THE MASLOV INDEX, THE SIGNATURE AND BAGELS

Andrew Ranicki (Edinburgh)
http://www.maths.ed.ac.uk/~aar

Göttingen, 22 December 2009

Introduction

- The original **Maslov index** appeared in the early 1960's work of the Russian mathematical physicist V.P.Maslov on the quantum mechanics of nanostructures and lasers; he has also worked on the tokamak (= magnetic field bagel with plasma filling). The Maslov index also appeared in the early 1960's work of J.B.Keller and H.M.Edwards.
- V.I.Arnold (1967) put the Maslov index on a mathematical footing, in terms of the intersections of paths in the space of **lagrangian** subspaces of a **symplectic form**.
- The **Maslov index** is the generic name for a very large number of inter-related invariants which arise in the topology of manifolds, symplectic geometry, mathematical physics, index theory, L^2-cohomology, surgery theory, knot theory, singularity theory, differential equations, group theory, representation theory, as well as the algebraic theory of quadratic forms and their automorphisms.
- **Maslov index**: 387 entries on Mathematical Reviews, 27,100 entries on Google Scholar, 45,000 entries on Google.

The 1-dimensional lagrangians

- **Definition** (i) Let $\mathbb{R}^2$ have the symplectic form

$$[\ ,\ ]\ :\ \mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}\ ;\ ((x_1,y_1),(x_2,y_2))\mapsto x_1y_2-x_2y_1\ .$$

- (ii) A subspace $L\subset\mathbb{R}^2$ is a **lagrangian** of $(\mathbb{R}^2,[\ ,\ ])$ if

$$L\ =\ L^{\perp}\ =\ \{x\in\mathbb{R}^2\,|\,[x,y]=0 \text{ for all } y\in L\}\ .$$

- **Proposition** A subspace $L\subset\mathbb{R}^2$ is a lagrangian of $(\mathbb{R}^2,[\ ,\ ])$ if and only if L is 1-dimensional,
- **Definition** (i) The **1-dimensional lagrangian Grassmannian** $\Lambda(1)$ is the space of lagrangians $L\subset(\mathbb{R}^2,[\ ,\ ])$, i.e. the Grassmannian of 1-dimensional subspaces $L\subset\mathbb{R}^2$.
- (ii) For $\theta\in\mathbb{R}$ let

$$L(\theta)\ =\ \{(r\cos\theta,r\sin\theta)\,|\,r\in\mathbb{R}\}\in\Lambda(1)$$

be the lagrangian with gradient $\tan\theta$.

The topology of $\Lambda(1)$

- **Proposition** The square function

  $$\Lambda(1) \to S^1 \ ; \ L(\theta) \mapsto e^{2i\theta}$$

  and the square root function

  $$\omega \ : \ S^1 \to \Lambda(1) \ = \ \mathbb{R}\mathbb{P}^1 \ ; \ e^{2i\theta} \mapsto L(\theta)$$

  are inverse diffeomorphisms, and

  $$\pi_1(\Lambda(1)) \ = \ \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

- **Proof** Every lagrangian L in $(\mathbb{R}^2, [\ ,\ ])$ is of the type $L(\theta)$, and

  $$L(\theta) \ = \ L(\theta') \text{ if and only if } \theta' - \theta \ = \ k\pi \text{ for some } k \in \mathbb{Z} \ .$$

  Thus there is a unique $\theta \in [0, \pi)$ such that $L = L(\theta)$. The loop $\omega : S^1 \to \Lambda(1)$ represents the generator

  $$\omega = 1 \in \pi_1(\Lambda(1)) \ = \ \mathbb{Z} \ .$$

The real Maslov index of a 1-dimensional lagrangian I.

- **Definition** The **real-valued Maslov index** of a lagrangian $L = L(\theta)$ in $(\mathbb{R}^2, [\ ,\ ])$ is

$$\tau(L(\theta)) \;=\; \begin{cases} 1 - \dfrac{2\theta}{\pi} & \text{if } 0 < \theta < \pi \\ 0 & \text{if } \theta = 0 \end{cases} \;\in \mathbb{R}\ .$$

- **Examples**

$$\tau(L(0)) \;=\; \tau(L(\pi/2)) \;=\; 0\,,\ \tau(L(\pi/4)) \;=\; 1/2\,,\ \tau(L(3\pi/4)) \;=\; -1/2\,.$$

- For $0 < \theta < \pi$

$$\begin{aligned} \tau(L(\theta)) &= 1 - 2\theta/\pi \\ &= -1 + 2(\pi - \theta)/\pi \\ &= -\tau(L(\pi - \theta)) \in \mathbb{R}\ . \end{aligned}$$

The real Maslov index of a 1-dimensional lagrangian II.

- Motivation in terms of the L^2-signature for $\mathbb{Z}$, with $0 < \theta < \pi$

$$\begin{aligned}
\tau(L(\theta)) &= \frac{1}{2\pi}\int_{\omega \in S^1} \mathrm{sgn}((1-\omega)e^{i\theta} + (1-\overline{\omega})e^{-i\theta})d\omega \\
&= \frac{1}{2\pi}\int_{\psi=0}^{2\pi} \mathrm{sgn}(\sin(\psi/2)\sin(\psi/2+\theta))d\psi \ (\omega = e^{i\psi}) \\
&= \frac{1}{2\pi}(\int_{\psi=0}^{2\pi-2\theta} d\psi - \int_{\psi=2\pi-2\theta}^{2\pi} d\psi) \\
&= \frac{1}{2\pi}(2\pi - 2\theta - 2\theta) \\
&= 1 - \frac{2\theta}{\pi} \in \mathbb{R}\ .
\end{aligned}$$

The real Maslov index

- Many other motivations!
- The real Maslov index formula

  $$\tau(L(\theta)) \ = \ 1 - \frac{2\theta}{\pi} \in \mathbb{R}$$

  has featured in many guises (e.g. as assorted η-, γ-, ρ-invariants and an L^2-signature) in the papers of Arnold (1967), Atiyah, Patodi and Singer (1975), Neumann (1978), Atiyah (1987), Cappell, Lee and Miller (1994), Bunke (1995), Nemethi (1995), Cochran, Orr and Teichner (2003), . . .
- Can be traced back to the failure of the Hirzebruch signature theorem and the Atiyah-Singer index theorem for manifolds with boundary.
- See http://www.maths.ed.ac.uk/˜aar/maslov.htm for detailed references.

The real Maslov index of a pair of 1-dimensional lagrangians

- **Definition** The **Maslov index** of a pair of lagrangians in $(\mathbb{R}^2,[\ ,\ ])$

$$(L_1,L_2)\ =\ (L(\theta_1),L(\theta_2))$$

is

$$\begin{aligned}\tau(L_1,L_2)\ &=\ \tau(L(\theta_2-\theta_1))\\ &=\ \begin{cases}1-\dfrac{2(\theta_2-\theta_1)}{\pi} & \text{if } 0\leqslant\theta_1<\theta_2<\pi \quad \in\mathbb{R}\\[2ex] -1+\dfrac{2(\theta_1-\theta_2)}{\pi} & \text{if } 0\leqslant\theta_2<\theta_1<\pi\\[2ex] 0 & \text{if } \theta_1=\theta_2\ .\end{cases}\end{aligned}$$

- $\tau(L_1,L_2)=-\tau(L_2,L_1)\in\mathbb{R}$.
- **Examples** $\tau(L)=\tau(\mathbb{R}\oplus 0,L)$, $\tau(L,L)=0$.

The integral Maslov index of a triple of 1-dimensional lagrangians

- **Definition** The **Maslov index** of a triple of lagrangians

  $$(L_1, L_2, L_3) \;=\; (L(\theta_1), L(\theta_2), L(\theta_3))$$

  in $(\mathbb{R}^2, [\;,\;])$ is

  $$\begin{aligned}\tau(L_1, L_2, L_3) \;&=\; \tau(L_1, L_2) + \tau(L_2, L_3) + \tau(L_3, L_1) \\ &\in \{-1, 0, 1\} \subset \mathbb{R}\;.\end{aligned}$$

- **Example** If $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \pi$ then

  $$\tau(L_1, L_2, L_3) \;=\; 1 \in \mathbb{Z}\;.$$

The integral Maslov index and the degree I.

- A pair of 1-dimensional lagrangians $(L_1, L_2) = (L(\theta_1), L(\theta_2))$ determines a path in $\Lambda(1)$ from L_1 to L_2

$$\omega_{12} \ : \ I \to \Lambda(1) \ ; \ t \mapsto L((1-t)\theta_1 + t\theta_2) \ .$$

- For any $L = L(\theta) \in \Lambda(1) \backslash \{L_1, L_2\}$

$$\begin{aligned} (\omega_{12})^{-1}(L) &= \{t \in [0,1] \,|\, L((1-t)\theta_1 + t\theta_2) = L\} \\ &= \{t \in [0,1] \,|\, (1-t)\theta_1 + t\theta_2 = \theta\} \\ &= \begin{cases} \left\{ \dfrac{\theta - \theta_1}{\theta_2 - \theta_1} \right\} & \text{if } 0 < \dfrac{\theta - \theta_1}{\theta_2 - \theta_1} < 1 \\ \emptyset & \text{otherwise .} \end{cases} \end{aligned}$$

- The degree of a loop $\omega : S^1 \to \Lambda(1) = S^1$ is the number of elements in $\omega^{-1}(L)$ for a generic $L \in \Lambda(1)$. In the geometric applications the Maslov index counts the number of intersections of a curve in a lagrangian manifold with the codimension 1 cycle of singular points.

The Maslov index and the degree II.

- **Proposition** A triple of lagrangians (L_1, L_2, L_3) determines a loop in $\Lambda(1)$
$$\omega_{123} \ = \ \omega_{12}\omega_{23}\omega_{31} \ : \ S^1 \to \Lambda(1)$$
with homotopy class the Maslov index of the triple
$$\omega_{123} \ = \ \tau(L_1, L_2, L_3) \in \{-1, 0, 1\} \subset \pi_1(\Lambda(1)) \ = \ \mathbb{Z} \ .$$

- **Proof** It is sufficient to consider the special case
$$(L_1, L_2, L_3) \ = \ (L(\theta_1), L(\theta_2), L(\theta_3))$$
with $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \pi$, so that
$$\begin{aligned} \det{}^2\omega_{123} \ &= \ 1 \ : \ S^1 \to S^1 \ , \\ \text{degree}(\det{}^2\omega_{123}) \ &= \ 1 \ = \ \tau(L_1, L_2, L_3) \in \mathbb{Z} \end{aligned}$$

The Euclidean structure on $\mathbb{R}^{2n}$

- The **phase space** is the $2n$-dimensional Euclidean space $\mathbb{R}^{2n}$, with preferred basis $\{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n\}$.
- The $2n$-dimensional phase space carries 4 additional structures.
- **Definition** The **Euclidean structure** on $\mathbb{R}^{2n}$ is the positive definite symmetric form over $\mathbb{R}$

$$(\ , \ ) \ : \ \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R} \ ; \ (v, v') \mapsto \sum_{j=1}^{n} x_j x'_j + \sum_{k=1}^{n} y_k y'_k \ ,$$

$$(v \ = \ \sum_{j=1}^{n} x_j p_j + \sum_{k=1}^{n} y_k q_k \ , \ v' \ = \ \sum_{j=1}^{n} x'_j p_j + \sum_{k=1}^{n} y'_k q_k \in \mathbb{R}^{2n}) \ .$$

- The automorphism group of $(\mathbb{R}^{2n}, (\ , \ ))$ is the **orthogonal group** $O(2n)$ of invertible $2n \times 2n$ matrices $A = (a_{jk})$ ($a_{jk} \in \mathbb{R}$) such that $A^*A = I_{2n}$ with $A^* = (a_{kj})$ the transpose.

The complex structure on $\mathbb{R}^{2n}$

- **Definition** The **complex structure** on $\mathbb{R}^{2n}$ is the linear map

$$J ~:~ \mathbb{R}^{2n} \to \mathbb{R}^{2n} ~;~ \sum_{j=1}^{n} x_j p_j + \sum_{k=1}^{n} y_k q_k \mapsto \sum_{j=1}^{n} x_j q_j - \sum_{k=1}^{n} y_k p_k$$

  such that

$$J(p_j) ~=~ q_j ~,~ J(q_k) ~=~ -p_k ~,~ J^2 ~=~ -1 ~:~ \mathbb{R}^{2n} \to \mathbb{R}^{2n} ~.$$

  Use J to regard $\mathbb{R}^{2n}$ as an n-dimensional complex vector space, with an isomorphism

$$\mathbb{R}^{2n} \to \mathbb{C}^n ~;~ v \mapsto (x_1 + iy_1, x_2 + iy_2, \dots, x_n + iy_n) ~.$$

- The automorphism group of $(\mathbb{R}^{2n}, J) = \mathbb{C}^n$ is the **complex general linear group** $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices (a_{jk}) $(a_{jk} \in \mathbb{C})$.

The symplectic structure on $\mathbb{R}^{2n}$

- **Definition** The **symplectic structure** on $\mathbb{R}^{2n}$ is the symplectic form

$$[\ ,\ ]\ :\ \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}\ ;$$

$$(v, v') \mapsto [v, v']\ =\ (Jv, v')\ =\ -[v', v]\ =\ \sum_{j=1}^{n}(x'_j y_j - x_j y'_j)$$

$$(v\ =\ \sum_{j=1}^{n} x_j p_j + \sum_{k=1}^{n} y_k q_k\ ,\ v'\ =\ \sum_{j=1}^{n} x'_j p_j + \sum_{k=1}^{n} y'_k q_k \in \mathbb{R}^{2n})\ .$$

- The automorphism group of $(\mathbb{R}^{2n}, [\ ,\ ])$ is the **symplectic group** $Sp(n)$ of invertible $2n \times 2n$ matrices $A = (a_{jk})$ $(a_{jk} \in \mathbb{R})$ such that

$$A^* \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} A\ =\ \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\ .$$

The *n*-dimensional lagrangians

- **Definition** Given a finite-dimensional real vector space V with a nonsingular symplectic form $[\ ,\ ] : V \times V \to \mathbb{R}$ let $\Lambda(V)$ be the set of lagrangian subspaces $L \subset V$, with

$$L \ = \ L^{\perp} \ = \ \{x \in V \,|\, [x,y] = 0 \in \mathbb{R} \text{ for all } y \in L\} \ .$$

- **Terminology** $\Lambda(\mathbb{R}^{2n}) = \Lambda(n)$.
- **Proposition** Every lagrangian $L \in \Lambda(n)$ has a canonical complement $JL \in \Lambda(n)$, with $L \oplus JL = \mathbb{R}^{2n}$.
- **Example** $\mathbb{R}^n$ and $J\mathbb{R}^n$ are lagrangian complements, with $\mathbb{R}^{2n} = \mathbb{R}^n \oplus J\,\mathbb{R}^n$.
- **Definition** The **graph** of a symmetric form $(\mathbb{R}^n, \phi)$ is the lagrangian

$$\Gamma_{(\mathbb{R}^n,\phi)} \ = \ \{(x,\phi(x)) \,|\, x = \sum_{j=1}^{n} x_j p_j,\ \phi(x) = \sum_{j=1}^{n}\sum_{k=1}^{n} \phi_{jk} x_j q_k\} \in \Lambda(n)$$

complementary to $J\,\mathbb{R}^n$.
- **Proposition** Every lagrangian complementary to $J\,\mathbb{R}^n$ is a graph.

The hermitian structure on $\mathbb{R}^{2n}$

- **Definition** The **hermitian inner product** on $\mathbb{R}^{2n}$ is defined by

$$\langle\ ,\ \rangle\ :\ \mathbb{R}^{2n}\times\mathbb{R}^{2n}\to\mathbb{C}\ ;$$

$$(v,v')\mapsto\langle v,v'\rangle\ =\ (v,v')+i[v,v']\ =\ \sum_{j=1}^{n}(x_j+iy_j)(x'_j-iy'_j)\ ,$$

$$(v\ =\ \sum_{j=1}^{n}x_jp_j+\sum_{k=1}^{n}y_kq_k\ ,\ \ v'\ =\ \sum_{j=1}^{n}x'_jp_j+\sum_{k=1}^{n}y'_kq_k\in\mathbb{R}^{2n})$$

  or equivalently by

$$\langle\ ,\ \rangle\ :\ \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}\ ;\ (z,z')\mapsto\langle z,z'\rangle\ =\ \sum_{j=1}^{n}z_j\overline{z}'_j\ .$$

- The automorphism group of $(\mathbb{C}^n,\langle\ ,\ \rangle)$ is the **unitary group** $U(n)$ of invertible $n\times n$ matrices $A=(a_{jk})$ $(a_{jk}\in\mathbb{C})$ such that $AA^*=I_n$, with $A^*=(\overline{a}_{kj})$ the conjugate transpose.

The general linear, orthogonal and unitary groups

- **Proposition** (Arnold, 1967) (i) The automorphism groups of $\mathbb{R}^{2n}$ with respect to the various structures are related by

$$O(2n) \cap GL(n,\mathbb{C}) \;=\; GL(n,\mathbb{C}) \cap Sp(n) \;=\; Sp(n) \cap O(2n) \;=\; U(n)\ .$$

- (ii) The determinant map $\det : U(n) \to S^1$ is the projection of a fibre bundle

$$SU(n) \to U(n) \to S^1\ .$$

- (iii) Every $A \in U(n)$ sends the standard lagrangian $\mathbb{R}^n$ of $(\mathbb{R}^{2n}, [\ ,\ ])$ to a lagrangian $A(\mathbb{R}^n)$. The unitary matrix $A = (a_{jk})$ is such that $A(\mathbb{R}^n) = \mathbb{R}^n$ if and only if each $a_{jk} \in \mathbb{R} \subset \mathbb{C}$, with

$$O(n) \;=\; \{A \in U(n) \,|\, A(\mathbb{R}^n) = \mathbb{R}^n\} \subset U(n)\ .$$

The lagrangian Grassmannian $\Lambda(n)$ I.

- $\Lambda(n)$ is the space of all lagrangians $L \subset (\mathbb{R}^{2n}, [\ ,\ ])$.
- **Proposition** (Arnold, 1967) The function

  $$U(n)/O(n) \to \Lambda(n) \ ; \ A \mapsto A(\mathbb{R}^n)$$

  is a diffeomorphism.
- $\Lambda(n)$ is a compact manifold of dimension

  $$\dim \Lambda(n) \ = \ \dim U(n) - \dim O(n) \ = \ n^2 - \frac{n(n-1)}{2} \ = \ \frac{n(n+1)}{2} \ .$$

  The graphs $\{\Gamma_{(\mathbb{R}^n,\phi)} \,|\, \phi^* = \phi \in M_n(\mathbb{R})\} \subset \Lambda(n)$ define a chart at $\mathbb{R}^n \in \Lambda(n)$.
- **Example** (Arnold and Givental, 1985)

  $$\begin{aligned} \Lambda(2)^3 \ &= \ \{[x,y,z,u,v] \in \mathbb{R}\,\mathbb{P}^4 \,|\, x^2+y^2+z^2 = u^2+v^2\} \\ &= \ S^2 \times S^1/\{(x,y) \sim (-x,-y)\} \ . \end{aligned}$$

The lagrangian Grassmannian $\Lambda(n)$ II.

- In view of the fibration

$$\Lambda(n) \ = \ U(n)/O(n) \to BO(n) \to BU(n)$$

  $\Lambda(n)$ classifies real n-plane bundles β with a trivialisation $\delta\beta : \mathbb{C} \otimes \beta \cong \epsilon^n$ of the complex n-plane bundle $\mathbb{C} \otimes \beta$.
- The canonical real n-plane bundle η over $\Lambda(n)$ is

$$E(\eta) \ = \ \{(L,\ell) \,|\, L \in \Lambda(n),\ \ell \in L\} \ .$$

  The complex n-plane bundle $\mathbb{C} \otimes \eta$

$$E(\mathbb{C} \otimes \eta) \ = \ \{(L,\ell_{\mathbb{C}}) \,|\, L \in \Lambda(n),\ \ell_{\mathbb{C}} \in \mathbb{C} \otimes_{\mathbb{R}} L\}$$

  is equipped with the canonical trivialisation $\delta\eta : \mathbb{C} \otimes \eta \cong \epsilon^n$ defined by

$$\delta\eta \ : \ E(\mathbb{C} \otimes \eta) \xrightarrow{\cong} E(\epsilon^n) \ = \ \Lambda(n) \times \mathbb{C}^n \ ;$$

$$(L,\ell_{\mathbb{C}}) \mapsto (L,(p,q)) \text{ if } \ell_{\mathbb{C}} = (p,q) \in \mathbb{C} \otimes_{\mathbb{R}} L = L \oplus JL = \mathbb{C}^n \ .$$

The fundamental group $\pi_1(\Lambda(n))$

- **Theorem** (Arnold, 1967) The square of the determinant function
$$\det{}^2 \ : \ \Lambda(n) \to S^1 \ ; \ L = A(\mathbb{R}^n) \mapsto \det(A)^2$$
induces an isomorphism
$$\det{}^2 \ : \ \pi_1(\Lambda(n)) \ \xrightarrow{\cong} \ \pi_1(S^1) \ = \ \mathbb{Z} \ .$$
- **Proof** By the homotopy exact sequence of the commutative diagram of fibre bundles
$$\begin{tikzcd}
SO(n) \arrow[r] \arrow[d] & O(n) \arrow[r, "\det"] \arrow[d] & O(1) = S^0 \arrow[d] \\
SU(n) \arrow[r] \arrow[d] & U(n) \arrow[r, "\det"] \arrow[d] & U(1) = S^1 \arrow[d, "z \mapsto z^2"] \\
S\Lambda(n) \arrow[r] & \Lambda(n) \arrow[r, "\det^2"] & \Lambda(1) = S^1
\end{tikzcd}$$

The real Maslov index for n-dimensional lagrangians I.

- Unitary matrices can be diagonalized. For every $A \in U(n)$ there exists $B \in U(n)$ such that

$$BAB^{-1} \;=\; D(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n})$$

is the diagonal matrix, with $e^{i\theta_j} \in S^1$ the eigenvalues, i.e. the roots of the characteristic polynomial

$$\mathrm{ch}_z(A) \;=\; \det(zI_n - A) \;=\; \prod_{j=1}^{n}(z - e^{i\theta_j}) \in \mathbb{C}[z] \;.$$

- **Definition** The **Maslov index** of $L \in \Lambda(n)$ is

$$\tau(L) \;=\; \sum_{j=1}^{n} \tau(L(\theta_j)) \;=\; \sum_{j=1,\theta_j \neq 0}^{n} (1 - 2\theta_j/\pi) \in \mathbb{R}$$

with $\theta_1, \theta_2, \dots, \theta_n \in [0, \pi)$ such that $\pm e^{i\theta_1}, \pm e^{i\theta_2}, \dots, \pm e^{i\theta_n}$ are the eigenvalues of any $A \in U(n)$ such that $A(\mathbb{R}^n) = L$.

The real Maslov index for n-dimensional lagrangians II.

- Given $L, L' \in \Lambda(n)$ define

  $$\tau(L, L') \ = \ \tau(A(\mathbb{R}^n)) \in \mathbb{R}$$

  if $A \in U(n)$ is such that $A(L) = L'$.
- $\tau(L', L) = -\tau(L, L') \in \mathbb{R}$, since if $A(L) = L'$ with eigenvalues $e^{i\theta_j}$ $(0 \leqslant \theta_j \leqslant \pi)$ then $L' = A^{-1}(L)$ with eigenvalues $e^{-i\theta_j} = -e^{i(\pi-\theta_j)}$, and

  $$1 - \frac{2\theta_j}{\pi} \ = \ -(1 - \frac{2(\pi - \theta_j)}{\pi}) \in \mathbb{R} \ .$$

- In general, $\tau(L, L') \neq \tau(L') - \tau(L) \in \mathbb{R}$.
- Given $L, L', L'' \in \Lambda(n)$ define

  $$\tau(L, L', L'') \ = \ \tau(L, L') + \tau(L', L'') + \tau(L'', L') \in \mathbb{Z} \ .$$

The integral signature

- The **integral signature** of a $4k$-dimensional manifold with boundary $(M, \partial M)$ is

  $\sigma(M) \ = \ \text{signature(symmetric intersection form } (H_{2k}(M;\mathbb{R}), \phi_M)) \in \mathbb{Z}$.
- For a triple union of codimension 0 submanifolds

  $$M^{4k} \ = \ M_1 \cup M_2 \cup M_3$$

  Wall (1967) expressed the difference

  $$\sigma(M; M_1, M_2, M_3) \ = \ \sigma(M) - (\sigma(M_1) + \sigma(M_2) + \sigma(M_3)) \in \mathbb{Z}$$

  as an invariant of the three lagrangians in the nonsingular symplectic intersection form $(H_{2k-1}(M_{123};\mathbb{R}), \phi_{123})$

  $$L_j \ = \ \ker(H_{2k-1}(M_{123};\mathbb{R}) \to H_{2k-1}(M_{j+1} \cap M_{j+2};\mathbb{R})) \ (j(\bmod 3)) \ .$$

  with $M_{123} = M_1 \cap M_2 \cap M_3$ the $(4k-2)$-dimensional triple intersection.
- Kashiwara and Schapira (1992) identified

  $$\sigma(M; M_1, M_2, M_3) \ = \ \tau(L_1, L_2, L_3) \in \mathbb{Z} \subset \mathbb{R} \ .$$

The real signature

- Let $(M, \partial M)$ be a $4k$-dimensional manifold with boundary, and let $P^{4k-2} \subset \partial M$ be a separating codimension 1 submanifold of the boundary, with $\partial M = N_1 \cup_P N_2$. Let $(H_{2k-1}(P;\mathbb{R}), \phi_P)$ be the nonsingular symplectic intersection form, $n = \dim_{\mathbb{R}}(H_{2k-1}(P;\mathbb{R}))/2$.
- Given a choice of isomorphism

$$J \ : \ (H_{2k-1}(P;\mathbb{R}), \phi_P) \ \cong \ (\mathbb{R}^{2n}, [\ ,\ ])$$

  define the **real signature**

$$\tau_J(M, N_1, N_2, P) \ = \ \sigma(M) + \tau(L_1, L_2) \in \mathbb{R}$$

  with $L_1, L_2 \subset \mathbb{R}^{2n}$ the images under J of the lagrangians

$$\ker(H_{2k-1}(P;\mathbb{R}) \to H_{2k-1}(N_j;\mathbb{R})) \subset (H_{2k-1}(P;\mathbb{R}), \phi_P) \ (j = 1, 2)$$

- By Wall and Kashiwara+Schapira have additivity of the real signature

$$\tau_J(M \cup M'; N_1, N_3, P) \ = \ \tau_J(M; N_1, N_2, P) + \tau_J(M'; N_2, N_3, P) \in \mathbb{R} \ .$$

- Note: in general $\tau_J \in \mathbb{R}$ depends on the choice of J.

The Maslov index, whichever way you slice it! I.

- The lagrangians $L \subset (\mathbb{R}^2, [\ ,\ ])$ are parametrized by $\theta \in \mathbb{R}$

  $$L(\theta) \ = \ \{(r\cos\theta, r\sin\theta) \mid r \in \mathbb{R}\} \subset \mathbb{R} \oplus \mathbb{R}$$

  with indeterminacy $L(\theta) = L(\theta + \pi)$. The map

  $$\det{}^2 \ : \ \Lambda(1) \ = \ U(1)/O(1) \to S^1 \ ; \ L(\theta) \mapsto e^{2i\theta}$$

  is a diffeomorphism.
- The canonical $\mathbb{R}$-bundle η over $\Lambda(1)$

  $$E(\eta) \ = \ \{(L, x) \mid L \in \Lambda(1)\, , \, x \in L\}$$

  is nontrivial = infinite Möbius band. The induced $\mathbb{C}$-bundle over $\Lambda(1)$ is

  $$E(\mathbb{C} \otimes_{\mathbb{R}} \eta) \ = \ \{(L, y) \mid L \in \Lambda(1)\, , \, y \in \mathbb{C} \otimes_{\mathbb{R}} L\}$$

  is equipped with the canonical trivialisation $\delta\eta : \mathbb{C} \otimes_{\mathbb{R}} \eta \cong \epsilon$ defined by

  $$\begin{aligned} &\delta\eta \ : \ E(\mathbb{C} \otimes_{\mathbb{R}} \eta) \xrightarrow{\cong} E(\epsilon) \ = \ \Lambda(1) \times \mathbb{C} \ ; \\ &(L, y) = (L(\theta), (u + iv)(\cos\theta, \sin\theta)) \mapsto (L(\theta), (u + iv)e^{i\theta}) \ . \end{aligned}$$

The Maslov index, whichever way you slice it! II.

- Given a bagel $B = S^1 \times D^2 \subset \mathbb{R}^3$ and a map $\lambda : S^1 \to \Lambda(1) = S^1$ slice B along
$$C \;=\; \{(x,y) \in B \,|\, y \in \lambda(x)\}\ .$$
- The slicing line $(x, \lambda(x)) \subset B$ is the fibre over $x \in S^1$ of the pullback $[-1,1]$-bundle
$$[-1,1] \to C = D(\lambda^*\eta) \to S^1$$
with boundary (where the knife goes in and out of the bagel)
$$\partial C \;=\; \{(x,y) \in C \,|\, y \in \partial\lambda(x)\}$$
a double cover of S^1. There are two cases:
    - C is a trivial $[-1,1]$-bundle over S^1 (i.e. an annulus), with ∂C two disjoint circles, which are linked in $\mathbb{R}^3$. The complement $B \backslash C$ has two components, with the same linking number.
    - C is a non trivial $[-1,1]$-bundle over S^1 (i.e. a Möbius band), with ∂C a single circle, which is self-linked in $\mathbb{R}^3$. The complement $B \backslash C$ is connected, with the same self-linking number ($=$ linking of ∂C and $S^1 \times \{(0,0)\} \subset C \subset \mathbb{R}^3$).

The Maslov index, whichever way you slice it! III.

- By definition, Maslov index(λ) $=$ degree(λ) $\in \mathbb{Z}$.
- degree : $\pi_1(S^1) \to \mathbb{Z}$ is an isomorphism, so it may be assumed that

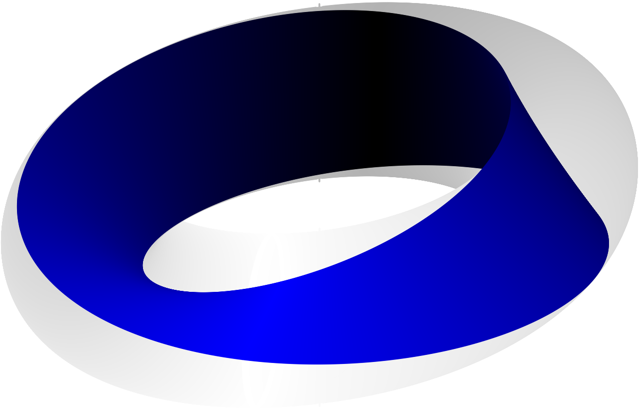
  $$\lambda \ : \ S^1 \to \Lambda(1) \ ; \ e^{2i\theta} \mapsto L(n\theta)$$

  with Maslov index $= n \geqslant 0$. The knife is turned through a total angle $n\pi$ as it goes round B. It may also be assumed that the bagel B is horizontal. The projection of ∂C onto the horizontal cross-section of B consists of $n = |\lambda^{-1}(L(0))|$ points. For $n > 0$ this corresponds to the angles $\theta = j\pi/n \in [0, \pi)$ $(0 \leqslant j \leqslant n-1)$ where $L(n\theta) = L(0)$, i.e. $\sin n\theta = 0$.
- The two cases are distinguished by:
  - If $n = 2k$ then ∂C is a union of two disjoint linked circles in $\mathbb{R}^3$. Each successive pair of points in the projection contributes 1 to the linking number $n/2 = k$.
  - If $n = 2k + 1$ then ∂C is a single self-linked circle in $\mathbb{R}^3$. Each point in the projection contributes 1 to the self-linking number $n = 2k + 1$. (Thanks to Laurent Bartholdi for explaining this case to me.)

Maslov index $= 0$, $C =$ annulus , linking number $= 0$

$$\lambda \ : \ S^1 \to S^1 \ ; \ z \mapsto 1 \ .$$

Maslov index = 1 , C = Möbius band , self-linking number = 1



$$\lambda \ : \ S^1 \to S^1 \ ; \ z \mapsto z \ .$$

Thanks to Clara Löh for this picture.

Maslov index = 2 , C = annulus , linking number = 1

$$\lambda \;:\; S^1 \to S^1 \;;\; z \mapsto z^2 \;.$$

http://www.georgehart.com/bagel/bagel.html