A COMPOSITION FORMULA FOR MANIFOLD STRUCTURES

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Homotopy equivalences and homeomorphisms

- Every homotopy equivalence of 2-dimensional manifolds is homotopic to a homeomorphism.
- For $n \geqslant 3$ a homotopy equivalence of n-dimensional manifolds $f : N \to M$ is not in general homotopic to a homeomorphism, e.g. lens spaces for $n = 3$.
- There are surgery obstructions to making the normal maps $f| : f^{-1}(L) \to L$ normal bordant to homotopy equivalences for every submanifold $L \subset M$. For $n \geqslant 5$ f is homotopic to a homeomorphism if and only if there exist such normal bordisms which are compatible with each other.
- Novikov (1964) used surgery to construct homotopy equivalences $f : N \to M = S^p \times S^q$ for certain $p, q \geqslant 2$, which are not homotopic to homeomorphisms.

The topological structure set $\mathcal{S}^{TOP}(M)$

- The **structure set** $\mathcal{S}^{TOP}(M)$ of an n-dimensional topological manifold M is the pointed set of equivalence classes of pairs (N, f) with N an n-dimensional topological manifold and $f : N \to M$ a homotopy equivalence.
- $(N, f) \sim (N', f')$ if $(f^{-1})f' : N' \to N$ is homotopic to a homeomorphism.
- Base point $(M, 1) \in \mathcal{S}^{TOP}(M)$.
- **Poincaré conjecture** $\mathcal{S}^{TOP}(S^n) = \{*\}$.
- **Borel conjecture** If M is aspherical then $\mathcal{S}^{TOP}(M) = \{*\}$.
- For $n \geqslant 5$ surgery theory expresses $\mathcal{S}^{TOP}(M)$ in terms of topological K-theory of the homotopy type of M and algebraic L-theory of $\mathbb{Z}[\pi_1(M)]$. The algebra gives $\mathcal{S}^{TOP}(M)$ a homotopy invariant functorial abelian group structure.

G/TOP

- G/TOP = the classifying space for fibre homotopy trivialized topological bundles, the homotopy fibre of $BTOP \to BG$.
- G/TOP has two H-space structures:
  1. The Whitney sum
  $$\oplus \ : \ G/TOP \times G/TOP \to G/TOP \ .$$
  2. The Sullivan 'characteristic variety' addition, or equivalently the Quinn disjoint union addition, or equivalently the direct sum of quadratic forms:
  $$+ \ : \ G/TOP \times G/TOP \to G/TOP \ .$$
- **Proposition** (R., 1978) G/TOP has the homotopy type of the space $\mathbb{L}_\bullet$ constructed algebraically from quadratic forms over $\mathbb{Z}$.
- $\pi_*(BTOP) \otimes \mathbb{Q} = \pi_*(BO) \otimes \mathbb{Q}$, $\pi_*(BG) \otimes \mathbb{Q} = 0$.
- $\pi_*(G/TOP) = L_*(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots$ the 4-periodic simply-connected surgery obstruction groups, given by the signature/8 and Arf invariant.

Why *TOP* and not *DIFF*?

- Surgery theory started in *DIFF*. The differentiable manifold structure set $\mathcal{S}^{DIFF}(M)$ can be defined for a differentiable manifold M, with $(N, f) \sim (N', f')$ if $(f^{-1})f' : N' \to N$ is homotopic to a diffeomorphism.
- $\mathcal{S}^{DIFF}(S^n) = \pi_n(TOP/O)$ is the Kervaire-Milnor group of exotic spheres, which fits into the exact sequence

$$\cdots \to \pi_{n+1}(G/O) \to L_{n+1}(\mathbb{Z}) \to \mathcal{S}^{DIFF}(S^n) \to \pi_n(G/O) \to \dots .$$

- Why not *DIFF*? In general $\mathcal{S}^{DIFF}(M)$ does not have a group structure. Essentially because G/O has a much more complicated homotopy structure than G/TOP.

Normal maps

- A manifold M has a stable normal bundle $\nu_M : M \to BTOP$.
- A **normal map** of n-dimensional manifolds $(f, b) : N \to M$ is a degree 1 map $f : N \to M$ together with a fibre homotopy trivialized topological bundle $\nu_b : M \to G/TOP$ and a bundle map $\nu_N \to \nu_M \oplus \nu_b$ over f.
- Let $\mathcal{T}^{TOP}(M)$ be the set of bordism classes of normal maps $(f, b) : N \to M$. The function

$$\mathcal{T}^{TOP}(M) \to [M, G/TOP] \; ; \; (f, b) \mapsto \nu_b$$

  is a bijection.
- The **normal invariant** of (f, b) is the class

$$(f, b) \; = \; \nu_b \in \mathcal{T}^{TOP}(M) \; = \; [M, G/TOP] \; .$$

The composition formula for degree 1 maps

- A degree 1 map $f : N \longrightarrow M$ of n-dimensional manifolds induces surjections $f_* : H_*(N) \twoheadrightarrow H_*(M)$ which are split by the **Umkehr** morphisms

$$f^! \ : \ H_*(M) \ \cong H^{n-*}(M) \xrightarrow{f^*} H^{n-*}(N) \ \cong \ H_*(N) \ .$$

- Similarly for the $\mathbb{Z}[\pi_1(M)]$-module homology of the universal cover $\widetilde{M}$ and the pullback cover $\widetilde{N} = f^*\widetilde{M}$ of N.
- The **kernel** $\mathbb{Z}[\pi_1(M)]$-modules

$$K_*(f) \ = \ H_*(f^!) \ = \ H_{*+1}(f)$$

are such that $H_*(\widetilde{N}) = H_*(\widetilde{M}) \oplus K_*(f)$.
- **The composition formula for degree 1 maps**
The composite of degree 1 maps $f : N \to M$, $g : P \to N$ is a degree 1 map $fg : P \to M$ with kernel $\mathbb{Z}[\pi_1(M)]$-modules

$$K_*(fg) = K_*(f) \oplus f_* K_*(g) \text{ where } f_* = \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} - \ .$$

$\mathcal{T}^{TOP}(M)$ has two abelian group structures, $\oplus$ and $+$.

- The Whitney sum of fibre homotopy trivialized topological bundles defines an abelian group structure

$$\oplus \ : \ \mathcal{T}^{TOP}(M) \times \mathcal{T}^{TOP}(M) \to \mathcal{T}^{TOP}(M) \ ; \ (\nu, \nu') \mapsto \nu \oplus \nu' \ .$$

- Define the disjoint union abelian group structure

$$\begin{aligned} + \ : \ \mathcal{T}^{TOP}(M) \times \mathcal{T}^{TOP}(M) &\to \mathcal{T}^{TOP}(M) \ ; \\ ((f,b) : N \to M, (f',b') : N' \to M) &\mapsto (f'',b'') \ . \end{aligned}$$

using a normal map $(f'', b'') : N'' \to M$ such that

$$K_*(f'') \ = \ K_*(f) \oplus K_*(f') \ .$$

A direct geometric construction requires surgery below the middle dimension and the Wall realization theorem.

Homotopy equivalences are normal maps

- **Proposition** A homotopy equivalence $f : N \to M$ of manifolds is automatically a normal map (f, ν_f) with

$$\nu_f \ = \ (f^{-1})^* \nu_N - \nu_M \ : \ M \to G/TOP \ .$$

- Proof uses the uniqueness of the Spivak normal fibration.
- The normal invariant defines a function

$$\eta \ : \ \mathcal{S}^{TOP}(M) \to \mathcal{T}^{TOP}(M) = [M, G/TOP] \ ; \ (N, f) \mapsto \nu_f \ .$$

- A homotopy equivalence f has $K_*(f) = 0$, so cannot use degree 1 map composition formula directly to prove a composition formula for manifold structures.
  But it is the key ingredient.

The composition of normal maps

- **Definition** The normal maps $(f,b):N\to M$, $(g,c):P\to N$ are **composable** if

$$\nu_c \in \mathrm{im}(f^*:[M,G/TOP]\rightarrowtail [N,G/TOP])\ ,$$

so $\nu_c = f^*(f^*)^{-1}(\nu_c)$ for a unique $(f^*)^{-1}(\nu_c)\in[M,G/TOP]$. In this case it is possible to define the composite normal map $(fg,bc):P\to M$.
- **Example** If $f:N\to M$ is a homotopy equivalence then f^* is a bijection, and (f,ν_f), (g,c) are composable for any (g,c).
- **Composition formula for the topological normal invariant** (Brumfiel, 1971) The normal invariant of composable normal maps $(f,b):N\to M$, $(g,c):P\to N$ is

$$\nu_{bc} \ = \ \nu_b \oplus (f^*)^{-1}(\nu_c)\in[M,G/TOP]\ .$$

Thus for homotopy equivalences f,g have

$$\nu_{fg} \ = \ \nu_f \oplus (f^*)^{-1}(\nu_g)\in[M,G/TOP]\ .$$

The surgery obstruction

- $L_*(\mathbb{Z}[\pi])$ = the 4-periodic Wall surgery obstruction groups, defined algebraically for any group π to be the Witt groups of quadratic forms over $\mathbb{Z}[\pi]$, and their automorphisms. Abelian.
- **Theorem** (Wall, 1970) An n-dimensional normal map $(f, b) : N \to M$ has a surgery obstruction

$$\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)]) \ .$$

  For $n \geqslant 5$ (f, b) is normal bordant to a homotopy equivalence if and only if $\sigma_*(f, b) = 0$.
- (R., 1980) Expression of L_* as the cobordism groups of chain complexes with Poincaré duality. Expression of the surgery obstruction $\sigma_*(f, b)$ as the cobordism class of chain complex C with $H_*(C) = K_*(f)$.

The topological surgery exact sequence

- **Theorem** (Browder-Novikov-Sullivan-Wall for *DIFF*, 1970 + Kirby-Siebenmann for *TOP*, 1970)
  For $n \geqslant 5$ the structure set of an n-dimensional topological manifold M fits into an exact sequence of pointed sets

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}^{TOP}(M) \xrightarrow{\eta} [M, G/TOP] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)])$$

  with η the normal invariant function, and σ_* the surgery obstruction.
- It is well-known that the surgery obstruction function σ_* is a homomorphism of abelian groups for $+$ on G/TOP but not for $\oplus$ on G/TOP. Thus the topological surgery exact sequence does not endow $\mathcal{S}^{TOP}(M)$ with an abelian group structure.

The algebraic surgery exact sequence

- **Theorem** (R., 1992) For any space M there is an exact sequence of abelian groups

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathcal{S}_{n+1}(M)$$
$$\to H_n(M;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])$$

  with $\mathbb{L}_\bullet$ an algebraically defined Ω-spectrum of quadratic forms over $\mathbb{Z}$, corresponding to the $+$ H-space structure. $\mathbb{L}_0 \simeq G/TOP$, $\pi_*(\mathbb{L}_\bullet) = L_*(\mathbb{Z})$. A is the assembly map.
- $H_n(M;\mathbb{L}_\bullet)$ is the cobordism group of sheaves Γ over M of n-dimensional $\mathbb{Z}$-module chain complexes with Poincaré duality. The assembly $A(\Gamma)$ is an n-dimensional $\mathbb{Z}[\pi_1(M)]$-module chain complex with Poincaré duality.
- $\mathcal{S}_{n+1}(M)$ is the cobordism group of sheaves Γ with $\mathbb{Z}[\pi_1(M)]$-contractible assembly $A(\Gamma)$.
- **Example** $\mathcal{S}_{n+1}(S^n) = 0$.

The symmetric L-theory spectrum of $\mathbb{Z}$

- The symmetric L-theory spectrum $\mathbb{L}^\bullet$ is an algebraically defined Ω-spectrum of symmetric forms over $\mathbb{Z}$, with 4-periodic homotopy groups

$$\pi_*(\mathbb{L}^\bullet) \;=\; L^*(\mathbb{Z}) \;=\; \mathbb{Z}, \mathbb{Z}_2, 0, 0, \ldots$$

  given by the signature and deRham invariant.
- $\mathbb{L}^\bullet$ is a ring spectrum with addition by direct sum $\oplus$ and product by tensor product $\otimes$ of symmetric forms over $\mathbb{Z}$.
- $\mathbb{L}_\bullet$ is an $\mathbb{L}^\bullet$-module spectrum.
- The symmetrization map $1 + T : \mathbb{L}_\bullet \to \mathbb{L}^\bullet$ is a homotopy equivalence away from 2.
- For any space M

$$H_n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H_{n-4*}(M; \mathbb{Q}) \;,$$
$$H^n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} \;=\; H^{n+4*}(M; \mathbb{Q})$$

  with $\otimes = \cup : H^p \times H^q \to H^{p+q}$.

The L-theory orientation of topology

- **Theorem** (R., 1992) (i) A topological bundle $\alpha : X \to BTOP(k)$ has an $\mathbb{L}^\bullet$-cohomology orientation, i.e. a Thom class
$$U_\alpha \in \tilde{H}^k(T(\alpha);\mathbb{L}^\bullet)\ ,$$
using $w_1(\alpha)$-twisted coefficients in the non-$\mathbb{Z}$-oriented case.
- (ii) An n-dimensional manifold M has a $\mathbb{L}^\bullet$-homology orientation, i.e. a fundamental class
$$[M]_{\mathbb{L}} \in H_n(M;\mathbb{L}^\bullet)$$
S-dual to U_{ν_M}, with quadratic L-theory Poincaré duality isomorphisms
$$[M]_{\mathbb{L}} \cap - \ :\ H^*(M;\mathbb{L}_\bullet) \xrightarrow{\cong} H_{n-*}(M;\mathbb{L}_\bullet)\ .$$

The algebraic normal invariant

- **Definition** (R., 1992) The **algebraic normal invariant** of an n-dimensional normal map $(f,b):N\to M$ is the $\mathbb{L}_\bullet$-homology class $t(f,b)\in H_n(M;\mathbb{L}_\bullet)$ with

$$(1+T)t(f,b) \;=\; f_*[N]_{\mathbb{L}}-[M]_{\mathbb{L}}\in H_n(M;\mathbb{L}^\bullet)\ .$$

- $t(f,b)$ is represented by the sheaf of simply-connected surgery obstructions of the restrictions $(f,b)|:f^{-1}(L)\to L$ for submanifolds $L\subset M$.
- **Proposition** (R., 1992) The Poincaré duality isomorphism

$$t=[M]_{\mathbb{L}}\cap -\ :\ H^0(M;\mathbb{L}_\bullet)=[M,G/TOP]\xrightarrow{\cong}H_n(M;\mathbb{L}_\bullet)$$

is given by $t:\nu_b\mapsto t(f,b)$, sending the topological normal invariant to the algebraic normal invariant.
- Assembly = the surgery obstruction

$$A(t(f,b)) \;=\; \sigma_*(f,b)\in L_n(\mathbb{Z}[\pi_1(M)])\ .$$

The Hirzebruch $\mathcal{L}$-genus

- The $\mathbb{L}^\bullet$-cohomology Thom class of $\alpha : X \to BTOP(k)$ is an integral version of the Hirzebruch $\mathcal{L}$-genus

$$U_\alpha \otimes \mathbb{Q} \;=\; \mathcal{L}(\alpha) \in H^{4*}(X;\mathbb{Q}) \;.$$

- The $\mathbb{L}^\bullet$-homology fundamental class of M is an integral version of the Poincaré dual of the Hirzebruch $\mathcal{L}$-genus

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} \;=\; [M] \cap \mathcal{L}(\nu_M) \;=\; \mathcal{L}(M) \in H_{n-4*}(M;\mathbb{Q}) \;.$$

- The algebraic normal invariant of an n-dimensional normal map $(f,b) : N \to M$ is such that

$$\begin{aligned} t(f,b) \otimes \mathbb{Q} \;&=\; f_*\mathcal{L}(N) - \mathcal{L}(M) \;=\; \mathcal{L}(\nu_M \oplus \nu_b) - \mathcal{L}(\nu_M) \\ &=\; \mathcal{L}(M) \cap (\mathcal{L}(\nu_b) - 1) \in H_n(M;\mathbb{L}_\bullet) \otimes \mathbb{Q} \;=\; H_{n-4*}(M;\mathbb{Q}) \;. \end{aligned}$$

- For $n = 4k$ $\langle [M]_{\mathbb{L}} \otimes \mathbb{Q}, [M] \rangle = \mathrm{sign}(M) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$ and $A(t(f,b)) = \sigma_*(f,b) = (\mathrm{sign}(N) - \mathrm{sign}(M))/8 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}$.

The isomorphism of surgery exact sequences, from topology to algebra

- A homotopy equivalence $f : N \to M$ of n-dimensional manifolds determines a sheaf Γ with contractible $A(\Gamma)$, with stalks $H_*(f^{-1}(x) \to \{x\})$ $(x \in M)$. The **algebraic structure invariant** of f is the cobordism class $s(f) = \Gamma \in \mathcal{S}_{n+1}(M)$.
- **Theorem** (R., 1992) For $n \geqslant 5$ the function

$$s \ : \ \mathcal{S}^{TOP}(M) \to \mathcal{S}_{n+1}(M) \ ; \ (N, f) \mapsto s(f) = \Gamma$$

is a bijection, which fits into an isomorphism from the topological sequence of pointed sets to the algebraic sequence of abelian groups

$$\begin{array}{ccccccccc}
\to L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \to & \mathcal{S}^{TOP}(M) & \xrightarrow{\eta} & [M, G/TOP] & \xrightarrow{\sigma_*} & L_n(\mathbb{Z}[\pi_1(M)]) \\
\| & & s\downarrow\cong & & t\downarrow\cong & & \| \\
\to L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \to & \mathcal{S}_{n+1}(M) & \longrightarrow & H_n(M;\mathbb{L}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1(M)])
\end{array}$$

The functorial nature of the algebraic structure sequence

- Can use the bijection $s : \mathcal{S}^{TOP}(M) \to \mathcal{S}_{n+1}(M)$ and the abelian group structure on $\mathcal{S}_{n+1}(M)$ to define a homotopy-invariant functorial abelian group structure on $\mathcal{S}^{TOP}(M)$.
- A map of spaces $f : N \to M$ induces a morphism of exact sequences of abelian groups

$$\begin{tikzcd}
\to L_{n+1}(\mathbb{Z}[\pi_1(N)]) \arrow[r] \arrow[d, "f_*"'] & \mathcal{S}_{n+1}(N) \arrow[r] \arrow[d, "f_*"'] & H_n(N;\mathbb{L}_\bullet) \arrow[r] \arrow[d, "f_*"'] & L_n(\mathbb{Z}[\pi_1(N)]) \to \\
\to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \arrow[r] & \mathcal{S}_{n+1}(M) \arrow[r] & H_n(M;\mathbb{L}_\bullet) \arrow[r] & L_n(\mathbb{Z}[\pi_1(M)]) \to
\end{tikzcd}$$

- If f is a homotopy equivalence then each f_* is an isomorphism.

The composition formula for manifold structures I.

- **Problem 1** (2006) Matthias Kreck and Wolfgang Lück asked for a composition formula for manifold structures, for use in their work on the Borel rigidity conjecture for non-aspherical manifolds.
- **Theorem** (R., 2006)
  (i) For any composable normal maps $(f,b):N\to M$, $(g,c):P\to N$

$$t(fg,cb) \;=\; t(f,b)+f_*t(g,c)\in H_n(M;\mathbb{L}_\bullet)\ .$$

  (ii) For homotopy equivalences $f:N\to M$, $g:P\to N$ of n-dimensional manifolds

$$(P,fg) \;=\; (N,f)+f_*(P,g)\in\mathcal{S}_{n+1}(M) \;=\; \mathcal{S}^{TOP}(M)\ .$$

- **Proof** (i) Local version of $K_*(fg)=K_*(f)\oplus f_*K(g)$.
  (ii) As for (i), noting that the surgery obstruction is additive for $+$.

The composition formula for manifold structures II.

- **Problem 2** (2007) Larry Taylor asked how the manifold structure composition formula matches with the 1971 Brumfiel normal invariant composition formula

$$\nu_{fg} \;=\; \nu_f \oplus (f^{-1})^*\nu_g \in [M, G/TOP] \;=\; \mathcal{T}^{TOP}(M)$$

given that $\oplus$ does not correspond to $+$.

- **Solution** The algebraic normal invariant bijection

$$t \;:\; \mathcal{T}^{TOP}(M) \to H_n(M;\mathbb{L}_\bullet)\;;\; \nu_b \mapsto t(f,b)$$

does not send the Whitney sum $\oplus$ in $\mathcal{T}^{TOP}(M)$ to the addition $+$ in $H_n(M;\mathbb{L}_\bullet)$. The image of $\nu_{fg} = \nu_f \oplus (f^*)^{-1}\nu_g \in \mathcal{T}^{TOP}(M)$ is

$$t(fg,\nu_{fg}) \;=\; t(f,\nu_f) + f_*t(g,\nu_g) \in H_n(M;\mathbb{L}\bullet)\;.$$

The product and Whitney sum of bundles

- The **product** of topological bundles $\alpha : X \to BTOP(j)$, $\beta : Y \to BTOP(k)$ is a topological bundle $\alpha \times \beta : X \times Y \to BTOP(j+k)$ with

$$\begin{aligned} T(\alpha \times \beta) &= T(\alpha) \wedge T(\beta) \ , \\ U_{\alpha\times\beta} &= U_\alpha \times U_\beta \in \tilde{H}^{j+k}(T(\alpha \times \beta); \mathbb{L}^\bullet) \ . \end{aligned}$$

- The **Whitney sum** of topological bundles $\alpha : X \to BTOP(j)$, $\beta : X \to BTOP(k)$ is the pullback of the product $\alpha \times \beta : X \times X \to BTOP(j+k)$ along the diagonal $\Delta : X \to X \times X; x \mapsto (x, x)$. Thus

$$\begin{aligned} \alpha \oplus \beta &= \Delta^*(\alpha \times \beta) \ : \ X \to BTOP(j+k) \ , \\ U_{\alpha\oplus\beta} &= \Delta^*(U_\alpha \times U_\beta) \in \tilde{H}^{j+k}(T(\alpha \oplus \beta); \mathbb{L}^\bullet) \ . \end{aligned}$$

The surgery product formula

- **Degree 1 product formula** The product of a degree 1 map $f : N \to M$ of n-dimensional manifolds and a degree 1 map $f' : N' \to M'$ of n'-dimensional manifolds is a degree 1 map $f \times f' : N \times N' \to M \times M'$ of $(n+n')$-dimensional manifolds with kernel $\mathbb{Z}[\pi_1(M) \times \pi_1(M')]$-modules

$$K_*(f \times f') \;=\; K_*(f) \otimes_{\mathbb{Z}} K_*(f') \oplus H_*(M) \otimes_{\mathbb{Z}} K_*(f') \oplus K_*(f) \otimes_{\mathbb{Z}} H_*(M') \;.$$

- **Proof** Chain level product of $H_*(N) = K_*(f) \oplus H_*(M)$, $H_*(N') = K_*(f') \oplus H_*(M')$.
- **The surgery product formula** (R., 1980) For normal maps (f, b), (f', b')

$$\begin{aligned}\sigma_*(f \times f', b \times b') &= \sigma_*(f, b) \otimes_{\mathbb{Z}} \sigma_*(f', b') \\ &\quad \oplus A([M]_{\mathbb{L}}) \otimes_{\mathbb{Z}} \sigma_*(f', b') \oplus \sigma_*(f, b) \otimes_{\mathbb{Z}} A([M']_{\mathbb{L}}) \;, \\ t(f \times f', b \times b') &= t(f, b) \otimes_{\mathbb{Z}} t(f', b') \\ &\quad \oplus [M]_{\mathbb{L}} \otimes_{\mathbb{Z}} t(f', b') \oplus t(f, b) \otimes_{\mathbb{Z}} [M']_{\mathbb{L}} \;.\end{aligned}$$

The $\mathbb{L}_\bullet$-coefficient intersection pairing

- Given an n-dimensional manifold M use the tensor product of quadratic forms over $\mathbb{Z}$ and the $\mathbb{L}_\bullet$-Poincaré duality

$$[M]_{\mathbb{L}} \cap \ : \ H^*(M;\mathbb{L}_\bullet) \cong H_{n-*}(M;\mathbb{L}_\bullet)$$

to define an intersection pairing

$$\otimes : H_p(M;\mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_q(M;\mathbb{L}_\bullet) \to H_{p+q-n}(M;\mathbb{L}_\bullet) \ .$$

- Only interested in the case $p = q = n$

$$\otimes : H_n(M;\mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_n(M;\mathbb{L}_\bullet) \to H_n(M;\mathbb{L}_\bullet) \ .$$

The Whitney sum of normal invariants

- **Proposition** The normal map $(f'', b'') : N'' \to M$ obtained from a product of normal maps

$$(f \times f', b \times b') : N \times N' \to M \times M$$

by transversality at $\Delta : M \subset M \times M$ has normal invariant $\nu_{b''} = \nu_b \oplus \nu_{b'} \in [M, G/TOP]$ and algebraic normal invariant

$$t(f'', b'') = t(f, b) + t(f', b') + t(f, b) \otimes t(f', b') \in H_n(M; \mathbb{L}_\bullet) .$$

- **Proof** Pull back the surgery product formula along $\Delta : M \subset M \times M$, use $\nu_\Delta = \tau_M$, $\nu_\Delta \oplus \nu_M \simeq *$ and the commutative square

$$\begin{array}{ccc}
H^0(M \times M; \mathbb{L}_\bullet) & \xrightarrow[\cong]{[M \times M]_{\mathbb{L}} \cap -} & H_{2n}(M \times M; \mathbb{L}_\bullet) \\
\downarrow \Delta^* & & \downarrow \\
H^0(M; \mathbb{L}_\bullet) & \xrightarrow[\cong]{[M]_{\mathbb{L}} \cap -} & H_{2n}(M \times M, M \times M \backslash \Delta(M); \mathbb{L}_\bullet) \cong H_n(M; \mathbb{L}_\bullet)
\end{array}$$

The two H-space structures on G/TOP

- **Proposition** (R., 1978) The two H-space structures

$$\oplus\ ,\ +\ :\ G/TOP \times G/TOP \to G/TOP$$

are related by

$$x \oplus y\ =\ x + y + x \otimes y$$

where $x \otimes y$ is given by the tensor product of quadratic forms over $\mathbb{Z}$ in the algebraic model $\mathbb{L}_0$ for G/TOP.

- **Corollary** For any manifold M there are two abelian group structures $\oplus$ and $+$ on

$$\mathcal{T}^{TOP}(M)\ =\ H_n(M;\mathbb{L}_\bullet)\ =\ [M, G/TOP]\ ,$$

which are also related by $x \oplus y = x + y + x \otimes y$.

The pushforward and the pullback

- A map $f : N \to M$ of n-dimensional manifolds induces
  (i) a function $f_* : \mathcal{T}^{TOP}(N) \to \mathcal{T}^{TOP}(M)$ which is a morphism of abelian groups with respect to $+$,
  (ii) a function $f^* : \mathcal{T}^{TOP}(M) \to \mathcal{T}^{TOP}(N)$ which is a morphism of abelian groups with respect to $\oplus$.
- If f is a homotopy equivalence then both the pushforward f_* and the pullback $(f^{-1})*$ are isomorphisms, but in general they are not the same! The square of bijections

$$\begin{tikzcd}
\mathcal{T}^{TOP}(N) = H^0(N;\mathbb{L}_\bullet) \arrow[r, "(f^{-1})^*", "\cong"'] \arrow[d, "{[N]_{\mathbb{L}}\cap}", "\cong"'] & H^0(M;\mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M) \arrow[d, "\cong"', "{[M]_{\mathbb{L}}\cap -}"] \\
\mathcal{T}^{TOP}(N) = H_n(N;\mathbb{L}_\bullet) \arrow[r, "f_*", "\cong"'] & H_n(M;\mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M)
\end{tikzcd}$$

  fails to commute by the cap product with

$$f_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}} \;=\; (1+T)t(f,\nu_f) \in H_n(M;\mathbb{L}^\bullet)\ .$$

- In general, $f_*\mathcal{L}(N) \neq \mathcal{L}(M) \in H_{n-4*}(M;\mathbb{Q})$.

The algebraic normal invariant of the pushforward of a manifold structure

- Let $f : N \to M$, $g : P \to N$ be homotopy equivalences of n-dimensional manifold, and let

$$\begin{aligned} f_*(P,g) &= (Q,h) \in \mathcal{S}^{TOP}(M) = \mathcal{S}_{n+1}(M) \ , \\ (f^{-1})^* t(g,\nu_g) &= t(f',b') \in \mathcal{T}^{TOP}(M) = [M, G/TOP] \ . \end{aligned}$$

- **Proposition** The algebraic normal invariants are such that

$$\begin{aligned} t(h,\nu_h) &= f_* t(g,\nu_g) = t(fg,\nu_{fg}) - t(f,\nu_f) \\ &= f_*[N]_{\mathbb{L}} \cap \nu_{b'} = [M]_{\mathbb{L}} \cap \nu_{b'} + t(f,\nu_f) \otimes t(f',b') \\ &= t(f',b') + t(f,\nu_f) \otimes t(f',b') \\ &\in \mathrm{im}(\mathcal{S}_{n+1}(M) \to H_n(M;\mathbb{L}_\bullet)) = \ker(A) \ . \end{aligned}$$

- In general, $\sigma_*(f',b') = A(t(f',b')) \neq 0 \in L_n(\mathbb{Z}[\pi_1(M)])$.

Recovering the Brumfiel normal invariant composition formula from the algebra

- **Proposition** The algebraic normal invariant of the composite $fg : P \to M$ of homotopy equivalences $f : N \to M$, $g : P \to N$ of n-dimensional manifolds is given by

$$\begin{aligned} t(fg, \nu_{fg}) &= t(f, \nu_f) + f_* t(g, \nu_g) \\ &= t(f, \nu_f) + t(f', b') + t(f, \nu_f) \otimes t(f', b') \\ &= t(f, \nu_f) \oplus t(f', b') \\ &= t(f, \nu_f) \oplus (f^{-1})^* t(g, \nu_g) \\ &\qquad \in \mathcal{T}^{TOP}(M) = [M, G/TOP] = H_n(M; \mathbb{L}_\bullet) \ . \end{aligned}$$

A worked example: $M = S^p \times S^q$, for $p, q \geqslant 2$

- The assembly map in quadratic L-theory is given by

$$A \ : \ H_{p+q}(M;\mathbb{L}_\bullet) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \to L_{p+q}(\mathbb{Z}) \ ; \qquad (x,y,z) \mapsto z \ .$$

- The addition $+$, intersection pairing $\otimes$ and Whitney sum $\oplus$ are given by

$$\begin{aligned}
(x,y,z) + (x',y',z') &= (x+x',y+y',z+z') \ , \\
(x,y,z) \otimes (x',y',z') &= (0,0,x \otimes y' + x' \otimes y) \ , \\
(x,y,z) \oplus (x',y',z') &= (x+x',y+y',x \otimes y' + x' \otimes y + z + z') \ .
\end{aligned}$$

- $\mathcal{S}^{TOP}(M) = \mathcal{S}_{p+q+1}(M) = \ker(A) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z})$.
- Case $p \equiv q \equiv 0 (\mathrm{mod}\ 4)$ detected by the $\mathcal{L}$-genus.

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