Exotic spheres and the Kervaire invariant

Addendum to the slides

Michel Kervaire's work in surgery and knot theory

http://www.maths.ed.ac.uk/~aar/slides/kervaire.pdf

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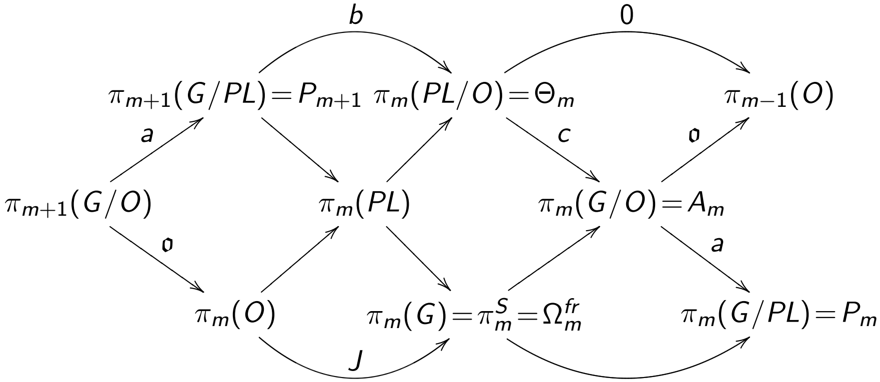
29th May, 2009

The Kervaire-Milnor braid for m I.

- For any $m \geqslant 5$ there is a commutative braid of 4 interlocking exact sequences (slide 46)



The Kervaire-Milnor braid for m II.

- Θ_m is the K-M group of oriented m-dimensional exotic spheres.
- $P_m = \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, \ldots$ is the m-dimensional simply-connected surgery obstruction group. These groups only depend on $m(\text{mod}\,4)$.
- $a : A_m = \pi_m(G/O) \to P_m$ sends an m-dimensional almost framed differentiable manifold M to the surgery obstruction of the corresponding normal map $(f, b) : M^m \to S^m$.
- For even m $b : P_m \to \Theta_{m-1}$ sends a nonsingular $(-)^{m/2}$-quadratic form over $\mathbb{Z}$ of rank r to the boundary $\Sigma^{m-1} = \partial W$ of the Milnor plumbing W of r copies of $\tau_{S^{m/2}}$ realizing the form.
- The image of b is the subgroup $bP_m \subseteq \Theta_{m-1}$ of the $(m-1)$-dimensional exotic spheres Σ^{m-1} which are the boundaries $\Sigma^{m-1} = \partial W$ of m-dimensional framed differentiable manifolds W.
- $c : \Theta_m \to \pi_m(G/O)$ sends an m-dimensional exotic sphere Σ^m to its fibre-homotopy trivialized stable normal bundle.

The Kervaire-Milnor braid for m III.

- $J : \pi_m(O) \to \pi_m(G) = \pi_m^S$ is the J-homomorphism sending $\eta : S^m \to O$ to the m-dimensional framed differentiable manifold (S^m, η).
- The map $\mathfrak{o} : \pi_m(G/O) = A_m \to \pi_{m-1}(O)$ sends an m-dimensional almost framed differentiable manifold M to the framing obstruction

$$\mathfrak{o}(M) \in \pi_m(BO) \ = \ \pi_{m-1}(O)\ .$$

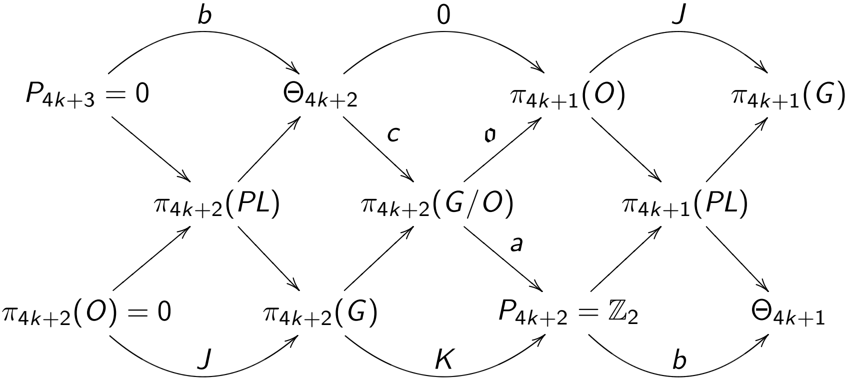
- The isomorphism $\pi_m(PL/O) \to \Theta_m$ sends a vector bundle $\alpha : S^m \to BO(k)$ (k large) with a PL trivialization $\beta : \alpha^{PL} \simeq * : S^m \to BPL(k)$ to the exotic sphere Σ^m such that $\Sigma^m \times \mathbb{R}^k$ is the smooth structure on the PL-manifold $E(\alpha)$ given by smoothing theory, with stable normal bundle

$$\nu_{\Sigma^m} : \Sigma^m \simeq S^m \xrightarrow{\alpha} BO(k)$$

- $\pi_m(PL) = \Theta_m^{fr}$ is the K-M group of framed n-dimensional exotic spheres.

The Kervaire-Milnor braid for $m = 4k+2$ I.

- For $m = 4k+2 \geqslant 5$ the braid is given by



with K the Kervaire invariant map.

The Kervaire-Milnor braid for $m = 4k+2$ II.

- K is the Kervaire invariant on the $(4k+2)$-dimensional stable homotopy group of spheres

$$\begin{aligned} K \ : \ \pi_{4k+2}(G) \ &= \ \pi^S_{4k+2} \ = \ \varinjlim_j \pi_{j+4k+2}(S^j) \\ &= \ \Omega^{fr}_{4k+2} \ = \ \{\text{framed cobordism}\} \to P_{4k+2} = \mathbb{Z}_2 \end{aligned}$$

- K is the surgery obstruction: $K = 0$ if and only if every $(4k+2)$-dimensional framed differentiable manifold is framed cobordant to a framed exotic sphere.
- The exotic sphere group Θ_{4k+2} fits into the exact sequence

$$0 \longrightarrow \Theta_{4k+2} \longrightarrow \pi_{4k+2}(G) \xrightarrow{K} \mathbb{Z}_2 \longrightarrow \ker(\pi_{4k+1}(PL) \to \pi_{4k+1}(G)) \longrightarrow 0$$

The Kervaire-Milnor braid for $m = 4k + 2$ III.

- $a : \pi_{4k+2}(G/O) \to \mathbb{Z}_2$ is the surgery obstruction map, sending a normal map $(f, b) : M^{4k+2} \to S^{4k+2}$ to the Kervaire invariant of M.
- $b : P_{4k+2} = \mathbb{Z}_2 \to \Theta_{4k+1}$ sends the generator $1 \in \mathbb{Z}_2$ to the boundary $b(1) = \Sigma^{4k+1} = \partial W$ of the Milnor plumbing W of two copies of $\tau_{S^{2k+1}}$ using the standard rank 2 quadratic form $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ over $\mathbb{Z}$ with Arf invariant 1.
- The image of b is the subgroup $bP_{4k+2} \subseteq \Theta_{4k+1}$ of the $(4k+1)$-dimensional exotic spheres Σ^{4k+1} which are the boundaries $\Sigma^{4k+1} = \partial W$ of framed $(4k+2)$-dimensional differentiable manifolds W. If k is such that $K = 0$ (e.g. $k = 2$) then $bP_{4k+2} = \mathbb{Z}_2 \subseteq \Theta_{4k+1}$, and if $\Sigma^{4k+1} = 1 \in bP_{4k+2}$ (as above) then $M^{4k+2} = W \cup_{\Sigma^{4k+1}} D^{4k+2}$ is the $(4k+2)$-dimensional Kervaire PL manifold without a differentiable structure.
- $c : \Theta_{4k+2} \to \pi_{4k+2}(G/O)$ sends a $(4k+2)$-dimensional exotic sphere Σ^{4k+2} to its fibre-homotopy trivialized stable normal bundle.

What if $K = 0$?

- For any $k \geqslant 1$ the following are equivalent:
  - $K : \pi_{4k+2}(G) = \pi^S_{4k+2} \to \mathbb{Z}_2$ is 0,
  - $\Theta_{4k+2} \cong \pi_{4k+2}(G)$,
  - $\ker(\pi_{4k+1}(PL) \to \pi_{4k+1}(G)) \cong \mathbb{Z}_2$,
  - Every simply-connected $(4k+2)$-dimensional Poincaré complex X with a vector bundle reduction $\tilde{\nu}_X : X \to BO$ of the Spivak normal fibration $\nu_X : X \to BG$ is homotopy equivalent to a closed $(4k+2)$-dimensional differentiable manifold.

When is $K \neq 0$?

- **Theorem** (Browder 1969)
  If $K \neq 0$ then $4k+2 = 2^j - 2$ for some $j \geqslant 2$.
- It is known that $K \neq 0$ for $4k+2 \in \{2, 6, 14, 30, 62\}$.
- **Theorem** (Hill-Hopkins-Ravenel 2009)
  If $K \neq 0$ then $4k+2 \in \{2, 6, 14, 30, 62, 126\}$.
- It is not known if $K = 0$ or $K \neq 0$ for $4k+2 = 126$.

The exotic spheres home page

http://www.maths.ed.ac.uk/~aar/exotic.htm

The Kervaire invariant home page

http://www.math.rochester.edu/u/faculty/doug/kervaire.html