THE GEOMETRIC HOPF INVARIANT IN SURGERY THEORY

Andrew Ranicki (Edinburgh)

Sussex, 15th June, 2006

Surgery

- A surgery on an m-dimensional manifold M^m uses an embedding $f : S^n \times D^{m-n} \hookrightarrow M$ to construct a new m-dimensional manifold

$$M' = (M - f(S^n \times D^{m-n})) \cup D^{n+1} \times S^{m-n-1} .$$

  The surgery kills the homology class $f_*[S^n] \in H_n(M)$.
- The trace of the surgery is the $(m+1)$-dimensional cobordism $(W; M, M')$ with

$$W^{m+1} = M \times [0,1] \cup D^{n+1} \times D^{m-n}$$

- In fact, every cobordism is a union of such elementary cobordisms.
- Note that surgery is very invasive, changing the homotopy type (e.g. connectivity).

Surgery theory

- The theory (1963-) addresses the basic questions:
  - When is a space homotopy equivalent to an m-dimensional manifold?
  - When is a homotopy equivalence of m-dimensional manifolds homotopic to a homeomorphism?
- The theory is most effective for $m \geqslant 5$, when these questions can be answered using a combination of bundle theory and quadratic forms.
- In general, the answers are no, but in an interesting way.
- The low dimensions $m = 3, 4$ are special, with a rich inner life below the level of the high dimensional theory.
- Surgery theory motivates a new homotopy theoretic understanding of double points of maps of manifolds, based on a geometric version of the classical Hopf invariant (1938) in the homotopy groups of spheres.

The intersection pairing λ

- Let M be an oriented m-dimensional manifold. The universal cover $\widetilde{M}$ has group of covering translations is the fundamental group $\pi = \pi_1(M)$. The diagonal map

$$\Delta_M \ : \ M = \widetilde{M}/\pi \to \widetilde{M} \times_\pi \widetilde{M} \ ; \ a \mapsto (\widetilde{a}, \widetilde{a})$$

  induces the <u>intersection pairing</u> in the homology $\mathbb{Z}[\pi]$-modules

$$\lambda \ : \ H_n(\widetilde{M}) \times H_{m-n}(\widetilde{M}) \to \mathbb{Z}[\pi]$$

  which is bilinear, and such that

$$\lambda(x, y) \ = \ (-1)^{n(m-n)}\overline{\lambda(y, x)} \in \mathbb{Z}[\pi] \ \ (\overline{g} = g^{-1} \in \pi).$$

- Submanifolds $K^n, L^{m-n} \subset \widetilde{M}$ represent homology classes $[K] \in H_n(\widetilde{M})$, $[L] \in H_{m-n}(\widetilde{M})$. If K, L intersect in a finite number of points (the generic case) can define

$$\lambda([K], [L]) \ = \ |K \cap L| \in \mathbb{Z}[\pi] \ .$$

$\lambda(f,f)=0$ is necessary but not sufficient for surgery

- For $n \geqslant 2$ the image of the Hurewicz map

$$h \ : \ \pi_n(M) = \pi_n(\widetilde{M}) \to H_n(\widetilde{M})$$

  is the $\mathbb{Z}[\pi_1(M)]$-submodule of the homology classes represented by maps $f : S^n \to M$.
- A homology class $x \in \mathrm{im}(h) \subseteq H_n(\widetilde{M})$ can be killed by surgery if and only if x is represented by an embedding $f : S^n \hookrightarrow M$ with an extension $b : S^n \times D^{m-n} \hookrightarrow M$.
- (Whitney, 1940s) For $2n < m$ every map of manifolds $f : N^n \to M^m$ is homotopic to an embedding.
- For $m = 2n$ it is necessary but not sufficient for x to be represented by a map $f : S^n \to M$ such that

$$\lambda(f,f) \ = \ \lambda(x,x) \ = \ 0 \in \mathbb{Z}[\pi_1(M)] \ .$$

Double points

- For any map $f : N \to M$ there is defined a $\mathbb{Z}_2$-equivariant map
$$f \times f \ : \ N \times N \to M \times M \ ; \ (x, y) \mapsto (f(x), f(y))$$
with the generator $T \in \mathbb{Z}_2$ acting by $T(x, y) = (y, x)$.
- The ordered double point set of f is the free $\mathbb{Z}_2$-set
$$\begin{aligned} \overline{D}_2(f) &= (f \times f)^{-1}(\Delta_M) - \Delta_N \\ &= \{(x, y) \in N \times N \,|\, x \neq y \in N,\ f(x) = f(y) \in M\} \ . \end{aligned}$$
The $\mathbb{Z}_2$-set $S = (f \times f)^{-1}(\Delta_M) \subseteq N \times N$ is the union
$S = S^{\mathbb{Z}_2} \cup (S - S^{\mathbb{Z}_2}) = \mathbb{Z}_2\text{-fixed points} \cup \text{free } \mathbb{Z}_2\text{-set} = \Delta_N \cup \overline{D}_2(f)$.
- $D_2(f) = \overline{D}_2(f)/\{(x, y) \sim (y, x)\}$ is the unordered double point set. The projection
$$\overline{D}_2(f) \to D_2(f) \ ; \ (x, y) \mapsto [x, y]$$
is a double cover.
- $f : N \to M$ is an embedding if and only if $D_2(f) = \emptyset$.

Immersions

- An immersion of manifolds $f : N^n \looparrowright M^m$ is a local embedding.
- A regular homotopy between immersions $f_0, f_1 : N \to M$ is a homotopy

$$f \ : \ N \times [0,1] \to M \ ; \ (x,t) \mapsto f_t(x)$$

  such that each $f_t : N \to M$ is an immersion.
- For $2n \leqslant m$ every map $f : N^n \to M^m$ is homotopic to an immersion.
- For $2n < m$ every immersion $f : N^n \looparrowright M^m$ is regular homotopic to an embedding.
- The double point set $D_2(f)$ of an immersion $f : N^n \looparrowright M^m$ is generically a $(2n-m)$-dimensional manifold, e.g. if there are no triple points. Thus if $m = 2n$ $D_2(f)$ is 0-dimensional.
- Surgery theory needs to know: when is an immersion $f : S^n \looparrowright M^{2n}$ regular homotopic to an embedding?

$\mu(f) = 0$ is necessary and sufficient for surgery

- Given a group π and $\epsilon = +1$ or -1 let

$$Q_\epsilon(\mathbb{Z}[\pi]) \;=\; \mathbb{Z}[\pi]/\{g - \epsilon g^{-1} \mid g \in \pi\} \;.$$

- Theorem (Wall, 1966) The unordered double point set of a generic immersion $f : N^n \looparrowright M^{2n}$ determines an element

$$\mu(f) \;=\; \sum_{[x,y] \in D_2(f)} g_{(x,y)} \in Q_{(-)^n}(\mathbb{Z}[\pi_1(M)]) \;.$$

with $g_{(x,y)} \in \pm\pi_1(M)$ an index associated to each ordered double point $(x, y) \in \overline{D}_2(f)$, such that

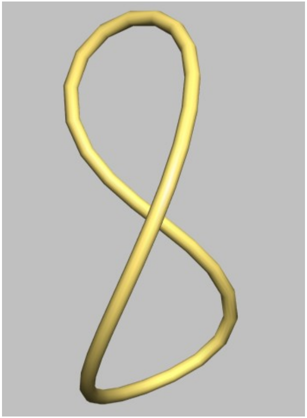
$$g_{(y,x)} \;=\; (-)^n g_{(x,y)}^{-1} \in \pm\pi_1(M) \;.$$

For $n \geqslant 3$ $\mu(f) = 0$ if and only if f is regular homotopic to an embedding. For $f : N = S^n \looparrowright M^{2n}$ with an extension $b : S^n \times D^n \looparrowright M^{2n}$ the condition $\mu(f) = 0$ is necessary and sufficient to kill $\widetilde{f}_*[S^n] \in H_n(\widetilde{M})$ by surgery.

The figure 8 immersion

- Example The figure 8 immersion $f : S^1 \looparrowright S^2$ has a single unordered double point, with

$$\mu(f) = 1 \in Q_-(\mathbb{Z}) = \mathbb{Z}_2 \ .$$



Capturing double points by homotopy theory

- Problem Compute $\mu(f)$ using homotopy theory!
- Programme (Crabb-R.) Capture the double points of immersions using $\mathbb{Z}_2$-equivariant stable homotopy theory, via the 'geometric Hopf invariant'.
  - A framed immersion $(f,b) : N^n \looparrowright M^m$ has a k-stable Umkehr map $F : \Sigma^k M^+ \to \Sigma^{k+m-n} N^+$ for some $k \geqslant 0$, with $\Sigma^k M^+$ the k-fold suspension of $M^+ = M \cup \{\infty\}$.
  - If (f,b) is regular homotopic to a framed embedding $(f_0,b_0) : N \hookrightarrow M$ then F is homotopic to the k-fold suspension $\Sigma^k F_0$ of an unstable map $F_0 : M^+ \to \Sigma^{m-n} N^+$.
  - The double points of (f,b) correspond to the generalized Hopf invariants of the Umkehr map F.
- These are hardly new ideas. But new constructions are needed to deal with the non-simply-connected case $\pi_1(M) \neq \{1\}$.

Stable homotopy theory

- The smash product of spaces X, Y with base points $x_0 \in X$, $y_0 \in X$ is the quotient space

$$X \wedge Y = (X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y) .$$

- The suspension of a pointed space X is $\Sigma X = X \wedge S^1$. For example, $S^k = \Sigma^k(S^0)$ $(k \geqslant 1)$.
- For $k \geqslant 0$ a k-stable map $F : X \nrightarrow Y$ of pointed spaces X, Y is a pointed map of k-fold suspensions $F : \Sigma^k X \to \Sigma^k Y$.
- Given pointed spaces X, Y let $[X, Y]$ be the set of homotopy classes of maps $X \to Y$. For example, $\pi_n(X) = [S^n, X]$.
- The stable homotopy group is

$$\{X; Y\} = \varinjlim_k [\Sigma^k X, \Sigma^k Y]$$

- The failure of the stabilization map $[X, Y] \to \{X; Y\}$ to be a bijection is detected by generalized Hopf invariants. E.g. $[S^2, S^1] = 0$, $[S^3, S^2] = \mathbb{Z}$, $\{S^2; S^1\} = \mathbb{Z}_2$.

Framed embeddings

- The Umkehr (or Pontrjagin-Thom) map of a codimension 0 embedding $X^n \hookrightarrow Y^n$ is the projection

$$F \ : \ Y/\partial Y \to Y/(Y - X) \ = \ X/\partial X \ .$$

- A framed embedding $(f, b) : N^n \hookrightarrow M^m$ is an embedding $f : N \hookrightarrow M$ together with an extension of f to a codimension 0 embedding $b : N \times D^{m-n} \hookrightarrow M$, with an Umkehr map

$$F \ : \ M^+ \to N \times D^{m-n}/N \times S^{m-n-1} \ = \ \Sigma^{m-n} N^+ \ .$$

- Pontrjagin-Thom isomorphism (1952) For any m-dimensional manifold M^m there is a 1-1 correspondence between the set of cobordism classes of framed embeddings $(f, b) : N^n \hookrightarrow M^m$ and $[M^+, S^{m-n}]$. The homotopy class of a generic map $g : M^+ \to S^{m-n}$ corresponds to the cobordism class of the framed embedding $(f, b) = g| : N^n = g^{-1}(*) \hookrightarrow M$ with

$$g \ = \ \mathrm{proj} \circ F \ : \ M^+ \to \Sigma^{m-n} N^+ \to S^{m-n} \ .$$

Framed immersions

- A framed immersion $(f, b) : N^n \looparrowright M^m$ is an immersion $f : N \looparrowright M$ together with an extension of f to a codimension 0 immersion $b : N \times D^{m-n} \looparrowright M$.
- General position For any framed immersion $(f, b) : N^n \looparrowright M^m$ and $k \geqslant 2n - m + 1$ the product framed immersion

$$(f, b) \times 0 \ : \ N \looparrowright M \times D^k \ ; \ x \mapsto (f(x), 0)$$

  is regular homotopic to a framed embedding $(f', b') : N \hookrightarrow M \times D^k$ with Umkehr k-stable map

$$F \ : \ M \times D^k/\partial \ = \ \Sigma^k M^+ \to N \times D^{k+m-n}/\partial \ = \ \Sigma^k(\Sigma^{m-n} N^+) \, .$$

  The stable homotopy class $F \in \{M^+; \Sigma^{m-n} N^+\}$ is a regular homotopy invariant of (f, b).
- If (f, b) is regular homotopic to a framed embedding $(f_0, b_0) : N \hookrightarrow M$ then F is homotopic to the k-fold suspension $\Sigma^k F_0$ of the Umkehr map $F_0 : M^+ \to \Sigma^{m-n} N^+$.

The stable $\mathbb{Z}_2$-equivariant homotopy groups

- Given pointed $\mathbb{Z}_2$-spaces X, Y let $[X, Y]_{\mathbb{Z}_2}$ be the set of $\mathbb{Z}_2$-equivariant homotopy classes of $\mathbb{Z}_2$-equivariant maps $X \to Y$.
- The stable $\mathbb{Z}_2$-equivariant homotopy group is

$$\{X; Y\}_{\mathbb{Z}_2} \;=\; \varinjlim_{U} \varinjlim_{V} [U^+ \wedge LV^+ \wedge X, U^+ \wedge LV^+ \wedge Y]_{\mathbb{Z}_2}$$

  with U, V running over finite-dimensional real vector spaces, and $LV = V$ with the $\mathbb{Z}_2$-action $T : LV \to LV; v \mapsto -v$.
- Example The $\mathbb{Z}_2$-equivariant Pontrjagin-Thom isomorphism identifies $\{S^0; S^0\}_{\mathbb{Z}_2}$ with the cobordism group of 0-dimensional framed $\mathbb{Z}_2$-manifolds ($=$ finite $\mathbb{Z}_2$-sets). The decomposition of finite $\mathbb{Z}_2$-sets as fixed $\cup$ free determines an isomorphism

$$\{S^0; S^0\}_{\mathbb{Z}_2} \;\cong\; \mathbb{Z} \oplus \mathbb{Z}\,;\; S = S^{\mathbb{Z}_2} \cup (S - S^{\mathbb{Z}_2}) \mapsto \left(|S^{\mathbb{Z}_2}|, \frac{|S| - |S^{\mathbb{Z}_2}|}{2}\right).$$

The quadratic construction in homotopy theory

- The quadratic construction on a pointed space X is defined for any inner product space V to be the pointed space

$$Q_V(X) \;=\; S(LV)^+ \wedge_{\mathbb{Z}_2} (X \wedge X)$$

with $S(LV) = \{v \in V \mid \|v\| = 1\}$ the unit sphere in V, and

$$\begin{aligned} T &: S(LV) \to S(LV) \; ; \; v \mapsto -v \; , \\ T &: X \wedge X \to X \wedge X \; ; \; (x,y) \mapsto (y,x) \; . \end{aligned}$$

The projection $\widetilde{Q}_V(X) = S(LV)^+ \wedge (X \wedge X) \to Q_V(X)$ is a double cover away from the base point. For $1 \leqslant k \leqslant \infty$ write

$$Q_k(X) \;=\; Q_{\mathbb{R}^k}(X) \, , \; \widetilde{Q}_k(X) \;=\; \widetilde{Q}_{\mathbb{R}^k}(X) \, , \; Q_\infty(X) = \varinjlim_k Q_k(X) \, .$$

- Example $Q_0(X) = \{\text{pt.}\}$, $Q_1(X) \;=\; X \wedge X$.
- Example $Q_k(Y^+) = (S^{k-1} \times_{\mathbb{Z}_2} (Y \times Y))^+$.

$\mathbb{Z}_2$-equivariant stable homotopy theory = fixed-point + fixed-point-free

- Theorem (Crabb, 1980) For any pointed spaces X, Y there is a split exact sequence of abelian groups

$$0 \to \{X; Q_\infty(Y)\} \longrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2} \xrightarrow{\rho} \{X; Y\} \to 0$$

with $T \in \mathbb{Z}_2$ acting by the identity on X and

$$\{X; Q_\infty(Y)\} = \varinjlim_V [\Sigma S(LV)^+ \wedge X, LV^+ \wedge Y \wedge Y]_{\mathbb{Z}_2} \text{ (S-duality)}.$$

- ρ is given by the $\mathbb{Z}_2$-fixed points, and is split by

$$\sigma : \{X; Y\} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} ; F \mapsto \Delta_Y F .$$

- The injection is induced by projection $(\varinjlim_k S^k)^+ \to 0^+$

$$\{X; Q_\infty(Y)\} = \{X; \widetilde{Q}_\infty(Y)\}_{\mathbb{Z}_2} \to \{X; Y \wedge Y\}_{\mathbb{Z}_2} .$$

The geometric Hopf invariant $h(F)$

- The geometric Hopf invariant of a k-stable map $F : \Sigma^k X \to \Sigma^k Y$ is
$$h(F) = (F \wedge F)\Delta_X - \Delta_Y F \in \ker(\rho : \{X; Y \wedge Y\}_{\mathbb{Z}_2} \to \{X; Y\})$$
$$= \operatorname{im}(\{X; Q_\infty(Y)\} \hookrightarrow \{X; Y \wedge Y\}_{\mathbb{Z}_2}) .$$
- Proposition (i) The function
$$h : \{X; Y\} \to \{X; Q_\infty(Y)\} ; \ F \mapsto h(F)$$
is nonadditive, being quadratic in nature:
$$h(F + G) = h(F) + h(G) + (F \wedge G)\Delta_X$$
(ii) If $F \in \operatorname{im}([X, Y] \to \{X; Y\})$ then $h(F) = 0$.
The stable $\mathbb{Z}_2$-equivariant homotopy class of $h(F)$ is the primary obstruction to the k-fold desuspension of F.
- Example Let $X = S^1$, $Y = S^0$. The geometric Hopf invariant of a k-stable map $F : \Sigma^k X = S^{k+1} \to \Sigma^k Y = S^k$ is
$$h(F) = \text{mod 2 Hopf invariant}(F) \in \{X; Q_\infty(Y)\} = \mathbb{Z}_2 .$$

The $\mathbb{Z}_2$-equivariant Umkehr map

- Let $(f,b) : N^n \looparrowright M^m$ be a framed immersion, and let

$$(f',b') \;:\; N \hookrightarrow M' \;=\; M \times D^k$$

  be a framed embedding regular homotopic to $(f,b)\times 0$, with Umkehr map $F : M'/\partial M' = \Sigma^k M^+ \to \Sigma^{k+m-n}N^+$.
- The $\mathbb{Z}_2$-equivariant product framed embedding

$$(f' \times f', b' \times b') \;:\; N \times N \hookrightarrow M' \times M'$$

  restricts to a $\mathbb{Z}_2$-equivariant framed embedding

$$(f' \times f', b' \times b')| \;:\; \overline{D}_2(f)^{2n-m} \hookrightarrow M \times D^k \times D^k$$

  with $\mathbb{Z}_2$-equivariant Umkehr map

$$G \;:\; M^+ \wedge (D^k \times D^k/\partial) \to \overline{D}_2(f)^+ \wedge (D^{k+m-n} \times D^{k+m-n})/\partial\,.$$

Double points via the geometric Hopf invariant

- Double Point Theorem (Crabb+R.) If $(f,b) : N^n \looparrowright M^m$ is a framed immersion the geometric Hopf invariant of the Umkehr map $F : \Sigma^k M^+ \to \Sigma^{k+m-n} N^+$ (k large) factors through the ordered double point set $\overline{D}_2(f)$

$$\begin{aligned} h(F) &= (1 \wedge i)G \\ &\in \ker\big(\rho : \{M^+; \Sigma^{m-n}N^+ \wedge \Sigma^{m-n}N^+\}_{\mathbb{Z}_2} \to \{M^+; \Sigma^{m-n}N^+\}\big) \\ &= \operatorname{im}\big(\{M^+; Q_\infty(\Sigma^{m-n}N^+)\} \hookrightarrow \{M^+; \Sigma^{m-n}N^+ \wedge \Sigma^{m-n}N^+\}_{\mathbb{Z}_2}\big) \end{aligned}$$

with $i : \overline{D}_2(f) \to N \times N$ the inclusion, and

$$M^+ \xrightarrow{G} \overline{D}_2(f)^+ \wedge (D^{m-n} \times D^{m-n})/\partial \xrightarrow{i \wedge 1} \Sigma^{m-n}N^+ \wedge \Sigma^{m-n}N^+$$

- The identity $(F \wedge F)\Delta_{M^+} = \Delta_{\Sigma^{m-n}N^+} + h(F)$ in $\{M^+; \Sigma^{m-n}N^+ \wedge \Sigma^{m-n}N^+\}_{\mathbb{Z}_2}$ is the stable $\mathbb{Z}_2$-equivariant homotopy version of the identity of $\mathbb{Z}_2$-sets

$$(f \times f)^{-1}(\Delta_M) = \Delta_N \cup \overline{D}_2(f) \subset N \times N\ .$$

An example

- Example Let $X = Y = S^0$. For any integer $d \geqslant 1$ define a framed immersion of 0-dimensional manifolds

$$(f, b) \ : \ N \ = \ \{1, 2, \dots, d\} \looparrowright M \ = \ \{1\}$$

with double point set

$$\overline{D}_2(f) \ = \ \{(i,j) \,|\, 1 \leqslant i, j \leqslant d,\ i \neq j\}$$

and Umkehr map

$$F \ = \ 1 \vee 1 \vee \cdots \vee 1 \ :$$
$$\Sigma X \ = \ \Sigma M^+ \ = \ S^1 \to \Sigma N^+ \ = \ S^1 \vee S^1 \vee \cdots \vee S^1 \ .$$

Let $p : \Sigma N^+ \to \Sigma Y = S^1$ be the projection. The geometric Hopf invariant of the degree d 1-stable map

$$pF \ : \ \Sigma X \ = \ S^1 \to \Sigma Y \ = \ S^1$$

is

$$h(pF) \ = \ |D_2(f)| \ = \ d(d-1)/2 \in \{X; Q_\infty(Y)\} \ = \ \mathbb{Z} \ .$$

The Wall self-intersection μ is a generalized Hopf invariant

- Let M be an m-dimensional manifold with universal cover $\widetilde{M}$. A framed immersion $(f,b) : N^n \looparrowright M^m$ lifts to a $\pi_1(M)$-equivariant framed immersion $(\widetilde{f},\widetilde{b}) : \widetilde{N} = f^*\widetilde{M} \looparrowright \widetilde{M}$ with a $\pi_1(M)$-equivariant Umkehr map $\widetilde{F} : \Sigma^k \widetilde{M}^+ \to \Sigma^{k+m-n}\widetilde{N}^+$.
- The geometric Hopf invariant $h(\widetilde{F}) : \widetilde{M}^+ \nrightarrow Q_\infty(\Sigma^{m-n}\widetilde{N}^+)$ is $\pi_1(M)$-equivariant. The non-simply-connected version of the Double Point Theorem factors $h(\widetilde{F})$ through the double point set $\overline{D}_2(\widetilde{f})$.
- In particular, for $m = 2n$ the factorization expresses the Wall quadratic self-intersection in terms of $h(\widetilde{F})$

$$\begin{aligned}\mu(f) \;&=\; h(\widetilde{F})/\pi_1(M) \\ &\in \{M^+; Q_\infty(\Sigma^n\widetilde{N}^+)/\pi_1(M)\}_{\mathbb{Z}_2} \;=\; Q_{(-)^n}(\mathbb{Z}[\pi_1(M)])\ .\end{aligned}$$

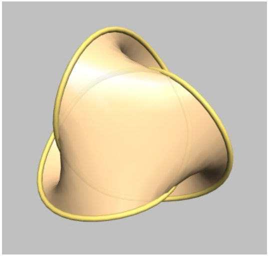
Surfaces

- Every connected surface M ($=$ 2-dimensional manifold) is of the type
$$M^2 \;=\; \text{closure}(\mathop{\#}_{g} S^1 \times S^1 - \bigcup_{h} D^2)$$
for some <u>genus</u> $g \geqslant 0$, with $\partial M = \bigcup_{h} S^1$ for some $h \geqslant 0$.
The homology $\mathbb{Z}$-module $H_1(M) = \mathbb{Z}^{2g}$ has basis $\{a_1, a_2, \dots, a_{2g-1}, a_{2g}\}$, with
$$\lambda \;:\; H_1(M) \times H_1(M) \to \mathbb{Z} \;;$$
$$(a_i, a_j) \mapsto \begin{cases} (-)^i \text{ if } \{i, j\} = \{2k-1, 2k\} \\ 0 \text{ otherwise} . \end{cases}$$
- If $g \geqslant 1$ every $a_i \in H_1(M)$ can be killed by surgery on M, resulting in a surface M' of genus $g-1$ and $\partial M' = \partial M$.
- Every surface M admits a framed embedding $(f, b) : M \hookrightarrow S^{n+2}$ ($n \geqslant 1$). In general, a surgery on M will not respect the framing.

Seifert surfaces

- For any knot $k : S^1 \subset S^3$ there exists a Seifert surface M^2 with boundary $\partial M = S^1$ and a framed embedding $(f, b) : M \hookrightarrow S^3$ such that $f(\partial M) = k(S^1) \subset S^3$.
- Example The trefoil knot $k : S^1 \subset S^3$ has a Seifert surface which is a punctured torus

$$M \;=\; \mathrm{cl.}(S^1 \times S^1 - D^2) \subset S^3$$



- Surgery theory for Seifert surfaces preserving the framing can be used to classify knots.

The traditional Hopf invariant

- The linking number of disjoint knots $k, k' : S^1 \subset S^3$ is

$$\mathrm{Lk}(k, k') \ = \ |M \cap k'(S^1)| \in \mathbb{Z}$$

  for any Seifert surface M for k.
- The Hopf invariant of a framed embedding $(f, b) : S^1 \subset S^3$ is the linking number

$$H(f, b) \ = \ \mathrm{Lk}(f, f_b) \in \mathbb{Z}$$

  of the disjoint knots defined by the restrictions of $b : S^1 \times D^2 \hookrightarrow S^3$

$$f = b| : S^1 \times (0,0) \hookrightarrow S^3 \ , \ f_b = b| : S^1 \times (1,0) \hookrightarrow S^3 \ .$$

- The Hopf invariant defines an isomorphism

$$H \ : \ \pi_3(S^2) = [S^3, S^2] \xrightarrow{\cong} \mathbb{Z} \ ; \ (f, b) \mapsto H(f, b)$$

The Hopf link

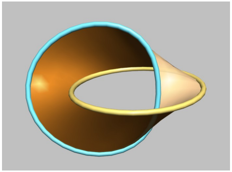
- Use the rotation homeomorphism

$$S^1 \to SO(2)\ ;\ x = (\cos t, \sin t) \mapsto R(x) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

to define a framed embedding $(f, b) : S^1 \subset S^3$ by

$$b\ :\ D^2 \times S^1 \to S^3\ =\ D^2 \times S^1 \cup S^1 \times D^2\ ;\ (y, x) \mapsto (R(x)(y), x)$$

with $f(S^1) = b((0,0) \times S^1),\ b((1,0) \times S^1) \subset S^3$ linked circles.



- (f, b) has Hopf invariant $H(f, b) = 1 \in \mathbb{Z}$, and Umkehr map

$$F\ :\ S^3 \to \Sigma^2(S^1)^+\ =\ S^3 \vee S^2\ ;\ z \mapsto \eta(z)$$

where $\eta : S^3 \to S^2$ is the Hopf map with $H(\eta) = h(\eta) = 1$.

The mod 2 Hopf invariant

- The mod 2 Hopf invariant of a framed embedding $(f, b) : S^1 \subset S^{n+1}$ is the homotopy class of the derivative $db : S^1 \to SO(n+1)$ (obtained using bundle theory)

$$H_2(f, b) \;=\; db \in \pi_1(SO(n+1)) \;=\; \mathbb{Z}_2 \; (n \geqslant 2) \; .$$

- Can also be detected from the Umkehr map $F : S^{n+1} \to S^n$ using the Steenrod square in the mod 2 cohomology

$$H_2(f, b) = Sq^2 \; : \; H^n(X; \mathbb{Z}_2) = \mathbb{Z}_2 \to H^{n+2}(X; \mathbb{Z}_2) = \mathbb{Z}_2$$

of the mapping cone $X = S^n \cup_F D^{n+2}$.

- The mod 2 Hopf invariant defines an isomorphism

$$H_2 \; : \; \pi_1^S = \{S^1; S^0\} = \varinjlim \pi_{n+1}(S^n) \;\cong\; \mathbb{Z}_2 \; .$$

The quadratic function of a framed surface

- Let $(f, b) : M^2 \hookrightarrow S^{n+2}$ be a framed surface, with Umkehr map $F : S^{n+2} \to \Sigma^n M^+$.
- Every element $x \in H_1(M)$ is represented by a framed immersion $(x, a) : S^1 \looparrowright M$ with Umkehr map

$$(x, a)^* \ : \ M^+ \twoheadrightarrow S^1 \times D^1/\partial \ = \ S^2 \vee S^1 \ .$$

- The quadratic function of (f, b)

$$\mu_b \ : \ H_1(M) \to \mathbb{Z}_2 \ ; \ ((x, a) : S^1 \looparrowright M) \mapsto$$
$$\text{mod 2 Hopf invariant of } \Sigma^n(x, a)^* F : S^{n+2} \to S^{n+1}$$

  is a quadratic refinement of the intersection pairing λ, with

$$\lambda(x, y) \ = \ \mu_b(x + y) - \mu_b(x) - \mu_b(y) \in \mathbb{Z}_2 \ .$$

- It is possible to kill $x \in H_1(M)$ by framed surgery on (M, b) if and only if x generates a direct summand of $H_1(M)$ and $\mu_b(x) = 0 \in \mathbb{Z}_2$.

The cobordism of framed surfaces

- Let Ω_2^{fr} be the cobordism group of framed surfaces $(f, b) : M^2 \hookrightarrow S^{n+2}$ (n large).
- Theorem (Pontrjagin, 1950) The Arf invariant function

$$\Omega_2^{fr} = \pi_2^S = \{S^2; S^0\} = \varinjlim \pi_{n+2}(S^n) \to \mathbb{Z}_2 \ ;$$
$$(M, b) \mapsto \mathrm{Arf}(M, b) = \sum_{k=1}^{g} \mu_b(a_{2k-1})\mu_b(a_{2k})$$

  is an isomorphism.
- The generator $(S^1 \times S^1, b) \in \Omega_2^{fr}$ has

$$H_1(S^1 \times S^1) \ = \ \mathbb{Z} \oplus \mathbb{Z} \ , \ a_1 \ = \ (1, 0) \ , \ a_2 \ = \ (0, 1) \ ,$$
$$\mu_b(x, y) \ = \ x + xy + y \in \mathbb{Z}_2 \ ,$$
$$\mathrm{Arf}(S^1 \times S^1, b) \ = \ \mu_b(a_1)\mu_b(a_2) \ = \ 1 \in \Omega_2^{fr} \ = \ \mathbb{Z}_2 \ .$$

  The exotic framing b of $S^1 \times S^1$ restricts to the framing of the punctured torus Seifert surface $M^2 \subset S^3$ of the trefoil knot.