ALGEBRAIC TRANSVERSALITY

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What is algebraic transversality?

- Geometric transversality is one of the most important properties of manifolds, dealing with the construction of submanifolds.
- Easy to establish for smooth manifolds (Thom, 1954)
- Hard to establish for topological manifolds (Kirby-Siebenmann, 1970), and that only for dimensions $\geq 5$.
- Algebraic transversality deals with the construction of subcomplexes of chain complexes over group rings.
- Algebraic transversality is needed to quantify geometric transversality, to control the algebraic topology of the submanifolds created by the geometric construction.
Codimension \( k \) subspaces

**Definition** A framed codimension \( k \) subspace of a space \( X \) is a closed subspace \( Y \subset X \) such that \( X \) has a decomposition

\[
X = X_0 \cup_{Y \times S^{k-1}} Y \times D^k ,
\]

with the complement

\[
X_0 = \text{cl.}(X \setminus Y \times D^k) \subset X
\]

a closed subspace homotopy equivalent to \( X \setminus Y \).
Geometric transversality

- **Theorem** (Thom, 1954) Every map $f : M^m \to X$ from a smooth $m$-dimensional manifold to a space $X$ with a framed codimension $k$ subspace $Y \subset X$ is homotopic to a smooth map (also denoted $f$) which is transverse regular at $Y \subset X$, so that

$$N^{m-k} = f^{-1}(Y) \subset M$$

is a framed codimension $k$ submanifold with

$$f = f_0 \cup g \times 1_{D^k} : M = M_0 \cup N \times D^k \to X = X_0 \cup Y \times D^k.$$ 

- Algebraic transversality studies analogous decompositions of chain complexes! Particularly concerned with homotopy equivalences and contractible chain complexes.
The infinite cyclic cover example of geometric transversality

$X = S^1$ has framed codimension 1 subspace $Y = \{\ast\} \subset S^1$
with complement $X_0 = I$

$$S^1 = I \cup \{\ast\} \times S^0 \{\ast\} \times D^1.$$ 

By geometric transversality every map $f : M^m \to S^1$ is
homotopic to a map transverse regular at $\{\ast\} \subset S^1$, with

$$N^{m-1} = f^{-1}(\ast) \subset M$$

a framed codimension 1 submanifold with complement

$M_0 = f^{-1}(I)$

$$M = M_0 \cup_{N \times S^0} N \times D^1.$$
The infinite cyclic cover example of geometric transversality II.

- The pullback infinite cyclic cover of $M$ has fundamental domain $(M_0; N, tN)$

$$\tilde{M} = f^*\mathbb{R} = \bigcup_{j=-\infty}^{\infty} t^j(M_0; N, tN).$$
The infinite cyclic cover example of algebraic transversality

- **Proposition** (Higman, Waldhausen, R.)
  For every finite f.g. free $\mathbb{Z}[t, t^{-1}]$-module chain complex $C$ there exists a finite f.g. free $\mathbb{Z}$-module subcomplex $C_0 \subset C$ with $D = C_0 \cap tC_0$ a finite f.g. free $\mathbb{Z}$-module chain complex, and the $\mathbb{Z}$-module chain maps

  \[ i_0 : D \to C_0 ; x \mapsto x \, , \]
  \[ i_1 : D \to C_0 ; x \mapsto t^{-1}x \]

  such that there is defined a short exact sequence of finite f.g. free $\mathbb{Z}[t, t^{-1}]$-module chain complexes

  \[ 0 \to D[t, t^{-1}] \xrightarrow{i_0 - ti_1} C_0[t, t^{-1}] \to C \to 0 \]

- Note that if $C$ is contractible then $C_0, D$ need not be contractible.
- Can replace $\mathbb{Z}$ by any ring $A$. 
Split homotopy equivalences

- **Definition** A homotopy equivalence $f : M \to X$ from a smooth $m$-dimensional manifold splits at a framed codimension $k$ subspace $Y \subset X$ if $f$ is transverse regular at $Y \subset X$, and the restrictions

  \[ g = f| : N = f^{-1}(Y) \to Y, \]

  \[ f_0 = f| : M_0 = M \setminus N \to X_0 = X \setminus Y \]

  also homotopy equivalences.

- **Definition** $f$ splits up to homotopy if it is homotopic to a homotopy equivalence (also denoted by $f$) which is split.

- In general, homotopy equivalences do not split up to homotopy. Surgery theory provides splitting obstructions.
The uniqueness of smooth manifold structures

- **Surgery Theory Question** Is a homotopy equivalence \( f : M \to X \) of smooth \( m \)-dimensional manifolds homotopic to a diffeomorphism?

- **Answer** No, in general. The Browder-Novikov-Sullivan-Wall theory (1960’s) provides obstructions in homotopy theory and algebra, and systematic construction of counterexamples. For \( X = S^m \) this is the Kervaire-Milnor classification of exotic spheres.

- **Example** Diffeomorphisms are split. If \( f \) is homotopic to a diffeomorphism then \( f \) splits up to homotopy at every submanifold \( Y \subset X \).

- **Contrapositive** If \( f \) does not split up to homotopy at a submanifold \( Y \subset X \) then \( f \) is not homotopic to a diffeomorphism.
The uniqueness of topological manifold structures

- **Surgery Theory Question** Is a homotopy equivalence $f : M \to X$ of topological $m$-dimensional manifolds homotopic to a homeomorphism?

- **Answer** No, in general. As in the smooth case, surgery theory provides systematic obstruction theory for $m \geq 5$. Need Kirby-Siebenmann (1970) structure theory for topological manifolds.

- **Example** Homeomorphisms are split. If $f$ is homotopic to a homeomorphism then $f$ splits up to homotopy at every submanifold $Y \subset X$.

- **Contrapositive** If $f$ does not split up to homotopy at a submanifold $Y \subset X$ then $f$ is not homotopic to a homeomorphism.
The Borel Conjecture

- **BC (1953)** Every homotopy equivalence $f : M \to X$ of smooth $m$-dimensional aspherical manifolds is homotopic to a homeomorphism.


- In the last 30 years the conjecture has been verified in many cases, using surgery theory, geometric group theory and differential geometry (Farrell-Jones, Lück).
The existence of smooth manifold structures

- A smooth $m$-dimensional manifold $M$ is a finite CW complex with $m$-dimensional Poincaré duality $H^{m-*}(M) \cong H_*(M)$

- **Surgery Theory Question** If $X$ is a finite CW complex with $m$-dimensional Poincaré duality isomorphisms

  \[ H^{m-*}(X) \cong H_*(X) \text{ (with } \mathbb{Z}[[\pi_1(X)]]\text{-coefficients)} \]

  is $X$ homotopy equivalent to a smooth $m$-dimensional manifold?

- The Browder-Novikov-Sullivan-Wall surgery theory deals with both existence and uniqueness, providing obstructions in terms of homotopy theory and algebra.

- Various examples of Poincaré duality spaces not of the homotopy type of smooth manifolds
The existence of topological manifold structures

- **Surgery Theory Question** If $X$ is a finite $CW$ complex with $m$-dimensional Poincaré duality isomorphisms is $X$ homotopy equivalent to a topological $m$-dimensional manifold?

- For $m \geq 5$ the Browder-Novikov-Sullivan-Wall surgery theory provides algebraic obstructions. The reduction to pure algebra makes use of algebraic transversality and codimension 1 splitting obstructions (R., 1992).
Obstructions to splitting homotopy equivalences up to homotopy

- In general, homotopy equivalences of manifolds are not split up to homotopy, in both the smooth and topological categories.
- There are algebraic $K$ and $L$-theory obstructions to splitting homotopy equivalences up to homotopy for $m - k \geq 5$ (Browder, Wall, Cappell 1960’s and 1970’s).
- Waldhausen (1970’s) dealt with the case $m = 3, k = 1$, motivated by the Haken theory of 3-manifolds.
- Cappell (1974) constructed homotopy equivalences
  \[ f : M^m \to X = \mathbb{RP}^m \# \mathbb{RP}^m \]
  which cannot be split up to homotopy, for $m \equiv 1 \pmod{4}$ with $m \geq 5$, and $Y = S^{m-1} \subset X$.
- Same algebraic $K$- and $L$-theory obstructions to decomposing Poincaré duality space as $X = X_0 \cup Y \times D^k$, with $Y$ codimension $k$ Poincaré. (R.)
Given a $CW$ complex $X$ and a regular cover $\tilde{X}$ with group of covering translations $\pi$ let $C(\tilde{X})$ be the cellular chain complex, a free $\mathbb{Z}[\pi]$-module chain complex with one generator for each cell of $X$.

A map $f : M \to X$ from a $CW$ complex induces a $\pi$-equivariant map $\tilde{f} : \tilde{M} = f^*\tilde{X} \to \tilde{X}$ of the covers, and hence a $\mathbb{Z}[\pi]$-module chain map $\tilde{f} : C(\tilde{M}) \to C(\tilde{X})$.

**Theorem** (J.H.C. Whitehead) A map $f : M \to X$ is a homotopy equivalence if and only if $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism and the algebraic mapping cone $C(\tilde{f})$ is chain contractible, with $\tilde{X}$ the universal cover of $X$ and $\pi = \pi_1(X)$. 
If \( i : Y \subset X \) is the inclusion of a framed codimension \( k \) subcomplex the decomposition \( X = X_0 \cup Y \times S^{k-1} Y \times D^k \) lifts to a \( \pi \)-equivariant decomposition
\[
\tilde{X} = \tilde{X}_0 \cup \tilde{Y} \times S^{k-1} \tilde{Y} \times D^k
\]
with \( \tilde{Y} = i^* \tilde{X} \) the pullback cover of \( Y \), a framed codimension \( k \) subcomplex of \( \tilde{X} \).

The \( \mathbb{Z}[\pi] \)-module chain complex of \( \tilde{X} \) has an algebraic decomposition
\[
C(\tilde{X}) = C(\tilde{X}_0) \cup C(\tilde{Y}) \otimes C(S^{k-1}) C(\tilde{Y}) \otimes C(D^k)
\]

Algebraic transversality studies \( \mathbb{Z}[\pi] \)-module chain complexes with such decompositions.

If \( f : M \to X \) is transverse at \( Y \subset X \) the algebraic mapping cone of \( \tilde{f} : \tilde{M} \to \tilde{X} \) has such a decomposition
\[
C(\tilde{f}) = C(\tilde{f}_0) \cup C(\tilde{g}) \otimes C(S^{k-1}) C(\tilde{g}) \otimes C(D^k)
\]
The fundamental groups in codimension 1

- If $X$ is a connected $CW$ complex and $Y \subset X$ is a connected framed codimension 1 subcomplex then

$$X = \begin{cases} X_1 \cup_{Y \times D^1} X_2 & \text{if } X_0 = X_1 \sqcup X_2 \text{ is disconnected} \\ X_0 \cup_{Y \times S^0} Y \times D^1 & \text{if } X_0 \text{ is connected} \end{cases}$$

according as to whether $Y$ separates $X$ or not.

- The fundamental group $\pi_1(X)$ is given by the Seifert-van Kampen theorem to be the amalgamated free product

$$\pi_1(X) = \begin{cases} \pi_1(X_1) \ast_{\pi_1(Y)} \pi_1(X_2) & \text{determined by the morphisms} \\ \pi_1(X_0) \ast_{\pi_1(Y)} \{t\} \end{cases}$$

$$\begin{cases} \pi_1(Y) \to \pi_1(X_1) , \pi_1(Y) \to \pi_1(X_2) \\ \pi_1(Y \times \{0\}) \to \pi_1(X_0) , \pi_1(Y \times \{1\}) \to \pi_1(X_0) \end{cases}$$
Separating and non-separating codimension 1 subspaces

$Y$ separates $X$

$Y$ does not separate $X$
Handle exchanges I.

- Will only deal with the separating case.
- Let $M$ be an $m$-dimensional manifold with a separating framed codimension 1 submanifold $N^{m-1} \subset M$, so that

$$M = M_1 \cup_N M_2.$$  

- A handle exchange uses an embedding

$$(D^r \times D^{m-r}, S^{r-1} \times D^{m-r}) \subset (M_i, N) \ (i = 1 \text{ or } 2)$$


to obtain a new decomposition

$$M = M'_1 \cup_{N'} M'_2,$$

with

$$N' = \text{cl.}(N\setminus S^{r-1} \times D^{m-r}) \cup D^r \times S^{m-r-1},$$

$$M'_i = \text{cl.}(M_i\setminus D^r \times D^{m-r}) ,$$

$$M'_{2-i} = M_{2-i} \cup D^r \times D^{m-r} .$$

- Initiated by Stallings ($m = 3$) and Levine in the 1960’s.
Handle exchanges II.

\[ X'_1 = X_1 \cup D^r \times D^{m-r} , \quad X'_2 = \text{cl.} \left( X_2 \setminus D^r \times D^{m-r} \right) . \]
Codimension 1 geometric transversality I.

Let \( X = X_1 \cup_Y X_2 \) be a connected CW complex with a separating connected framed codimension 1 subspace \( Y \subset X \) such that \( \pi_1(Y) \to \pi_1(X) \) is injective. Then

\[
\pi_1(X) = \pi_1(X_1) \ast_{\pi_1(Y)} \pi_1(X_2) = \pi
\]

with injective morphisms

\[
\pi_1(Y) = \rho \to \pi_1(X_1) = \pi_1, \quad \pi_1(Y) = \rho \to \pi_1(X_2) = \pi_2.
\]

The Bass-Serre tree \( T \) is a contractible non-free \( \pi \)-space with

\[
T^{(0)} = [\pi : \pi_1] \cup [\pi : \pi_2], \quad T^{(1)} = [\pi : \rho], \quad T/\pi = I.
\]

The universal cover of \( X \) decomposes as

\[
\tilde{X} = [\pi : \pi_1] \times \tilde{X}_1 \cup_{[\pi : \rho] \times \tilde{Y}} [\pi : \pi_2] \times \tilde{X}_2
\]

with \( \tilde{X}_1, \tilde{X}_2, \tilde{Y} \) the universal covers of \( X_1, X_2, Y \), and

\[
\tilde{Y} = \tilde{X}_1 \cap \tilde{X}_2 \subset \tilde{X}.
\]
The universal cover $\tilde{X}$ of $X = X_1 \cup_Y X_2$
Codimension 1 geometric transversality II.

If $X$ is finite the cellular f.g. free $\mathbb{Z}[\pi]$-module chain complex $C(\tilde{X})$ has f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $C(\tilde{X}_i) \subset C(X)$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex $C(\tilde{Y}) = C(\tilde{X}_1) \cap C(\tilde{X}_2) \subset C(\tilde{X})$

with a short exact Mayer-Vietoris sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

$$0 \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} C(\tilde{Y}) \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} C(\tilde{X}_1) \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} C(\tilde{X}_2) \to C(\tilde{X}) \to 0.$$ 

If $f : M \to X$ is a homotopy equivalence of $m$-dimensional manifolds there is no obstruction to making $f$ transverse regular at $Y \subset X$, but there are algebraic $K$- and $L$-theory obstructions to splitting $f$ up to homotopy, involving the MV sequence of the contractible $\mathbb{Z}[\pi]$-module chain complex $C(f : \tilde{M} \to \tilde{X})$ and algebraic handle exchanges.
Codimension 1 algebraic transversality

- Let $\pi = \pi_1 \ast \rho \pi_2$ be an injective amalgamated free product.

- **Proposition** (Waldhausen, R.) For any f.g. free $\mathbb{Z}[\pi]$-module chain complex $C$ there exist f.g. free $\mathbb{Z}[\pi_i]$-module chain subcomplexes $D_i \subset C$ and a f.g. free $\mathbb{Z}[\rho]$-module chain subcomplex $E = D_1 \cap D_2 \subset C$ with a short exact MV sequence of f.g. free $\mathbb{Z}[\pi]$-module chain complexes

\[ 0 \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\rho]} E \to \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_1]} D_1 \oplus \mathbb{Z}[\pi] \otimes_{\mathbb{Z}[\pi_2]} D_2 \to C \to 0. \]

Any two such choices $(D_1, D_2, E)$ are related by a finite sequence of algebraic handle exchanges. If $C$ is contractible there is an algebraic $K$-theory obstruction to choosing $D_1, D_2, E$ to be contractible.

- **Corollary** (Cappell, R.) If $C$ has $m$-dimensional Poincaré duality there is an algebraic $L$-theory obstruction to choosing $(D_i, \mathbb{Z}[\pi_i] \otimes_{\mathbb{Z}[\rho]} E)$ to have $m$-dimensional Poincaré-Lefschetz duality and $E$ to have $(m - 1)$-dimensional Poincaré duality.
**Universal transversality**

- Let $X$ be a finite simplicial complex, with barycentric subdivision $X'$ and dual cells

$$D(\sigma) = \{\hat{\sigma}_0\hat{\sigma}_1 \ldots \hat{\sigma}_r \mid \sigma \leq \sigma_0 < \sigma_1 < \ldots < \sigma_r\}.$$

- A map $f : M \to |X| = |X'|$ from an $m$-dimensional manifold is **universally transverse** if each inverse image

$$M(\sigma) = f^{-1}(D(\sigma)) \subset M$$

is a framed codimension $|\sigma|$ submanifold with boundary

$$\partial M(\sigma) = \bigcup_{\tau > \sigma} M(\tau).$$

- The algebraic obstruction theory for the existence and uniqueness of topological manifold structures in a homotopy type uses algebraic universal transversality.
References

  Exact sequences in the algebraic theory of surgery,

  Algebraic $L$-theory and topological manifolds,
  Cambridge University Press (1992)

  On the Novikov conjecture,

  Algebraic and combinatorial codimension 1 transversality,

- [http://www.dailymotion.com/playlist/x2v26c_Carmen_Rovi_transversality-algebra-topology](http://www.dailymotion.com/playlist/x2v26c_Carmen_Rovi_transversality-algebra-topology)
  Videos of 4 lectures on transversality in algebra and topology, Edinburgh (2013)