A COMPOSITION FORMULA FOR MANIFOLD STRUCTURES

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Homotopy equivalences and homeomorphisms

▶ Every homotopy equivalence of 2-dimensional manifolds is homotopic to a homeomorphism.

▶ For \( n \geq 3 \) a homotopy equivalence of \( n \)-dimensional manifolds \( f : N \to M \) is not in general homotopic to a homeomorphism, e.g. lens spaces for \( n = 3 \).

▶ There are surgery obstructions to making the normal maps \( f| : f^{-1}(L) \to L \) normal bordant to homotopy equivalences for every submanifold \( L \subset M \). For \( n \geq 5 \) \( f \) is homotopic to a homeomorphism if and only if there exist such normal bordisms which are compatible with each other.

▶ Novikov (1964) used surgery to construct homotopy equivalences \( f : N \to M = S^p \times S^q \) for certain \( p, q \geq 2 \), which are not homotopic to homeomorphisms.
The topological structure set $S^{TOP}(M)$

- The **structure set** $S^{TOP}(M)$ of an $n$-dimensional topological manifold $M$ is the pointed set of equivalence classes of pairs $(N, f)$ with $N$ an $n$-dimensional topological manifold and $f : N \to M$ a homotopy equivalence.

- $(N, f) \sim (N', f')$ if $(f^{-1})f' : N' \to N$ is homotopic to a homeomorphism.

- Base point $(M, 1) \in S^{TOP}(M)$.

- **Poincaré conjecture** $S^{TOP}(S^n) = \{*\}$.

- **Borel conjecture** If $M$ is aspherical then $S^{TOP}(M) = \{*\}$.

- For $n \geq 5$ surgery theory expresses $S^{TOP}(M)$ in terms of topological $K$-theory of the homotopy type of $M$ and algebraic $L$-theory of $\mathbb{Z}[\pi_1(M)]$. The algebra gives $S^{TOP}(M)$ a homotopy invariant functorial abelian group structure.
$G/TOP$

$G/TOP = \text{the classifying space for fibre homotopy trivialized topological bundles, the homotopy fibre of } BTOP \to BG.$

$G/TOP$ has two $H$-space structures:

1. The Whitney sum

$$\oplus : \ G/TOP \times G/TOP \to G/TOP.$$  

2. The Sullivan ‘characteristic variety’ addition, or equivalently the Quinn disjoint union addition, or equivalently the direct sum of quadratic forms:

$$+ : \ G/TOP \times G/TOP \to G/TOP.$$  

**Proposition** (R., 1978) $G/TOP$ has the homotopy type of the space $\mathbb{L}_\bullet$ constructed algebraically from quadratic forms over $\mathbb{Z}$.

$\pi_*(BTOP) \otimes \mathbb{Q} = \pi_*(BO) \otimes \mathbb{Q}, \pi_*(BG) \otimes \mathbb{Q} = 0.$

$\pi_*(G/TOP) = L_*(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots$ the 4-periodic simply-connected surgery obstruction groups, given by the signature/8 and Arf invariant.
Why \textit{TOP} and not \textit{DIFF}?

- Surgery theory started in \textit{DIFF}. The differentiable manifold structure set $S^{\text{DIFF}}(M)$ can be defined for a differentiable manifold $M$, with $(N, f) \sim (N', f')$ if $(f^{-1})f': N' \to N$ is homotopic to a diffeomorphism.

- $S^{\text{DIFF}}(S^n) = \pi_n(\text{TOP}/O)$ is the Kervaire-Milnor group of exotic spheres, which fits into the exact sequence

$$
\cdots \to \pi_{n+1}(G/O) \to L_{n+1}(\mathbb{Z}) \to S^{\text{DIFF}}(S^n) \to \pi_n(G/O) \to \cdots \, .
$$

- Why not \textit{DIFF}? In general $S^{\text{DIFF}}(M)$ does not have a group structure. Essentially because $G/O$ has a much more complicated homotopy structure than $G/\text{TOP}$. 

Normal maps

- A manifold $M$ has a stable normal bundle $\nu_M : M \to BTOP$.
- A **normal map** of $n$-dimensional manifolds $(f, b) : N \to M$ is a degree 1 map $f : N \to M$ together with a fibre homotopy trivialized topological bundle $\nu_b : M \to G/TOP$ and a bundle map $\nu_N \to \nu_M \oplus \nu_b$ over $f$.
- Let $\mathcal{T}^{TOP}(M)$ be the set of bordism classes of normal maps $(f, b) : N \to M$. The function

$$\mathcal{T}^{TOP}(M) \to [M, G/TOP] ; (f, b) \mapsto \nu_b$$

is a bijection.
- The **normal invariant** of $(f, b)$ is the class

$$(f, b) = \nu_b \in \mathcal{T}^{TOP}(M) = [M, G/TOP].$$
The composition formula for degree 1 maps

- A degree 1 map $f : N \to M$ of $n$-dimensional manifolds induces surjections $f_* : H_*(N) \to H_*(M)$ which are split by the Umkehr morphisms

$$f! : H_*(M) \cong H^{n-*}(M) \xrightarrow{f^*} H^{n-*}(N) \cong H_*(N).$$

- Similarly for the $\mathbb{Z}[\pi_1(M)]$-module homology of the universal cover $\tilde{M}$ and the pullback cover $\tilde{N} = f^*\tilde{M}$ of $N$.

- The kernel $\mathbb{Z}[\pi_1(M)]$-modules

$$K_*(f) = H_*(f!) = H_{*+1}(f)$$

are such that $H_*(\tilde{N}) = H_*(\tilde{M}) \oplus K_*(f)$.

- The composition formula for degree 1 maps

The composite of degree 1 maps $f : N \to M$, $g : P \to N$ is a degree 1 map $fg : P \to M$ with kernel $\mathbb{Z}[\pi_1(M)]$-modules

$$K_*(fg) = K_*(f) \oplus f_*K_*(g)$$

where $f_* = \mathbb{Z}[\pi_1(M)] \otimes \mathbb{Z}[\pi_1(N)]$. 
\( \mathcal{I}^{TOP}(M) \) has two abelian group structures, \( \oplus \) and \( + \).

- The Whitney sum of fibre homotopy trivialized topological bundles defines an abelian group structure

\[ \oplus : \mathcal{I}^{TOP}(M) \times \mathcal{I}^{TOP}(M) \to \mathcal{I}^{TOP}(M) ; (\nu, \nu') \mapsto \nu \oplus \nu'. \]

- Define the disjoint union abelian group structure

\[ + : \mathcal{I}^{TOP}(M) \times \mathcal{I}^{TOP}(M) \to \mathcal{I}^{TOP}(M) ; \\
((f, b) : N \to M, (f', b') : N' \to M) \mapsto (f'', b'') \]

using a normal map \((f'', b'') : N'' \to M\) such that

\[ K_\ast(f'') = K_\ast(f) \oplus K_\ast(f') \]

A direct geometric construction requires surgery below the middle dimension and the Wall realization theorem.
Homotopy equivalences are normal maps

- **Proposition** A homotopy equivalence $f : N \to M$ of manifolds is automatically a normal map $(f, \nu_f)$ with

  $$\nu_f = (f^{-1})^* \nu_N - \nu_M : M \to G/\text{TOP}.$$ 

- Proof uses the uniqueness of the Spivak normal fibration.
- The normal invariant defines a function

  $$\eta : S^{\text{TOP}}(M) \to T^{\text{TOP}}(M) = [M, G/\text{TOP}] ; (N, f) \mapsto \nu_f.$$ 

- A homotopy equivalence $f$ has $K_\ast(f) = 0$, so cannot use degree 1 map composition formula directly to prove a composition formula for manifold structures. But it is the key ingredient.
The composition of normal maps

Definition The normal maps \((f, b): N \to M, (g, c): P \to N\) are composable if
\[
\nu_c \in \text{im}(f^*: [M, G/TOP] \to [N, G/TOP]) ,
\]
so \(\nu_c = f^* (f^*)^{-1} (\nu_c)\) for a unique \((f^*)^{-1} (\nu_c) \in [M, G/TOP]\). In this case it is possible to define the composite normal map \((fg, bc): P \to M\).

Example If \(f: N \to M\) is a homotopy equivalence then \(f^*\) is a bijection, and \((f, \nu_f), (g, c)\) are composable for any \((g, c)\).

Composition formula for the topological normal invariant (Brumfiel, 1971) The normal invariant of composable normal maps \((f, b): N \to M, (g, c): P \to N\) is
\[
\nu_{bc} = \nu_b \oplus (f^*)^{-1} (\nu_c) \in [M, G/TOP] .
\]
Thus for homotopy equivalences \(f, g\) have
\[
\nu_{fg} = \nu_f \oplus (f^*)^{-1} (\nu_g) \in [M, G/TOP] .
\]
The surgery obstruction

- $L_*(\mathbb{Z}[\pi]) = \text{the 4-periodic Wall surgery obstruction groups, defined algebraically for any group } \pi \text{ to be the Witt groups of quadratic forms over } \mathbb{Z}[\pi], \text{ and their automorphisms. Abelian.}$
- **Theorem** (Wall, 1970) An $n$-dimensional normal map $(f, b) : N \to M$ has a surgery obstruction

\[ \sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)]) \, . \]

For $n \geq 5$ $(f, b)$ is normal bordant to a homotopy equivalence if and only if $\sigma_*(f, b) = 0$.

- (R., 1980) Expression of $L_*$ as the cobordism groups of chain complexes with Poincaré duality. Expression of the surgery obstruction $\sigma_*(f, b)$ as the cobordism class of chain complex $C$ with $H_*(C) = K_*(f)$. 
The topological surgery exact sequence

**Theorem** (Browder-Novikov-Sullivan-Wall for \(\text{DIFF} \), 1970 + Kirby-Siebenmann for \(\text{TOP} \), 1970)

For \( n \geq 5 \) the structure set of an \( n \)-dimensional topological manifold \( M \) fits into an exact sequence of pointed sets

\[
\cdots \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) \rightarrow S^{\text{TOP}}(M) \rightarrow [M, G/\text{TOP}] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)])
\]

with \( \eta \) the normal invariant function, and \( \sigma_* \) the surgery obstruction.

**It is well-known** that the surgery obstruction function \( \sigma_* \) is a homomorphism of abelian groups for \( + \) on \( G/\text{TOP} \) but not for \( \oplus \) on \( G/\text{TOP} \). Thus the topological surgery exact sequence does not endow \( S^{\text{TOP}}(M) \) with an abelian group structure.
The algebraic surgery exact sequence

- **Theorem** (R., 1992) For any space $M$ there is an exact sequence of abelian groups

$$
\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S_{n+1}(M) \to H_n(M; \mathbb{L}_\bullet) \to A \to L_n(\mathbb{Z}[\pi_1(M)])
$$

with $\mathbb{L}_\bullet$ an algebraically defined $\Omega$-spectrum of quadratic forms over $\mathbb{Z}$, corresponding to the $+ H$-space structure. $\mathbb{L}_0 \simeq G/TOP$, $\pi_*(\mathbb{L}_\bullet) = L_*(\mathbb{Z})$. $A$ is the assembly map.

- $H_n(M; \mathbb{L}_\bullet)$ is the cobordism group of sheaves $\Gamma$ over $M$ of $n$-dimensional $\mathbb{Z}$-module chain complexes with Poincaré duality. The assembly $A(\Gamma)$ is an $n$-dimensional $\mathbb{Z}[\pi_1(M)]$-module chain complex with Poincaré duality.

- $S_{n+1}(M)$ is the cobordism group of sheaves $\Gamma$ with $\mathbb{Z}[\pi_1(M)]$-contractible assembly $A(\Gamma)$.

- **Example** $S_{n+1}(S^n) = 0$. 
The symmetric $L$-theory spectrum of $\mathbb{Z}$

- The symmetric $L$-theory spectrum $\mathbb{L}^\bullet$ is an algebraically defined $\Omega$-spectrum of symmetric forms over $\mathbb{Z}$, with 4-periodic homotopy groups

$$\pi_\ast(\mathbb{L}^\bullet) = L^\ast(\mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, 0, \ldots$$

given by the signature and deRham invariant.

- $\mathbb{L}^\bullet$ is a ring spectrum with addition by direct sum $\oplus$ and product by tensor product $\otimes$ of symmetric forms over $\mathbb{Z}$.

- $\mathbb{L}^\bullet$ is an $\mathbb{L}^\bullet$-module spectrum.

- The symmetrization map $1 + T : \mathbb{L}_\ast \to \mathbb{L}^\bullet$ is a homotopy equivalence away from 2.

- For any space $M$

$$H_n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} = H_{n-4\ast}(M; \mathbb{Q}) ,$$

$$H^n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} = H^{n+4\ast}(M; \mathbb{Q})$$

with $\otimes = \cup : H^p \times H^q \to H^{p+q}$.
The $L$-theory orientation of topology

- **Theorem (R., 1992)** (i) A topological bundle $\alpha : X \to BTOP(k)$ has an $\mathbb{L}^\bullet$-cohomology orientation, i.e. a Thom class
  \[ U_\alpha \in \tilde{H}^k(T(\alpha); \mathbb{L}^\bullet), \]
  using $w_1(\alpha)$-twisted coefficients in the non-$\mathbb{Z}$-oriented case.

- (ii) An $n$-dimensional manifold $M$ has a $\mathbb{L}^\bullet$-homology orientation, i.e. a fundamental class
  \[ [M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet) \]
  $S$-dual to $U_{\nu M}$, with quadratic $L$-theory Poincaré duality isomorphisms
  \[ [M]_{\mathbb{L}} \cap - : H^*(M; \mathbb{L}_\bullet) \xrightarrow{\cong} H_{n-*}(M; \mathbb{L}_\bullet). \]
The algebraic normal invariant

Definition (R., 1992) The **algebraic normal invariant** of an $n$-dimensional normal map $(f, b) : N \to M$ is the $\mathbb{L} \cdot$-homology class $t(f, b) \in H_n(M; \mathbb{L} \cdot)$ with

$$(1 + T)t(f, b) = f_*[N]_\mathbb{L} - [M]_\mathbb{L} \in H_n(M; \mathbb{L} \cdot).$$

$t(f, b)$ is represented by the sheaf of simply-connected surgery obstructions of the restrictions $(f, b)| : f^{-1}(L) \to L$ for submanifolds $L \subset M$.

Proposition (R., 1992) The Poincaré duality isomorphism

$$t = [M]_\mathbb{L} \cap - : H^0(M; \mathbb{L} \cdot) = [M, G/\text{TOP}] \xrightarrow{\cong} H_n(M; \mathbb{L} \cdot)$$

is given by $t : \nu_b \mapsto t(f, b)$, sending the topological normal invariant to the algebraic normal invariant.

Assembly = the surgery obstruction

$$A(t(f, b)) = \sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)]) .$$
The Hirzebruch $L$-genus

- The $\mathbb{L}^{\bullet}$-cohomology Thom class of $\alpha : X \to BTOP(k)$ is an integral version of the Hirzebruch $L$-genus
  \[ U_\alpha \otimes \mathbb{Q} = L(\alpha) \in H^{4\ast}(X; \mathbb{Q}) . \]

- The $\mathbb{L}^{\bullet}$-homology fundamental class of $M$ is an integral version of the Poincaré dual of the Hirzebruch $L$-genus
  \[ [M]_{\mathbb{L}} \otimes \mathbb{Q} = [M] \cap L(\nu_M) = L(M) \in H_{n-4\ast}(M; \mathbb{Q}) . \]

- The algebraic normal invariant of an $n$-dimensional normal map $(f, b) : N \to M$ is such that
  \[ t(f, b) \otimes \mathbb{Q} = f_\ast L(N) - L(M) = L(\nu_M \oplus \nu_b) - L(\nu_M) \]
  \[ = L(M) \cap (L(\nu_b) - 1) \in H_n(M; \mathbb{L}^{\bullet}) \otimes \mathbb{Q} = H_{n-4\ast}(M; \mathbb{Q}) . \]

- For $n = 4k$ $\langle [M]_{\mathbb{L}} \otimes \mathbb{Q}, [M] \rangle = \text{sign}(M) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$ and
  \[ A(t(f, b)) = \sigma_\ast(f, b) = (\text{sign}(N) - \text{sign}(M))/8 \in L_{4k}(\mathbb{Z}) = \mathbb{Z} . \]
The isomorphism of surgery exact sequences, from topology to algebra

A homotopy equivalence $f : N \to M$ of $n$-dimensional manifolds determines a sheaf $\Gamma$ with contractible $A(\Gamma)$, with stalks $H_*(f^{-1}(x) \to \{x\}) \ (x \in M)$. The algebraic structure invariant of $f$ is the cobordism class $s(f) = \Gamma \in S_{n+1}(M)$.

Theorem (R., 1992) For $n \geq 5$ the function

$$s : S^{TOP}(M) \to S_{n+1}(M) \ ; \ (N, f) \mapsto s(f) = \Gamma$$

is a bijection, which fits into an isomorphism from the topological sequence of pointed sets to the algebraic sequence of abelian groups

$$\begin{array}{c}
\to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S^{TOP}(M) \xrightarrow{\eta} [M, G/TOP] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)]) \\
\downarrow \cong s \downarrow \cong t \downarrow \cong \\
\to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to S_{n+1}(M) \to H_n(M; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])
\end{array}$$
The functorial nature of the algebraic structure sequence

- Can use the bijection $s : S^{TOP}(M) \rightarrow S_{n+1}(M)$ and the abelian group structure on $S_{n+1}(M)$ to define a homotopy-invariant functorial abelian group structure on $S^{TOP}(M)$.

- A map of spaces $f : N \rightarrow M$ induces a morphism of exact sequences of abelian groups

$$\rightarrow L_{n+1}(\mathbb{Z}[\pi_1(N)]) \rightarrow S_{n+1}(N) \rightarrow H_n(N; \mathbb{Z}_\bullet) \rightarrow L_n(\mathbb{Z}[\pi_1(N)]) \rightarrow$$

$$\rightarrow f_* \rightarrow f_* \rightarrow f_* \rightarrow f_*$$

$$\rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) \rightarrow S_{n+1}(M) \rightarrow H_n(M; \mathbb{Z}_\bullet) \rightarrow L_n(\mathbb{Z}[\pi_1(M)]) \rightarrow$$

- If $f$ is a homotopy equivalence then each $f_*$ is an isomorphism.
The composition formula for manifold structures I.


Theorem (R., 2006)
(i) For any composable normal maps \((f, b) : N \to M, (g, c) : P \to N\)

\[ t(fg, cb) = t(f, b) + f_* t(g, c) \in H_n(M; \mathbb{L}_\bullet) . \]

(ii) For homotopy equivalences \(f : N \to M, g : P \to N\) of \(n\)-dimensional manifolds

\[ (P, fg) = (N, f) + f_* (P, g) \in S_{n+1}(M) = S^{TOP}(M) . \]

Proof (i) Local version of \(K_*(fg) = K_*(f) \oplus f_* K(g)\).
(ii) As for (i), noting that the surgery obstruction is additive for \(+\).
The composition formula for manifold structures II.

Problem 2 (2007) Larry Taylor asked how the manifold structure composition formula matches with the 1971 Brumfiel normal invariant composition formula

\[ \nu_{fg} = \nu_f \oplus (f^{-1})^*\nu_g \in [M, G/TOP] = \mathcal{T}^{TOP}(M) \]

given that \( \oplus \) does not correspond to \( + \).

Solution The algebraic normal invariant bijection

\[ t : \mathcal{T}^{TOP}(M) \rightarrow H_n(M; \mathbb{L}_\bullet) ; \nu_b \mapsto t(f, b) \]

does not send the Whitney sum \( \oplus \) in \( \mathcal{T}^{TOP}(M) \) to the addition \( + \) in \( H_n(M; \mathbb{L}_\bullet) \). The image of \( \nu_{fg} = \nu_f \oplus (f^*)^{-1}\nu_g \in \mathcal{T}^{TOP}(M) \) is

\[ t(fg, \nu_{fg}) = t(f, \nu_f) + f_*t(g, \nu_g) \in H_n(M; \mathbb{L}_\bullet) \].
The product and Whitney sum of bundles

- The **product** of topological bundles $\alpha : X \to BTOP(j)$, $\beta : Y \to BTOP(k)$ is a topological bundle $\alpha \times \beta : X \times Y \to BTOP(j + k)$ with

  $$
  T(\alpha \times \beta) = T(\alpha) \wedge T(\beta), \\
  U_{\alpha \times \beta} = U_\alpha \times U_\beta \in \tilde{H}^{j+k}(T(\alpha \times \beta); \mathbb{L}^\bullet).
  $$

- The **Whitney sum** of topological bundles $\alpha : X \to BTOP(j)$, $\beta : X \to BTOP(k)$ is the pullback of the product $\alpha \times \beta : X \times X \to BTOP(j + k)$ along the diagonal $\Delta : X \to X \times X; x \mapsto (x, x)$. Thus

  $$
  \alpha \oplus \beta = \Delta^*(\alpha \times \beta) : X \to BTOP(j + k), \\
  U_{\alpha \oplus \beta} = \Delta^*(U_\alpha \times U_\beta) \in \tilde{H}^{j+k}(T(\alpha \oplus \beta); \mathbb{L}^\bullet).
  $$
The surgery product formula

- **Degree 1 product formula** The product of a degree 1 map $f : N \rightarrow M$ of $n$-dimensional manifolds and a degree 1 map $f' : N' \rightarrow M'$ of $n'$-dimensional manifolds is a degree 1 map $f \times f' : N \times N' \rightarrow M \times M'$ of $(n + n')$-dimensional manifolds with kernel $\mathbb{Z}[\pi_1(M) \times \pi_1(M')]$-modules

$$K_*(f \times f') = K_*(f) \otimes \mathbb{Z} K_*(f') \oplus H_*(M) \otimes \mathbb{Z} K_*(f') \oplus K_*(f) \otimes \mathbb{Z} H_*(M').$$

- **Proof** Chain level product of $H_*(N) = K_*(f) \oplus H_*(M)$, $H_*(N') = K_*(f') \oplus H_*(M').$

- **The surgery product formula** (R., 1980) For normal maps $(f, b), (f', b')$

$$\sigma_*(f \times f', b \times b') = \sigma_*(f, b) \otimes \mathbb{Z} \sigma_*(f', b')$$
$$\oplus A([M]_L) \otimes \mathbb{Z} \sigma_*(f', b') \oplus \sigma_*(f, b) \otimes \mathbb{Z} A([M']_L),$$
$$t(f \times f', b \times b') = t(f, b) \otimes \mathbb{Z} t(f', b')$$
$$\oplus [M]_L \otimes \mathbb{Z} t(f', b') \oplus t(f, b) \otimes \mathbb{Z} [M']_L.$$
The $\mathbb{L}_\bullet$-coefficient intersection pairing

Given an $n$-dimensional manifold $M$ use the tensor product of quadratic forms over $\mathbb{Z}$ and the $\mathbb{L}_\bullet$-Poincaré duality

$$[M]_{\mathbb{L}} \cap : H^*(M; \mathbb{L}_\bullet) \cong H_{n-\bullet}(M; \mathbb{L}_\bullet)$$

to define an intersection pairing

$$\otimes : H_p(M; \mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_q(M; \mathbb{L}_\bullet) \to H_{p+q-n}(M; \mathbb{L}_\bullet).$$

Only interested in the case $p = q = n$

$$\otimes : H_n(M; \mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_n(M; \mathbb{L}_\bullet) \to H_n(M; \mathbb{L}_\bullet).$$
The Whitney sum of normal invariants

**Proposition** The normal map \((f'', b'') : N'' \to M\) obtained from a product of normal maps

\[(f \times f', b \times b') : N \times N' \to M \times M\]

by transversality at \(\Delta : M \subset M \times M\) has normal invariant \(\nu_{b''} = \nu_b \oplus \nu_{b'} \in [M, G/TOP]\) and algebraic normal invariant

\[t(f'', b'') = t(f, b) + t(f', b') + t(f, b) \otimes t(f', b') \in H_n(M; \mathbb{L}_\bullet).\]

**Proof** Pull back the surgery product formula along \(\Delta : M \subset M \times M\), use \(\nu_\Delta = \tau_M\), \(\nu_\Delta \oplus \nu_M \simeq \ast\) and the commutative square

\[
\begin{array}{ccc}
H^0(M \times M; \mathbb{L}_\bullet) & \xrightarrow{[M \times M]_{L \cap -}} & H_{2n}(M \times M; \mathbb{L}_\bullet) \\
\downarrow \Delta^* & & \downarrow \\
H^0(M; \mathbb{L}_\bullet) & \xrightarrow{[M]_{L \cap -}} & H_{2n}(M \times M, M \times M \setminus \Delta(M); \mathbb{L}_\bullet) \cong H_n(M; \mathbb{L}_\bullet)
\end{array}
\]
The two $H$-space structures on $G/TOP$

**Proposition** (R., 1978) The two $H$-space structures

\[ \oplus, + : G/TOP \times G/TOP \to G/TOP \]

are related by

\[ x \oplus y = x + y + x \otimes y \]

where $x \otimes y$ is given by the tensor product of quadratic forms over $\mathbb{Z}$ in the algebraic model $\mathbb{L}_0$ for $G/TOP$.

**Corollary** For any manifold $M$ there are two abelian group structures $\oplus$ and $+$ on

\[ \mathcal{T}^{TOP}(M) = H_n(M; \mathbb{L}_\bullet) = [M, G/TOP], \]

which are also related by $x \oplus y = x + y + x \otimes y$. 
The pushforward and the pullback

- A map $f : N \to M$ of $n$-dimensional manifolds induces
  (i) a function $f_* : \mathcal{T}^{TOP}(N) \to \mathcal{T}^{TOP}(M)$ which is a morphism of
      abelian groups with respect to $+$,
  (ii) a function $f^* : \mathcal{T}^{TOP}(M) \to \mathcal{T}^{TOP}(N)$ which is a morphism of
       abelian groups with respect to $\oplus$.

- If $f$ is a homotopy equivalence then both the pushforward $f_*$
  and the pullback $(f^{-1})^*$ are isomorphisms, but in general they
  are not the same! The square of bijections

$$
\begin{align*}
\mathcal{T}^{TOP}(N) & = H^0(N; \mathbb{L}_\bullet) \xrightarrow{(f^{-1})^*} H^0(M; \mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M) \\
[N]_{\mathbb{L}} \cap & \cong \\
\downarrow & \cong \\
\mathcal{T}^{TOP}(N) & = H_n(N; \mathbb{L}_\bullet) \xrightarrow{f_*} H_n(M; \mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M)
\end{align*}
$$

fails to commute by the cap product with

$$
f_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}} = (1 + T)t(f, \nu_f) \in H_n(M; \mathbb{L}_\bullet) .
$$

- In general, $f_*\mathcal{L}(N) \neq \mathcal{L}(M) \in H_{n-4*}(M; \mathbb{Q})$. 

The algebraic normal invariant of the pushforward of a manifold structure

Let \( f : N \to M, \ g : P \to N \) be homotopy equivalences of \( n \)-dimensional manifold, and let

\[
f_*(P, g) = (Q, h) \in S^{TOP}(M) = S_{n+1}(M),
\]

\[
(f^{-1})^* t(g, \nu_g) = t(f', b') \in \mathcal{T}^{TOP}(M) = [M, G/TOP].
\]

**Proposition** The algebraic normal invariants are such that

\[
t(h, \nu_h) = f_* t(g, \nu_g) = t(fg, \nu_{fg}) - t(f, \nu_f)
\]

\[
= f_* [N]_{\mathbb{L}} \cap \nu_{b'} = [M]_{\mathbb{L}} \cap \nu_{b'} + t(f, \nu_f) \otimes t(f', b')
\]

\[
= t(f', b') + t(f, \nu_f) \otimes t(f', b')
\]

\[
\in \text{im}(S_{n+1}(M) \to H_n(M; \mathbb{L} \cdot)) = \ker(A).
\]

**In general,** \( \sigma_*(f', b') = A(t(f', b')) \neq 0 \in L_n(\mathbb{Z}[\pi_1(M)]) \).
Recovering the Brumfiel normal invariant composition formula from the algebra

**Proposition** The algebraic normal invariant of the composite \( fg : P \to M \) of homotopy equivalences \( f : N \to M \), \( g : P \to N \) of \( n \)-dimensional manifolds is given by

\[
\begin{align*}
t(fg, \nu_{fg}) &= t(f, \nu_f) + f_\ast t(g, \nu_g) \\
&= t(f, \nu_f) + t(f', b') + t(f, \nu_f) \otimes t(f', b') \\
&= t(f, \nu_f) \oplus t(f', b') \\
&= t(f, \nu_f) \oplus (f^{-1})_\ast t(g, \nu_g) \\
&\in \mathcal{T}^{TOP}(M) = [M, G/\text{TOP}] = H_n(M; \mathbb{L}_\bullet)
\end{align*}
\]
A worked example: \( M = S^p \times S^q, \text{ for } p, q \geq 2 \)

- The assembly map in quadratic \( L \)-theory is given by

\[
A : H_{p+q}(M; \mathbb{L}_\bullet) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \to L_{p+q}(\mathbb{Z}) ;
\]

\[
(x, y, z) \mapsto z.
\]

- The addition \(+\), intersection pairing \(\otimes\) and Whitney sum \(\oplus\) are given by

\[
(x, y, z) + (x', y', z') = (x + x', y + y', z + z'),
\]

\[
(x, y, z) \otimes (x', y', z') = (0, 0, x \otimes y' + x' \otimes y),
\]

\[
(x, y, z) \oplus (x', y', z') = (x + x', y + y', x \otimes y' + x' \otimes y + z + z').
\]

- \( S^{TOP}(M) = S_{p+q+1}(M) = \ker(A) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}). \)

- Case \( p \equiv q \equiv 0(\text{mod } 4) \) detected by the \( L \)-genus.
References


5. ——, *Algebraic L-theory and topological manifolds*, Tracts in Mathematics 102, Cambridge (1992)