P \to P$ of an $A$-module $P$ is **nilpotent** if $\nu^N = 0 : P \to P$ for some $N \geq 0$.

If $\nu$ is nilpotent then $1 + \nu : P \to P$ is an isomorphism with

$$(1 + \nu)^{-1} = 1 - \nu + \nu^2 - \cdots + (-)^{N-1}\nu^{N-1} : P \to P.$$ 

For an indeterminate $z$ let $A[z]$ be the polynomial extension, and let $A[[z]]$ be the ring of formal power series.

**Proposition 1** Let $f, g : P \to Q$ be morphisms of f.g. projective $A$-modules. The $A[z]$-module morphism $f + gz : P[z] \to Q[z]$ is an isomorphism if and only if $f : P \to Q$ is an isomorphism and $f^{-1}g : P \to P$ is nilpotent.

**Remark 1** Proposition 1 is false if $P$ is not f.g., for example if $f = 1, g = y : P = A[[y]] \to P = A[[y]]$

with $(f + gz)^{-1} = \sum_{j=0}^{\infty} (-)^j g^j z^j : P[z] \to P[z]$. 

Near-projections

Let $A[z, z^{-1}]$ be the Laurent polynomial extension of $A$.

An endomorphism $\rho : P \to P$ of an $A$-module $P$ is a **near-projection** if $\rho(1 - \rho) : P \to P$ is nilpotent.

**Example 1** If $\nu$ is nilpotent then $\nu$ is a near-projection.

**Example 2** If $\nu$ is nilpotent then $1 - \nu$ is a near-projection.

**Proposition 2** Let $f, g : P \to Q$ be morphisms of f.g. projective $A$-modules. The $A[z, z^{-1}]$-module morphism $f + gz : P[z, z^{-1}] \to Q[z, z^{-1}]$ is an isomorphism if and only if $f + g : P \to Q$ is an isomorphism and $(f + g)^{-1}g : P \to P$ is a near-projection.

**Remark 2** Proposition 2 is false if $P$ is not f.g. – same counterexample as in Remark 1.
Why is $1 - \rho + \rho z$ an isomorphism for a near-projection $\rho$?

Given a near-projection $\rho : P \to P$ let $\nu = \rho(1 - \rho) : P \to P$, so that $\nu^N = 0$ for some $N \geq 0$. Define the projection

$$\pi = (\rho^N + (1 - \rho)^N)^{-1}\rho^N$$

$$= \rho + (1/2)(2\rho - 1)((1 - 4\nu)^{-1/2} - 1)$$

$$= \rho + (2\rho - 1)(\nu + 3\nu^2 + 10\nu^3 + \ldots) : P \to P$$

The near-projection splits as

$$\rho = \rho_+ \oplus \rho_- : P = P_+ \oplus P_- \to P = P_+ \oplus P_-$$

with $P_+ = (1 - \pi)(P)$, $P_- = \pi(P)$ and the endomorphisms

$$\rho_+ = \rho| : P_+ \to P_+ , 1 - \rho_- = (1 - \rho)| : P_- \to P_-$$

nilpotent.

The endomorphism of $(P_+ \oplus P_-)[z, z^{-1}]$

$$1 - \rho + \rho z = (1 + \rho_+(z - 1)) \oplus z(1 + (1 - \rho_-)(z^{-1} - 1))$$

is an isomorphism, by a double application of Proposition 1.
**Algebraic $K$-theory**

- The **algebraic $K$-groups** of $A$ are the algebraic $K$-groups of the exact category $\text{Proj}(A)$ of f.g. projective $A$-modules

$$K_\ast(A) = K_\ast(\text{Proj}(A)) .$$

- The **nilpotent $K$-groups** of $A$ are the algebraic $K$-groups of the exact category $\text{Nil}(A)$ of f.g. projective $A$-modules $P$ with a nilpotent endomorphism $\nu : P \to P$

$$\text{Nil}_\ast(A) = K_\ast(\text{Nil}(A)) = K_\ast(A) \oplus \text{Nil}_\ast(A) .$$

- **Proposition 3** Let $\text{Near}(A)$ be the exact category of f.g. projective $A$-modules $P$ with a near-projection $\rho : P \to P$. The equivalence of exact categories

$$\text{Near}(A) \xrightarrow{\sim} \text{Nil}(A) \times \text{Nil}(A) ; (P, \rho) \mapsto (P_+, \rho_+) \times (P_-, 1 - \rho_-)$$

induces an isomorphism of algebraic $K$-groups
The Bass-Heller-Swan Theorem

**Theorem** (B-H-S 1965 for \( n \leq 1 \), Quillen 1972 for \( n \geq 2 \))

For any ring \( A \) there are natural splittings

\[
K_n(A[z]) = K_n(A) \oplus \tilde{\text{Nil}}_{n-1}(A),
\]

\[
K_n(A[z, z^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A).
\]

**Original proof** (i) Use Higman linearization to represent every \( \tau \in K_1(A[z]) \) by a linear invertible \( k \times k \) matrix \( B = B_0 + zB_1 \in \text{GL}_k(A[z]) \) with \( B_0 \in M_k(A) \) invertible and \( (B_0)^{-1}B_1 \in M_k(A) \) nilpotent.

(ii) Represent every \( \tau \in K_1(A[z, z^{-1}]) \) by \( B = B_0 + zB_1 \in \text{GL}_k(A[z, z^{-1}]) \) with \( B_0 + B_1 \in M_k(A) \) invertible and \( (B_0 + B_1)^{-1}B_1 \in M_k(A) \) a near-projection.

(iii) For \( n \in \mathbb{Z} \) apply the algebraic \( K \)-theory commutative localization exact sequence for \( A[z] \to \{z\}^{-1}A[z] = A[z, z^{-1}] \).
The Farrell-Hsiang splitting theorem

**Theorem** (1968)

A homotopy equivalence \( h : M^n \to X^{n-1} \times S^1 \) with \( M \) an \( n \)-dimensional manifold and \( X \) an \((n-1)\)-dimensional manifold has a splitting obstruction

\[
\Phi(h) \in \text{Nil}_0(\mathbb{Z}[\pi_1(X)])/\text{Nil}_0(\mathbb{Z}) = \tilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]).
\]

**\( \Phi(h) = 0 \) if (and for \( n \geq 6 \) only if) \( h \) is \( h \)-cobordant to a split homotopy equivalence \( h : M \to X \times S^1 \), with the restriction

\[
h : V^{n-1} = h^{-1}(X \times \{*\}) \to X
\]

also a homotopy equivalence.

**\( \Phi(h) \) is a component of the Whitehead torsion

\[
\tau(h) = (-)^{n-1} \tau(h)^* \in \text{Wh}(\pi_1(X) \times \mathbb{Z})
\]

\[
= \text{Wh}(\pi_1(X)) \oplus \tilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]).
\]
**Geometric transversality over \( S^1 \)**

- Given a map \( h : M \to X \times S^1 \) let \( \overline{M} = h^*(X \times \mathbb{R}) \) be the pullback infinite cyclic cover of \( M \), with \( z : \overline{M} \to \overline{M} \) a generating covering translation.

- Assuming \( M \) is an \( n \)-dimensional manifold make \( h \) transverse regular at \( X \times \{\ast\} \subset X \times S^1 \), with

\[
V^{n-1} = h^{-1}(X \times \{\ast\}) \subset M^n
\]

a 2-sided codimension 1 submanifold. Cutting \( M \) at \( V \subset M \) there is obtained a fundamental domain \((W; z^{-1}V, V)\) for \( \overline{M} \)

\[
\overline{M} = \bigcup_{k=-\infty}^{\infty} z^k(W; z^{-1}V, V).
\]

| \( \overline{M} \) | \( z^{-1}V \) | \( W \leftarrow f \) | \( V \xrightarrow{zg} zW \) | \( zV \) | \( z^2W \) | \( z^2V \) |
Algebraic transversality over $S^1$

Let $C(V), C(W)$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complexes of the pullbacks to $V, W$ of the universal cover $\tilde{X}$ of $X$.

Identify $\mathbb{Z}[\pi_1(X \times S^1)] = \mathbb{Z}[\pi_1(X)][z, z^{-1}]$ and let $C(M)$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$-module chain complex of the pullback to $M$ of the universal cover $\tilde{X} \times \mathbb{R}$ of $X \times S^1$.

The decomposition $\overline{M} = \bigcup_{k=-\infty}^{\infty} z^k W$ determines a Mayer-Vietoris presentation of $C(\overline{M})$

\[
0 \longrightarrow C(V)[z, z^{-1}] \xrightarrow{f - zg} C(W)[z, z^{-1}] \longrightarrow C(\overline{M}) \longrightarrow 0
\]

with $f, g : C(V) \to C(W)$ the left and right inclusions.

For any ring $A$ every finite f.g. free $A[z, z^{-1}]$-module chain complex $C$ has a Mayer-Vietoris presentation.
The two ends of $\overline{M}$

- *Everything has an end, except a sausage which has two!*
- The infinite cyclic cover of $M$ is a union

$$\overline{M} = \overline{M}^+ \cup_V \overline{M}^-$$

with

$$\overline{M}^+ = \bigcup_{k=1}^{\infty} z^k W , \quad \overline{M}^- = \bigcup_{k=-\infty}^{0} z^k W .$$

| $\overline{M}^-$ | $V$ | $\overline{M}^+$ |
Chain homotopy nilpotence

- An $A$-module chain complex $C$ is **finitely dominated** if it is chain equivalent to a finite f.g. projective $A$-module chain complex.
- An $A$-module chain map $\nu : C \to C$ is **chain homotopy nilpotent** if $\nu^N \simeq 0 : C \to C$ for some $N \geq 0$.
- If $h : M^n \to X \times S^1$ is a homotopy equivalence then
  
  $$C(M^+, V) \oplus C(M^-, V) \to C(V \to X)$$

  is a chain equivalence with $C(V \to X)$ a finite f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex.
- The free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(M^+, V)$ is finitely dominated.
- The $\mathbb{Z}[\pi_1(X)]$-module chain map
  
  $$\nu^+ : C(M^+, V) \to C(M^+, zW) \cong C(zM^+, zV) \cong C(M^+, V)$$

  is chain homotopy nilpotent.
\[ \partial x V \]

\[ \overline{M}^+ = zW \cup z\overline{M}^+ \]

\[ \partial x zW \]

\[ \partial x \]

\[ zV \]

\[ x = y + z\nu^+(x) \]

\[ \partial \nu^+(x) \]

\[ \nu^+(x) \]
The F-H splitting obstruction from the chain complex point of view

- For a homotopy equivalence \( h : M^n \to X \times S^1 \) the contractible finite based f.g. free \( \mathbb{Z}[\pi_1(X)][z, z^{-1}] \)-module chain complex \( C(h : \overline{M} \to X \times \mathbb{R}) \) fits into a short exact sequence

\[
0 \to C(V, X)[z, z^{-1}] \xrightarrow{f - zg} C(W, X \times I)[z, z^{-1}] \to C(h) \to 0
\]

- The splitting obstruction of \( h \) is the nilpotent class

\[
\Phi(h) = (C(\overline{M}^+, V), \nu^+) \in \text{Nil}_0(\mathbb{Z}[\pi_1(X)]) / \text{Nil}_0(\mathbb{Z})
\]

where

\[
C(\overline{M}^+, V) = \text{coker}(f - zg : zC(V, X)[z] \to C(W, X \times I)[z]).
\]

- \( \Phi(h) = 0 \) if and only if \( (C(\overline{M}^+, V), \nu^+) \) is equivalent to 0 by a finite sequence of algebraic handle exchanges.

- For \( n \geq 6 \) can realize algebraic handle exchanges by geometric handle exchanges.
Universal localization

- (P.M.Cohn, 1971) Given a ring $R$ and a set $\Sigma$ of morphisms $\sigma : P \to Q$ of f.g. projective $R$-modules there exists a universal localization $\Sigma^{-1}R$, a ring with a morphism $R \to \Sigma^{-1}R$ universally inverting each $\sigma$.

- **Universal property** For any ring morphism $R \to S$ such that $1 \otimes \sigma : S \otimes_R P \to S \otimes_R Q$ is an $S$-module isomorphism for each $\sigma \in \Sigma$ there is a unique factorization $R \to \Sigma^{-1}R \to S$.

- **Warning 1** $R \to \Sigma^{-1}R$ need not be injective.

- **Warning 2** $\Sigma^{-1}R$ could be 0.

- **Gerasimov-Malcolmson normal form** An element $q\sigma^{-1}p \in \Sigma^{-1}R$ is an equivalence class of triples

\[
((\sigma : P \to Q) \in \Sigma, p \in P, q \in Q^* = \text{Hom}_R(Q, R))
\]
The algebraic $K$-theory localization exact sequence

- Assume $R \to \Sigma^{-1}R$ is injective.
- An $(R, \Sigma)$-torsion module is an $R$-module $T$ such that

$$0 \to P_1 \xrightarrow{d} P_0 \to T \to 0$$

with $P_0, P_1$ f.g. projective $R$-modules and $1 \otimes d : \Sigma^{-1}P_1 \to \Sigma^{-1}P_0$ a $\Sigma^{-1}R$-module isomorphism.

- **Theorem** (Neeman+R., 2004) For an injective universal localization $R \to \Sigma^{-1}R$ such that
  $$\text{Tor}_\ast^R(\Sigma^{-1}R, \Sigma^{-1}R) = 0$$ (stable flatness)

there is a long exact sequence of algebraic $K$-groups

$$\cdots \to K_n(R) \to K_n(\Sigma^{-1}R) \to K_{n-1}(H(R, \Sigma)) \to K_{n-1}(R) \to \cdots$$

with $H(R, \Sigma)$ the exact category of $(R, \Sigma)$-torsion modules.
Triangular matrix rings

- Given rings $R_1, R_2$ and an $(R_2, R_1)$-bimodule $Q$ define the triangular matrix ring

\[
R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix}.
\]

- **Proposition 4** (i) The category of $R$-modules is equivalent to the category of triples

\[
M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2)
\]

with $M_i$ $R_i$-modules ($i = 1, 2$), $\mu$ an $R_2$-module morphism.

- (ii) An $R$-module $M$ is f.g. projective if and only if $M_1$ is a f.g. projective $R_1$-module, $\mu$ is injective, and $\text{coker}(\mu)$ is a f.g. projective $R_2$-module.

- (iii) $K_*(R) = K_*(R_1) \oplus K_*(R_2)$. 
Full matrix rings

Let $R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix}$, $P_1 = \begin{pmatrix} R_1 \\ Q \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix}$.

The $R$-modules $P_1$, $P_2$ are f.g. projective, since $P_1 \oplus P_2 = R$.

If $R \to S$ is a ring morphism with $S \otimes_R P_1 \cong S \otimes_R P_2$ then

$$S = M_2(T)$$

with $T = \text{End}_S(S \otimes_R P_1) = \text{End}_S(S \otimes_R P_2)$.

Morita equivalence

$$\{S\text{-modules}\} \xrightarrow{\cong} \{T\text{-modules}\}; \quad N \mapsto (T \ T) \otimes_S N.$$

The induced functor

$$\{R\text{-modules}\} \to \{S\text{-modules}\} \xrightarrow{\cong} \{T\text{-modules}\};$$

$M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2) \mapsto$

$$(T \ T) \otimes_R M = \text{coker}(T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2)$$

is an assembly map, i.e. local-to-global.
\((R, \Sigma)\)-torsion modules

**Proposition 5** The universal localization of

\[
R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix} = P_1 \oplus P_2
\]

inverting a set \(\Sigma\) of \(R\)-module morphisms \(\sigma : P_2 \to P_1\) is \(\Sigma^{-1}R = M_2(T)\) with \(T = \text{End}_{\Sigma^{-1}R}(\Sigma^{-1}P_1)\).

**Proposition 6** Assume that \(R \to \Sigma^{-1}R = M_2(T)\) is injective, and that \(Q\) is a flat right \(R_1\)-module.

An \(R\)-module \(M = (M_1, M_2, \mu)\) is \((R, \Sigma)\)-torsion if and only if

1. \(\cdots \to 0 \to Q \otimes_{R_1} M_1 \xrightarrow{\mu} M_2 \to \cdots\) is homology equivalent to a 1-dimensional f.g.projective \(R_1\)-module chain complex,
2. \(M_2\) is an h.d. 1 \(R_2\)-module,
3. the assembly

\[
T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2
\]

is a \(T\)-module isomorphism.
Polynomial extensions as universal localizations

For any ring $A$ let

$$R = \begin{pmatrix} A & 0 \\ A & A \end{pmatrix}, \quad P_1 = \begin{pmatrix} A \\ A & A \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

and let $\sigma_+, \sigma_- : P_2 \to P_1$ be the two inclusions.

Proposition 7 (Schofield, 1985)

(i) The universal localization of $R$ inverting $\Sigma_+ = \{\sigma_+\}$ is

$$\Sigma_+^{-1}R = M_2(A[z]) .$$

(ii) The universal localization of $R$ inverting $\Sigma = \{\sigma_+, \sigma_-\}$ is

$$\Sigma^{-1}R = M_2(A[z, z^{-1}]) .$$
Torsion = nilpotence

Let $R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix}$. An $R$-module $M = (P, Q, f, g)$ is defined by $A$-modules $P, Q$ and $A$-module morphisms $f, g : P \to Q$.

**Proposition 8** (i) The assembly of $M = (P, Q, f, g)$ with respect to $\Sigma^{-1}R = M_2(A[z])$ is the $A[z]$-module

$$(A[z] \oplus A[z]) \otimes_R M = \text{coker}(f + g z : P[z] \to Q[z]).$$

$M$ is an $(R, \Sigma_+)$-module if and only if $P, Q$ are f.g. projective $A$-modules and $f + g z$ is an $A[z]$-module isomorphism. Thus

$$\text{Nil}(A) \to H(A[z], \Sigma_+) ; (P, \nu) \mapsto (P, P, 1, \nu)$$

is an equivalence of exact categories, by Proposition 1.

(ii) Likewise for $\Sigma^{-1}A[z] = M_2(A[z, z^{-1}])$, with

$$\text{Near}(A) \to H(A[z], \Sigma) ; (P, \rho) \mapsto (P, P, \rho, 1 - \rho)$$

an equivalence of exact categories by Proposition 2.
Universal localization proof of B-H-S theorem

Apply the universal localization exact sequence

\[ \cdots \rightarrow K_n(R) \rightarrow K_n(\Sigma^{-1}R) \rightarrow K_{n-1}(H(R, \Sigma)) \rightarrow K_{n-1}(R) \rightarrow \cdots \]

to the stably flat universal localizations of \( R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix} \)

\[ \Sigma_+^{-1}R = M_2(A[z]) \), \( \Sigma^{-1}R = M_2(A[z, z^{-1}]) \). \]

Identify

\[
K_\ast(R) = K_\ast(A) \oplus K_\ast(A), \\
K_\ast(\Sigma_+^{-1}R) = K_\ast(A[z]) , \ H(R, \Sigma_+) = \text{Nil}(A), \\
K_\ast(\Sigma^{-1}R) = K_\ast(A[z, z^{-1}]), \\
H(R, \Sigma) = \text{Near}(A) = \text{Nil}(A) \times \text{Nil}(A)
\]

to recover

\[
K_n(A[z]) = K_n(A) \oplus \tilde{\text{Nil}}_{n-1}(A), \\
K_n(A[z, z^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A).
\]
Generalized free products

- A group $\pi$ is a **generalized free product** if it is
  - either amalgamated free product $\pi = \pi_1 \ast_{\rho} \pi_2$,
  - or an HNN extension $\pi = \pi_1 \ast_{\rho} \{t\}$.

- (Bass-Serre, 1970) A group $\pi$ is a generalized free product if and only if $\pi$ acts on a tree $T$ with $T/\pi = [0, 1]$ or $S^1$.


- Nilpotence = torsion also in the generalized free product case.

- Also in algebraic $L$-theory, with the Cappell (1974) UNil-groups.
“There -- now I’ve taught you everything I know about codimension 1 splitting”