NONCOMMUTATIVE LOCALIZATION AND MANIFOLD TRANSVERSALITY

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- A group is a generalized free product if it is either an HNN extension or an amalgamated free product. Likewise for rings.

- The noncommutative localization of triangular matrix rings can be used to relate :

  1. the topology of manifolds with fundamental group a generalized free product, and

  2. the algebraic properties of modules and quadratic forms over a generalized free product.

The Seifert-van Kampen theorem

Let W be a space with a decomposition

$$W = X \times [0,1] \cup_{X \times \{0,1\}} Y$$

such that W and X are connected.

Theorem Y has either 1 or 2 components, and the fundamental group $\pi_1(W)$ is a generalized free product :

1. If Y is connected then $\pi_1(W)$ is an HNN extension

$$\begin{aligned} \pi_1(W) &= \pi_1(Y) *_{i_1,i_2} \{z\} \\ &= \pi_1(Y) * \{z\}/\{i_1(x)z = zi_2(x) \,|\, x \in \pi_1(X)\} \end{aligned}$$

with $i_1, i_2 : \pi_1(X) \to \pi_1(Y)$ induced by the two inclusions $i_1, i_2 : X \to Y$.

2. If Y is disconnected, $Y = Y_1 \cup Y_2$, then $\pi_1(W)$ is an amalgamated free product

$$\pi_1(W) = \pi_1(Y_1) *_{\pi_1(X)} \pi_1(Y_2)$$

with $i_1 : \pi_1(X) \to \pi_1(Y_1)$, $i_2 : \pi_1(X) \to \pi_1(Y_2)$ induced by the inclusions $i_1 : X \to Y_1$, $i_2 : X \to Y_2$.

Mayer-Vietoris and transversality

- The homology groups of $W = X \times [0,1] \cup Y$ fit into the Mayer-Vietoris exact sequence

$$\cdots \to H_n(X) \xrightarrow{i_1 - i_2} H_n(Y) \to H_n(W) \xrightarrow{\partial} H_{n-1}(X) \to \dots .$$

- A map $f : V \to W$ is <u>transverse</u> at $X \subset W$ if

$$V = T \times [0,1] \cup U$$

with $T = f^{-1}(X)$, $U = f^{-1}(Y)$.

- A homotopy equivalence $f : V \to W$ <u>splits</u> if it is homotopic to a transverse map such that the restrictions $f| : T \to X$, $f| : U \to Y$ are also homotopy equivalences.

Geometric transversality

- Every map $f : V^n \to W = X \times [0,1] \cup Y$ from a manifold V is homotopic to a transverse map, with $T^{n-1} = f^{-1}(X) \subset V^n$ a codimension 1 submanifold.

- A homotopy equivalence of manifolds

$$f : V^n \to W^n = X^{n-1} \times [0,1] \cup Y^n$$

  does not split in general.

- For $n \geqslant 6$ a homotopy equivalence of manifolds splits if and only if certain algebraic K- and L-theory obstructions vanish.

- If a homotopy equivalence is homotopic to a homeomorphism then it splits. There exist homotopy equivalences which do not split, and are therefore not homotopic to homeomorphisms.

Splitting obstruction theory

- The algebraic K-groups of $\mathbb{Z}[\pi_1(W)]$ for $W = X \times [0,1] \cup Y$ with

$$\boxed{\pi_1(X) \to \pi_1(W) \text{ injective}}$$

  fit into almost-Mayer-Vietoris exact sequence

$$\cdots \to K_n(\mathbb{Z}[\pi_1(X)]) \xrightarrow{i_1 - i_2} K_n(\mathbb{Z}[\pi_1(Y)]) \to$$

$$K_n(\mathbb{Z}[\pi_1(W)]) \xrightarrow{\partial} \widetilde{\text{Nil}}_{n-1} \oplus K_{n-1}(\mathbb{Z}[\pi_1(X)]) \to \dots$$

  with $\widetilde{\text{Nil}}_*$ the exotic algebraic K-groups of nilpotent endomorphisms of f.g. projective modules.

- The K-theory splitting obstruction of a homotopy equivalence $f : V \to W$ is $\partial(\tau(f))$ with $\tau(f) =$ Whitehead torsion $\in K_1(\mathbb{Z}[\pi_1(W)])$.

- Similarly for the algebraic L-groups of quadratic forms.

Algebraic transversality

- Let $W = X \times [0,1] \cup Y$, so that $\pi_1(W)$ is a generalized free product of $\pi_1(X)$, $\pi_1(Y)$.

- A <u>Mayer-Vietoris presentation</u> of a finite f.g. free $\mathbb{Z}[\pi_1(W)]$-module chain complex C is an exact sequence

$$0 \to \mathbb{Z}[\pi_1(W)] \otimes_{\mathbb{Z}[\pi_1(X)]} D \xrightarrow{i_1 - i_2} \mathbb{Z}[\pi_1(W)] \otimes_{\mathbb{Z}[\pi_1(Y)]} E \to C \to 0$$

with D a finite f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex, E a finite f.g. free $\mathbb{Z}[\pi_1(Y)]$-module chain complex, and i_1, i_2 $\mathbb{Z}[\pi_1(Y)]$-module chain maps $\mathbb{Z}[\pi_1(Y)] \otimes_{\mathbb{Z}[\pi_1(X)]} D \to E$.

- If $\pi_1(X) \to \pi_1(W)$ is injective every C admits a Mayer-Vietoris presentation.

The universal cover and fundamental domains

- The fundamental group $\pi_1(W)$ of a connected space W acts on the universal cover $\widetilde{W}$ by covering translations $\pi_1(W) \times \widetilde{W} \to \widetilde{W}$. The homology $\mathbb{Z}[\pi_1(W)]$-modules are

$$H_*(\widetilde{W}) \;=\; H_*(C(\widetilde{W}))$$

  with $C(\widetilde{W})$ a free $\mathbb{Z}[\pi_1(W)]$-module chain complex (simplicial, cellular, singular, ...).

- For $W = X \times [0,1] \cup Y$ the universal cover $\widetilde{W}$ has fundamental domain the universal cover $\widetilde{Y}$ of Y, with adjoining translates intersecting in copies of the universal cover $\widetilde{X}$ of X. Geometry gives the Mayer-Vietoris presentation

$$0 \to \mathbb{Z}[\pi_1(W)] \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X}) \xrightarrow{i_1 - i_2} \mathbb{Z}[\pi_1(W)] \otimes_{\mathbb{Z}[\pi_1(Y)]} C(\widetilde{Y}) \to C(\widetilde{W}) \to 0$$

Morita theory

For any ring D and $k \geqslant 1$ let $M_k(D)$ be the ring of $k \times k$ matrices in D.

Proposition (i) The functors

$$\{D\text{-modules}\} \to \{M_k(D)\text{-modules}\} ;$$

$$M \mapsto \begin{pmatrix} D \\ D \\ \vdots \\ D \end{pmatrix} \otimes_D M ,$$

$$\{M_k(D)\text{-modules}\} \to \{D\text{-modules}\} ;$$

$$N \mapsto (D \ D \ \ldots \ D) \otimes_{M_k(D)} N$$

are inverse equivalences of categories.
(ii) $K_*(M_k(D)) = K_*(D)$.

The Mayer-Vietoris localization

Key idea: for $W = X \times [0,1] \cup Y$ the expression of $\pi_1(W)$ as a generalized free product motivates the construction of a triangular matrix ring A with a noncommutative localization the matrix ring

$$\Sigma^{-1}A \;=\; M_k(\mathbb{Z}[\pi_1(W)])$$

where $k = (\text{no. of components of } Y) + 1$.
The localization functor

$$\{A\text{-modules}\} \to \{\Sigma^{-1}A\text{-modules}\} \; ; \; M \mapsto \Sigma^{-1}M$$

is an algebraic analogue of the forgetful functor

$$\{\text{transverse maps } V \to W\} \to \{\text{maps } V \to W\} \; .$$

For any map $V \to W$ $C(\widetilde{V})$ is a $\Sigma^{-1}A$-module chain complex, up to Morita equivalence. For a transverse map $V = T \times [0,1] \cup U \to W$ the Mayer-Vietoris presentation of $C(\widetilde{V})$ is an A-module chain complex Γ such that

$$\Sigma^{-1}\Gamma \;=\; C(\widetilde{V}) \; .$$

A polynomial extension is a noncommutative localization

For any ring R define the triangular matrix ring

$$A = \begin{pmatrix} R & 0 \\ R \oplus R & R \end{pmatrix} .$$

An A-module is a quadruple

$$M = (K, L, \mu_1, \mu_2 : K \to L)$$

with K, L R-modules and μ_1, μ_2 R-module morphisms. The localization of A inverting

$$\Sigma = \{\sigma_1, \sigma_2 : \begin{pmatrix} 0 \\ R \end{pmatrix} \to \begin{pmatrix} R \\ R \oplus R \end{pmatrix}\}$$

is a ring morphism

$$A \to \Sigma^{-1}A = M_2(D) , \ D = R[z, z^{-1}]$$

such that

$$\{A\text{-modules}\} \to \{M_2(D)\text{-modules}\} \approx \{D\text{-modules}\}$$

sends an A-module M to the D-module

$$(D \ D) \otimes_A M$$
$$= \text{coker}(\mu_1 - z\mu_2 : K[z, z^{-1}] \to L[z, z^{-1}])$$

Manifolds over S^1

- Given a manifold V^n and map $f : V \to W = S^1$ which is transverse at $X = \{\text{pt.}\} \subset S^1$ cut V along the codimension 1 submanifold $T^{n-1} = f^{-1}(X) \subset V$ to obtain

$$V \;=\; T \times [0,1] \cup_{T \times \{0,1\}} U \;.$$

  The cobordism $(U; T_0, T_1)$ is a fundamental domain for the infinite cyclic cover $\overline{V} = f^*\mathbb{R}$ of V, with T_0, T_1 copies of T.

- $A \;=\; \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $\;\Sigma^{-1}A \;=\; M_2(\mathbb{Z}[z, z^{-1}])$.

  The A-module chain complex

$$\Gamma \;=\; (C(T), C(U), \mu_1, \mu_2)$$

  induces the $\mathbb{Z}[z, z^{-1}]$-module chain complex

$$\begin{aligned} &(D\ \ D) \otimes_A \Gamma \\ &=\; \text{coker}(\mu_1 - z\mu_2 : C(T)[z, z^{-1}] \to C(U)[z, z^{-1}]) \\ &=\; C(\overline{V}) \;. \end{aligned}$$

The lifting problem for chain complexes

- Let $\Sigma^{-1}A$ be the localization of A inverting a set Σ of morphisms of f.g. projective A-modules.

- A lift of a finite f.g. free $\Sigma^{-1}A$-module chain complex C is a finite f.g. projective A-module chain complex B with a chain equivalence $\Sigma^{-1}B \simeq C$.

- Every n-dimensional f.g. free $\Sigma^{-1}A$-module chain complex C can be lifted if $n \leqslant 2$. For $n \geqslant 3$ there are lifting obstructions in $\text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A)$ for $i \geqslant 1$.

- Definition A localization $\Sigma^{-1}A$ of a ring A inverting a set Σ of morphisms of f.g. projective A-modules is stably flat if

$$\text{Tor}_i^A(\Sigma^{-1}A, \Sigma^{-1}A) = 0 \ (i \geqslant 1) .$$

Theorem of Neeman + R.

If $A \to \Sigma^{-1}A$ is injective and stably flat then :

- have 'fibration sequence of exact categories'

$$T(A,\Sigma) \to P(A) \to P(\Sigma^{-1}A)$$

  with $P(A)$ the category of f.g. projective A-modules and $T(A,\Sigma)$ the category of h.d. 1 Σ-torsion A-modules, and

- every finite f.g. free $\Sigma^{-1}A$-module chain complex can be lifted, and

- there is a long exact sequence in algebraic K-theory

$$\cdots \to K_{n+1}(\Sigma^{-1}A) \to K_n(T(A,\Sigma)) \to K_n(A) \to K_n(\Sigma^{-1}A) \to \dots$$

- http://arXiv.org/abs/math.RA/0109118

Modules over a triangular matrix ring

Given rings A_1, A_2 and an (A_2, A_1)-bimodule B define the triangular matrix ring

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix}$$

with f.g. projectives $P_1 = \begin{pmatrix} A_1 \\ B \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 \\ A_2 \end{pmatrix}$.

Proposition (i) The category of A-modules is equivalent to the category of triples

$$M = (M_1, M_2, \mu : B \otimes_{A_1} M_1 \to M_2)$$

with M_1 an A_1-module, M_2 an A_2-module and μ an A_2-module morphism.
(ii) $K_*(A) = K_*(A_1) \oplus K_*(A_2)$.
(iii) If $A \to C$ is a ring morphism such that there is a C-module isomorphism $C \otimes_A P_1 \cong C \otimes_A P_2$ then $C = M_2(D)$ with $D = \mathrm{End}_C(C \otimes_A P_1)$,

$\{A\text{-modules}\} \to \{C\text{-modules}\} \approx \{D\text{-modules}\};$
$M \mapsto (D\ \ D) \otimes_A M$
$= \mathrm{coker}(D \otimes_{A_2} B \otimes_{A_1} M_1 \to D \otimes_{A_1} M_1 \oplus D \otimes_{A_2} M_2)$

The stable flatness theorem

Theorem Let

$$A = \begin{pmatrix} A_1 & 0 \\ B & A_2 \end{pmatrix} \to A_\Sigma = M_2(D)$$

be the localization inverting a set Σ of A-module morphisms $\sigma : \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \to \begin{pmatrix} A_1 \\ B \end{pmatrix}$. If B and D are flat A_1-modules and D is a flat A_2-module then A_Σ is stably flat.

Proof The A-module $E = \begin{pmatrix} D \\ D \end{pmatrix}$ has a 1-dimensional flat A-module resolution

$$\begin{aligned} 0 \to & \begin{pmatrix} 0 \\ B \end{pmatrix} \otimes_{A_1} D \\ & \to \begin{pmatrix} A_1 \\ B \end{pmatrix} \otimes_{A_1} D \oplus \begin{pmatrix} 0 \\ A_2 \end{pmatrix} \otimes_{A_2} D \to E \to 0 \end{aligned}$$

and hence so does $A_\Sigma = E \oplus E$.

HNN extensions

The HNN extension of ring morphisms $i_1, i_2 : R \to S$ is the ring

$$S *_{i_1,i_2} \{z\} \;=\; S * \mathbb{Z}/\{i_1(x)z = zi_2(x) \,|\, x \in R\} \ .$$

<u>Corollary 1.</u> Let $A = \begin{pmatrix} R & 0 \\ S_1 \oplus S_2 & S \end{pmatrix}$, with $S_j = S$ the (S,R)-bimodule

$$S \times S_j \times R \to S_j \ ; \ (s,t,u) \mapsto sti_j(u) \ .$$

The localization of A inverting the inclusions $\Sigma = \{\sigma_1, \sigma_2 : \begin{pmatrix} 0 \\ S \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \oplus S_2 \end{pmatrix}\}$ is

$$\Sigma^{-1}A \;=\; M_2(S *_{i_1,i_2} \{z\}) \ .$$

If $i_1, i_2 : R \to S$ are split injections and S_1, S_2 are flat R-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has

$$\begin{aligned} K_n(A) \;&=\; K_n(R) \oplus K_n(S) \ , \\ K_n(\Sigma^{-1}A) \;&=\; K_n(S *_{i_1,i_2} \{z\}) \ , \\ K_n(T(A,\Sigma)) \;&=\; K_n(R) \oplus K_n(R) \oplus \widetilde{\mathrm{Nil}}_n \ . \end{aligned}$$

Amalgamated free products

The amalgamated free product $S_1 *_R S_2$ is defined for ring morphisms $R \to S_1$, $R \to S_2$.

<u>Corollary 2.</u> The localization of the ring

$$A = \begin{pmatrix} R & 0 & 0 \\ S_1 & S_1 & 0 \\ S_2 & 0 & S_2 \end{pmatrix}$$

inverting the inclusions

$$\Sigma = \{\sigma_1 : \begin{pmatrix} 0 \\ S_1 \\ 0 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}, \sigma_2 : \begin{pmatrix} 0 \\ 0 \\ S_2 \end{pmatrix} \to \begin{pmatrix} R \\ S_1 \\ S_2 \end{pmatrix}\}$$

is the 3×3 matrix ring

$$\Sigma^{-1}A = M_3(S_1 *_R S_2) .$$

If $R \to S_1$, $R \to S_2$ are split injections with S_1, S_2 flat R-modules then $A \to \Sigma^{-1}A$ is injective and stably flat. The algebraic K-theory localization exact sequence has

$$\begin{aligned} K_n(A) &= K_n(R) \oplus K_n(S_1) \oplus K_n(S_2) , \\ K_n(\Sigma^{-1}A) &= K_n(S_1 *_R S_2) , \\ K_n(T(A, \Sigma)) &= K_n(R) \oplus K_n(R) \oplus \widetilde{\mathrm{Nil}}_n . \end{aligned}$$

A localization which is not stably flat

- Given a ring extension $R \subset S$ and an S-module M let

$$K(M) \ = \ \ker(S \otimes_R M \to M) \ .$$

- **Theorem** (Neeman, R. and Schofield)
(i) The localization of the triangular matrix ring

$$A = \begin{pmatrix} R & 0 & 0 \\ S & R & 0 \\ S & S & R \end{pmatrix} = P_1 \oplus P_2 \oplus P_3$$

inverting $\Sigma = \{P_3 \subset P_2, P_2 \subset P_1\}$ is

$$\Sigma^{-1} A = M_3(S)$$

(ii) If S is flat as an R-module then

$$\mathrm{Tor}_2^A(\Sigma^{-1} A, \Sigma^{-1} A) \ = \ M_3(K^3(S)) \ .$$

(iii) If R is a field and $S = R^d$ then

$$K^3(S) \ = \ R^{(d-1)^3 d} \ .$$