

A COMPOSITION FORMULA FOR MANIFOLD STRUCTURES

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Homotopy equivalences and homeomorphisms

- ▶ Every homotopy equivalence of 2-dimensional manifolds is homotopic to a homeomorphism.
- ▶ For $n \geq 3$ a homotopy equivalence of n -dimensional manifolds $f : N \rightarrow M$ is not in general homotopic to a homeomorphism, e.g. lens spaces for $n = 3$.
- ▶ There are surgery obstructions to making the normal maps $f|_L : f^{-1}(L) \rightarrow L$ normal bordant to homotopy equivalences for every submanifold $L \subset M$. For $n \geq 5$ f is homotopic to a homeomorphism if and only if there exist such normal bordisms which are compatible with each other.
- ▶ Novikov (1964) used surgery to construct homotopy equivalences $f : N \rightarrow M = S^p \times S^q$ for certain $p, q \geq 2$, which are not homotopic to homeomorphisms.

The topological structure set $\mathcal{S}^{TOP}(M)$

- ▶ The **structure set** $\mathcal{S}^{TOP}(M)$ of an n -dimensional topological manifold M is the pointed set of equivalence classes of pairs (N, f) with N an n -dimensional topological manifold and $f : N \rightarrow M$ a homotopy equivalence.
- ▶ $(N, f) \sim (N', f')$ if $(f^{-1})f' : N' \rightarrow N$ is homotopic to a homeomorphism.
- ▶ Base point $(M, 1) \in \mathcal{S}^{TOP}(M)$.
- ▶ **Poincaré conjecture** $\mathcal{S}^{TOP}(S^n) = \{*\}$.
- ▶ **Borel conjecture** If M is aspherical then $\mathcal{S}^{TOP}(M) = \{*\}$.
- ▶ For $n \geq 5$ surgery theory expresses $\mathcal{S}^{TOP}(M)$ in terms of topological K -theory of the homotopy type of M and algebraic L -theory of $\mathbb{Z}[\pi_1(M)]$. The algebra gives $\mathcal{S}^{TOP}(M)$ a homotopy invariant functorial abelian group structure.

G/TOP

- ▶ G/TOP = the classifying space for fibre homotopy trivialized topological bundles, the homotopy fibre of $BTOP \rightarrow BG$.
- ▶ G/TOP has two H -space structures:

1. The Whitney sum

$$\oplus : G/TOP \times G/TOP \rightarrow G/TOP .$$

2. The Sullivan 'characteristic variety' addition, or equivalently the Quinn disjoint union addition, or equivalently the direct sum of quadratic forms:

$$+ : G/TOP \times G/TOP \rightarrow G/TOP .$$

- ▶ **Proposition** (R., 1978) G/TOP has the homotopy type of the space \mathbb{L}_\bullet constructed algebraically from quadratic forms over \mathbb{Z} .
- ▶ $\pi_*(BTOP) \otimes \mathbb{Q} = \pi_*(BO) \otimes \mathbb{Q}$, $\pi_*(BG) \otimes \mathbb{Q} = 0$.
- ▶ $\pi_*(G/TOP) = L_*(\mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \dots$ the 4-periodic simply-connected surgery obstruction groups, given by the signature/8 and Arf invariant.

Why *TOP* and not *DIFF*?

- ▶ Surgery theory started in *DIFF*. The differentiable manifold structure set $\mathcal{S}^{DIFF}(M)$ can be defined for a differentiable manifold M , with $(N, f) \sim (N', f')$ if $(f^{-1})f' : N' \rightarrow N$ is homotopic to a diffeomorphism.
- ▶ $\mathcal{S}^{DIFF}(S^n) = \pi_n(TOP/O)$ is the Kervaire-Milnor group of exotic spheres, which fits into the exact sequence

$$\cdots \rightarrow \pi_{n+1}(G/O) \rightarrow L_{n+1}(\mathbb{Z}) \rightarrow \mathcal{S}^{DIFF}(S^n) \rightarrow \pi_n(G/O) \rightarrow \dots$$

- ▶ Why not *DIFF*? In general $\mathcal{S}^{DIFF}(M)$ does not have a group structure. Essentially because G/O has a much more complicated homotopy structure than G/TOP .

Normal maps

- ▶ A manifold M has a stable normal bundle $\nu_M : M \rightarrow BTOP$.
- ▶ A **normal map** of n -dimensional manifolds $(f, b) : N \rightarrow M$ is a degree 1 map $f : N \rightarrow M$ together with a fibre homotopy trivialized topological bundle $\nu_b : M \rightarrow G/TOP$ and a bundle map $\nu_N \rightarrow \nu_M \oplus \nu_b$ over f .
- ▶ Let $\mathcal{T}^{TOP}(M)$ be the set of bordism classes of normal maps $(f, b) : N \rightarrow M$. The function

$$\mathcal{T}^{TOP}(M) \rightarrow [M, G/TOP] ; (f, b) \mapsto \nu_b$$

is a bijection.

- ▶ The **normal invariant** of (f, b) is the class

$$(f, b) = \nu_b \in \mathcal{T}^{TOP}(M) = [M, G/TOP] .$$

The composition formula for degree 1 maps

- ▶ A degree 1 map $f : N \rightarrow M$ of n -dimensional manifolds induces surjections $f_* : H_*(N) \twoheadrightarrow H_*(M)$ which are split by the **Umkehr** morphisms

$$f^! : H_*(M) \cong H^{n-*}(M) \xrightarrow{f^*} H^{n-*}(N) \cong H_*(N) .$$

- ▶ Similarly for the $\mathbb{Z}[\pi_1(M)]$ -module homology of the universal cover \tilde{M} and the pullback cover $\tilde{N} = f^*\tilde{M}$ of N .
- ▶ The **kernel** $\mathbb{Z}[\pi_1(M)]$ -modules

$$K_*(f) = H_*(f^!) = H_{*+1}(f)$$

are such that $H_*(\tilde{N}) = H_*(\tilde{M}) \oplus K_*(f)$.

- ▶ **The composition formula for degree 1 maps**

The composite of degree 1 maps $f : N \rightarrow M$, $g : P \rightarrow N$ is a degree 1 map $fg : P \rightarrow M$ with kernel $\mathbb{Z}[\pi_1(M)]$ -modules

$$K_*(fg) = K_*(f) \oplus f_*K_*(g) \text{ where } f_* = \mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(N)]} - .$$

$\mathcal{T}^{TOP}(M)$ has two abelian group structures, \oplus and $+$.

- ▶ The Whitney sum of fibre homotopy trivialized topological bundles defines an abelian group structure

$$\oplus : \mathcal{T}^{TOP}(M) \times \mathcal{T}^{TOP}(M) \rightarrow \mathcal{T}^{TOP}(M) ; (\nu, \nu') \mapsto \nu \oplus \nu' .$$

- ▶ Define the disjoint union abelian group structure

$$\begin{aligned} + : \mathcal{T}^{TOP}(M) \times \mathcal{T}^{TOP}(M) &\rightarrow \mathcal{T}^{TOP}(M) ; \\ ((f, b) : N \rightarrow M, (f', b') : N' \rightarrow M) &\mapsto (f'', b'') . \end{aligned}$$

using a normal map $(f'', b'') : N'' \rightarrow M$ such that

$$K_*(f'') = K_*(f) \oplus K_*(f') .$$

A direct geometric construction requires surgery below the middle dimension and the Wall realization theorem.

Homotopy equivalences are normal maps

- ▶ **Proposition** A homotopy equivalence $f : N \rightarrow M$ of manifolds is automatically a normal map (f, ν_f) with

$$\nu_f = (f^{-1})^* \nu_N - \nu_M : M \rightarrow G/TOP .$$

- ▶ Proof uses the uniqueness of the Spivak normal fibration.
- ▶ The normal invariant defines a function

$$\eta : \mathcal{S}^{TOP}(M) \rightarrow \mathcal{T}^{TOP}(M) = [M, G/TOP] ; (N, f) \mapsto \nu_f .$$

- ▶ A homotopy equivalence f has $K_*(f) = 0$, so cannot use degree 1 map composition formula directly to prove a composition formula for manifold structures.
But it is the key ingredient.

The composition of normal maps

- ▶ **Definition** The normal maps $(f, b) : N \rightarrow M$, $(g, c) : P \rightarrow N$ are **composable** if

$$\nu_c \in \text{im}(f^* : [M, G/TOP] \rightarrow [N, G/TOP]) ,$$

so $\nu_c = f^*(f^*)^{-1}(\nu_c)$ for a unique $(f^*)^{-1}(\nu_c) \in [M, G/TOP]$. In this case it is possible to define the composite normal map $(fg, bc) : P \rightarrow M$.

- ▶ **Example** If $f : N \rightarrow M$ is a homotopy equivalence then f^* is a bijection, and (f, ν_f) , (g, c) are composable for any (g, c) .
- ▶ **Composition formula for the topological normal invariant** (Brumfiel, 1971) The normal invariant of composable normal maps $(f, b) : N \rightarrow M$, $(g, c) : P \rightarrow N$ is

$$\nu_{bc} = \nu_b \oplus (f^*)^{-1}(\nu_c) \in [M, G/TOP] .$$

Thus for homotopy equivalences f, g have

$$\nu_{fg} = \nu_f \oplus (f^*)^{-1}(\nu_g) \in [M, G/TOP] .$$

The surgery obstruction

- ▶ $L_*(\mathbb{Z}[\pi])$ = the 4-periodic Wall surgery obstruction groups, defined algebraically for any group π to be the Witt groups of quadratic forms over $\mathbb{Z}[\pi]$, and their automorphisms. Abelian.
- ▶ **Theorem** (Wall, 1970) An n -dimensional normal map $(f, b) : N \rightarrow M$ has a surgery obstruction

$$\sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)]) .$$

For $n \geq 5$ (f, b) is normal bordant to a homotopy equivalence if and only if $\sigma_*(f, b) = 0$.

- ▶ (R., 1980) Expression of L_* as the cobordism groups of chain complexes with Poincaré duality. Expression of the surgery obstruction $\sigma_*(f, b)$ as the cobordism class of chain complex C with $H_*(C) = K_*(f)$.

The topological surgery exact sequence

- ▶ **Theorem** (Browder-Novikov-Sullivan-Wall for *DIFF*, 1970 + Kirby-Siebenmann for *TOP*, 1970)

For $n \geq 5$ the structure set of an n -dimensional topological manifold M fits into an exact sequence of pointed sets

$$\begin{aligned} \cdots \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) &\rightarrow \mathcal{S}^{TOP}(M) \\ &\xrightarrow{\eta} [M, G/TOP] \xrightarrow{\sigma_*} L_n(\mathbb{Z}[\pi_1(M)]) \end{aligned}$$

with η the normal invariant function, and σ_* the surgery obstruction.

- ▶ It is well-known that the surgery obstruction function σ_* is a homomorphism of abelian groups for $+$ on G/TOP but not for \oplus on G/TOP . Thus the topological surgery exact sequence does not endow $\mathcal{S}^{TOP}(M)$ with an abelian group structure.

The algebraic surgery exact sequence

- ▶ **Theorem** (R., 1992) For any space M there is an exact sequence of abelian groups

$$\begin{aligned} \cdots \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) &\rightarrow \mathcal{S}_{n+1}(M) \\ &\rightarrow H_n(M; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)]) \end{aligned}$$

with \mathbb{L}_\bullet an algebraically defined Ω -spectrum of quadratic forms over \mathbb{Z} , corresponding to the $+$ H -space structure. $\mathbb{L}_0 \simeq G/TOP$, $\pi_*(\mathbb{L}_\bullet) = L_*(\mathbb{Z})$. A is the assembly map.

- ▶ $H_n(M; \mathbb{L}_\bullet)$ is the cobordism group of sheaves Γ over M of n -dimensional \mathbb{Z} -module chain complexes with Poincaré duality. The assembly $A(\Gamma)$ is an n -dimensional $\mathbb{Z}[\pi_1(M)]$ -module chain complex with Poincaré duality.
- ▶ $\mathcal{S}_{n+1}(M)$ is the cobordism group of sheaves Γ with $\mathbb{Z}[\pi_1(M)]$ -contractible assembly $A(\Gamma)$.
- ▶ **Example** $\mathcal{S}_{n+1}(S^n) = 0$.

The symmetric L -theory spectrum of \mathbb{Z}

- ▶ The symmetric L -theory spectrum \mathbb{L}^\bullet is an algebraically defined Ω -spectrum of symmetric forms over \mathbb{Z} , with 4-periodic homotopy groups

$$\pi_*(\mathbb{L}^\bullet) = L^*(\mathbb{Z}) = \mathbb{Z}, \mathbb{Z}_2, 0, 0, \dots$$

given by the signature and deRham invariant.

- ▶ \mathbb{L}^\bullet is a ring spectrum with addition by direct sum \oplus and product by tensor product \otimes of symmetric forms over \mathbb{Z} .
- ▶ \mathbb{L}_\bullet is an \mathbb{L}^\bullet -module spectrum.
- ▶ The symmetrization map $1 + T : \mathbb{L}_\bullet \rightarrow \mathbb{L}^\bullet$ is a homotopy equivalence away from 2.
- ▶ For any space M

$$H_n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} = H_{n-4*}(M; \mathbb{Q}),$$

$$H^n(M; \mathbb{L}^\bullet) \otimes \mathbb{Q} = H^{n+4*}(M; \mathbb{Q})$$

with $\otimes = \cup : H^p \times H^q \rightarrow H^{p+q}$.

The L -theory orientation of topology

- **Theorem** (R., 1992) (i) A topological bundle $\alpha : X \rightarrow BTOP(k)$ has an \mathbb{L}^\bullet -cohomology orientation, i.e. a Thom class

$$U_\alpha \in \tilde{H}^k(T(\alpha); \mathbb{L}^\bullet) ,$$

using $w_1(\alpha)$ -twisted coefficients in the non- \mathbb{Z} -oriented case.

- (ii) An n -dimensional manifold M has a \mathbb{L}^\bullet -homology orientation, i.e. a fundamental class

$$[M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet)$$

S -dual to U_{ν_M} , with quadratic L -theory Poincaré duality isomorphisms

$$[M]_{\mathbb{L}} \cap - : H^*(M; \mathbb{L}_\bullet) \xrightarrow{\cong} H_{n-*}(M; \mathbb{L}_\bullet) .$$

The algebraic normal invariant

- ▶ **Definition** (R., 1992) The **algebraic normal invariant** of an n -dimensional normal map $(f, b) : N \rightarrow M$ is the \mathbb{L}_\bullet -homology class $t(f, b) \in H_n(M; \mathbb{L}_\bullet)$ with

$$(1 + T)t(f, b) = f_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}} \in H_n(M; \mathbb{L}^\bullet) .$$

- ▶ $t(f, b)$ is represented by the sheaf of simply-connected surgery obstructions of the restrictions $(f, b)| : f^{-1}(L) \rightarrow L$ for submanifolds $L \subset M$.
- ▶ **Proposition** (R., 1992) The Poincaré duality isomorphism

$$t = [M]_{\mathbb{L}} \cap - : H^0(M; \mathbb{L}_\bullet) = [M, G/TOP] \xrightarrow{\cong} H_n(M; \mathbb{L}_\bullet)$$

is given by $t : \nu_b \mapsto t(f, b)$, sending the topological normal invariant to the algebraic normal invariant.

- ▶ Assembly = the surgery obstruction

$$A(t(f, b)) = \sigma_*(f, b) \in L_n(\mathbb{Z}[\pi_1(M)]) .$$

The Hirzebruch \mathcal{L} -genus

- ▶ The \mathbb{L}^\bullet -cohomology Thom class of $\alpha : X \rightarrow BTOP(k)$ is an integral version of the Hirzebruch \mathcal{L} -genus

$$U_\alpha \otimes \mathbb{Q} = \mathcal{L}(\alpha) \in H^{4*}(X; \mathbb{Q}) .$$

- ▶ The \mathbb{L}^\bullet -homology fundamental class of M is an integral version of the Poincaré dual of the Hirzebruch \mathcal{L} -genus

$$[M]_{\mathbb{L}} \otimes \mathbb{Q} = [M] \cap \mathcal{L}(\nu_M) = \mathcal{L}(M) \in H_{n-4*}(M; \mathbb{Q}) .$$

- ▶ The algebraic normal invariant of an n -dimensional normal map $(f, b) : N \rightarrow M$ is such that

$$\begin{aligned} t(f, b) \otimes \mathbb{Q} &= f_* \mathcal{L}(N) - \mathcal{L}(M) = \mathcal{L}(\nu_M \oplus \nu_b) - \mathcal{L}(\nu_M) \\ &= \mathcal{L}(M) \cap (\mathcal{L}(\nu_b) - 1) \in H_n(M; \mathbb{L}_\bullet) \otimes \mathbb{Q} = H_{n-4*}(M; \mathbb{Q}) . \end{aligned}$$

- ▶ For $n = 4k$ $\langle [M]_{\mathbb{L}} \otimes \mathbb{Q}, [M] \rangle = \text{sign}(M) \in L^{4k}(\mathbb{Z}) = \mathbb{Z}$ and $A(t(f, b)) = \sigma_*(f, b) = (\text{sign}(N) - \text{sign}(M))/8 \in L_{4k}(\mathbb{Z}) = \mathbb{Z}$.

The isomorphism of surgery exact sequences, from topology to algebra

- ▶ A homotopy equivalence $f : N \rightarrow M$ of n -dimensional manifolds determines a sheaf Γ with contractible $A(\Gamma)$, with stalks $H_*(f^{-1}(x) \rightarrow \{x\})$ ($x \in M$). The **algebraic structure invariant** of f is the cobordism class $s(f) = \Gamma \in \mathcal{S}_{n+1}(M)$.
- ▶ **Theorem** (R., 1992) For $n \geq 5$ the function

$$s : \mathcal{S}^{TOP}(M) \rightarrow \mathcal{S}_{n+1}(M) ; (N, f) \mapsto s(f) = \Gamma$$

is a bijection, which fits into an isomorphism from the topological sequence of pointed sets to the algebraic sequence of abelian groups

$$\begin{array}{ccccccc}
 \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \rightarrow & \mathcal{S}^{TOP}(M) & \xrightarrow{\eta} & [M, G/TOP] & \xrightarrow{\sigma_*} & L_n(\mathbb{Z}[\pi_1(M)]) \\
 \parallel & & \downarrow s \cong & & \downarrow t \cong & & \parallel \\
 \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \rightarrow & \mathcal{S}_{n+1}(M) & \rightarrow & H_n(M; \mathbb{L}_\bullet) & \xrightarrow{A} & L_n(\mathbb{Z}[\pi_1(M)])
 \end{array}$$

The functorial nature of the algebraic structure sequence

- ▶ Can use the bijection $s : \mathcal{S}^{TOP}(M) \rightarrow \mathcal{S}_{n+1}(M)$ and the abelian group structure on $\mathcal{S}_{n+1}(M)$ to define a homotopy-invariant functorial abelian group structure on $\mathcal{S}^{TOP}(M)$.
- ▶ A map of spaces $f : N \rightarrow M$ induces a morphism of exact sequences of abelian groups

$$\begin{array}{ccccccc}
 \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(N)]) & \rightarrow & \mathcal{S}_{n+1}(N) & \rightarrow & H_n(N; \mathbb{L}_\bullet) & \rightarrow & L_n(\mathbb{Z}[\pi_1(N)]) \rightarrow \\
 & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
 \rightarrow L_{n+1}(\mathbb{Z}[\pi_1(M)]) & \rightarrow & \mathcal{S}_{n+1}(M) & \rightarrow & H_n(M; \mathbb{L}_\bullet) & \rightarrow & L_n(\mathbb{Z}[\pi_1(M)]) \rightarrow
 \end{array}$$

- ▶ If f is a homotopy equivalence then each f_* is an isomorphism.

The composition formula for manifold structures I.

- ▶ **Problem 1** (2006) Matthias Kreck and Wolfgang Lück asked for a composition formula for manifold structures, for use in their work on the Borel rigidity conjecture for non-aspherical manifolds.

- ▶ **Theorem** (R., 2006)

(i) For any composable normal maps $(f, b) : N \rightarrow M$,
 $(g, c) : P \rightarrow N$

$$t(fg, cb) = t(f, b) + f_*t(g, c) \in H_n(M; \mathbb{L}_\bullet) .$$

(ii) For homotopy equivalences $f : N \rightarrow M$, $g : P \rightarrow N$ of n -dimensional manifolds

$$(P, fg) = (N, f) + f_*(P, g) \in \mathcal{S}_{n+1}(M) = \mathcal{S}^{TOP}(M) .$$

- ▶ **Proof** (i) Local version of $K_*(fg) = K_*(f) \oplus f_*K(g)$.
- (ii) As for (i), noting that the surgery obstruction is additive for $+$.

The composition formula for manifold structures II.

- ▶ **Problem 2** (2007) Larry Taylor asked how the manifold structure composition formula matches with the 1971 Brumfiel normal invariant composition formula

$$\nu_{fg} = \nu_f \oplus (f^{-1})^* \nu_g \in [M, G/TOP] = \mathcal{T}^{TOP}(M)$$

given that \oplus does not correspond to $+$.

- ▶ **Solution** The algebraic normal invariant bijection

$$t : \mathcal{T}^{TOP}(M) \rightarrow H_n(M; \mathbb{L}_\bullet) ; \nu_b \mapsto t(f, b)$$

does not send the Whitney sum \oplus in $\mathcal{T}^{TOP}(M)$ to the addition $+$ in $H_n(M; \mathbb{L}_\bullet)$. The image of $\nu_{fg} = \nu_f \oplus (f^*)^{-1} \nu_g \in \mathcal{T}^{TOP}(M)$ is

$$t(fg, \nu_{fg}) = t(f, \nu_f) + f_* t(g, \nu_g) \in H_n(M; \mathbb{L}_\bullet) .$$

The product and Whitney sum of bundles

- ▶ The **product** of topological bundles $\alpha : X \rightarrow BTOP(j)$, $\beta : Y \rightarrow BTOP(k)$ is a topological bundle $\alpha \times \beta : X \times Y \rightarrow BTOP(j+k)$ with

$$T(\alpha \times \beta) = T(\alpha) \wedge T(\beta) ,$$

$$U_{\alpha \times \beta} = U_\alpha \times U_\beta \in \tilde{H}^{j+k}(T(\alpha \times \beta); \mathbb{L}^\bullet) .$$

- ▶ The **Whitney sum** of topological bundles $\alpha : X \rightarrow BTOP(j)$, $\beta : X \rightarrow BTOP(k)$ is the pullback of the product $\alpha \times \beta : X \times X \rightarrow BTOP(j+k)$ along the diagonal $\Delta : X \rightarrow X \times X; x \mapsto (x, x)$. Thus

$$\alpha \oplus \beta = \Delta^*(\alpha \times \beta) : X \rightarrow BTOP(j+k) ,$$

$$U_{\alpha \oplus \beta} = \Delta^*(U_\alpha \times U_\beta) \in \tilde{H}^{j+k}(T(\alpha \oplus \beta); \mathbb{L}^\bullet) .$$

The surgery product formula

- ▶ **Degree 1 product formula** The product of a degree 1 map $f : N \rightarrow M$ of n -dimensional manifolds and a degree 1 map $f' : N' \rightarrow M'$ of n' -dimensional manifolds is a degree 1 map $f \times f' : N \times N' \rightarrow M \times M'$ of $(n + n')$ -dimensional manifolds with kernel $\mathbb{Z}[\pi_1(M) \times \pi_1(M')]$ -modules

$$K_*(f \times f') = K_*(f) \otimes_{\mathbb{Z}} K_*(f') \oplus H_*(M) \otimes_{\mathbb{Z}} K_*(f') \oplus K_*(f) \otimes_{\mathbb{Z}} H_*(M').$$

- ▶ **Proof** Chain level product of $H_*(N) = K_*(f) \oplus H_*(M)$, $H_*(N') = K_*(f') \oplus H_*(M')$.
- ▶ **The surgery product formula** (R., 1980) For normal maps $(f, b), (f', b')$

$$\begin{aligned} \sigma_*(f \times f', b \times b') &= \sigma_*(f, b) \otimes_{\mathbb{Z}} \sigma_*(f', b') \\ &\quad \oplus A([M]_{\mathbb{L}}) \otimes_{\mathbb{Z}} \sigma_*(f', b') \oplus \sigma_*(f, b) \otimes_{\mathbb{Z}} A([M']_{\mathbb{L}}), \\ t(f \times f', b \times b') &= t(f, b) \otimes_{\mathbb{Z}} t(f', b') \\ &\quad \oplus [M]_{\mathbb{L}} \otimes_{\mathbb{Z}} t(f', b') \oplus t(f, b) \otimes_{\mathbb{Z}} [M']_{\mathbb{L}}. \end{aligned}$$

The \mathbb{L}_\bullet -coefficient intersection pairing

- ▶ Given an n -dimensional manifold M use the tensor product of quadratic forms over \mathbb{Z} and the \mathbb{L}_\bullet -Poincaré duality

$$[M]_{\mathbb{L}} \cap : H^*(M; \mathbb{L}_\bullet) \cong H_{n-*}(M; \mathbb{L}_\bullet)$$

to define an intersection pairing

$$\otimes : H_p(M; \mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_q(M; \mathbb{L}_\bullet) \rightarrow H_{p+q-n}(M; \mathbb{L}_\bullet) .$$

- ▶ Only interested in the case $p = q = n$

$$\otimes : H_n(M; \mathbb{L}_\bullet) \otimes_{\mathbb{Z}} H_n(M; \mathbb{L}_\bullet) \rightarrow H_n(M; \mathbb{L}_\bullet) .$$

The Whitney sum of normal invariants

- **Proposition** The normal map $(f'', b'') : N'' \rightarrow M$ obtained from a product of normal maps

$$(f \times f', b \times b') : N \times N' \rightarrow M \times M$$

by transversality at $\Delta : M \subset M \times M$ has normal invariant $\nu_{b''} = \nu_b \oplus \nu_{b'} \in [M, G/TOP]$ and algebraic normal invariant $t(f'', b'') = t(f, b) + t(f', b') + t(f, b) \otimes t(f', b') \in H_n(M; \mathbb{L}_\bullet)$.

- **Proof** Pull back the surgery product formula along $\Delta : M \subset M \times M$, use $\nu_\Delta = \tau_M$, $\nu_\Delta \oplus \nu_M \simeq *$ and the commutative square

$$\begin{array}{ccc} H^0(M \times M; \mathbb{L}_\bullet) & \xrightarrow[\cong]{[M \times M]_{\mathbb{L} \cap -}} & H_{2n}(M \times M; \mathbb{L}_\bullet) \\ \downarrow \Delta^* & & \downarrow \\ H^0(M; \mathbb{L}_\bullet) & \xrightarrow[\cong]{[M]_{\mathbb{L} \cap -}} & H_{2n}(M \times M, M \times M \setminus \Delta(M); \mathbb{L}_\bullet) \cong H_n(M; \mathbb{L}_\bullet) \end{array}$$

The two H -space structures on G/TOP

- **Proposition** (R., 1978) The two H -space structures

$$\oplus, + : G/TOP \times G/TOP \rightarrow G/TOP$$

are related by

$$x \oplus y = x + y + x \otimes y$$

where $x \otimes y$ is given by the tensor product of quadratic forms over \mathbb{Z} in the algebraic model \mathbb{L}_0 for G/TOP .

- **Corollary** For any manifold M there are two abelian group structures \oplus and $+$ on

$$\mathcal{T}^{TOP}(M) = H_n(M; \mathbb{L}_\bullet) = [M, G/TOP],$$

which are also related by $x \oplus y = x + y + x \otimes y$.

The pushforward and the pullback

- ▶ A map $f : N \rightarrow M$ of n -dimensional manifolds induces
 - a function $f_* : \mathcal{T}^{TOP}(N) \rightarrow \mathcal{T}^{TOP}(M)$ which is a morphism of abelian groups with respect to $+$,
 - a function $f^* : \mathcal{T}^{TOP}(M) \rightarrow \mathcal{T}^{TOP}(N)$ which is a morphism of abelian groups with respect to \oplus .
- ▶ If f is a homotopy equivalence then both the pushforward f_* and the pullback $(f^{-1})_*$ are isomorphisms, but in general they are not the same! The square of bijections

$$\begin{array}{ccc}
 \mathcal{T}^{TOP}(N) = H^0(N; \mathbb{L}_\bullet) & \xrightarrow[\cong]{(f^{-1})_*} & H^0(M; \mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M) \\
 \downarrow [N]_{\mathbb{L}} \cap - \cong & & \cong \downarrow [M]_{\mathbb{L}} \cap - \\
 \mathcal{T}^{TOP}(N) = H_n(N; \mathbb{L}_\bullet) & \xrightarrow[\cong]{f_*} & H_n(M; \mathbb{L}_\bullet) = \mathcal{T}^{TOP}(M)
 \end{array}$$

fails to commute by the cap product with

$$f_*[N]_{\mathbb{L}} - [M]_{\mathbb{L}} = (1 + T)t(f, \nu_f) \in H_n(M; \mathbb{L}^\bullet).$$

- ▶ In general, $f_*\mathcal{L}(N) \neq \mathcal{L}(M) \in H_{n-4*}(M; \mathbb{Q})$.

The algebraic normal invariant of the pushforward of a manifold structure

- ▶ Let $f : N \rightarrow M$, $g : P \rightarrow N$ be homotopy equivalences of n -dimensional manifold, and let

$$\begin{aligned} f_*(P, g) &= (Q, h) \in \mathcal{S}^{TOP}(M) = \mathcal{S}_{n+1}(M), \\ (f^{-1})^*t(g, \nu_g) &= t(f', b') \in \mathcal{T}^{TOP}(M) = [M, G/TOP]. \end{aligned}$$

- ▶ **Proposition** The algebraic normal invariants are such that

$$\begin{aligned} t(h, \nu_h) &= f_*t(g, \nu_g) = t(fg, \nu_{fg}) - t(f, \nu_f) \\ &= f_*[N]_{\mathbb{L}} \cap \nu_{b'} = [M]_{\mathbb{L}} \cap \nu_{b'} + t(f, \nu_f) \otimes t(f', b') \\ &= t(f', b') + t(f, \nu_f) \otimes t(f', b') \\ &\in \text{im}(\mathcal{S}_{n+1}(M) \rightarrow H_n(M; \mathbb{L}_\bullet)) = \ker(A). \end{aligned}$$

- ▶ In general, $\sigma_*(f', b') = A(t(f', b')) \neq 0 \in L_n(\mathbb{Z}[\pi_1(M)])$.

Recovering the Brumfiel normal invariant composition formula from the algebra

- **Proposition** The algebraic normal invariant of the composite $fg : P \rightarrow M$ of homotopy equivalences $f : N \rightarrow M$, $g : P \rightarrow N$ of n -dimensional manifolds is given by

$$\begin{aligned}
 t(fg, \nu_{fg}) &= t(f, \nu_f) + f_* t(g, \nu_g) \\
 &= t(f, \nu_f) + t(f', b') + t(f, \nu_f) \otimes t(f', b') \\
 &= t(f, \nu_f) \oplus t(f', b') \\
 &= t(f, \nu_f) \oplus (f^{-1})^* t(g, \nu_g) \\
 &\in \mathcal{T}^{TOP}(M) = [M, G/TOP] = H_n(M; \mathbb{L}_\bullet) .
 \end{aligned}$$

A worked example: $M = S^p \times S^q$, for $p, q \geq 2$

- ▶ The assembly map in quadratic L -theory is given by

$$A : H_{p+q}(M; \mathbb{L}_\bullet) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z}) \oplus L_{p+q}(\mathbb{Z}) \rightarrow L_{p+q}(\mathbb{Z}) ;$$

$$(x, y, z) \mapsto z .$$

- ▶ The addition $+$, intersection pairing \otimes and Whitney sum \oplus are given by

$$(x, y, z) + (x', y', z') = (x + x', y + y', z + z') ,$$

$$(x, y, z) \otimes (x', y', z') = (0, 0, x \otimes y' + x' \otimes y) ,$$

$$(x, y, z) \oplus (x', y', z') = (x + x', y + y', x \otimes y' + x' \otimes y + z + z') .$$

- ▶ $\mathcal{S}^{TOP}(M) = \mathcal{S}_{p+q+1}(M) = \ker(A) = L_p(\mathbb{Z}) \oplus L_q(\mathbb{Z})$.
- ▶ Case $p \equiv q \equiv 0 \pmod{4}$ detected by the \mathcal{L} -genus.

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