Jan. '70. Simple Homotopy Theory. A.J. Casson

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## §1. Projective modules & automorphisms.

We construct $\mathbb{Z}[\pi]$ the integral group ring of the group $\pi$. This consists of formal linear combinations $\sum_{g\in\pi} n_g g$ ($n_g \in \mathbb{Z}$, $n_g = 0$ for all but finitely many $g$).

$$\sum n_g g + \sum m_g g = \sum (n_g + m_g) g.$$

$$\left(\sum m_g g\right)\left(\sum n_g g\right) = \sum_g \left(\sum_{hk=g} m_h n_k\right) g$$

Ring $R$ (associative, has 1, not nec. comm.)

(Left) R-module is Abelian group $A$ with $ra$ defined in $A$ $\forall r \in R, a \in A$. s.t.

$$r(a+b) = ra + rb$$
$$(r+s)a = ra + sa$$
$$(rs)a = r(sa)$$
$$1a = a$$

R-homomorphism $f : A \to B$ is group homo. with $f(ra) = rf(a)$ $\forall r \in R, a \in A$.

$A_i$: R-module ($i \in I$) $\bigoplus_{i\in I} A_i$ consists of formal sums $\sum_{i\in I} a_i$ with $a_i \in A_i$ & $a_i = 0$ almost all $i$.

$A$ is free if it is isomorphic to a direct sum of copies of $R$; equivalently, $A$ has a basis $\{a_i\}_{i\in I}$ s.t. $\forall a \in A$ $\exists$ unique $r_i \in R$ st. $a = \sum r_i a_i$ ($r_i = 0$ almost all $i$).

$A$ is projective if, given R-modules $B, C$ & R-homs $\phi : B \to C$, $f : A \to C$ with $\phi$ onto, $\exists$ $g : A \to B$ st $\phi g = f$.

**Lemma 1.1** $A$ is projective iff it is a direct summand of a free module.

$A$ is finitely generated if $\exists$ finite subset $\{a_1, .., a_n\}$ of $A$ which spans $A$.

**Corollary 1.2** A f.g. projective module is a direct summand of a f.g. free module.

$R$ any ring. Define $K_0(R)$ to be Abelian group with one generator $[A]$ for each isomorphism class of f.g. projective $R$-modules, subject to relations

$$[A] + [B] = [A \oplus B].$$

Define $\tilde{K}_0(R) = K_0(R) / (\text{subgp gen. by } [R])$
projective class group of $R$.

Examples

1) $R = \mathbb{Z}$. f.g. proj. $\mathbb{Z}$-modules all free.

~~$\tilde{K}_0(R) = 0$, K~~
$\tilde{K}_0(\mathbb{Z}) = 0$, $K_0(\mathbb{Z}) \cong \mathbb{Z}$

2) $R$ = field. f.g. proj. $R$-modules are f.d. vector spaces.
$\tilde{K}_0(R) = 0$, $K_0(R) \cong \mathbb{Z}$.

3) $p, q$ distinct primes. $K_0(\mathbb{Z}_{pq}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $\tilde{K}_0(\mathbb{Z}_{pq}) \cong \mathbb{Z}$

4) $R$ = ring of algebraic integers in some alg. number field.
$K_0(R) = \mathbb{Z} \oplus$ (ideal class group of $R$), $\tilde{K}_0(R) \cong$ ideal class group.

**Lemma 1.3** Any element of $K_0(R)$ can be expressed as $[A] - [B]$, where $A, B$ are f.g. proj. modules;
$[A] - [B] = [C] - [D]$ iff $\exists$ f.g. proj. $X$ st $A \oplus D \oplus X \cong B \oplus C \oplus X$.

Proof: Consider ordered pairs of f.g. proj. modules $(A, B)$. define $(A, B) \sim (C, D)$ if $A \oplus D \oplus X \cong B \oplus C \oplus X$ for some $X$, let $G$ be set of equivalence classes.

Addition in $G$: $(A, B) + (C, D)$ rep by $(A \oplus C, B \oplus D)$.

$G$ is a group.

Define $\phi : K_0(R) \to G$, $\psi : G \to K_0(R)$ by

$$\phi[A] = (A, 0), \quad \psi(A, B) = [A] - [B].$$

**Corollary 1.4.** Any element of $\tilde{K}_0(R)$ can be expressed as $[A]$; $[A] = [B]$ iff $A \oplus F \cong B \oplus G$ for some f.g. free $F, G$.

Proof: Any el. of $\tilde{K}_0(R)$ can be expressed as $[A] - [B]$.

~~Any~~ $B$ f.g. proj $\Rightarrow \exists X$ st $B \oplus X$ is f.g. free.

$\therefore$ Any el. of $K_0(R)$ is of form $[A \oplus X] - [B \oplus X]$.

$\therefore$ Any el. of $\tilde{K}_0(R)$ is of form $[A \oplus X]$.

Suppose $[A] = [B]$ in $\tilde{K}_0(R)$.

So $[A] - [B]$ in $K_0(R) \in$ subgp gen by $[R]$.

$\therefore$ $[A] - [B] = [F] - [G]$ ; $F, G$ f.g. free.

So $A \oplus G \oplus X \cong B \oplus F \oplus X$ some fg proj. $X$.

$X \oplus Y$ is fg free some $Y$.

$\therefore$ $A \oplus (G \oplus X \oplus Y) \cong B \oplus (F \oplus X \oplus Y)$

$A \oplus F \cong B \oplus G \Rightarrow [A] - [B] = [G] - [F]$ in $K_0$

$\Rightarrow [A] = [B]$ in $\tilde{K}_0$.

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## Tensor products.

Let $A$ be a right $R$-module, $B$ a left $R$-module. $A \otimes_R B$ is the universal Abelian group of bilinear maps $\phi : A \times B \longrightarrow G$ s.t.

$\phi(ar, b) = \phi(a, rb)$.

If $A$ is an $(S, R)$-bimodule [ie left $S$-module, right $R$-module s.t. $(sa)r = s(ar)$], then $A \otimes_R B$ inherits structure of left $S$-module.

[ $s \in S$ induced by $A \times B \longrightarrow A \otimes_R B$
$(a, b) \longmapsto sa \otimes b$ ]

If $A$ is an $(S, R)$-bimodule & $B$ is an $(R, T)$-bimodule, then $A \otimes_R B$ is an $(S, T)$-bimodule.

$R \xrightarrow{f} S$ ring homomorphism preserving $1$.

Construct $f_* : K_0(R) \to K_0(S)$.

Regard $S$ as $(S, R)$-bimodule ; $S$ acts on $S$ by left-multiplication, $R$ acts on $S$ on right by $s.r = sf(r)$

$A$ is left $R$-module $\Rightarrow$ $S \otimes_R A$ is a left $S$-module.

__Lemma 1.5__ $S \otimes_R (A \oplus B) \cong (S \otimes_R A) \oplus (S \otimes_R B)$
and if $A$ is f.g. projective $R$-module then $S \otimes_R A$ is f.g. projective $S$-module.

__Proof__: First part obvious.
Note that $S \otimes_R R \cong S$
$\therefore$ $S \otimes_R$ (f.g. free module) is f.g. free.
If $A$ is f.g. proj. $R$-module, then $A \oplus X$ is f.g. free for some $X$.
$\therefore$ $(S \otimes_R A) \oplus (S \otimes_R X)$ is f.g. free.
$\therefore$ $S \otimes_R A$ is f.g. proj $S$-module.

Define $f_* : K_0(R) \longrightarrow K_0(S)$ by $f_*[A] = [S \otimes_R A]$;
this gives homomorphism by 1.5. $(fg)_* = f_* g_*$, $1_* = 1$.

__Theorem 1.6__ $K_0$ and $\tilde{K}_0$ are covariant functors from the category of rings and ring homomorphisms (preserving 1) to the category of Abelian groups and homomorphisms.

__Examples__:

1) Suppose $\exists$ homomorphism $R \to K$, $K$ a field.
then $\mathbb{Z} \longrightarrow R \longrightarrow K$ induce homomorphisms

$$\begin{tikzcd} K_0(\mathbb{Z}) \arrow[r] \arrow[d, equal] & K_0(R) \arrow[r] & K_0(K) \arrow[d, equal] \\ \mathbb{Z} \arrow[rr, "\cong"] & & \mathbb{Z} \end{tikzcd}$$

$\therefore$ $K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$

In particular, this holds for commutative rings, and integral group rings. $\left( \sum n_g g \longrightarrow \sum n_g \right)$

2) $K_0(M_n(R)) \cong K_0(R)$
$R^n$ can be regarded as an $(R, M_n(R))$-bimodule
$r(x_1, \dots, x_n) = (rx_1, \dots, rx_n)$
$(x_1, \dots, x_n)\, a_{ij} = \left(\sum x_i a_{i1}, \dots, \sum x_i a_{in}\right)$
or an $(M_n(R), R)$-bimodule.

$R^n \otimes_R R^n \cong M_n(R)$ as an $M_n(R)$ bimodule.

$R^n \otimes_{M_n(R)} R^n \cong R$ as an $R$-bimodule.

If $A$ is left $M_n(R)$-module, $A^* = R^n \otimes_{M_n(R)} A$ left $R$-mod; $B$ ... $R$-module, $B_* = R^n \otimes_R B$ left $M_n(R)$-mod.

$^*$, $_*$ preserve $\oplus$ & f.g. projectives; $(A^*)_* \cong A$ and $(B_*)^* \cong B$

$\therefore$ defines inverse isomorphisms $K_0(M_n(R)) \cong K_0(R)$.

In general $\tilde{K}_0(M_n(R)) \ncong \tilde{K}_0(R)$

eg. $\tilde{K}_0(M_n(\mathbb{Z})) \cong \mathbb{Z}_n$

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Any ring $R$; $GL(n,R)$ = group of invertible $n \times n$ matrices $/R$.

Regard $GL(n,R)$ as subgp of $GL(n+1,R)$

$M \in GL(n,R)$ identified with $\begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix} \in GL(n+1,R)$.

$$GL(1,R) \subset GL(2,R) \subset \cdots \subset GL(n,R) \subset GL(n+1,R) \subset \cdots$$

Define $GL(R) = \bigcup_{n=1}^{\infty} GL(n,R)$.

Alter as $\infty \times \infty$ matrices, $a_{ij} = \delta_{ij}$ for all but finitely many $i,j$.

Let $e_{ij}$ be the matrix with 1 in $(i,j)$ th place, zero elsewhere.

If $i \neq j$ and $r \in R$, then $1 + re_{ij} \in GL(R)$, inverse $1 - re_{ij}$.

Let $E(R)$ be the group generated by these elementary ~~matrices~~ matrices.

<u>Lemma 1.7</u> (J.H.C. Whitehead).

$E(R)$ is the commutator subgroup of $GL(R)$.

<u>Proof</u>: Suppose $i,j,k$ distinct. Then

$(1+re_{ij})(1+se_{jk})(1-re_{ij})(1-se_{jk}) =$

$(1 + r e_{ij} + s e_{jk} + rs e_{ik})(1 - r e_{ij} - s e_{jk} + rs e_{ik})$

$= \quad 1 + rs e_{ik}$

$\therefore$ All elementary matrices are commutators.

Let $X, Y \in GL(n, R)$ ; then in $GL(R)$ we have

$$XYX^{-1}Y^{-1} = \begin{pmatrix} XYX^{-1}Y^{-1} & 0 \\ 0 & I_n \end{pmatrix}$$

$$= \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} (YX)^{-1} & 0 \\ 0 & YX \end{pmatrix}$$

$$\begin{pmatrix} Z & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Z & 1 \end{pmatrix} \begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\therefore \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -Z & 1 \end{pmatrix}$ are products of elem. matrices

$$\begin{pmatrix} 1 & 0 \\ -Z & 1 \end{pmatrix} = \prod_{\substack{n+1 \le i \le 2n \\ 1 \le j \le n}} (1 - z_{ij} e_{ij})$$

$$\therefore \quad E(R) \cong GL(R)'$$

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Define $K_1(R) = GL(R) / E(R)$ ; this is Abelian, usually written additively.

Let $A$ be f.g. projective, and let $\alpha : A \to A$ be an automorphism of $A$. Define $\tau(\alpha) \in K_1(R)$ (the whitehead determinant of $\alpha$) as follows.

If $A$ is free, pick basis + represent $\alpha$ by invertible matrix $M$.

Then $\tau(\alpha) =$ image of $M$ in $K_1(R)$ ; independent of basis as im $M =$ im $S^{-1}MS$.

If $A$ is f.g. proj., pick $X$ st $A \oplus X$ is f.g. free.

Define $\tau(\alpha) = \tau(\alpha \oplus 1_X)$ (already defined).

Ex independent of $X$.

1) $\tau(\alpha\beta) = \tau(\alpha) + \tau(\beta)$ if $\alpha, \beta$ autos of $A$

2) $\tau(\alpha \oplus \beta) = \tau(\alpha) + \tau(\beta)$ if $\alpha$ auto of $A$, $\beta$ auto of $B$.

In fact, $\tau$ is universal with respect to 1) and 2).

Let $\pi$ be any group. $g \in \pi \Rightarrow \underset{1\times 1 \text{ matrix}}{[\pm g]} \in GL(1, \mathbb{Z}[\pi]) \subset GL(\mathbb{Z}[\pi])$

<u>Def</u>$^n$: $Wh[\pi] = K_1(\mathbb{Z}[\pi]) / \{\tau(\pm g) : g \in \pi\}$

the <u>Whitehead group</u> of $\pi$

$f: R \to S$ induces homomorphism $f_*: GL(R) \to GL(S)$.

By Abelianising, get $f_*: K_1(R) \to K_1(S)$

<u>Theorem 1.8</u> – $K_1$ is a covariant functor from the category of rings and ring homomorphisms to the category of Abelian groups and homomorphisms.

Analogous result for $Wh$.

<u>Examples</u>

1) If $R$ is commutative, $\det : GL(R) \to U(R)$ = gp of units of $R$.

$$U(R) = GL(1, R) \subset GL(R) \longrightarrow K_1(R) \xrightarrow{\det} U(R)$$

$$u \longmapsto \begin{pmatrix} u & & 0 \\ & 1 & \\ 0 & & 1 \ddots \end{pmatrix} \longmapsto u$$

$\therefore K_1(R) \cong U(R) \oplus SK_1(R)$ for commutative $R$.

2) $Wh(C_5) \neq 0$. Enough to find a unit in $\mathbb{Z}[C_5]$ not of form $\pm g$ $(g \in C_5)$ $\quad Wh(\pi) \cong \dfrac{U(\mathbb{Z}[\pi])}{\pm\pi} \oplus SK_1$

$t$ generates $C_5$.

$1 - t - t^4$ is a unit in $\mathbb{Z}[C_5]$ inverse $1 - t^2 - t^3$

In fact $Wh(C_5) \cong \mathbb{Z}$ generated by $1 - t - t^4$ (hard to prove)

3) $K_1(\mathbb{Z}) \cong \mathbb{Z}_2 \cong U(\mathbb{Z})$ . $SK_1(\mathbb{Z}) = 0$.
Implies that Wh (trivial group) = 0.

$A \in GL(n, \mathbb{Z})$ with $\det A = 1$.
RTP that $A$ is a product of elementary matrices.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Simplify $(a_{11} \ldots a_{1n})$ by Euclidean algorithm. Suppose $a_{1r}$ has maximal modulus in top row.
Suppose $a_{1s} \neq 0$ for some $s \neq r$

Pick $\lambda \in \mathbb{Z}$ such that $|a_{1r} - \lambda a_{1s}| < |a_{1s}|$

$A(1 - \lambda e_{sr})$ has same top row as $A$ except that $a_{1r}$ is replaced by $a_{1r} - \lambda a_{1s}$.
Repeat until the top row has only one non-zero element — must be $\pm 1$. If $n \geq 2$, can make top row $(1, 0, \ldots, 0)$.

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ | & A' \end{bmatrix}$$

Premultiply by elementary matrices to kill first column.

$$\begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A' \\ 0 & \end{bmatrix}$$

$\therefore A \equiv$ some element of $GL(n-1, \mathbb{Z})$ $(\bmod E(\mathbb{Z}))$

Continue until $A \equiv \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$

But $\det A = 1$, so $A \equiv I \mod E(\mathbb{Z})$

4) If $R$ is a field then $K_1(R) \cong R^* = U(R)$
Similar to above, but easier.

5) $K_1(M_n(R)) \cong K_1(R)$

$GL(k, M_n(R)) \cong GL(nk, R)$ (partioned matrices)

$\therefore GL(M_n(R)) \cong GL(R)$

Abelianise $\Rightarrow K_1(M_n(R)) \cong K_1(R)$

**Lemma 1.9** $\pi$ group. If $\gamma : \pi \to \pi$, $x \mapsto gxg^{-1}$ is conjugation by some $g \in \pi$, then $\gamma_* : K_i(\mathbb{Z}[\pi]) \to K_i(\mathbb{Z}[\pi])$ is the identity $(i = 0, 1)$.

Proof: If $A$ is f.g. projective over $\mathbb{Z}[\pi]$, then $\gamma_*[A]$ represented by $C \otimes_{\mathbb{Z}[\pi]} A$ where $C = \mathbb{Z}[\pi]$ as left $\mathbb{Z}[\pi]$-module with right $\mathbb{Z}[\pi]$-action given by $c \cdot r = cgrg^{-1}$ ($c \in C$, $r \in \mathbb{Z}[\pi]$, $\cdot$ denotes right action on $C$)

Define $\phi : C \to \mathbb{Z}[\pi]$ by $\phi(c) = cg$

Left $\mathbb{Z}[\pi]$-module isomorphism, and

$$\phi(c \cdot r) = \phi(cgrg^{-1}) = cgr$$

$$\phi(c) r = cgr$$

$\therefore$ $\phi$ is a bimodule isomorphism, so $C \otimes_{\mathbb{Z}[\pi]} A \cong A$

$\therefore$ ~~$\phi$ is a b~~ $\gamma_* : K_0(\mathbb{Z}[\pi]) \to K_0(\mathbb{Z}[\pi])$ is identity

If $M \in GL(n, \mathbb{Z}[\pi])$, then $\gamma_* M = (gI_n) M (gI_n)^{-1}$

$\therefore$ $\gamma_* M \equiv M \mod E(\mathbb{Z}[\pi])$, so $\gamma_* : K_1 \to K_1$ is identity

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$Wh(\pi)$ is f.g. if $\pi$ is finite (Bass).

$\tilde{K}_0(\mathbb{Z}[C_\infty \times C_{p^2}])$ not f.g.

$\tilde{K}_0(\mathbb{Z}[\pi])$ is summand of $Wh(\pi \times C_\infty)$.

$Wh(\pi) = \tilde{K}_0(\pi) = 0$ if $\pi$ free or free Ab.

## §2 Chain Complexes.

Consider chain complexes of left $R$-modules.

$$C_*: \quad \dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \dots \to C_1 \to C_0 \to 0$$

$\partial$ is an $R$-homomorphism such that $\partial^2 = 0$.

$H_n(C_*)$ is a left $R$-module.

$C_*$ is free / proj / ~~f.g.~~ $\iff$ ~~$\bigoplus$~~ $C_n$ is free / proj $\forall n$

$C_*$ is f.g. $\iff \bigoplus_{n=0}^{\infty} C_n$ is f.g.

Example   $X$ a (simplicial) complex, fundamental group $\pi$, and universal cover $\tilde{X}$ triangulated canonically. Chain complex $C_*(\tilde{X})$ (finite simplicial chains). $\pi$ acts on $\tilde{X}$, so $C_*(\tilde{X})$ is chain complex of $\mathbb{Z}[\pi]$-modules. Free: one basis element for each simplex of $X$.

If $X$ dominated by finite complex $X \xrightarrow{f} K \xrightarrow{g} X$. $gf \simeq 1$.

$C_*(\tilde{X}) \to C_*(\tilde{K}) \to C_*(\tilde{X})$ with $g_* f_* \simeq 1$.
f.g. free.

**Lemma 2.1.** If $C_*$ is projective and acyclic, then $\exists$ $R$-homomorphisms $\Gamma_i : C_i \to C_{i+1}$ such that $\partial\Gamma + \Gamma\partial = 1$. ($\Gamma_*$ is a contraction of $C_*$).

Proof: $C_1 \xrightarrow{\partial} C_0$ onto, $C_0$ projective, so $\exists\, \Gamma_0 : C_0 \to C_1$ with $\partial\Gamma_0 = 1$.

Suppose inductively that $\Gamma_0, \dots, \Gamma_{n-1}$ defined.

$x \in C_n$;  $\partial x = (\partial\Gamma_{n-1} + \Gamma_{n-2}\partial)\partial x = \partial\Gamma\partial x$

$\therefore\ (1 - \Gamma_{n-1}\partial)x \in Z_n = \ker\partial : C_n \to C_{n-1}$

$Z_n = \operatorname{im} \partial : C_{n+1} \to C_n = B_n$

$C_n$ projective $\Rightarrow \exists\, \Gamma_n : C_n \to C_{n+1}$ s.t. $\partial\Gamma_n = 1 - \partial\Gamma_{n-1}$

ie $\partial\Gamma_n + \Gamma_{n-1}\partial = 1$ completes induction step.

$f : C_* \longrightarrow D_*$ chain map.

<u>Algebraic mapping cylinder</u> $M_*$ of $f$ has

$M_n = C_n \oplus C_{n-1} \oplus D_n$ with $\partial : M_n \longrightarrow M_{n-1}$

defined by $\partial(x, y, z) = (\partial x - y, -\partial y, \partial z + f y)$

Check $\partial^2 = 0$.

Chain maps $\lambda : C_* \longrightarrow M_*$, $\mu : M_* \longrightarrow D_*$

$x \longmapsto (x, 0, 0)$, $(x, y, z) \longmapsto z + fx$

$\mu\lambda = f$ and $\mu$ is a chain equivalence.

Inverse $\bar{\mu} : D_* \longrightarrow M_*$ ; $z \longmapsto (0, 0, z)$.

$\mu\bar{\mu} = 1$. homotopy $\bar{\mu}\mu \simeq 1$ given by

$\Delta_n : M_n \longrightarrow M_{n+1}$

$(x, y, z) \longmapsto (0, x, 0)$

$$
\begin{aligned}
(\partial\Delta + \Delta\partial)(x, y, z) &= (-x, -\partial x, fx) + (0, \partial x - y, 0) \\
&= (-x, -y, fx) \\
&= (\bar{\mu}\mu - 1)(x, y, z)
\end{aligned}
$$

<u>Algebraic mapping cone</u> $Q_* = M_* / \operatorname{im} \lambda$

$\therefore Q_n = C_{n-1} \oplus D_n$, $\partial(y, z) = (-\partial y, \partial z + fy)$

$$
\begin{tikzcd}
0 \arrow[r] & C_* \arrow[r, "\lambda"] \arrow[dr, "f"'] & M_* \arrow[r, "\pi"] \arrow[d, "\mu"] & Q_* \arrow[r] & 0 \\
& & D_* & &
\end{tikzcd}
$$

Commutes, top row exact.

Define $H_n(f) = H_n(Q_*)$ ; get exact homology sequence of $f$.

$$
H_n(C_*) \xrightarrow{f_*} H_n(D_*) \longrightarrow H_n(f) \longrightarrow H_{n-1}(C_*) \xrightarrow{f_*}
$$

<u>Lemma 2.2</u> : If $f : C_* \longrightarrow D_*$ induces homology group isomorphisms, and $C_*$, $D_*$ projective, then $f$ is a chain equivalence.

<u>Proof</u> : $M_*$, $Q_*$ mapping cylinder and cone of $f$.

Enough to show $\lambda : C_* \longrightarrow M_*$ is equivalence.
$Q_*$ is acyclic + projective $\therefore$ by 2.1 $\exists$ contraction $\Gamma_*$

Put $M_n = C_n \oplus Q_n$ in obvious way.
Put $\Delta_n = 0 \oplus \Gamma_n : M_n \longrightarrow M_{n+1}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_i & \xrightarrow{\lambda} & M_i & \xrightarrow{\pi} & Q_i & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \Delta_i} & & \downarrow{\scriptstyle \Gamma_i} & & \\ 0 & \longrightarrow & C_{i+1} & \xrightarrow{\lambda} & M_{i+1} & \xrightarrow{\pi} & Q_{i+1} & \longrightarrow & 0 \end{array}$$

commutes.

$\pi(1-\partial\Delta-\Delta\partial) = (1-\partial\Gamma-\Gamma\partial)\pi = 0$

$\therefore$ $\exists$ unique $\bar\lambda : \cancel{C}\, M_* \longrightarrow C_*$ such that $\lambda\bar\lambda = 1-\partial\Delta-\Delta\partial$

$\lambda\bar\lambda \simeq 1$

$\lambda\bar\lambda\lambda(x) = \cancel{\lambda}(1-\partial\Delta-\Delta\partial)\lambda(x) = \lambda(x)$

$\lambda$ mono $\Rightarrow$ $\bar\lambda\lambda = 1$.

So $\bar\lambda$ chain inverse to $\lambda$ as required.

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$C_*$ <u>dominated</u> by $D_*$ if $\exists\, f : C_* \longrightarrow D_*$, $g : D_* \to C_*$, $gf \simeq 1$. <u>dimension</u> of $C_*$ is $\dim(C_*) = \sup\{n : C_n \neq 0\}$.

<u>Theorem 2.3</u> (C.T.C. Wall).
If $C_*, D_*$ projective, $D_*$ dominates $C_*$, and $D_*$ is f.g., then $C_*$ is equivalent to a f.g. projective complex of dimension $\leq \dim(D_*)$.

<u>Def$^n$</u>: $C_*$ is of finite type if $C_n$ is f.g. $\forall n$.

<u>Lemma 2.4</u> If $C_*, D_*$ projective, $D_*$ dominates $C_*$, and $D_*$ is of finite type, then $C_* \simeq$ some complex of finite type.

<u>Proof</u>: $\cancel{\exists}$ $f : C_* \longrightarrow D_*$, $g : D_* \longrightarrow C_*$, $gf \simeq 1$.
Suppose inductively that $H_i(f) = 0$ for $i < n$. (start with $n = 0$).
First step: $H_n(f)$ is f.g.
Homology sequence of $f$:

$$0 \longrightarrow H_i(C_*) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_i(D_*) \longrightarrow H_i(f) \longrightarrow 0 \qquad (*)$$

Let $r = fg : D_* \longrightarrow D_*$

$f, g, r$ induce homology isomorphisms in dimensions $< n$.

Exact sequence of $r$:

$$H_n(D_*) \xrightarrow{r_*} H_n(D_*) \longrightarrow H_n(r) \longrightarrow 0$$

$r_* = f_* g_*$, $f_* = r_* f_* \implies \operatorname{im} r_* = \operatorname{im} f_*$

$\implies H_n(f) = H_n(r)$

Let $Q_*$ be mapping cone of $r$.

$H_i(Q_*) = 0$ for $i < n$.

Exact sequence. $0 \to Z_n(Q_*) \subset\!\!\to Q_n \xrightarrow{\partial} Q_{n-1} \xrightarrow{\partial} \to Q_0 \to 0$

$Q_i$ is projective so argument of 2.1 $\Rightarrow \exists$ contraction $\Gamma$

(don't use $Z_n$ projective).

$\Gamma_n | Z_n(Q_*) = 1$, so $Z_n$ is a direct summand of $Q_n$.

$\therefore Z_n$ is f.g., $\therefore H_n(f) \cong H_n(Q_*)$ is f.g.

From $(*)$, $H_n(f) \cong \ker g_* : H_n(D_*) \longrightarrow H_n(C_*)$

Pick f.g. projective $E$ & epimorphism $e : E \longrightarrow \ker g_*$

$\exists\, d$ s.t.
$$\begin{array}{ccc} E & \overset{d}{\dashrightarrow} & Z_n(D_*) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle proj} \\ \ker g_* & \xrightarrow[\text{inc.}]{} & H_n(D_*) \end{array}$$
commutes.

$$\begin{array}{ccccccc} \xrightarrow{\partial} C_{n+2} & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & \\ f \downarrow \uparrow g & & f\oplus 0 \downarrow \uparrow g\oplus c & & f \downarrow \uparrow g & & (\dagger) \\ \xrightarrow{\partial} D_{n+2} & \xrightarrow{\partial \oplus 0} & D_{n+1} \oplus E & \xrightarrow{\partial \oplus d} & D_n & \xrightarrow{\partial} & \end{array}$$

To choose $c$, note that $gd(C_n) \subset B_n(C_*)$.

since $e(E) \subset \ker g_*$.

$E$ projective, so $\exists\, c : E \to C_{n+1}$ s.t. $\partial c = gd$.

Replace $D_*$ by bottom row of $(\dagger)$ : chain complex of finite type. Haven't changed $gf$, so $D_*$ still dominates $C_*$

& $g$ induces homology isomorphisms in dimensions $\leq n$.
$\therefore$ $f$ does too. $\therefore$ $H_i(f) = 0$ for $i \leq n$.

Only changed $D_{n+1}$.

Iterate infinitely, obtain complex $D'_*$ & map $f' : C_* \to D'_*$ inducing homology isomorphisms in all dimensions. $\therefore$ By 2.2., $C_* \simeq D'_*$, which is of finite type.

---

<u>Proof of Th 2.3</u> By L 2.4, replace $C_*$ by an equivalent complex of finite type. $f : C_* \to D_*$, $g : D_* \to C_*$ s.t. $gf \simeq 1$, say $1 - gf = \partial\Delta + \Delta\partial$ where $\Delta_i : C_i \to C_{i+1}$.

Let $n = \dim D_*$. Then $gf : C_{n+1} \to C_{n+1}$ is zero.

$\therefore$ $\partial\Delta_{n+1} + \Delta_n\partial = 1_{C_{n+1}} \Rightarrow \partial\Delta_n\partial = \partial$

$\therefore$ Have map $\partial\Delta_n : C_n \to B_n$ such that $\partial\Delta_n | B_n = 1$.

$\therefore$ $B_n$ is a direct summand of $C_n$.

$\therefore$ $C_n / B_n$ is f.g. projective.

Let $E_*$ be complex

$$0 \to C_n/B_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0$$

Projection : $C_* \to E_*$ induces homology isomorphisms (clear for dimensions $\leq n$, and $H_i(D_*) = 0$ for $i > n$, from $H_i(C_*) \leftrightarrows H_i(D_*)$).

$\therefore$ $C_* \simeq E_*$ by 2.2; and $E_*$ is f.g. proj., $\dim E_* = \dim D_*$.

---

Let $C_*$ be f.g. projective. Define <u>Wall invariant</u> $\sigma(C_*)$ to be $\sum_i (-1)^i [C_i] \in \tilde{K}_0(R)$.

<u>Lemma 2.5</u> If $C_* \simeq D_*$, then $\sigma(C_*) = \sigma(D_*)$ (where $C_*$, $D_*$ are f.g. projective).

**Proof**: Let $Q_*$ be mapping cone of a chain equivalence $C_* \longrightarrow D_*$. Then $Q_*$ is acyclic, so $\exists$ contraction $\Gamma_*$.

$$\therefore \quad 0 \longrightarrow B_n \xrightarrow{\subset} Q_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0 \quad \text{splits}$$

$$\therefore \quad B_n \oplus B_{n-1} \cong Q_n \cong C_{n-1} \oplus D_n$$

$$\therefore \quad \sigma(C_*) - \sigma(D_*) = \sum_n (-1)^{n-1}\{[C_{n-1}] + [D_n]\}$$
$$= \sum (-1)^{n-1}\{[B_n] + [B_{n-1}]\}$$
$$= 0.$$

Can generalise definition of $\sigma(C_*)$ to case when $C_*$ is projective and dominated by a f.g. proj. complex. For such a $C_* \simeq$ f.g. proj. complex $E_*$ (by 2.3) and define $\sigma(C_*)$ to be $\sigma(E_*)$; well defined by L.2.5.

**Theorem 2.6** A f.g. projective complex $C_*$ is equivalent to a f.g. free complex of dimension at most $\dim C_*$ iff $\sigma(C_*) = 0$.

**Proof**: "Only if" is clear.

"If": Suppose $\sigma(C_*) = 0$.

Suppose inductively that $C_i$ free for $i < n$.

$C_n$ f.g. proj $\Rightarrow \exists$ $R$-module $E$, f.g. proj., s.t. $C_n \oplus E$ is free.

Replace $C_*$ by complex

$$\xrightarrow{\partial} C_{n+2} \xrightarrow{\partial \oplus 0} C_{n+1} \oplus E \xrightarrow{\partial \oplus 1} C_n \oplus E \xrightarrow{\partial \oplus 0} C_{n-1} \xrightarrow{\partial} C_{n-2} \longrightarrow$$

which is equivalent to $C_*$ by L 2.2.

This completes the induction; only had to alter $C_n$ and $C_{n+1}$.

Let $m = \dim C_*$: continue this process until $C_i$ is free, $i < m$. (doesn't increase $\dim C_*$).

$\sigma(C_*) = 0$ but $\sigma(C_*) = (-1)^m [C_m]$.

$\therefore$ $\exists$ f.g. free $F, G$ s.t. $C_m \oplus F \cong G$.

Replace $C_*$ by complex

$$0 \longrightarrow C_m \oplus F \xrightarrow{\partial \oplus 1} C_{m-1} \oplus F \xrightarrow{\partial \oplus 0} C_{m-2} \xrightarrow{\partial}$$

which is $\simeq C_*$ by 2.2.; and it is f.g. free of dim $m$

---

## Whitehead Torsion.

Hypothesis (for rest of § 2): $R$ is such that free modules $R^m$, $R^n$ are isomorphic iff $m = n$.

Examples 1) if $R$ any ring: $R^\infty$ = free left $R$-module on countably many generators. $S = \mathrm{End}_R(R^\infty)$. If $A$ is any left $R$-module, $\mathrm{Hom}_R(A, R^\infty)$ is a left $S$-module. But, as left $S$-modules

$$S = \mathrm{Hom}_R(R^\infty, R^\infty) \cong \mathrm{Hom}_R(R^\infty \oplus R^\infty, R^\infty)$$
$$\cong S \oplus S$$

so hypothesis doesn't hold for $S$.

2) Hypothesis does hold ~~for~~ if $R$ can be mapped homomorphically into a field.
eg commutative rings, $\mathbb{Z}[\pi]$.

---

Let $A$ be a f.g. free $R$-module, and let $b = (b_1, \dots, b_m)$, $c = (c_1, \dots, c_n)$ be bases for $A$. Then $m = n$, so $\exists$ unique square matrix $[a_{ij}] \in GL(n, R)$ s.t. $c_i = \sum a_{ij} b_j$.
Write $[c/b]$ for $\tau[a_{ij}] \in K_1(R)$.

A based chain complex is a f.g. free chain complex $C_*$ together with a basis $c_n = (c_n^{(1)}, \dots, c_n^{(d_n)})$ of $C_n$, $\forall n$.

Let $C_*$ be based and acyclic. By 2.1 $\exists$ contraction $\Gamma_*$.

Exact sequence

$$\begin{tikzcd} 0 \arrow[r] & B_n \arrow[r, "\subset"] \arrow[d, "1"] & C_n \arrow[r, "\partial"] \arrow[d, "\partial \oplus \partial \Gamma_n"] & B_{n-1} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & B_n \arrow[r, "\subset"] & B_{n-1} \oplus B_n \arrow[r, "p_1"] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$ splits

commutative diagram. Five lemma $\Rightarrow \partial \oplus \partial \Gamma_n$ isomorphism $= \gamma_n$

$$\gamma_n : C_n \longrightarrow B_{n-1} \oplus B_n$$

Let $\gamma = (\oplus \gamma_{2i})^{-1} (\oplus \gamma_{2i+1}) : \oplus C_{2i+1} \longrightarrow \oplus C_{2i}$

Bases $\oplus c_{2i}$, $\gamma(\oplus c_{2i+1})$ for $\oplus C_{2i}$

Define $\tau(C_*)$ to be $[\gamma(\oplus c_{2i+1}) / \oplus c_{2i}]$.

Re-ordering bases: $\tau\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau(-1)$, so that re-ordering bases adds $\tau(\pm 1)$ to $\tau(C_*)$.

Define $\bar{K}_1(R) = K_1(R)/\{\tau(\pm 1)\}$

$$= \operatorname{coker}(K_1(\mathbb{Z}) \longrightarrow K_1(R)).$$

Torsions of chain complexes will be regarded as elements of $\bar{K}_1(R)$.

<u>Lemma 2.7</u> The torsion $\tau(C_*)$ depends only on $C_*$ and bases $c_*$.

Proof: Let $\Gamma'_*$ be another contraction giving isomorphisms $\gamma'_n : C_n \longrightarrow B_{n-1} \oplus B_n$.

Let $\beta_n = \gamma'_n \gamma_n^{-1} : B_{n-1} \oplus B_n \longrightarrow B_{n-1} \oplus B_n$.

Enough to prove $\tau(\beta) = 0$.

Commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & B_n \arrow[r] \arrow[d, "1"] & B_{n-1} \oplus B_n \arrow[r] \arrow[d, "\beta_n"] & B_{n-1} \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & B_n \arrow[r] & B_{n-1} \oplus B_n \arrow[r] & B_{n-1} \arrow[r] & 0 \end{tikzcd}$$

$B_{n-1}, B_n$ are f.g. projective: $\exists X_{n-1}, X_n$ s.t. $X_{n-1} \oplus B_{n-1}$, $B_n \oplus X_n$ f.g. free. Let $F_n = B_n \oplus X_n$.

$$\phi_n = 1 \oplus \beta_n \oplus 1 : F_{n-1} \oplus F_n \longrightarrow F_{n-1} \oplus F_n$$

$$\tau(\phi_n) = \tau(\beta_n)$$

$$\begin{array}{ccccccccc} 0 \longrightarrow & F_n & \longrightarrow & F_{n-1} \oplus F_n & \longrightarrow & F_{n-1} & \longrightarrow 0 \\ & \circlearrowleft 1 & & \circlearrowleft \phi_n & & \circlearrowleft 1 & \end{array}$$

Wrt bases for $F_{n-1}, F_n$, $\phi_n$ has matrix $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix}$ which is a product of elementary matrices.

$\therefore \tau(\beta_n) = \tau(\phi_n) = 0$. as required.

---

$C_*, D_*$ based $f : C_* \longrightarrow D_*$ chain map, mapping cone $Q_*$ : $Q_n = C_{n-1} \oplus D_n$ : basis $q_n = c_{n-1} \oplus d_n$

$Q_*$ is based and acyclic if $f$ is a chain equivalence

Define $\tau(f) = \tau(Q_*)$.

Call $f$ a simple equivalence if $\tau(f) = 0$

Theorem 2.8 : If $f : C_* \longrightarrow D_*$ is a chain equivalence of based chain complexes, and $g \simeq f$, then $\tau(g) = \tau(f)$.

Proof : $f - g = \partial\Delta + \Delta\partial$

Let $Q_*^f, Q_*^g$ be the mapping cones of $f, g$.

$Q_n^f = Q_n^g = C_{n-1} \oplus D_n$, $\quad q_n^f = q_n^g = c_{n-1} \oplus d_n$

$\partial^f(y, z) = (-\partial y, \partial z + fy)$

$\partial^g(y, z) = (-\partial y, \partial z + gy)$

Define $\phi : Q_*^f \longrightarrow Q_*^g$ by $\phi(y, z) = (y, z + \Delta y)$

Chain map : $\phi\partial^f(y, z) = (-\partial y, \partial z + fy - \Delta\partial y)$

$\partial^g\phi(y, z) = (-\partial y, \partial z + \partial\Delta y + gy)$

$\phi$ is an isomorphism of chain complexes.

In fact, $\phi_n : C_{n-1} \oplus D_n \longrightarrow C_{n-1} \oplus D_n$ is a product of elementary automorphisms, so $[\phi(q_n)/q_n] = 0$

$\therefore \tau(Q_*^f) = \tau(Q_*^g)$ as required.

---

**Lemma 2.9** : Let $0 \longrightarrow C'_* \xrightarrow{i} C_* \xrightarrow{j} C''_* \longrightarrow 0$ be a s.e.s. of based acyclic complexes. Suppose $i, j$ preserve bases, in the sense that $i(c'_n) \subset c_n$ and $j(c_n - i(c'_n)) = c''_n$. Then $\tau(C_*) = \tau(C'_*) + \tau(C''_*)$

*Proof* : Claim $\exists$ contractions $\Gamma_*, \Gamma'_*, \Gamma''_*$ such that

$$\begin{tikzcd}
0 \arrow[r] & C'_n \arrow[r, "i"] \arrow[d, "\Gamma'_n"] & C_n \arrow[r, "j"] \arrow[d, "\Gamma_n"] & C''_n \arrow[r] \arrow[d, "\Gamma''_n"] & 0 \\
0 \arrow[r] & C'_{n+1} \arrow[r, "i"] & C_{n+1} \arrow[r, "j"] & C''_{n+1} \arrow[r] & 0
\end{tikzcd}$$

commutes.

Let $\Gamma''_*$ be any contraction of $C''_*$.

$C_n$ free $\Longrightarrow \exists\, \Delta_n : C_n \longrightarrow C_{n+1}$ s.t. $j\Delta_n = \Gamma''_n j$

$\therefore \; j(1 - \partial\Delta - \partial\Delta) = (1 - \partial\Gamma'' - \Gamma''\partial)j$

$\exists$ unique $k : C_* \longrightarrow C'_*$ such that $ik = 1 - \partial\Delta - \Delta\partial : C_* \to C_*$

$C'_*$ contractible, so $k \simeq 0$, say $k = \partial\Delta' + \Delta'\partial$,

$\Delta'_n : C_n \longrightarrow C'_{n+1}$.

Put $\Gamma_n = \Delta_n + i\Delta'_n$ ; then

$\partial\Gamma + \Gamma\partial = 1$; $\Gamma_*$ contraction.

$j\Gamma_n = j\Delta_n = \Gamma''_n j$.

Diagram chasing $\Longrightarrow$ $0 \to B'_n \xrightarrow{i} B_n \xrightarrow{j} B''_n \to 0$ exact.

$$\begin{tikzcd}
0 \arrow[r] & C'_n \arrow[r, "i"] \arrow[d, "\gamma'_n"] & C_n \arrow[r, "j"] \arrow[d, "\gamma_n"] & C''_n \arrow[r] \arrow[d, "\gamma''_n"] & 0 \\
0 \arrow[r] & B'_{n-1} \oplus B'_n \arrow[r] & B_{n-1} \oplus B_n \arrow[r] & B''_{n-1} \oplus B''_n \arrow[r] & 0
\end{tikzcd}$$

$$\partial + \partial\Gamma = \gamma_n : C_n \longrightarrow B_{n-1} \oplus B_n$$

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus C'_{2r+1} \arrow[r] \arrow[d, "\gamma'"] & \bigoplus C_{2r+1} \arrow[r] \arrow[d, "\gamma"] & \bigoplus C''_{2r+1} \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & \bigoplus C'_{2r} \arrow[r] & \bigoplus C_{2r} \arrow[r] & \bigoplus C''_{2r} &
\end{tikzcd}$$

both commute.

Let $M, M', M''$ be matrices of $\gamma, \gamma', \gamma''$ wrt given bases.

$i, j$ preserve bases. Re-order bases $c_n$ of $C_n$ to bring $M$ into form $\begin{pmatrix} M' & X \\ 0 & M'' \end{pmatrix} = \begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix}\begin{pmatrix} 1 & (M')^{-1}X \\ 0 & 1 \end{pmatrix}$

$$\therefore \quad \tau(M) \equiv \tau(M') + \tau(M'') \quad \text{mod } \tau(\pm 1)$$

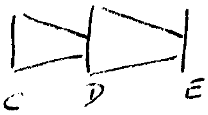
$$\therefore \quad \tau(C_*) = \tau(C'_*) + \tau(C''_*) \in \bar{K}_1(R)$$

---

<u>Theorem 2.10</u> If $f : C_* \longrightarrow D_*$, $g : D_* \longrightarrow E_*$ are chain equivalences of based complexes, then $\tau(gf) = \tau(g) + \tau(f)$.

Proof: Let $Q^f_*, Q^g_*, Q^{gf}_*$ be mapping cones.

Define $S_*$ by



squash C.

$$S_n = \underset{y}{C_{n-1}} \oplus \underset{z}{D_n} \oplus \underset{v}{D_{n-1}} \oplus \underset{w}{E_n}$$

bases $s_n = c_{n-1} \oplus d_n \oplus d_{n-1} \oplus e_n$

$$\partial(y, z, v, w) = (-\partial y, \partial z + fy - v, -\partial v, \partial w + gv)$$

Based exact sequence

$$0 \longrightarrow Q^f_* \longrightarrow S_* \longrightarrow Q^g_* \longrightarrow 0$$

$$(y, z) \longmapsto (y, z, 0, 0)$$

$$(y, z, v, w) \longmapsto (v, w)$$

$$\tau(S_*) = \tau(f) + \tau(g) \quad \text{by } 2.9.$$

Define $i : Q^{gf}_* \longrightarrow S_*$ by $i(y, w) = (y, 0, fy, w)$ chain map.

Define complex $T_*$ by $T_n = D_n \oplus D_{n-1}$ basis $t_n = d_n \oplus d_{n-1}$

$$\partial(z, v) = (\partial z - v, -\partial v)$$

$$0 \longrightarrow Q^{gf}_* \xrightarrow{i} S_* \xrightarrow{j} T_* \longrightarrow 0$$

$$(y, z, v, w) \longrightarrow (z, v - fy)$$

This is not based.

New basis for $S_n$ : $s'_n = i(c_{n-1} \oplus e_n) \cup d_n \oplus d_{n-1}$

In fact, $[s'_n / s_n] = 0 \in \bar{K}_1(R)$ related to $s_n$ by transformation $(y, z, v, w) \longmapsto (y, z, v + fy, w)$.

By L 2.9, $\tau(gf) + \tau(T_*) = \tau(S_*) = \tau(f) + \tau(g)$.

$T_n = D_n \oplus D_{n-1}$ $\qquad \partial(z, v) = (\partial z - v, -\partial v)$, $t_n = d_n \oplus d_{n-1}$

Define $T'_*$ by $T'_n = T_n$, $t'_n = t_n$, $\partial'(z, v) = (-v, 0)$.

Define $\phi : T_* \to T'_*$ by $\phi(z, v) = (z, v - \partial z)$ chain map.

~~$\phi \partial(z, v) =$~~

$\phi$ is elementary automorphism of $T_n$.

$[\phi t_n / t_n] = 0$

$\therefore \quad \tau(T_*) = \tau(T'_*)$

To calculate $\tau(T'_*)$, use contraction $\Gamma'_*$, with

$\Gamma'(z, v) = (0, -z)$

Matrix of $\gamma : \oplus T'_{2i+1} \longrightarrow \oplus T'_{2i}$ has integer coefficients. $\bar{K}_1(\mathbb{Z}) = 0$, so $\gamma$ has zero torsion. $\therefore \tau(T'_*) = 0$.

---

**Corollary 2.11** Let $0 \longrightarrow C'_* \xrightarrow{i} C_* \xrightarrow{j} C''_* \longrightarrow 0$ be an exact sequence of based complexes. Suppose $i$ is a chain equivalence, and $i, j$ preserve bases. Then $\tau(i) = \tau(C''_*)$.

~~Proof~~ : Let $Q_*$ be the mapping cone of $i$; let $Q'_*$ be the mapping cone of $1_{C'_*}$.

Then $\tau(Q_*) = \tau(i)$ and $\tau(Q'_*) = 0$ by (2.10).

Define $u : Q'_* \longrightarrow Q_*$ by $u(y, z) = (y, i(z))$.

Define $v : Q_* \longrightarrow C''_*$ by $v(y, z) = j(z)$.

preserve bases.

Exact sequence: $0 \longrightarrow Q'_* \longrightarrow Q_* \longrightarrow C''_* \longrightarrow 0$

By L.2.9., $\tau(i) = \tau(Q_*) = \tau(C''_*)$.

$f : C_* \longrightarrow D_*$ any chain map of based complexes.

$M_*$ = mapping cylinder : $M_n = C_n \oplus C_{n-1} \oplus D_n$,

bases $m_n = c_n \oplus c_{n-1} \oplus d_n$

$$\partial(x,y,z) = (\partial x - y, -\partial y, \partial z + fy).$$

Chain equivalence $\mu : M_* \longrightarrow D_*$

<u>Corollary 2.12</u> $\mu$ is a simple equivalence, ie $\tau(\mu) = 0$.

<u>Proof</u>: Recall from 2.2 that a chain inverse of $\mu$ is given by $\bar{\mu}(z) = (0,0,z)$.

Define $T_*$ by $T_n = C_n \oplus C_{n-1}$, basis $t_n = c_n \oplus c_{n-1}$

$\partial(x,y) = (\partial x - y, -\partial y)$.

Based exact sequence

$$0 \longrightarrow D_* \xrightarrow{\bar{\mu}} M_* \longrightarrow T_* \longrightarrow 0$$
$$(x,y,z) \longmapsto (x,y)$$

$\therefore$ $\tau(\bar{\mu}) = \tau(T_*) = 0$ as in proof of 2.10.

$\because$ $\mu\bar{\mu} = 1$, so $\tau(\mu) = 0$ by 2.10.

---

An <u>elementary</u> based chain complex of dimension $n$ is one of form

$$0 \to \ldots \to 0 \to E_n \to E_{n-1} \to 0 \to \ldots$$

with $E_i = 0$ if $i \neq n, n-1$.

$E_n = E_{n-1} = R$, $e_n = e_{n-1} = 1$.

$\partial : E_n \longrightarrow E_{n-1}$ is $\pm$ identity.

<u>Example</u>: $K, L$ (finite) simplicial complexes.

Suppose $K \searrow L$ by elementary simplicial collapse.

$\tilde{K}, \tilde{L}$ universal covers.

Exact sequence

$$0 \longrightarrow C_*(\tilde{L}) \xrightarrow{\subset_*} C_*(\tilde{K}) \longrightarrow E_* \longrightarrow 0$$

where $E_*$ is elementary, of same dimension as collapse.

Suppose $C_*$, $D_*$ are based, and there is a based exact sequence $0 \longrightarrow C_* \xrightarrow{i} D_* \longrightarrow E_* \longrightarrow 0$

with $E_*$ elementary.

Then $i$ is called an <u>elementary</u> <u>expansion</u>.

By 2.2, $i$ is a homotopy equivalence.

Any chain inverse is called an <u>elementary</u> <u>collapse</u>.

<u>Theorem 2.13</u> A chain map $f : C_* \to D_*$ is a simple equivalence iff it can be factored into finitely many elementary expansions and collapses.

<u>Proof</u>: The torsion of an elementary complex is $0$; by Lemma 2.11, an elementary expansion or collapse has torsion zero.

<u>Lemma 2.14</u> A based acyclic complex with zero torsion can be reduced to $0$ by finitely many elementary expansions and collapses.

<u>Proof</u>: $C_*$ based acyclic, $n = \dim C_*$.

First we show how to alter basis $c_{n-1} = (c^1, .., c^d)$ of $C_{n-1}$ by an elementary matrix $1 + \lambda e_{ij}$.

$$
\begin{tikzcd}
 & 0 \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d, "i_1"] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & C_{n-2} \arrow[r, "\partial"] \arrow[d, "1"] & {} \\
 & 0 \arrow[r] \arrow[d] & C_n \oplus R \arrow[r, "\partial^2"] \arrow[d, "i_{2,3}"] & C_{n-1} \oplus R \arrow[r, "\partial'"] \arrow[d, "1"] & C_{n-2} \arrow[r, "\partial"] \arrow[d, "1"] & {} \\
0 \arrow[r] & C_n \arrow[r] & C_n \oplus C_n \oplus R \arrow[r, "\partial^3"] & C_{n-1} \oplus R \arrow[r, "\partial'"] & C_{n-2} \arrow[r, "\partial"] & {} \\
 & 0 \arrow[r] \arrow[u] & C_n \oplus R \arrow[r, "\partial^4"] \arrow[u, "i_{1,3}"] & C_{n-1} \oplus R \arrow[r, "\partial'"] \arrow[u, "1"] & C_{n-2} \arrow[r, "\partial"] \arrow[u, "1"] & {} \\
 & 0 \arrow[r] & C_n \arrow[r, "\partial"] \arrow[u, "i_1"] & C_{n-1} \arrow[r, "\partial"] \arrow[u, "\phi"] & C_{n-2} \arrow[r, "\partial"] \arrow[u, "1"] & {}
\end{tikzcd}
$$

where

$$
\begin{aligned}
\partial'(z, r) &= \partial z + r(c^j + \lambda c^i) \\
\partial^2(y, r) &= (\partial y - r(c^j + \lambda c^i), r) \\
\partial^3(x, y, r) &= \partial^2(x + y, r + (\partial x)_j) \\
\partial^4(x, r) &= \partial^3(x, 0, r) \\
\partial^5(x) &= (x, -x, -(\partial x)_j) \\
\phi(z) &= (z - (z)_j (c^j + \lambda c^i), (z)_j)
\end{aligned}
\qquad \partial x = \sum (\partial x)_r c^r
$$

Vertical maps define elementary expansions; except that $\phi$ isn't based.
To make it based, have to replace $(c^1, \dots, c^d)$ by $(c^1, \dots, c^{j-1}, c^j + \lambda c^i, c^{j+1}, \dots, c^d)$.
But this is the change we wanted to produce.

If $n \geq 2$, make expansion

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & C_{n-2} & \xrightarrow{\partial} & \\ & & \downarrow 1 & & \downarrow i_1 & & \downarrow i_1 & & \\ 0 & \longrightarrow & C_n & \xrightarrow{\partial \oplus 0} & C_{n-1} \oplus C_n & \xrightarrow{\partial \oplus 1} & C_{n-2} \oplus C_n & \xrightarrow{\partial \oplus 0} & \end{array}$$

This makes $B_{n-2}$ (bottom row) $\cong$ $B_{n-2}$ (top row) $\oplus C_n$

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow B_{n-2} \longrightarrow 0 \quad \text{splits.}$$

ie it makes $B_{n-2}$ free.

Bases $c_n, c_{n-1}$ for $C_n, C_{n-1}$.
From the exact sequence $0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} B_{n-2} \longrightarrow 0$
and freeness of $B_{n-2}$, can extend $\partial c_n$ to a basis $\overline{\partial c_n}$ of $C_{n-1}$.
$\exists$ matrix $M \in GL(k, R)$ ($k$ = rank of $C_{n-1}$).
s.t. $\overline{\partial c_n} = M c_{n-1}$. Make another expansion

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & C_{n-2} & \xrightarrow{\partial} & \\ & & \downarrow 1 & & \downarrow i_1 & & \downarrow i_1 & & \\ 0 & \longrightarrow & C_n & \xrightarrow{\partial \oplus 0} & C_{n-1} \oplus R^k & \xrightarrow{\partial \oplus 1} & C_{n-2} \oplus R^k & \xrightarrow{\partial \oplus 0} & . \end{array}$$

Extend $c_{n-1}$ to bases of $C_{n-1} \oplus R^k$ by adjoining standard basis $(e^1, \dots, e^k)$ of $R^k$.
Extend $\overline{\partial c_n}$ to basis of $C_{n-1} \oplus R^k$ by adjoining $(M^{-1}e^1, \dots, M^{-1}e^k)$
Now $\overline{\partial c_n} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} c_{n-1}$ and $\begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}$ is product of elementary matrices.
So we can change $c_{n-1}$ into $\overline{\partial c_n}$ by elementary expansions and collapses.
Then $\partial : C_n \longrightarrow C_{n-1}$ is based injection, so we can collapse $C_*$ onto $0 \longrightarrow \frac{C_{n-1}}{\partial C_n} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} C_{n-3} \xrightarrow{\partial}$
This reduces dim $C_*$.

• Continue until $\dim C_* = 1$.

$$0 \longrightarrow C_1 \xrightarrow[\cong]{\partial} C_0 \longrightarrow 0$$

Since $\tau(C_*) = 0$, $\partial$ is given ~~by~~ (w.r.t bases ...) by matrix $M$ with $\tau(M) = 0$. Expand until $M$ is a product of elementary matrices.
Change basis of $C_0$ to make $\partial$ based (by expansions & collapses as above). Now $C_*$ can be collapsed to $0$.
This proves the lemma.

Proof of Th 2.13 $\quad f : C_* \to D_*$ is simple equiv, $C_*, D_*$ based. $M_*$ = mapping cylinder of $f$.

~~$0 \longrightarrow C_*$~~ $\qquad \bar{\mu} : D_* \longrightarrow M_*$

$D_n \ni z \longmapsto (0,0,z) \in C_n \oplus C_{n-1} \oplus D_n$

Exercise: $\bar{\mu} : D_* \to M_*$ is a product of elementary expansions.

$$\begin{array}{ccccccc}
\xrightarrow{\partial} D_2 & \xrightarrow{\partial} & D_1 & \xrightarrow{\partial} & D_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow i_2 & & \downarrow i_2 & & \\
\xrightarrow{\partial} D_2 & \longrightarrow & C_0 \oplus D_1 & \longrightarrow & C_0 \oplus D_0 & \longrightarrow & 0 \\
 & & (y,z) & \longmapsto & (y, fy + \partial z) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow C_1 \oplus D_2 & \longrightarrow & C_1 \oplus C_0 \oplus D_1 & \longrightarrow & C_0 \oplus D_0 & \longrightarrow & 0
\end{array}$$

Replace $f : C_* \longrightarrow D_*$ by a based injection.
Exact sequence $\quad 0 \longrightarrow C_* \xrightarrow{f} D_* \xrightarrow{\pi} A_* \longrightarrow 0$ based.

$\tau(A_*) = \tau(f) = 0. \qquad A_*$ acyclic.

$\therefore$ Can reduce $A_*$ to $0$ by Lemma 2.14.
We show how to "cover" expansions & collapses of $A_*$ by corresponding expansions & collapses of $D_*$.

If $A_* \longrightarrow A'_*$ is an elementary collapse
then $D_* \longrightarrow D'_* = \pi^{-1}(A'_*)$
Let $A_* \longrightarrow A'_*$ be an elementary expansion.
Let $h : A'_* \longrightarrow A_*$ be a collapse. Then $h/A_*$ is

chain homotopic to 1. Extend homotopy to get collapse $g : A'_* \longrightarrow A_*$ with $g | A_* = 1$.

($A_n$ direct summand of $A'_n$).

Define $D'_* = \{ (x, y) \in D_* \oplus A'_* : \pi(x) = g(y) \}$;

$$\partial(x, y) = (\partial x, \partial y).$$

$x \longmapsto (x, \pi(x))$ is a based injection $D_* \longrightarrow D'_*$.

Extend basis of $D_*$ to basis of $D'_*$ suitably; then $D_* \longrightarrow D'_*$ is elementary expansion.

Still have exact sequence $0 \to C_* \xrightarrow{f'} D'_* \xrightarrow{\pi'} A'_* \to 0$ (based).

This finishes the proof of Th 2.13

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<u>Exercise</u> 1) Can get from $D_*$ to $C_*$ by ~~to~~ expansions and collapses of dimension at most $\max(\dim C_* + 1, \dim D_*) + 1$.

2) In lemma 2.14, we can get from $C_*$ to $0$ by expansions and collapses of dimension $\geq 2$.

# §3. CW complexes.

$e^n$ closed $n$-cell.

C.W. complex is Hausdorff space $X$ with maps $\phi_\alpha : e^n \to X$ $(\alpha \in A_n)$

i) if $X^n = \bigcup_{r \leq n} \bigcup_{\alpha \in A_r} \phi_\alpha(e^r)$, then $X = \bigcup X^n$ and ~~$\phi_\alpha(\partial e^n) \subset X^{n-1}$~~

$\phi_\alpha(\partial e^n) \subset X^{n-1}$

ii) $\phi_\alpha(\text{int } e^n) \cap \phi_\beta(\text{int } e^m) = \emptyset$ unless $\alpha = \beta$ and $n = m$.

ie $\phi_\alpha | \text{int } e^n$ is 1-1.

iii) $\forall \alpha$, $\phi_\alpha(e^n)$ = finite union of interiors of cells.

iv) $C \subset X$ closed $\iff$ $\phi_\alpha^{-1}(C)$ closed in $e^n$ for all $\alpha$.

<u>Lemma 3.1</u> Any CW complex has the homotopy type of a simplicial complex.

<u>Proof</u>: Suppose $\simeq$ equiv $f : X^{n-1} \to K^{n-1}$ = simp. complex.

$A_n$ discrete topology.

$\phi : A_n \times \partial e^n \to X^{n-1}$ given by $\phi(\alpha, x) = \phi_\alpha(x)$

Let $\psi$ be simplicial approximation of $f\phi$

$f\phi : A_n \times \partial e^n \to K^{n-1}$.

By homotopy theory, $X^n = X^{n-1} \cup_\phi (A_n \times e^n)$

$$\simeq K^{n-1} \cup_\psi (A_n \times e^n)$$
$$= K^n$$

which can be triangulated

<u>Corollary</u> Any CW complex is locally path connected and weakly locally simply connected. (ie $\forall x \in X$ $\exists$ nhd $U$ of $x$ s.t. any loop in $U$ is null-homotopic in $X$).

Let $X$ be a connected CW complex, $x_0 \in X$, $G \subset \pi_1(X, x_0)$. Then $\exists$ covering space, $p : \tilde{X} \to X$, with $\tilde{x}_0 \in \tilde{X}$ s.t. $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = G$; $\tilde{X}$ connected.

A *covering translation* of $p : \tilde{X} \to X$ is a homeo

$h : \tilde{X} \to \tilde{X}$ with $ph = p$.

*Example* $\mathbb{R}^n$ is a cover of $n$-fold torus $T^n$.

$$(x_1, \ldots, x_n) \longmapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$$

Group of covering translations is $\mathbb{Z}^n$.

*Lemma 3.3* : If $G$ is normal in $\pi_1(X, x_0)$ [*regular* cover] then the group of covering translations is $\cong \pi = \dfrac{\pi_1(X, x_0)}{G}$.

*Proof* : Suppose covering translation $h : \tilde{X} \to \tilde{X}$.

$\exists$ path $f : I \to \tilde{X}$ with $f(0) = \tilde{x}_0$, $f(1) = h(\tilde{x}_0)$.

$pf : I \to X$ is a loop in $X$, representing

$\eta(h) \in \pi$. Well defined; homomorphism.

*Injective* : suppose $\eta(h) = 1$.

$pf$ represents element of $G = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\therefore$ $pf \simeq pl$, $l$ some loop in $\tilde{X}$, $\tilde{x}_0$ ends.

Lift this homotopy to $\tilde{X}$ to prove $f(0) = f(1)$, so $h(\tilde{x}_0) = \tilde{x}_0$.

$\therefore$ $h = 1$.

*Surjective* : take loop $l : I \to X$

Lift to path $\tilde{l} : I \to \tilde{X}$, $\tilde{l}(0) = \tilde{x}_0$.

$p_*(\pi_1(\tilde{X}, \tilde{l}(1))) = G$ (since $G$ normal)

$$\left[\ \begin{tikzcd} A \arrow[r, dashed, "\tilde{u}"] \arrow[dr, "u"'] & \tilde{X} \arrow[d, "p"] \\ & X \end{tikzcd} \right.$$

$\exists$ unique $\tilde{u} : A \to \tilde{X}$ with $p\tilde{u} = u$ and $\tilde{u}(a_0) = \tilde{x}$

provided $u_* \pi_1(A, a_0) \subset p_*(\pi_1(\tilde{X}, \tilde{x}))$. $]$

By covering space theory, $\exists\ \tilde{h} : \tilde{X} \to \tilde{X}$ with

$\tilde{h}(\tilde{x}_0) = \tilde{l}(1)$. Clearly $\eta(h)$ rep by $l$.

---

*Lemma 3.4* If $X$ is a connected CW complex, then any covering $\tilde{X}$ of $X$ has the structure of a CW complex.

*Proof* : Any map $\phi : e^n \to X$ has a lift (non-unique) $\tilde{\phi} : e^n \to \tilde{X}$ with $p\tilde{\phi} = \phi$.

Two lifts $\tilde{\phi}_1, \tilde{\phi}_2$ with $\tilde{\phi}_1(x) = \tilde{\phi}_2(x)$ for some $x \in e^n$ are equal everywhere.

Take for n-cells of $\tilde{X}$ all lifts of all $\phi_\alpha : e^n \to X$ $(\alpha \in A_n)$. Easy to check that this is CW complex.

Example : $P^2 = S^1 \cup_2 e^2 = e^0 \cup e^1 \cup_2 e^2$

$S^2$ = universal cover of $P^2$
$= (e^0 \cup e^0) \cup (e^1 \cup e^1) \cup (e^2 \cup e^2)$

If $\tilde{X} \xrightarrow{p} X$ is a regular cover of CW complex $X$, with $\pi$ = group of translations, then $\pi$ permutes cells of $\tilde{X}$ freely. ($g \in \pi$, $e^n_\alpha$ cell of $\tilde{X}$, $g e^n_\alpha = e^n_\alpha \Rightarrow g = 1$).

$\pi$ permutes n-cells of $p^{-1}$(n-cell of $X$) transitively.

Cellular homology

$H_*(X, Y)$ = singular homology.

CW complex $X$ ; define $C_n(X) = H_n(X^n, X^{n-1})$.

$\partial_n : C_n(X) \to C_{n-1}(X)$ defined as composite

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{j_*} H_{n-1}(X^{n-1}, X^{n-2})$$

$(X^n, X^{n-1})$ $\qquad$ $(X^{n-1}, X^{n-2})$

$\partial^2 = 0$ ; chain complex $C_*(X)$.

Lemma 3.5 $C_*(X)$ is free Abelian with one generator for each n-cell of $X$. $C_*(X)$ is chain equivalent to the singular chain complex $S_*(X)$.

Proof : By excision and homotopy properties of singular homology.

$$H_m(X^n, X^{n-1}) \cong H_m(A_n \times e^n, A_n \times \partial e^n)$$
$$\cong 0 \text{ for } m \neq n.$$

$\therefore$ $C_n(X) \cong$ free Abelian with one generator for each n-cell.

It follows that $H_m(X^{m-1}) \cong H_m(X^{m-2}) \cong H_m(X^{m-3})$
$\cong \dots \cong H_m(X^0) = 0.$

and ~~$H_m(X^{m+1}) \cong H_m(X^{m+}$~~

$$H_m(X^{m+1}) \cong H_m(X^{m+2}) \cong H_m(X^{m+3}) \cong \dots \cong H_m(X)$$

$$Z_m(C_*(X)) = \ker(j_*\partial : H_m(X^m, X^{m-1}) \to H_{m-1}(X^{m-1}, X^{m-2}))$$
$$= \ker(\partial : H_m(X^m, X^{m-1}) \to H_{m-1}(X^{m-1})) \quad \text{as } j_* \text{ mono}$$
$$= \operatorname{im} j_*$$

$$Z_m / B_m \cong H_m(X^m) / j_*^{-1}(B_m).$$
$$\cong H_m(X^m) / j_*^{-1}(\operatorname{im} j_* \partial)$$
$$= H_m(X^m) / \operatorname{im} \partial$$

Exact sequence $H_m(X^{m+1}, X^m) \xrightarrow{\partial} H_m(X^m) \to H_m(X^{m+1}) \to 0.$
gives $H_m(X^m)/\operatorname{im}\partial \cong H_m(X^{m+1}) \cong H_m(X).$

Cycle $z \in C_*(X) = H_m(X^m, X^{m-1})$
Put $z = j_* y$, $y \in H_m(X^m)$.
Now image of $y$ in $H_m(X)$ is image of homology class of $z$ in $H_m(X)$.

$e^n_\alpha$ = basis element of $C_n(X)$ corresponding to $n$-cell $\phi_\alpha : e^n \to X$.

Seek map $\theta : C_*(X) \longrightarrow S_*(X)$ s.t. $\theta\partial = \partial\theta$,
$\theta(C_*(X^n)) \subset S_*(X^n)$, $\theta(e^n_\alpha)$ represents $e^n_\alpha \in H_n(X^n, X^{n-1})$

Define inductively; for $n = 0$, define ~~$\theta$~~
$\theta(e^0_\alpha)$ = 0-simplex at $e^0_\alpha$.
Suppose $\theta : C_*(X^{n-1}) \longrightarrow S_*(X^{n-1})$ defined.
If $e^n_\alpha$ is a basis element of $C_n(X)$, $\theta(\partial e^n_\alpha)$ already defined, represents $\partial e^n_\alpha$ in $H_{n-1}(X^{n-1}, X^{n-2})$.
Also, $\partial\theta(\partial e^n_\alpha) = 0$ as chain.
Pick chain $c^n_\alpha \in S_n(X^n)$ representing $e^n_\alpha$ in $H_n(X^n, X^{n-1})$ [so $\partial c^n_\alpha \in S_{n-1}(X^{n-1})$].
Now $\partial c^n_\alpha - \theta(\partial e^n_\alpha)$ represents $0$ in $H_{n-1}(X^{n-1}, X^{n-2})$.
But $j_* : H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$ is mono so
$\partial c^n_\alpha - \theta(\partial e^n_\alpha)$ represents $0$ in $H_{n-1}(X^{n-1})$

$\therefore \exists\ d^n_\alpha \in S_n(X^{n-1})$ s.t. $\partial c^n_\alpha - \theta(\partial e^n_\alpha) = \partial d^n_\alpha$

Put $\theta(e^n_\alpha) = c^n_\alpha - d^n_\alpha$. Then $\partial\theta(e^n_\alpha) = \theta\partial(e^n_\alpha)$.

$\theta(e^n_\alpha)$ represents $e^n_\alpha$ in $H_n(X^n, X^{n-1})$, because $d^n_\alpha \in S_n(X^{n-1})$.

This completes the induction.

It follows that $\theta$ induces homology isomorphisms given above.

$$\begin{array}{lll} \underset{\text{cycle}}{z} \in C_n(X) = H_n(X^n, X^{n-1}) \xleftarrow{j_*} & H_n(X^n) & \theta(z) \\ & \downarrow & j_*[\theta(z)] = z. \\ & H_n(X) & \end{array}$$

Theorems from homotopy theory:

<u>Whitehead Theorem</u>: Let $X, Y$ be connected CW complexes and let $f: X \to Y$ be a map inducing <u>homology</u> isomorphisms in all dimensions; then $f$ is a homotopy equivalence.

<u>Hurewicz Theorem</u>. Let $X, Y$ be connected, simply connected CW complexes and let $f: X \to Y$ be a map. If $H_r(f) = 0$ for all $r < n$, then $\pi_r(f) = 0$ for all $r < n$, and the natural map $\pi_n(f) \to H_n(f)$ is an isomorphism.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Phi \Uparrow \quad \uparrow & & \uparrow \phi \\ S^{n-1} & \subset & D^n \end{array} \qquad \begin{array}{l} \pi_n(f) = \text{homotopy classes of } \Phi. \\ \cong \pi_n(M_f, X) \end{array}$$

Connected CW complex $X$; $\tilde{X} \to X$ regular covering, group $\pi$. $C_*(\tilde{X})$ is a ~~covering~~ complex of free $\mathbb{Z}[\pi]$-modules.

$\sum n_g g \in \mathbb{Z}[\pi]$, $f^n_\alpha$ is a cell of $\tilde{X}$

Define $(\sum n_g g)(f^n_\alpha) = \sum n_g (g \cdot f^n_\alpha) \in C_n(\tilde{X})$.

$\partial$ is a $\mathbb{Z}[\pi]$-homomorphism.

For each cell $e^n_\alpha$ of $X$, pick lift $\tilde{e}^n_\alpha$ in $\tilde{X}$.

Then $\{\tilde{e}^n_\alpha\}$ is a basis for $C_n(\tilde{X})$ over $\mathbb{Z}[\pi]$.

(Any $n$-cell in $\tilde{X}$ can be expressed uniquely as $g\,\tilde{e}^n_\alpha$.)

Similarly, $S_*(\tilde{X})$ is a free chain complex over $\mathbb{Z}[\pi]$. Slight modification of 3.5 shows that $C_*(\tilde{X}) \simeq S_*(\tilde{X})$ over $\mathbb{Z}[\pi]$. (Actually get canonical homotopy class of equivalences $C_* \simeq S_*$).

CW complexes $X, Y$ (connected). $f : X \to Y$, $f$ induces $\pi_1$ surjection. Let $G = \ker f_* : \pi_1(X) \to \pi_1(Y)$, let $\tilde{X}$ be covering of $X$ corr. to $G$, let $\tilde{Y}$ be universal cover of $Y$. Then $\exists$ lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of $f$. If $\tilde{f}'$ is another lift, then $\tilde{f}' = g\tilde{f}$ for some covering translation $g$ of $\tilde{Y}$. If $h$ is a covering translation of $\tilde{X}$, then $\tilde{f}h = \tilde{h}\tilde{f}$ for some unique translation $\tilde{h}$ of $\tilde{Y}$. $h \mapsto \tilde{h}$ defines isomorphism, translation gp of $\tilde{X} \to$ gp of $\tilde{Y} \cong \pi_1(Y)$. Use this isomorphism to identify the groups.

Now $\tilde{f}_* : S_*(\tilde{X}) \to S_*(\tilde{Y})$ is a chain map over $\cancel{\mathbb{Z}[\pi]}$ $\mathbb{Z}[\pi_1(Y)]$. whence

$$C_*(\tilde{X}) \xrightarrow{\simeq} S_*(\tilde{X}) \xrightarrow{\tilde{f}_*} S_*(\tilde{Y}) \xleftarrow{\simeq} C_*(\tilde{Y})$$

So we obtain $\tilde{f}_* : C_*(\tilde{X}) \to C_*(\tilde{Y})$, but defined only up to chain homotopy.

A cellular map $f : X \to Y$ is one with $f(X^n) \subset Y^n$ $\forall n$. Then we obtain a unique $\tilde{f}_* : C_*(\tilde{X}) \to C_*(\tilde{Y})$ (unique up to covering translations).

<u>Lemma 3.6</u> If connected CW complex $X$ is dominated by a finite CW complex $K$, then $C_*(\tilde{X})$, $S_*(\tilde{X})$ are dominated by a finitely generated free $\mathbb{Z}[\pi_1(X)]$-complex. ($\tilde{X}$ = universal cover).

<u>Proof</u>: $X \xrightarrow{f} K \xrightarrow{g} X$, $gf \simeq 1_X$. Wlog $K$ connected. Let $G = \ker g_* : \pi_1 K \to \pi_1 X$, let $\tilde{K}$ be covering of $K$ corresp to $G$; let $\tilde{X}$ be universal cover of $X$. Lift $f, g$ to $\tilde{f} : \tilde{X} \to \tilde{K}$, $\tilde{g} : \tilde{K} \to \tilde{X}$.

Lift $gf \simeq 1$ to get $\tilde{g}\tilde{f} \simeq$ covering translation of $\tilde{X}$: choose $\tilde{g}$ to make $\tilde{g}\tilde{f} \simeq 1_{\tilde{X}}$.

$\tilde{g}_* : C_*(\tilde{K}) \longrightarrow C_*(\tilde{X})$ ; also $\tilde{f}_* : C_*(\tilde{X}) \longrightarrow C_*(\tilde{K})$ and $\tilde{g}_* \tilde{f}_* \simeq 1_{C_*(\tilde{X})}$ ; so $C_*(\tilde{K})$ dominates $C_*(\tilde{X})$; hence also $S_*(\tilde{X})$. $C_*(\tilde{K})$ f.g.-free.

By Th 2.3, $C_*(\tilde{X}) \simeq$ f.g. proj $\mathbb{Z}[\pi_1 X]$-complex $E_*$

Define Wall invariant $\sigma(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1 X])$ to be $\sigma(E_*)$.

By Th 2.5, $\sigma(X)$ depends only on homotopy type of $X$.

By Th 1.9, $\sigma(X)$ doesn't depend on base point of $X$.

**Theorem 3.7** Let $X$ be a connected CW complex, $A_*$ a free $\mathbb{Z}[\pi_1 X]$-complex, and let $\varphi : A_* \to C_*(\tilde{X})$ be a chain equivalence, s.t. $\varphi_i : A_i \longrightarrow C_i(\tilde{X})$ is bijective for $i \leq 2$. Then $\exists$ a CW complex $Z$, a cellular homotopy equivalence $Z \xrightarrow{f} X$ and chain equivalence $\alpha : C_*(\tilde{Z}) \longrightarrow A_*$ s.t. $\tilde{f}_* = \varphi\alpha$ and $\alpha : C_i(\tilde{Z}) \longrightarrow A_i$ is bijective for all $i$.

Proof: Suppose inductively that $Z^{n-1}$, $f | Z^{n-1} \to X$, $\alpha | C_*(\tilde{Z}^{n-1}) \to A_*$ already constructed, with $f$ cellular, $\alpha : C_i(\tilde{Z}^{n-1}) \longrightarrow A_i$ bijective for $i < n$ and $\tilde{f}_* = \varphi\alpha$.

Induction starts with $n = 3$. $Z^2 = X^2$, $f = \text{incl} : Z^2 \longrightarrow X$ ; $\alpha = \varphi^{-1} : C_i(\tilde{Z}) \to A_i$ $(i \leq 2)$. Note that $\pi_1(X^2) \cong \pi_1(X)$, so that all complexes are over $\mathbb{Z}[\pi_1 X]$.

$f$ induces map $g : Z^{n-1} \longrightarrow X^n$, $\alpha$ induces $\beta : C_*(\tilde{Z}^{n-1}) \to A_*^n$ the "n-skeleton" of $A_*$.

$$\begin{tikzcd} C_*(\tilde Z^{n-1}) \arrow[r, "1"] \arrow[d, "\beta"] & C_*(\tilde Z^{n-1}) \arrow[r, "\varphi|A_*^{n-1}"] \arrow[d, "\tilde g_*"] & C_*(\tilde X^{n-1}) \\ A_*^n \arrow[r, "\varphi|A_*^n"] & C_*(\tilde X^n) \arrow[r] & C_*(\tilde X^n) \end{tikzcd}$$

induces maps $(\varphi|A^n)_* : H_i(\beta) \longrightarrow H_i(\tilde g_*)$, isomorphisms for $i<n$ (because $\varphi : A_* \longrightarrow C_*(\tilde X)$ was chain equiv.)

But $H_i(\beta) = 0$ for $i<n$, $H_n(\beta) = A_n$

$\therefore$ $H_i(\tilde g_*) = 0$ for $i<n$, get map $\theta : A_n \longrightarrow H_n(\tilde g_*)$

[Note that composition $A_n \xrightarrow{\theta} H_n(\tilde g_*) \longrightarrow H_n(\tilde X^n, \tilde X^{n-1}) = C_n(\tilde X)$ is just $\varphi$ ]

By Hurewicz th$^m$ applied to $\tilde g : \tilde Z^{n-1} \longrightarrow \tilde X^{n-1}$

$$H_n(\tilde g_*) \cong \pi_n(\tilde g) \cong \pi_n(g).$$

Pick basis $\{a_t\}_{t\in T}$ for $A_n$; we can represent $\theta(a_t) \in H_n(\tilde g_*)$ by diagram

$$\begin{tikzcd} Z^{n-1} \arrow[r] & X^n \\ \partial e^n \arrow[r] \arrow[u, "v_t"] & e^n \arrow[u, "u_t"] \end{tikzcd}$$

Give $T$ discrete topology, define $v : T\times\partial e^n \longrightarrow Z^{n-1}$ by $v(t,x) = v_t(x)$. Let $Z^n = Z^{n-1} \cup_v (T\times e^n)$, define $f|T\times e^n \longrightarrow X^n$ by $f(t,x) = u_t(x)$ extends $g$ to a map $f : Z^n \longrightarrow X^n$.

Define $\alpha : C_n(\tilde Z^n) \longrightarrow A_n$ by $\alpha(\tilde e_t^n) = a_t$, where $\tilde e_t^n$ is a lift of cell $t\times e^n$ in $\tilde Z^n$. (Choose lift to make this a chain map.)

But $\tilde f_*(\tilde e_t^n)$ is represented by
$$\begin{tikzcd} \tilde X^{n-1} \arrow[r, "\text{inc}"] & \tilde X^n \\ \partial e^n \arrow[r] \arrow[u, "\tilde f\tilde v_t"] & e^n \arrow[u, "\tilde u_t"] \end{tikzcd}$$

But this is $\tilde f_* \theta(a_t) = \varphi(a_t)$

$$= \varphi\alpha(\tilde e_t^n)$$

$\therefore$ $\tilde f_* = \varphi\alpha$

A group $\pi$ is finitely presented if it is defined by a finite set of generators and relations $\{g_1, \dots, g_n : r_1(\underline{g}) = \dots = r_k(\underline{g}) = 1\}$

Group $H$ is a <u>retract</u> of $G$ if $\exists$ homomorphisms $\varphi : H \to G$, $\psi : G \to H$ with $\psi\varphi = 1_H$.

<u>Lemma 3.8</u>: A retract of a finitely presented group is finitely presented.

<u>Proof</u>: $G$ finitely presented as $\{g_i : r_j(\underline{g}) = 1\}$.

$\varphi : H \to G$, $\psi : G \to H$ s.t. $\psi\varphi = 1_H$.

$\varphi\psi(g_i) = w_i(\underline{g})$ for some word $w_i$.

Let $L = \{g_i : r_j(\underline{g}) = 1,\ w_i(\underline{g}) = g_i\}$.

$\exists$ homomorphism $\pi : G \to L$, $\pi(g_i) = g_i$

$\theta : L \to H$, $\theta(g_i) = \psi(g_i)$

[ well defined since

$$\theta(w_i(\underline{g})) = \psi(w_i(\underline{g})) = \psi\varphi\psi(g_i) = \psi(g_i) = \theta(g_i)$$

$$= w_i(\theta(\underline{g})) = w_i(\psi(\underline{g}))$$

]

$\theta$ is isomorphism with inverse $\pi\varphi : H \to L$

$$\pi\varphi\theta(g_i) = \pi\varphi\psi(g_i) = \pi w_i(\underline{g}) = w_i(\pi\underline{g}) = w_i(\underline{g}) \in L$$

$$= g_i \in L$$

~~and~~ $\{\psi(g_i)\}$ is set of generators for $H$.

$$\theta\pi\varphi(\psi g_i) = \theta\pi w_i(\underline{g}) = \theta w_i(\pi\underline{g}) = \theta w_i(\underline{g}) = \theta(g_i) = \psi(g_i)$$

$\therefore \pi\varphi\theta = 1$, $\theta\pi\varphi = 1$, so $H \cong L$ which is finitely presented.

---

<u>Lemma 3.9</u> If connected CW complex $X$ is dominated by a finite complex, then $X \simeq$ CW complex $Y$ with $Y^2$ finite.

<u>Proof</u>: Let $f : X \to K$, $g : K \to X$ be s.t. $K$ is finite, $gf \simeq 1_X$.

$f_* : \pi_1(X) \to \pi_1(K)$, $g_* : \pi_1(K) \to \pi_1(X)$

with $g_* f_* = 1$.

· $\exists\ r_1, \dots, r_\ell \in \ker g_* : \pi_1(K) \longrightarrow \pi_1(X)$ s.t.

$$\pi_1(K)/\{r_1, \dots, r_\ell\} \cong \pi_1(X)$$

Let $v_j : \partial e^2 \longrightarrow K^1$ represent $r_j \in \pi_1(K)$

Let $u_j : e^2 \longrightarrow X$ be a null-homotopy of $g v_j$.

Define $Y^2 = K^2 \cup_{v_1} e_1^2 \cup \dots \cup_{v_\ell} e_\ell^2$

Define $g | e_j^2 : \to X$ to be $u_j$.

Then we have $g : Y^2 \longrightarrow X$. induces bijection
$g_* : \pi_1 Y^2 \longrightarrow \pi_1 X$, and $g_* : \pi_2 Y^2 \longrightarrow \pi_2 X$ is onto.

$$\begin{tikzcd} \pi_2 Y^2 \arrow[r] & \pi_2 X \\ \pi_2(K^2) \arrow[u] \arrow[r, "\text{onto}"'] & \pi_2(K) \arrow[u, "g_*"', "\text{onto}"] \end{tikzcd}$$

so $\pi_i(g) = 0$ for $i \leq 2$.

Suppose we have $Y^{n-1} \supset Y^2$ as 2-skeleton,
and $g : Y^{n-1} \longrightarrow X$ with $\pi_i(g) = 0$ for $i < n$

Let $\{\xi_t\}_{t \in T}$ be a (not nec. finite) set of generators of $\pi_n(g)$

Represent $\xi_t$ by

$$\begin{tikzcd} Y^{n-1} \arrow[r, "g"] & X \\ \partial e^n \arrow[u, "v_t"] \arrow[r] & e^n \arrow[u, "u_t"'] \end{tikzcd}$$

Use $v_t$ to attach $n$-cells to $Y^{n-1}$, giving $Y^n$,
$u_t$ to extend $g$ to $g : Y^n \longrightarrow X$, so that $\pi_i(g) = 0$, $i \leq n$.
~~Continue~~ Construct $Y^2 \subset Y^3 \subset Y^4 \subset \dots$ with union $Y$,
map $g : Y \longrightarrow X$ with $\pi_*(g) = 0$.
$\therefore$ $g$ is a homotopy equiv.

A gap in the proof of Th 3.7.
$\alpha : C_*(\tilde{Z}^n) \longrightarrow A_*$, chain map in dims $< n$.
1. commutes if lift $\tilde{e}_t^n$ of $t \times e^n$ is carefully chosen.

$$\begin{tikzcd}
A_n \arrow[rr, "\partial"] \arrow[d, "\alpha^{-1}"] \arrow[dr, "\theta"] \arrow[d, phantom, shift left=6, "1."] & & A_{n-1} \arrow[d, "\alpha^{-1}"] \\
C_n(\tilde{Z}^n) \arrow[r] \arrow[d, equal] & H_n(\tilde{g}) \arrow[r] \arrow[d, "\partial"] \arrow[ur, phantom, "2."] & C_{n-1}(\tilde{Z}^n) \arrow[d, equal] \\
H_n(\tilde{Z}^n, \tilde{Z}^{n-1}) \arrow[r, "\partial"] \arrow[ur] & H_{n-1}(\tilde{Z}^{n-1}) \arrow[r] & H_{n-1}(\tilde{Z}^{n-1}, \tilde{Z}^{n-2})
\end{tikzcd}$$

2. commutes

$$\begin{tikzcd}
A_*^{n-2} \arrow[r, phantom, "\subset"] \arrow[d, "\alpha^{-1}"] & A_*^{n-1} \arrow[r, phantom, "\subset"] \arrow[d, "\alpha^{-1}"] & A_*^n \arrow[d, "\phi"] \\
C_*(\tilde{Z}^{n-2}) \arrow[r, phantom, "\subset"] & C_*(\tilde{Z}^{n-1}) \arrow[r, "\tilde{g}"] & C_*(\tilde{X}^n)
\end{tikzcd}$$

Homology sequence of triples

$$\begin{tikzcd}
A_n \arrow[r, "\partial"] \arrow[d, "\theta"] & A_{n-1} \arrow[d, "\alpha^{-1}"] \\
H_n(\tilde{g}) \arrow[r] & C_{n-1}(\tilde{Z}^n)
\end{tikzcd}$$

---

<u>Theorem 3.10</u> If the connected CW complex $X$ is dominated by a finite complex $K$, and $\sigma(X) = 0$, then $X \simeq$ finite complex of dimension $\leq \max(4, \dim K)$

<u>Remark</u>: 4 can be replaced by 3. [C T C Wall; Finiteness conditions I].

<u>Proof</u>: By 3.9, we can assume $X^2$ finite.
By 3.6, 2.3, 2.6, $C_*(\tilde{X})$ is equivalent to a f.g. free complex $E_*$, by maps $f: C_*(\tilde{X}) \to E_*$, $g: E_* \to C_*(\tilde{X})$, inverse equivalences.
Define complex $A_*$ suitable for 3.7 as follows.

$A_*^2 = C_*(\tilde{X}^2)$ — f.g. free.

$A_n = E_n, \quad n \geq 4$ — f.g. free.

$$\begin{tikzcd}
\arrow[r] & A_5 \arrow[r] \arrow[d, "1"] & A_4 \arrow[r, "\partial_4"] \arrow[d, "1"] & A_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"] & A_2 \arrow[r] \arrow[d, "f"] & A_1 \arrow[r] \arrow[d, "f"] & A_0 \arrow[r] \arrow[d, "f"] & 0 \\
\arrow[r] & E_5 \arrow[r] & E_4 \arrow[r] & E_3 \arrow[r] & E_2 \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & 0
\end{tikzcd}$$

Let $Q_*$ be mapping cone of $f | A_*^2 \to E_*$. This has $H_i(Q_*) = 0$ for $i \leq 2$.

$$0 \to Z_3(Q_*) \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to 0$$ exact.

Define $A_3 = Z_3(Q_*)$ — f.g. proj.

$$A_3 = \{(y, z) \in A_2 \oplus E_3 : \partial y = 0,\ \partial z = -fy\}.$$

Define $\partial_4(x) = (0, \partial x)$

$f_3(y, z) = z$

$\partial_3(y, z) = -y$

$A_*$ is a chain complex, and vertical maps induce homology isomorphisms.

So $f : A_* \to E_*$ is chain equivalence.

$gf : A_* \to C_*(\tilde{X})$ chain equivalence.

$gf \mid A^2_* \simeq$ inclusion.

$\therefore gf \simeq \phi : A_* \to C_*(\tilde{X})$ with $\phi \mid A^2_*$ bijective.

~~Now apply 3.7 to get $Y$, $h : Y \simeq Z$, with~~

~~$C_*(Y) \cong A_*$~~.

$A_*$ is f.g. projective, free except in dim 3, $\sigma(A_*) = 0$.

Enlarge $A_3, A_4$ to replace $A_*$ by $\simeq$ equivalent free complex.

By 3.7, $X \simeq Y$ with $C_*(\tilde{Y}) \cong A_*$

In particular, $Y$ finite, $\dim Y = \max(\dim E_*, 4)$.

By 2.3, can choose $E_*$ s.t. $\dim E_* = \dim K$.

---

__Exercise__ : Use Th 3.7 and methods of 3.9, 3.10, to show : (Milnor) : If $X$ is simply connected CW complex, and $H_n(X; \mathbb{Z})$ has rank $\beta_n$ and has $\tau_n$ "torsion coefficients", then $X \simeq$ CW complex with $\beta_n + \tau_{n-1} + \tau_n$ n-cells for each $n$.

**Theorem 3.11** Given finitely presented group $\pi$ and element $\sigma \in \tilde{K}_0(\mathbb{Z}\pi)$, $\exists$ connected CW-complex $X$ with $\pi_1 X \cong \pi$ and $\sigma(X) = \sigma$, $X$ dominated by a finite complex.

*Proof*: $\exists$ finite complex $K$ with $\pi_1 K \cong \pi$,
$n > \dim K$, $n > 1$.
Let $Y = K \vee \bigvee_{r=1}^{\infty} S_r^n$, let $p : Y \to K$ send $S_r^n$'s to base-pt.
Exact homotopy sequence of $p$ is split by $K \subset Y$;

$$0 \to \pi_{n+1}(p) \to \pi_n(Y) \xleftrightarrow{} \pi_n(K) \to 0$$

Apply Hurewicz Thm to $\tilde{p} : \tilde{Y} \to \tilde{K}$:

$$\pi_{n+1}(p) \cong \bigoplus_{r=1}^{\infty} \mathbb{Z}[\pi] \text{ as } \mathbb{Z}[\pi]\text{-module.}$$

f.g. projective $\mathbb{Z}[\pi]$-module $P$ representing $\sigma$.
$\exists Q$ s.t. $P \oplus Q$ f.g. free.

$$\begin{array}{ccc}
\pi_{n+1}(p) & \xrightarrow{\cong} & (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\
\downarrow \phi & & \downarrow \text{inc.} \\
\pi_{n+1}(p) & \xrightarrow{\cong} & P \oplus (Q \oplus P \oplus Q \oplus \cdots) \\
 & & \shortparallel \\
 & & (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots
\end{array}$$

$\phi$ mono, $\operatorname{coker} \phi \cong P$.
Let $\xi_r$ generate $r$th summand in $\bigoplus_{r=1}^{\infty} \mathbb{Z}[\pi] \cong \pi_{n+1}(p)$.
Represent $\phi(\xi_r)$ by

$$\begin{array}{ccc}
Y & \xrightarrow{p} & K \\
\uparrow v_r & & \uparrow u_r \\
\partial e_r^{n+1} & \subset & e_r^{n+1}
\end{array}$$

Use $v_r$ to attach $e_r^{n+1}$ to $Y$, $\forall r$, giving complex $X$.
Use $u_r$ to extend $p$ to map $p : X \to K$
Chain complex of $\tilde{X}$ has form

$$0 \to \pi_{n+1}(p) \xrightarrow{\phi} \pi_{n+1}(p) \to C_{n-1}(\tilde{K}) \to \cdots$$

chain equivalent to $0 \longrightarrow P \xrightarrow{i} C_{n+1}(\tilde{K}) \xrightarrow{\partial} \cdots$
which has Wall invariant $[P] = \sigma$.

Attach more $(n+1)$-cells & $n$-cells to $X$, [illegible] $Z$, [illegible]
retraction $p : Z \to X$, with $C_*(\tilde{Z})$ equivalent to [illegible]

$$0 \longrightarrow P \oplus Q \longrightarrow C_{n+1}(\tilde{K}) \xrightarrow{\partial} \cdots$$

which is f.g. free.

So $Z \simeq$ finite complex by 3.7. and $Z$ dominates $X$.

## §4. Torsion for CW complexes

$\pi$ any group. A f.g. free $\mathbb{Z}[\pi]$-module, $(a_1, .., a_k)$ basis. $(a_1', \ldots, a_k')$ is <u>equivalent</u> to $(a_1, .., a_k)$ if $a_i' = \pm g_i a_i$ where $g_i \in \pi$ (so $\pm g_i \in \mathbb{Z}[\pi]$).

Chain complexes $C_*, D_*$ (based), $f : C_* \to D_*$ chain equiv. Then image of $\tau(f)$ in $Wh(\pi)$ depends only on equiv. classes of bases of $C_*, D_*$.

$K$ finite CW complex. Equivalence class of basis of $C_n(\tilde{K})$ $(\tilde{e}_1^n, .., \tilde{e}_k^n)$ depends only on cell structure of $K$, not on choice of lifts $\tilde{e}_k^n$ or on orientation of cells.

$f : K \to L$ homotopy equivalence of finite CW complexes ~~depends~~ define $\tau(f)$ = image of $\tau(\tilde{f}_* : C_*(\tilde{K}) \to C_*(\tilde{L}))$ in $Wh(\pi)$.

This depends only on cell structures of $K, L$ and homotopy class of $f$ (by 2.8).

<u>Theorem 4.1</u> : If $f : K \to L$, $g : L \to M$ are homotopy equivalences of finite CW complexes, then $\tau(gf) = \tau(g) + \tau(f) \quad \in Wh(\pi_1 K = \pi_1 L = \pi_1 M = \pi)$.

<u>Problem</u>: Is $\tau(f)$ a topological invariant of $K, L, f$ ?
Yes if $K, L$ are compact manifolds.

$X$ any CW complex. Complex $X'$ is a subdivision of $X$ if $|X'| = |X|$ and the interior of each cell in $X'$ is contained in the interior of some cell in $X$.

Identity map $\chi : X \to X'$ is cellular.

<u>Theorem 4.2</u> : $\chi : X \to X'$ is a simple homotopy equivalence, ie $\tau(\chi) = 0$.

Proof: $X$ finite CW complex. $\exists$ subcomplexes

$$\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_k = X$$

s.t. $X_i - X_{i-1}$ consists of just one cell.

Let $X_i'$ be subdivision of $X_i$ induced by $X'$.

Let $Y_i = X_i' \cup (\text{cells of } X - X_i)$

Maps. $X = Y_{-1} \xrightarrow{\lambda} Y_0 \xrightarrow{\lambda} Y_1 \xrightarrow{\lambda} \cdots \xrightarrow{\lambda} Y_k = X'$.

Enough to prove $\lambda : Y_{i-1} \longrightarrow Y_i$ is s.h.e. ie $\tau(\lambda) = 0$.

Choose lift $\tilde{e}$ for each cell $e$ of $X$.

If $e'$ is a cell in $X'$, $\operatorname{int} e' \subset \operatorname{int} e$ for some unique cell $e$ in $X$. Choose lift $\tilde{e}'$ of $e'$ so that $\operatorname{int} \tilde{e}' \subset \operatorname{int} \tilde{e}$.

Exact sequence

(*) $0 \longrightarrow C_*(\tilde{Y}_{i-1}) \xrightarrow{\tilde{\lambda}_*} C_*(\tilde{Y}_i) \longrightarrow D_* \longrightarrow 0$ (defines $D_*$)

~~$\tilde{\lambda}_*$ maps cells of~~

Let $X_i - X_{i-1} = e_i^n$.

Then $\tilde{\lambda}_*$ maps each cell $\tilde{e}$ of $\tilde{Y}_{i-1}$ to a cell of $\tilde{Y}_i$, except that $\tilde{\lambda}_*(\tilde{e}_i^n) = \tilde{f}_1^n + \cdots + \tilde{f}_r^n$ where $\tilde{f}_1^n, \ldots, \tilde{f}_r^n$ are the $n$-cells of $\tilde{Y}_i$ with $\operatorname{int} \tilde{f}_j^n \subset \operatorname{int} \tilde{e}_i^n$.

Change basis of $C_n(\tilde{Y}_i)$ by replacing $\tilde{f}_1^n$ by $\tilde{f}_1^n + \cdots + \tilde{f}_r^n$. (leave other basis elements alone). This is an elementary operation, so it doesn't affect the torsion of $\tilde{\lambda}_*$.

But now (*) is a based exact sequence, so

$\tau(\tilde{\lambda}_*) = \tau(D_*)$.

~~But~~ Boundary maps of $D_*$ have matrices with integer coefficients (by the choice of lifts we never need to translate by an element of $\pi$).

$\therefore$ Torsion of $D_*$ is in image of $\bar{K}(\mathbb{Z}) = 0$.

$\therefore$ $\tau(\lambda) = 0$, as required.

**Corollary 4.3** : If $f : X \to Y$ is a homotopy equivalence of compact polyhedra, then $\tau(f)$ is well defined. (ie independant of PL triangulations chosen for $X, Y$).

**Theorem 4.4** : Given finite CW complex $K$ with fundamental group $\pi$, and element $\tau \in Wh(\pi)$, $\exists$ finite CW complex $L$ and homotopy equiv $f : K \to L$ with $\tau(f) = \tau$.

Proof : Represent $\tau$ by a matrix $M \in GL(k, \mathbb{Z}[\pi])$

Let $Y = K \vee \bigvee_{i=1}^{k} S_i^n$ , where $n \geq \dim K \geq 2$.

$p : Y \to K$ sends $S_i^n$ to base-pt.

As in 3.11, $\pi_{n+1}(p) \cong \bigoplus_{i=1}^{k} \mathbb{Z}[\pi]$ , one generator for each $S_i^n$ ; let $\xi_i$ be $i^{th}$.

Let $\phi : \pi_{n+1}(p) \to \pi_{n+1}(p)$ have matrix $M$.

Represent image of $\phi(\xi_i)$ in $\pi_n(Y)$ by map $v_i : \partial e_i^{n+1} \to Y$. Use the $v_i$'s to attach $e_1^{n+1}, \ldots, e_r^{n+1}$ to $Y$, giving complex $L \supset K$.

Then $C_*(\tilde{L})$ has form

$$0 \to \pi_{n+1}(p) \xrightarrow{\phi} \pi_{n+1}(p) \xrightarrow{0} C_{n-1}(\tilde{K}) \xrightarrow{\partial} \cdots$$

By 2.3 and Whitehead theorem, inclusion $K \subset L$ is homotopy equivalence.

$$0 \to C_*(\tilde{K}) \to C_*(\tilde{L}) \to (0 \to \pi_{n+1}(p) \xrightarrow{\phi} \pi_{n+1}(p) \to 0) \to 0$$

By 2.11, $\tau(f) = \tau(\phi) = \tau$.

---

Let $\Delta^n$ be an $n$-simplex, let $\Delta_0$ be an $(n-1)$ face, let $\Lambda = \overline{\partial\Delta - \Delta_0}$.

$K$ finite CW complex, $f : \Lambda, \partial\Lambda \to K^{n-1}, K^{n-2}$

Let $L = K \cup_f \Delta$ ; this is CW complex with cells of $K$ and $\Delta_0^{n-1}$; $\Delta^n$.

Then $K \subset L$ is called an elementary expansion of dimension $n$, and a homotopy inverse is an elementary collapse.

Both are homotopy equivalences, and have zero torsion.

<u>Example 4.5</u> (Milnor) There exist finite complexes $K, L$, which are homeomorphic but don't have isomorphic subdivisions. Thus $\exists$ compact polyhedra $|K|, |L|$, which are homeomorphic but not PL homeomorphic. (Hauptvermutung is false).

Proof: Group $\pi$ with $Wh(\pi) \neq 0$. eg $C_5$.
$\pi$ finitely presented.

$\exists$ finite simplicial complex $X_1$ with $\pi_1(X_1) \cong \pi$

By method of 4.4, $\exists$ finite simplicial complex $X_2 \supset X_1$ with s.t. inclusion $X_1 \subset X_2$ has torsion $\tau \neq 0$.

$\exists$ finite simplicial complex $X_3 \supset X_2$ s.t. $X_3 \searrow X_1$.
(eg take $k$ large, and extend $X_1 \longrightarrow X_1 \times \Delta^k$ to an embedding $X_2 \longrightarrow X_1 \times \Delta^k$ by general position)
$\exists$ finite simplicial complex $X_4 \supset X_3$ s.t. $X_4 \searrow X_2$.
Embed $X_4$ in some $\mathbb{R}^n$.
Let $W_4$ be a reg. nhd of $X_4$ in $\mathbb{R}^n$.
$X_i$ $W_{i+1}$, $i = 3, 2, 1$.
$W_i$

$W_4$ is reg nhd of $X_4$, $X_4 \searrow X_2$. $\therefore$ $W_4$ is reg nhd of $X_2$.
$W_2$ is reg nhd of $X_2$, $W_2 \subset \operatorname{int} W_4$

$\therefore$ $\overline{W_4 - W_2} \cong \partial W_2 \times I$

Similarly, $\overline{W_3 - W_1} \cong \partial W_1 \times I$. (†)
Let $V = \overline{W_2 - W_1}$, $V' = \overline{W_3 - W_2}$
$V$ is a cobordism from $M = \partial W_1$ to $N = \partial W_2$

$V'$ is a cobordism from $N$ to $\partial W_3 \cong \partial W_1$ by (†)

Now $V \cup V' \cong M \times I$

$$V \cong V \cup (V' \cup \overline{W_4 - W_3})$$
$$\cong (V \cup V') \cup (\overline{W_4 - W_3})$$
$$\cong \overline{W_4 - W_3}$$

$\therefore$ $V' \cup V \cong V' \cup \overline{W_4 - W_3} \cong N \times I$

$V$ is an <u>invertible</u> cobordism.

$M \hookrightarrow V$ has torsion $\tau$.

<u>Theorem 4.6</u> (Mazur).

If $V$ is an invertible cobordism from $M$ to $N$, then $V - N \cong M \times [0,\infty)$.

<u>Proof</u>: Let ~~$\mathrm{U}$~~ $V'$ = inverse of $V$.

Let $\mathcal{U} = V \cup_N V' \cup_M V \cup_N V' \cup_M \cdots$

$$\mathcal{U} \cong (V \cup V') \cup (V \cup V') \cup \cdots$$
$$\cong (M \times I) \cup (M \times I) \cup \cdots$$
$$\cong M \times [0,\infty).$$

But $\mathcal{U} \cong V \cup (V' \cup V) \cup (V' \cup V) \cup \cdots$
$$\cong V \cup (N \times I) \cup (N \times I) \cup \cdots$$
$$\cong V \cup_N (N \times [0,\infty)).$$

$\exists$ collar nbd $C$ of $N$ in $V$, so $\mathcal{U} \cong V - N$.

---

Take $K = (M \times I) \cup (\text{cone on } M \times 1)$

$L = V \cup (\text{cone on } N)$

Topologically, $K =$ 1-pt compactification of $M \times [0,\infty)$

$L =$ 1-pt compactification of $V \cup (N \times [0,\infty))$
$\cong M \times [0,\infty)$.

$\therefore$ $K \cong L$ topologically.

Suppose $K', L'$ are isomorphic subdivisions of $K, L$.

Let $a, b$ be vertices of the cones in $K, L$.

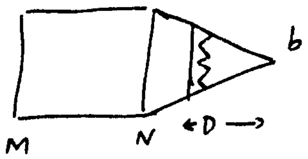
Let $P = \overline{K' - st(a, K')}$ , $Q = \overline{L' - st(b, L')}$

then ~~$M \times 0 \subset P$~~

$M \times 0 \subset M \times I \subset P$ , $M \subset V \subset Q$ ;

and $(P, M\times 0) \cong (M \times I, M)$ , $(Q, M) \cong (V, M)$

For




$CN$ = cone on $N = \{tx + (1-t)b : x \in N, t \in I\}$

Let $D = \{tx + (1-t)b : t \geq \tfrac{1}{2}\}$.

$\exists$ pseudo radial homeomorphism

~~$CN \to CN$~~

$CN \longrightarrow CN$ fixing $N$, $b$ & taking $st(b, K')$ onto $D$. Extends to a PL homeomorphism $L \longrightarrow L$ fixing $M$ and taking $Q$ onto $\overline{L-D} \cong V \cup (N \times I) \cong V$.

Similarly $(P, M\times 0) \cong (M \times I, M \times 0)$. Isomorphism $K' \xrightarrow{h} L'$ must take $a$ onto $b$. (for these are the only points with non-simply connected links). ($n \geq 2 + \dim X_4$).

$\therefore$ must take $P$ onto $Q$ by PL homeomorphism.

Now

$$\begin{array}{ccc} M & \subset & P \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ M & \subset & Q \end{array}$$

Vertical maps are PL homeos $\therefore$ have zero torsion.
$M \subset P$ has torsion zero
$M \subset Q$ has torsion $\tau \neq 0$.

Contradiction Th 4.2.

---

Remarks Every invertible cobordism $V$ from $M$ to $N$ is an $h$-cobordism.

Stallings proved that any $h$-cobordism $V$ of dimension $\geq 5$ is invertible. $\therefore V - N \cong M \times [0, \infty)$.

$s$-cobordism is an $h$-cobordism in which $M \subset V$, $N \subset V$ are simple homotopy equivalences.

Smale, Barden Mazur Stallings:

s-cobordism theorem : If $V^n$ is an $s$-cobordism and $n \geq 6$, then $V \cong M \times I$.

<u>Exercise</u> $M^n$ closed PL manifold, $n \geq 4$. Then, if $\tau \in Wh(\pi_1 M)$, $\exists$ an h-cobordism $W$ on $M$ with torsion $\tau(W, M) = \tau$.
[eg take $W$ = reg nbd of $M \cup$ suitable 2-complex in $M \times I$]

<u>Theorem 4.7</u> A cellular homotopy equivalence $f : K \to L$ between finite CW complexes has $\tau(f) = 0$ iff $f$ can be factored into finitely many elementary expansions and collapses.

<u>Proof</u>: From now on, "elementary collapse" means retraction $L \to K$ where $K \subset L$ is an elementary expansion.

Elementary expansions & collapses have zero torsion.

<u>Converse</u>: First note that $L \subset M_f$ is a composite of expansions. Put $M_f^i$ = mapping cylinder of ~~$f|K^i$~~ $f|K^i \to L$.

$$L \subset M_f^0 \subset M_f^1 \subset \dots \subset M_f^k = M_f \qquad (k = \dim K).$$

$M_f^{i-1} \subset M_f^i$ is composite of elementary expansions of dimension $i+1$, one for each $i$-cell of $K$. So we can replace $L$ by $M_f$, and $f : K \to L$ by an inclusion.
Assume from now on that $f$ is an inclusion.

<u>Lemma 4.8</u>. If $f : K \to L$ is a composite of elementary expansions and collapses, and $\phi : \partial e^n \to K^{n-1}$ is a map, then
$$f \cup 1 : K \cup_\phi e^n \to L \cup_{f\phi} e^n$$
is a composite of expansions and collapses.

<u>Proof</u>: Enough to consider case when $f$ is elem. expansion or collapse.

Expansion case is trivial, so suppose $f : K \to L$ is a collapse.

$\exists$ cellular homotopy $H : 1 \simeq f$, rel $L$.

Let $h = H\phi \cup 1 : (\partial e^n \times I) \cup (e^n \times 1) \to K \cup_\phi e^n$

Let $J = (K \cup_\phi e^n) \cup_h (e^n \times I)$, regard as CW complex with cells of $K$, $e^n\times 1$, $e^n\times 0$, $e^n\times I$.

$K\cup_\phi e^n \subset J$ is elem. expansion (add cells $e^n\times 0$, $e^n\times I$).

$K\cup_{f\phi} e^n \subset J$ is elem expansion (add cells $e^n\times 1$, $e^n\times I$).

Now $L\cup_{f\phi} e^n \subset K\cup_{f\phi} e^n$ is elem. expansion.

Hence result.

Proof of 4.7 : $f: K \to L$ inclusion, $\tau(f)=0$.

Assume inductively that $L-K$ has no cells of dimension $< r$. We modify $L$ keeping $K$ fixed so that $L-K$ has no cells of dimension $\le r$.

Let $e^r$ be an $r$-cell of $L-K$.

$\pi_r(L^{r+1}, K) \cong \pi_r(L,K) = 0$

$\exists$ cellular homotopy $H: e^r\times I \to L$ s.t.

$H_0 =$ inclusion, $H_1(e^r) \subset K$, $H_t|\partial e^r$ independant of $t$.

Let $e^{r+2} = e^r\times I\times I$, $e^{r+1} = \partial(e^r\times I\times I) - (e^r\times I\times 0)$

$h: e^r\times I\times 0 \to L$ induced by $H$.

Let $M = L\cup_h e^{r+2}$: CW complex with cells of $L$ together with $e^{r+1}$, $e^{r+2}$.

Now $K\cup e^r\cup e^{r+1}$ is a subcomplex of $M$, collapsing onto $K$. By repeated use of Lemma 4.8 (once for each cell of $M-(K\cup e^r\cup e^{r+1})$) we obtain a complex $L'\supset K$, obtained from $L$ by elementary expansions and collapses, such that $L'-K$ has fewer $r$-cells than $L-K$ (we have removed $e^r$, but introduced $e^{r+2}$). Repeat until $L-K$ has no $r$-cells, completing induction. Continue until $L-K$ has $n$-cells & $(n-1)$ cells only, with $n > \dim K$.

We show how to alter basis of $C_n(\tilde{L})$ by elementary matrix $1+ae_{ij}$ ($a\in \mathbb{Z}[\pi,K]$). Let $\tilde{e}_i^n$, $\tilde{e}_j^n$ be $n$-cells of $\tilde{L}$. By Hurewicz theorem, $H_n(\tilde{L}^n,\tilde{L}^{n-1}) \cong \pi_n(\tilde{L}^n,\tilde{L}^{n-1})$

$\therefore \exists$ map $\varphi: e^n,\partial e^n \to L^n, L^{n-1}$ rep. class $\tilde{e}_j^n + a\tilde{e}_i^n$

$\exists$ homotopy $G : \partial e^n \times I \longrightarrow L^n$ with

$G_0$ = attaching map of $e_j^n$.

$G_1 = \varphi / \partial e^n \longrightarrow L$.

Define $\psi = 1 \cup G : (e^n \times 0) \cup (\partial e^n \times I) \longrightarrow L^n$.

Let $M = L ~~K~~ \cup_\psi (e^n \times I)$ with cells of ~~K~~ $L$ and $e^n \times 1$, $e^n \times I$.

Then ~~K~~ $L$ expands to $L \cup_\psi (e^n \times I)$ which collapses onto $(L - e_j^n) \cup_\varphi e^n$. This performs desired change of basis.

Since $\tau(L \subset K) = 0$, we may expand (to increase chain groups) and then reduce matrix of $\partial : C_n(\tilde{L}, \tilde{K}) \longrightarrow C_{n-1}(\tilde{L}, \tilde{K})$ to 1.

(May also have to change lifts + orientations)

Let $\tilde{e}_i^n$ be an $n$-cell of $L$, so $\partial \tilde{e}_i^n = \tilde{e}_i^{n-1}$ in $H_{n-1}(\tilde{L}^{n-1}; \tilde{L}^{n-2} \cup \tilde{K})$

Let $\varphi : \partial e^n \longrightarrow L$ be attaching map.

Claim that $\varphi$ is homotopic to map $\psi : \partial e^n \longrightarrow L$ such that $\psi(\partial e^n) \cap (L - K) = e_i^{n-1}$ and $\psi | \psi^{-1}(\text{int } e_i^{n-1})$ 1-1 and $\overline{\psi^{-1}(\text{int } e_i^{n-1})} \cong n-1$ cell.

For let $\theta : \partial e_i^{n-1} \longrightarrow K$ be attaching map of $e_i^{n-1}$.

$\pi_{n-1}(L, K) = 0$, so $\exists$ homotopy $H : e_i^{n-1} \longrightarrow K$ such that $H | \partial e_i^{n-1} = \theta$.

Then $1 \cup H : \underbrace{e_i^{n-1} \cup_\partial e_i^{n-1}}_{\cong S^{n-1}} \longrightarrow L$ represents same element of $H_{n-1}(\tilde{L}^{n-1}, \tilde{K})$ as $\varphi$.

$\therefore$ $1 \cup H$ represents same element of $\pi_{n-1}(\tilde{L}^{n-1}; \tilde{K})$ as $\varphi$.

$$\pi_{n-1}(\tilde{K}) \longrightarrow \pi_{n-1}(\tilde{L}^{n-1}) \longrightarrow \pi_{n-1}(\tilde{L}^{n-1}; \tilde{K})$$

$\therefore$ $\varphi$ represents same element of $\pi_{n-1}(\tilde{L}^{n-1})$ as $\psi = (1 \cup H) + (\text{some element } \xi \text{ of } \pi_{n-1}(\tilde{K}))$ and $\psi$ has ~~sp~~ required properties.

By the trick used above for elementary change of basis, $L^{n-1} ~~K~~ \cup_\psi e_i^n$ is obtained from $L^{n-1} ~~K~~ \cup_\varphi e_i^n$ by elementary expansion and collapse. Also, $K \cup e_i^{n-1} \cup_\psi e_i^n$ collapses to $K$, so we can reduce number of cells in $L - K$.

Continue until $L-K$ has no cells; then we have obtained $K$ from $L$ by elementary moves.

# §5. Open Manifolds

$X$ any Hausdorff space.

An end of $X$ is a collection $\mathcal{E}$ of non-empty open sets in $X$, such that

i) $U \in \mathcal{E} \Rightarrow U$ connected and $Fr(U)$ compact.

ii) $U, V \in \mathcal{E} \Rightarrow \exists\, W \in \mathcal{E}$ with $W \subset U \cap V$.

iii) $\bigcap \{\bar{U}\,;\, U \in \mathcal{E}\} = \phi$.

iv) $\mathcal{E}$ maximal wrt i) – iii).

Example : $\mathbb{R}$ has just two ends, namely

$$\{(a, \infty)\,;\, a \in \mathbb{R}\} \quad \text{and} \quad \{(-\infty, b)\,;\, b \in \mathbb{R}\}.$$

Lemma 5.1 : Suppose $\mathcal{E}'$ satisfies i) – iii), and $A \subset X$ has compact frontier. Then $\exists\, U \in \mathcal{E}'$ such that either $\bar{U} \cap A = \phi$ or $\bar{U} \subset A$.

Proof : Since $Fr(A)$ is compact, and $\bigcap_{U \in \mathcal{E}'} (\bar{U} \cap Fr(A)) = \phi$,

$\exists\, U_1, \ldots, U_k \in \mathcal{E}'$ such that $\bar{U}_1 \cap \ldots \cap \bar{U}_k \cap Fr(A) = \phi$.

By ii), $\exists\, U \in \mathcal{E}'$ s.t. $U \subset U_1 \cap \ldots \cap U_k$, so

$\bar{U} \subset \bar{U}_1 \cap \ldots \cap \bar{U}_k$, so $\bar{U} \subset X - Fr(A)$.

Since $U$ connected, $\bar{U}$ connected, so $\bar{U} \subset A$ or $X - A$.

Corollary 5.2 : If $\mathcal{E}'$ satisfies i) – iii), then $\mathcal{E}'$ is contained in a unique end of $X$.

Proof : Let $\mathcal{E}$ be the collection of all non-empty connected open sets $V$ s.t. $V \supset U$ for some $U \in \mathcal{E}'$ and $Fr(V)$ compact. Then $\mathcal{E}$ satisfies i) – iii).

Suppose $\mathcal{E}'' \supset \mathcal{E}'$ also satisfies i) – iii). Then if $V \in \mathcal{E}''$, $\exists\, U \in \mathcal{E}'$ st $\bar{U} \cap V = \phi$ or $\bar{U} \subset V$. $\bar{U} \cap V$ impossible by i, ii) so $\bar{U} \subset V$, so $V \in \mathcal{E}$. So $\mathcal{E}'' \subset \mathcal{E}$, so $\mathcal{E}$ is unique end containing $\mathcal{E}'$.

A neighbourhood of $\mathcal{E}$ is a set $N$ containing some $U \in \mathcal{E}$.

Corollary 5.3 : Distinct ends of $X$ have disjoint neighbourhoods.

*Proof*: Suppose $\mathcal{E}, \mathcal{E}'$ are ends without disjoint nbds.
Choose $U \in \mathcal{E}$; $\exists V \in \mathcal{E}'$ s.t. $\bar{V} \subset U$ (by 5.1)
By maximality of $\mathcal{E}'$, $U \in \mathcal{E}'$, so $\mathcal{E} \subset \mathcal{E}'$. Similarly $\mathcal{E}' \subset \mathcal{E}$.
$\therefore \mathcal{E} = \mathcal{E}'$.

---

**Def$^n$**: A space $X$ is $\sigma$-compact if it is the union of countable many compact subspaces.

**Theorem 5.4** (Freudenthal). Let $X$ be locally connected, locally compact, connected, $\sigma$-compact, Hausdorff.
Then $X$ has an end iff $X$ is not compact.

*Proof*. A compact space has no ends, by ii) & iii) for ends.
Conversely; $X = \cup C_i$ where $C_i$ is compact,
$C_1 \subset C_2 \subset \dots$, $X$ non-compact,

$X$ locally compact, so $C_i$ has compact nbd $D_i$ in $X$. Every component $V$ of $X - C_i$ is open ($X$ is loc. connected), and meets $D_i$ ($\because X$ connected: if $V \cap D_i = \phi$, then $\bar{V} - V \subset C_i$ and $\bar{V} \subset X - C_i$; $\therefore \bar{V} - V = \phi$, so $V$ open & closed in $X$, contradiction).

$Fr(D_i)$ compact, so covered by finitely many components $V_i^1, \dots, V_i^k$ of $X - C_i$;
$X = D_i \cup V_i^1 \cup \dots \cup V_i^k$
$\therefore$ some $V_i^j$ has non-compact closure.
Choose inductively $U_1, U_2, \dots$ s.t. $U_i$ is a component of $X - C_i$, $\bar{U}_i$ non-compact, $U_i \subset U_{i-1}$.
$Fr(U_i) \subset C_i$ ~~because~~ $X$ connected,
$\therefore \{U_1, U_2, \dots\} = \mathcal{E}'$ satisfies i) – iii), so contained in an end of $X$.

---

**Examples**

i) $\mathbb{R}^n$, $n \geq 2$, has just one end.

$B^n_\lambda$ = closed ball radius $\lambda$.

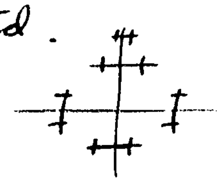
$\{ \mathbb{R}^n - B^n_\lambda : \lambda \in \mathbb{R} \}$ defines an end $\mathcal{E}$ of $\mathbb{R}^n$.

If $\mathcal{E}'$ is another end, $\exists$ disjoint nhds $U \in \mathcal{E}$, $V \in \mathcal{E}'$.

$Fr(U) \cup Fr(V)$ is compact, ~~tae~~ $\mathbb{R}^n - (Fr(U) \cup Fr(V))$ has at least two unbounded components $U, V$, which is impossible.

An end $\mathcal{E}$ is <u>isolated</u> if it has a neighbourhood $U$ which is not a nhd of any other end. It follows that $U$ ~~is a~~ has just one end.

<u>Ex</u> The universal cover of $S^1 \vee S^1$ has infinitely many ends, none of which is isolated.



An <u>open</u> manifold is a non-compact manifold without boundary.

If $W$ is an open manifold, a <u>completion</u> of $W$ is a homeomorphism (PL) from $W$ onto $\overline{W} - \partial \overline{W}$ $(= \mathrm{int}\, \overline{W})$ where $\overline{W}$ is a compact PL manifold.

<u>Theorem 5.5</u>

An open PL manifold has a completion iff it has finitely many ends, each of which has a collar.

A <u>collar</u> of an end $\mathcal{E}$ of $W$ is a submanifold $V$ of $W$ s.t. $\mathrm{int}\, V \in \mathcal{E}$, $V \cong \partial V \times [0, \infty)$

<u>Proof</u>: Suppose $W$ ~~homeomorphic~~ to $\mathrm{int}\, \overline{W}$, where $\overline{W}$ is compact.

Let $M_1, \ldots, M_k$ be components of $\partial \overline{W}$.

Let $\gamma_i : M_i \times I \longrightarrow \overline{W}$ be a collar nhd of $M_i$ in $\overline{W}$

s.t. $\operatorname{im}\gamma_i \cap \operatorname{im}\gamma_j = \phi$ if $i \neq j$.

Then $\{\gamma_i(M_i \times (a,1)) : a \in (0,1)\}$ defines an end $\mathcal{E}_i$ of $W$. $\mathcal{E}_1, \dots, \mathcal{E}_k$ are the only ends of $W$. If $\mathcal{E}$ were another, with nhd $U \in \mathcal{E}$ disjoint from $\gamma_i(M_i \times (a_i,1))$, so $\bar{U}$ = closure of $U$ in $\operatorname{int}\bar{W} \subset \operatorname{int}\bar{W} - \bigcup \gamma_i(M_i \times (a_i,1))$ which is compact *.

Converse by similar argument.

A <u>0-neighbourhood</u> of ~~end~~ an end $\mathcal{E}$ of $W$ is a submanifold $V$ of $W$ such that $\operatorname{int} V \in \mathcal{E}$, $V$ has just one end, $V$ is closed in $W$, and $\partial V$ is connected.

<u>Theorem 5.6</u> Any isolated end of an open manifold $W$ has a 0-neighbourhood.

<u>Proof</u>: $\exists$ neighbourhood $U \in \mathcal{E}$ which isn't a nhd of any other end. $Fr(U)$ is compact.

$\exists$ compact polyhedron $K \overset{PL}{\hookrightarrow} W$ which is nhd of $Fr(U)$. Now let $N$ be a regular nhd of $K$ in $W$. Now $\overline{U-N}$ has a non-compact component $V$ [because $\overline{U-N}$ has an end]. $V$ is connected, $Fr(V) \subset N$ (because $U$ connected), so $Fr(V)$ compact. $V$ is a PL submanifold with $\partial V = Fr(V)$, and it is a nhd of $\mathcal{E}$.

Let $M_1, M_2$ be two components of $\partial V$.

$\exists$ PL arc (embedded) $A \subset V$ with ends in $M_1, M_2$. [Possible for $\dim W \geq 3$ by general position For $\dim W \leq 2$, easy]

Now let $H$ be a reg. nhd of $A$ in $V$.

Replace $V$ by $\overline{V-H}$, which is still a nhd of $\mathcal{E}$, contained in $U$, PL manifold, connected; but with fewer boundary components than $V$.

Repeat the process until we get a 0-nbd of $\mathcal{E}$.

---

Remark: This process gives a 0-nbd in $U$, so it gives arbitrarily small 0-nbds of $\mathcal{E}$.

---

Inverse sequence of groups $\cdots\cdots \xrightarrow{f_4} G_3 \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1$ is stable if $\exists$ a subsequence $\cdots \xrightarrow{g_{n_3}} G_{n_2} \xrightarrow{g_{n_2}} G_{n_1}$ such that $g_{n_i}$ induces an isomorphism $\operatorname{im} g_{n_{i+1}} \longrightarrow \operatorname{im} g_{n_i}$, $\forall i$.

Then $\varprojlim G_n$ has $\operatorname{im} g_{n_i}$ for inverse limit.

Note: $g_{n_i} = f_{n_{i-1}+1} f_{n_{i-1}+2} \cdots f_{n_i}$

Let $\mathcal{E}$ be an end of $X$. $\pi_1$ is stable at $\mathcal{E}$ if $\exists$ path-connected nbds $U_1 \supset U_2 \supset U_3 \supset \cdots$ of $\mathcal{E}$ with $\bigcap \bar{U}_i = \phi$, with base pts $u_i \in U_i$, paths $p_i$ from $u_i$ to $u_i$. (in $U_i$) such that

$$\cdots \longrightarrow \pi_1(U_3, u_3) \longrightarrow \pi_1(U_2, u_2) \longrightarrow \pi_1(U_1, u_1)$$

is stable.

Lemma 5.7: If $\pi_1$ is stable at $\mathcal{E}$, and $V_1 \supset V_2 \supset \cdots$ is sequence of path-conn nbds of $\mathcal{E}$ with $\bigcap \bar{V}_i = \phi$, (and with base pts and paths), then

$$\longrightarrow \pi_1(V_3) \longrightarrow \pi_1(V_2) \longrightarrow \pi_1(V_1) \text{ is stable,}$$

with inverse limit equal to $\varprojlim \pi_1(U_i)$

Proof: Suppose wlog that $\rightarrow \pi_1(U_3) \xrightarrow{f_3} \pi_1(U_2) \xrightarrow{f_2} \pi_1(U_1)$ has $f_n$ inducing an isomorphism $\operatorname{im} f_{n+1} \cong \operatorname{im} f_n$

Choose $V_{n_1} \subset U_1$, $U_{r_1} \subset V_{n_1}$, $V_{n_2} \subset U_{r_1}$, $U_{r_2} \subset V_{n_2}$, etc.

Choose paths joining the base pts.

Have diagram

$$\begin{tikzcd}
\pi_1(U_{r_4}) \arrow[r, "h_4"] \arrow[d] & \pi_1(U_{r_3}) \arrow[r, "h_3"] \arrow[d] & \pi_1(U_{r_2}) \arrow[r, "h_2"] \arrow[d] & \pi_1(U_{r_1}) \arrow[d] \\
\pi_1(V_{n_4}) \arrow[r, "g_4"'] \arrow[ur] & \pi_1(V_{n_3}) \arrow[r, "g_3"'] \arrow[ur] & \pi_1(V_{n_2}) \arrow[r, "g_2"'] \arrow[ur] & \pi_1(V_{n_1})
\end{tikzcd}$$

Then $\operatorname{im} h_3 h_4 \longrightarrow \operatorname{im} g_3 g_4 \longrightarrow \operatorname{im} h_2$

whose composite is an isomorphism.

But $\operatorname{im} \cancel{h} g_3 g_4 \longrightarrow \operatorname{im} h_2$ is 1-1.

$\therefore$ $\operatorname{im} h_3 h_4 \longrightarrow \operatorname{im} g_3 g_4$ is iso.

So $\longrightarrow \pi_1(V_{n_6}) \xrightarrow{g_5 g_6} \pi_1(V_{n_4}) \xrightarrow{g_3 g_4} \pi_1(V_{n_2})$

has same inverse limit as $\longrightarrow \pi_1(U_{r_6}) \xrightarrow{h_5 h_6} \pi_1(U_{r_4}) \xrightarrow{h_3 h_4} \pi_1(U_{r_2})$

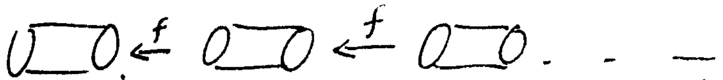
so $\varprojlim \pi_1(U_i) = \varprojlim \pi_1(V_i)$.

---

An end $E$ of $X$ is <u>tame</u> if $\pi_1$ is stable at $E$, and $E$ has arbitrarily small open nbds dominated by finite CW complexes, and $E$ is isolated.

<u>Examples</u>: i) Let $f: S^1 \to S^1$ be squaring map

$$X = S^1 \times I \underset{\overleftarrow{f}}{\cup} S^1 \times I \underset{\overleftarrow{f}}{\cup} S^1 \times I \cup \dots$$




just one end. Let $U_i = X -$ union of first $i$ $S^1 \times I$'s. $\simeq X \simeq S^1$ is dominated by a finite complex.

But $\pi_1$ isn't stable at $E$.

$\dots\dots \pi_1(U_3) \longrightarrow \pi_1(U_2) \longrightarrow \pi_1(U_1)$ is the same as

$\dots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

2) $X = S^2 \times S^2 \# S^2 \times S^2 \# \cdots$

Just one end, with arbitrarily small simply connected nbds
$\therefore \pi_1$ stable.
No nbd of end is dominated by a finite CW complex.

3) If $W$ is an open manifold with a completion, then all ends of $W$ are tame.

$V = \partial V \times [0, \infty)$. Look at $\partial V \times [n, \infty)$.

If $\mathcal{E}$ is an end of $X$ at which $\pi_1$ is stable, define $\pi_1(\mathcal{E}) = \varprojlim \pi_1(U_i)$ where $U_1 \supset U_2 \supset \cdots$ is sequence of path-connected nbds of $\mathcal{E}$ with $\bigcap \bar{U}_i = \emptyset$.

Let $\mathcal{E}$ be an isolated end of $W$ at which $\pi_1$ is stable. A <u>1-neighbourhood</u> of $\mathcal{E}$ is a 0-nbd $V$ with extra properties:

1) $\pi_1(\partial V) \cong \pi_1(V)$ (induced by inclusion)
2) The ~~map~~ natural map $\pi_1(\mathcal{E}) \to \pi_1(V)$ is an isomorphism.

<u>Theorem 5.8</u> Suppose $\mathcal{E}$ is an isolated tame end of an open manifold $W$. If $\dim W \geq 5$, then $\mathcal{E}$ has a 1-nbd.

<u>Proof</u>: First show that $\pi_1 \mathcal{E}$ is finitely presented.

Choose 0-nbds $V_1 \supset V_2 \supset \cdots$ of $\mathcal{E}$ with $\bigcap \bar{V}_n = \emptyset$ and such that $g_n : \pi_1(V_n) \to \pi_1(V_{n-1})$ induces iso $\mathrm{im}(g_{n+1}) \to \mathrm{im}(g_n)$.

$\exists$ nbd $U$ of $\mathcal{E}$, $U \subset V_1$, and $U$ dominated by finite complex $K$.

$\exists\, n$ s.t. $V_n \subset U$ : we have

$\mathrm{im}\, g_{n+1} \longrightarrow \pi_1(U) \longrightarrow \mathrm{im}\, g_2 \subset \pi_1(V_1)$

Composite is an isomorphism, so $\pi_1(\mathcal{E}) \cong \mathrm{im}(g_2)$

is a retract of $\pi_1(U)$, which is a retract of finitely presented group $\pi_1(K)$. By Lemma 3.8, $\pi_1(E)$ is finitely presented.

Let $E_n$ be image of map $\pi_1(E) \longrightarrow \pi_1(V_n)$

Seek $V' \in \mathrm{Int}\, V_3$ such that $\pi_1(\partial V') \longrightarrow E_2$ is onto.

$E_2$ is finitely generated : represent finite set of generators by arcs $A_1, \ldots, A_k$ embedded in $V_3$ with ends in $\partial V_4$. By general pos$^n$, $A_i \cap \partial V_4$ is finite set of pts.

Subdivide $A_1, \ldots, A_k$ into ~~art~~ arcs $B'_1, \ldots, B'_\ell$ such that $B'_j \cap \partial V_4 = \partial B'_j$ ; say $B'_1, \ldots, B'_p$ in $V_4$, and $B'_{p+1}, \ldots, B'_\ell \subset \overline{V_3 - V_4}$.

Adjust $B'_j$ slightly to obtain disjoint arcs $B_1, \ldots, B_\ell$.

Let $H_1, \ldots, H_p$ be reg nbds of $B_1, \ldots, B_p$ in $V_4$

$H_{p+1}, \ldots, H_\ell$ — — — $B_{p+1}, \ldots, B_\ell$ in $\overline{V_3 - V_4}$

Replace $V_4$ by $V' = \overline{V_4 - H_1 \cup \ldots \cup H_p} \cup H_{p+1} \cup \ldots \cup H_\ell$

This has the desired effect : $\pi_1(\partial V') \longrightarrow \pi_1(V_3) \longrightarrow E_2$ is onto.

Now we modify $V'$ further to make $\pi_1(\partial V') \xrightarrow{\varphi} E_2$ an isomorphism. [ It will then be a 1-nbd ].

<u>Lemma 5.9</u> Let $\pi$, $E$ be finitely presented groups and let $\varphi : \pi \longrightarrow E$ be an epimorphism. Then $\ker \varphi$ is the normal closure of a finite subset of $\pi$.

<u>Proof</u> : Let $\{ g_i ; r_j(\underline{g}) = 1 \}$, $\{ h_i : s_j(\underline{h}) = 1 \}$ be finite presentations of $\pi$, $E$. $\exists$ words $w_i$ s.t. $\varphi(g_i) = w_i(\underline{h})$. Since $\varphi$ is onto, $\exists$ words $v_i$ s.t. $h_i = \varphi(v_i(\underline{g})) = v_i(\varphi(\underline{g}))$. Now

$$E \cong \{ h_i : s_j(\underline{h}) = 1 \ , \ r_j(\underline{w}(\underline{h})) = 1, \ \ h_i = v_i(\underline{w}(\underline{h})) \}$$

$$\cong \{ h_i, g'_i : s_j(\underline{h}) = 1, \ g'_i = w_i(\underline{h}), \ r_j(\underline{g}') = 1, \ h_i = v_i(\underline{g}') \}$$

$E \cong \{ g_i' : s_j(v(\underline{g}')) = 1,\ r_j(\underline{g}') = 1,\ g_i' = w_i(\underline{v}(\underline{g}')) \}$.

$\varphi : \pi \to E$ has $\varphi(g_i) = w_i(\underline{h}) = g_i'$, so $\ker \varphi$ is normal closure of $\{ s_j(v(\underline{g}')) \} \cup \{ g_i^{-1} w_i(\underline{v}(\underline{g})) \}$ as required.

---

So $\varphi : \pi_1(\partial V') \to E_2$ onto, $\ker \varphi$ = normal closure of finite set $\{ z_1, \ldots, z_k \}$.

Represent $z_i$ by embedded circle $S_i$ in $\partial V'$

Since $\varphi(z_1) = 0$, $S_1$ bounds a disc $D$ in $V_2$

By general position ($\dim W \geq 5$), we can suppose $D$ embedded, and $\mathrm{int}\, D \cap \partial V'$ = finite union of circles.

Suppose first that $\mathrm{int}\, D \cap \partial V' = \emptyset$, so $D \subset V'$ or $D \subset \overline{V_2 - V'}$. Let $H$ be a reg nbd of $D$ in $V'$ or $\overline{V_2 - V'}$.

Replace $V'$ by $V_1'$ ~~by either~~ $= \overline{V' - H}$ or $V' \cup H$.

$$\begin{tikzcd} \pi_1(\partial V') \arrow[r] \arrow[dr] & \pi_1((\partial V') \cup H) \arrow[d] & \pi_1(\partial V_1') \arrow[l, "j_*"'] \arrow[dl] \\ & \pi_1(V_2) & \end{tikzcd}$$

Now $j_*$ is composite
$$\begin{tikzcd} \pi_1(\partial V_1') \arrow[r, Rightarrow, "\cong"] & \pi_1(((\partial V') \cup H) - D) \arrow[d, Rightarrow, "i"] \\ & \pi_1((\partial V') \cup H) \end{tikzcd}$$

isomorphism since $\dim H = \dim W \geq 5$, $\dim D = 2$.

So $j_*$ is an isomorphism. So $\pi_1(\partial V_1') \cong \dfrac{\pi_1(\partial V')}{(\text{normal closure of } z_1)}$

so we have killed $z_1$. Describe this process as swapping the disc $D$ across $\partial V$.

In general, $(\mathrm{int}\, D) \cap \partial V'$ = finite union of circles $S_1, \ldots, S_\ell$.

$S_i$ bounds a disc $D_i$ in $\mathrm{int}\, D$

Label $S_i$ so that $D_i \subset D_j \implies i \leq j$

Swap $D_1$ across $\partial V'$; this reduces the number of ~~interior~~ components of $(\text{int } D) \cap \partial V'$.

Repeat the process until $(\text{int } D) \cap \partial V' = \emptyset$;

now swap $D$ across $\partial V'$, killing $\gamma_1$.

Repeat to kill $\gamma_2, \ldots, \gamma_k$; then $\varphi : \pi_1(\partial V') \to E_2$ is isomorphism.

$\therefore \varphi : \pi_1(\partial V') \longrightarrow E_1$ is also isomorphism

$(\pi_1(V_2) \to \pi_1(V_1)$ induces iso $E_2 \to E_1)$.

$\therefore \pi_1(V') \longrightarrow \pi_1(V_2) \longrightarrow \pi_1(V_1)$ maps onto $E_1$.

Suppose $z \in$ kernel of $\psi : \pi_1(V') \longrightarrow \pi_1(V_1)$.

Represent $z$ by a circle $S$ in $V'$

$S$ bounds a disc $D$ in $V_1$: by general position, embedded with $D \cap \partial V' = S_1 \cup \ldots \cup S_\ell$ (circles).

Let $S_1$ be innermost circle, bounding disc $D_1 \subset D$.

$S_1 \subset \partial V'$ is null-homotopic in $V_1$, since $\pi_1(\partial V') \longrightarrow \pi_1(V_1)$ is 1-1, $S_1$ is null homotopic in $\partial V'$.

Let $D_1'$ be a small disc nbd of $D_1$ in $D$, not meeting $S_2, \ldots, S_\ell$.

$\partial D_1' \subset V'$ or $\overline{V_1 - V'}$; use the null-homotopy of $S_1$ in $\partial V'$ to span $\partial D_1'$ by a disc $D_1''$ in $V'$ or $\overline{V_1 - V'}$; by general position $D_1''$ is embedded and disjoint from $\partial V'$. Replace $D$ by $\overline{D - D_1'} \cup D_1''$, which meets $\partial V'$ in fewer components than $D$. Repeat until $D \cap \partial V'$ is empty; then $S$ bounds disc $D$ in $V'$

$\therefore S$ is null-homotopic in $V'$, so $z = 0$.

So $\psi : \pi_1(V') \longrightarrow \pi_1(V_1)$ is 1-1.

$\therefore \pi_1(V') \longrightarrow E_1$ is isomorphism.

But $\pi_1(E) \longrightarrow \pi_1(V') \longrightarrow E_1$ is an isomorphism.

$\therefore \pi_1(E) \longrightarrow \pi_1(V')$ is iso, and $\pi_1(\partial V') \to \pi_1(V')$ is iso. $\therefore V'$ is 1-nbd of $E$; in fact $\exists$ arb$^y$ small 1-nbds.

$\mathcal{E}$ tame end of $W$.

$\pi = \pi_1(\mathcal{E}) \cong \pi_1(\partial V) \cong \pi_1(V)$ for any 1-nbhd of $\mathcal{E}$.

$\tilde{V}, \partial\tilde{V}$ will be universal coverings, $C_*$ = singular chain group.

<u>Lemma 5.10</u> If $V$ is a sufficiently small 1-nbhd of $\mathcal{E}$, then $C_*(\tilde{V}, \partial\tilde{V})$ is homotopy equivalent to a f.g. projective complex over $\mathbb{Z}[\pi]$.

Proof: $\mathcal{E}$ tame, so $\exists$ open path-connected nbhd $U$ of $\mathcal{E}$ which is dominated by a finite complex. Let $V$ be any 1-nbhd with $\bar{V} \subset U$. Let $X = \overline{U - V}^{\,\text{in } U}$, so $U = X \cup V$, $X \cap V = \partial V$. (all CW complexes).

$C_*(\tilde{U}, \partial\tilde{V}) \simeq C_*(\tilde{V}, \partial\tilde{V}) \oplus C_*(\tilde{X}, \partial\tilde{V})$ (by excision + homotopy)

$\therefore$ $C_*(\tilde{V}, \partial\tilde{V})$ is dominated by $C_*(\tilde{U}, \partial\tilde{V})$.

$U$ is dominated by finite complex, so by 3.6, $C_*(\tilde{U})$ is equivalent to a f.g. proj complex, say $f : C_*(\tilde{U}) \xrightarrow{\simeq} D_*$

$\partial V$ is a finite complex, so $C_*(\partial\tilde{V})$ is equivalent to a f.g. free complex, say
$g : C_*(\partial\tilde{V}) \xrightarrow{\simeq} E_*$.

$$\begin{tikzcd} 0 \arrow[r] & C_*(\partial\tilde{V}) \arrow[r] \arrow[d, "g"'] & C_*(\tilde{U}) \arrow[r] \arrow[d, "f"] & C_*(\tilde{U}, \partial\tilde{V}) \arrow[r] & 0 \\ & E_* \arrow[r, "\varphi"'] & D_* & & \end{tikzcd}$$

commutes up to homotopy for suitable $\varphi$.

Follows that $C_*(\tilde{U}, \partial\tilde{V}) \simeq$ mapping cone of $\varphi$ $(E_*)$ which is f.g. proj.

$C_*(\tilde{V}, \partial\tilde{V})$ is dominated by $C_*(\tilde{U}, \partial\tilde{V})$, hence by f.g. proj complex.

$\therefore$ By Th 2.3, $C_*(\tilde{V}, \partial\tilde{V})$ is equiv to a f.g. proj. complex.

Def$^n$. A $k$-neighbourhood of end $\mathcal{E}$ of open manifold $W$ is a 1-nhd $V$ such that $H_i(\tilde{V}, \partial\tilde{V}) = 0$ for $i \leq k \geq 2$

Theorem 5.11 A tame end $\mathcal{E}$ of a manifold $W$ of dimension $n \geqslant 5$ has arbitrarily small $(n-3)$-nhds.

Proof: Suppose inductively that $\mathcal{E}$ has arbitrarily small $(k-1)$-nhds. Start with $k = 2$; suppose $k \leq n-3$. Let $V$ be a $(k-1)$-nhd.

$C_*(\tilde{V}, \partial\tilde{V})$ is equivalent to a f.g. proj. complex, say $E_*$. Since $H_i(E_*) = 0$, $i < k$, $\exists$ exact sequence

$$0 \longrightarrow Z_k(E_*) \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \to 0.$$

$\therefore$ $Z_k(E_*)$ is f.g. projective (as in 2.3).

$\therefore$ $H_k(E_*) \cong H_k(\tilde{V}, \partial\tilde{V})$ is f.g.

Let $\{x_1, \ldots, x_m\}$ be finite set of generators.

Lemma 5.12 Let $V$ be a $(k-1)$-nhd of end $\mathcal{E}$, and suppose $\mathcal{E}$ has arbitrarily small $(k-1)$-nhds. Then any element of $H_k(\tilde{V}, \partial\tilde{V})$ can be represented by a PL embedded disc $(D^k, \partial D^k) \subset (V, \partial V)$, provided $k \leq n-3$.

Completion of proof of 5.11.

Represent $x_1$ by an embedded disc $(D^k, \partial D^k) \subset$ $\subset (V, \partial V)$. Let $H$ be a reg. nhd of $D^k$ in $V$, and replace $V$ by $V' = \overline{V - H}$.

$V'$ is still a 1-nhd, for $\pi_1(V') \cong \pi_1(V - D) \underset{n-k \geq 3}{\cong} \pi_1(V)$

$\pi_1(\partial V') \cong \pi_1((\partial V \cup H) - D) \underset{n-k \geq 3}{\cong} \pi_1(\partial V) \cong \pi_1(V) \cong \pi_1(V')$

Homology exact sequence of $(\tilde{V}, \widetilde{\partial V \cup H}, \partial\tilde{V})$ gives

$$H_i(\widetilde{(\partial V) \cup H}, \partial\tilde{V}) \to H_i(\tilde{V}, \partial\tilde{V}) \to H_i(\tilde{V}', \partial\tilde{V}') \to H_{i-1}(\widetilde{(\partial V) \cup H}, \partial\tilde{V})$$

$\therefore \quad H_i(\tilde{V}', \partial\tilde{V}') = 0$ for $i < k$, so $V'$ is $(k-1)$-nbd.

$H_k(\tilde{V}, \partial\tilde{V}) \to H_k(\tilde{V}', \partial\tilde{V}')$ is onto, and kernel contains $x_1$.

Repeat process to kill off $x_2, .., x_r$; we finish with a $k$-nbd of $E$. Continue to get an $(n-3)$ nbd.

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Proof of lemma.

Represent $x \in H_k(\tilde{V}, \partial\tilde{V})$ by a map $\varphi : D^k, \partial D^k \to V, \partial V$ by the Hurewicz thm.

Image of $\varphi$ is compact, so $\exists$ small $(k-1)$ nbd $V' \subset V$ so that $\operatorname{Im}\varphi \subset V - V'$

Then $x \in$ image of $\psi : H_k(\widetilde{V-V'}, \partial\tilde{V}) \to H_k(\tilde{V}, \partial\tilde{V})$

say $x = \psi(y)$. Let $U = \overline{V-V'}$, $y \in H_k(\tilde{U}, \partial\tilde{V})$.

$\partial\tilde{V} \subset \tilde{V}$, $\tilde{U} \subset \tilde{V}$ induce isomorphisms of homology up to dimension $k-2$. (since $V, V'$ are $(k-1)$-nbds).

$\therefore \partial\tilde{V} \subset \tilde{U}$ induces ~~iso~~ homology isos in dims $\leq k-2$

$\therefore \quad H_i(\tilde{U}, \partial\tilde{V}) = 0$ for $i \leq k-2$.

Take handle decomposition of $U$ based on $\partial V$.

We can remove 1-handles, & and cancel handles of dimension $\leq k-2$, so there are no handles of dim $\leq \max(1, k-2)$.

Let $X$ = regular nbd of union of $(k-1)$-handles in $U$.

Let $Y = \overline{U-X}$, let $Z = X \cap Y$.

Let $\bar{y}$ = image of $y$ in $H_k(\tilde{U}, \tilde{X}) \cong H_k(\tilde{Y}, \tilde{Z})$

Let $h_1, .., h_r$ be the $k$-handles in $Y$.

Let $\eta_1, .., \eta_r$ be the homology classes in $H_k(\tilde{Y}, \tilde{Z})$ represented by $h_1, .., h_r$. Wlog $\eta_r = 0$ (otherwise introduce irrelevant $k$ & $k+1$ handles which cancel; then irrelevant $k$-handle reps $0$ in $H_k(\tilde{Y}, \tilde{Z})$.).

Then $\eta_1, .., \eta_r$ generate $H_k(\tilde{Y}, \tilde{Z})$ as $\mathbb{Z}[\pi]$-module.

Let $\bar{y} = \sum_{i=1}^{r} \rho_i \eta_i \quad (\rho \in \mathbb{Z}[\pi]) $, wlog $\rho_r = 1$.

Start with $(D^k, \partial D^k) \subset (Y, Z)$ as the core of $h_r$

Apply handle addition theorem to add on translates of $h_1, .., h_{r-1}$ to obtain disc $(D^k, \partial D^k) \subset (Y, Z)$ representing $\bar{y}$.

Suppose $k = 2$; since there are no 1-handles, no $(k-1)$-handles, so $X$ is a collar nbd of $\partial V$ in $V$, so we are home.

Suppose now $k \geq 3$, so $n \geq k+3 \geq 6$

$X$ is a collar nbd of $\partial V \cup (k-1)$-handles.

Let $X' = \partial V \cup (k-1)$-handles.

Let $h'$ be a $(k-1)$-handle.

$X$ is a collar nbd of $\partial X'$ in $V$, so we can replace $D^k$ by a disc $\bar{D}$ with $\partial \bar{D} \subset \partial X'$.

$h' \cong D^{k-1} \times D^{n-k+1}$. Let $S' =$ image of $0 \times S^{n-k}$ ie cocore bdy.

~~Ex~~

By general position, $\partial D \cap S'$ is a finite union of points, $P_1, \ldots, P_j$; each intersection transverse.

Choose path $p_i$ from $P_1$ to $P_i$ in $\partial D$,
path $p_i'$ from $P_1$ to $P_i$ in $S'$.

Let $g_i$ = element of $\pi_1(Z) \cong \pi$ represented by $p_i \circ \bar{p}_i'$ (out along $p_i$, back along $p_i'$).

Let $\varepsilon_i$ be sign of intersection at $P_i$ (depends on orientation of spheres $S'$, $\partial D$)

Now $\sum \varepsilon_i g_i \in \mathbb{Z}[\pi]$ is coefficient of $h'$ in $\partial \bar{y}$, which is $0$.

$\therefore$ We can pair off $P_1, .., P_j$ so that, if $P_s, P_t$ are paired, then $g_s = g_t$ and $\varepsilon_s = -\varepsilon_t$. ~~Now represent~~ ~~$p_i \circ \bar{p}_i$ by embedded $S'$ in $S' \cup \partial D$; since $g_i =$~~

Now choose a path $p$ from $P_s$ to $P_t$ in $\partial D$, path $p'$ from

$P_s$ to $P_t$ in $S'$. Then loop $\rho o \bar{\rho}'$ is null-homotopic in $Z$. So we can apply Whitney argument to remove intersections at $P_s, P_t$. (need $n \geq 6$). This reduces the number of intersections of $S', \partial D$; repeat until $S' \cap \partial D = \phi$.

Now deform $\partial D$ until it doesn't meet $h'$, by an isotopy. Do this for all $(k-1)$-handles $h'$: then $(D, \partial D) \subset (V, \partial V)$, and represents the right homology class $x$.

<u>Lemma 5.13</u> : Let $E$ be a tame end of manifold $W$, dim $W \geq 5$. If $V, V'$ are 1-nhds of $E$, then the Wall invariants $\sigma(C_*(\tilde{V}, \partial\tilde{V}))$, $\sigma(C_*(\tilde{V}', \partial\tilde{V}'))$ are equal. If $V$ is an $(n-3)$-nhd, then

$H_i(\tilde{V}, \partial\tilde{V}) = 0$ for $i \neq n-2$, and

$H_{n-2}(\tilde{V}, \partial\tilde{V})$ is a f.g. projective module,

representing $(-1)^n \sigma(C_*(\tilde{V}, \partial\tilde{V}))$ in $\tilde{K}_0(\mathbb{Z}[\pi])$.

<u>Proof</u> : By Th 5.8, it is enough to consider case $V' \subset \operatorname{int} V$. Let $U = \overline{V - V'}$. Exact sequence

$$\cancel{0 \longrightarrow C_*(\tilde{U}, \partial\tilde{V}) \longrightarrow C_* \longrightarrow C_*(\tilde{V}', \partial\tilde{V}') \longrightarrow 0}$$

$$0 \longrightarrow C_*(\tilde{U}, \partial\tilde{V}) \longrightarrow C_*(\tilde{V}, \partial\tilde{V}) \longrightarrow C_*(\tilde{V}, \tilde{U}) \longrightarrow 0$$

By excision, $C_*(\tilde{V}', \partial\tilde{V}') \cong C_*(\tilde{V}, \tilde{U})$.

$\exists$ chain equivalences $f : C_*(\tilde{V}, \partial\tilde{V}) \longrightarrow D_*$

$g : C_*(\tilde{U}, \partial\tilde{V}) \longrightarrow E_*$

with $E_*$ f.g. free, $D_*$ f.g. projective.

$\varphi : E_* \longrightarrow D_*$ making diagram below commute, up to chain homotopy

$$\begin{array}{ccccccccc} 0 \to & C_*(\tilde{U},\partial\tilde{V}) & \to & C_*(\tilde{V},\partial\tilde{V}) & \to & C_*(\tilde{V},\tilde{U}) & \to 0 \\ & \downarrow g & & \downarrow f & & & \\ & E_* & \xrightarrow{\varphi} & D_* & & & \end{array}$$

Now $C_*(\tilde{V},\tilde{U})$ is chain equivalent to the mapping cone of $\varphi$, say $Q_*$.

$\therefore\ \sigma(C_*(\tilde{V}',\partial\tilde{V}')) = \sigma(C_*(\tilde{V},\tilde{U})) = \sigma(Q_*) = \sigma(D_*)$
$= \sigma(C_*(\tilde{V},\partial\tilde{V}))$

since $E_*$ is f.g. free.

Define the <u>Siebenmann invariant</u> $\sigma(E)$ to be $\sigma(C_*(\tilde{V},\partial\tilde{V}))$ for any 1-nhd $V$ of $E$.

Now let $V$ be an $(n-3)$-nhd of $E$. By Th 5.8, $\exists$ 1-nhds $V_n$ of $E$ with $V_0 = V$, $\bigcap V_n = \phi$, and $V_{n+1} \subset \operatorname{int} V_n$. Let $U_n = \overline{V_n - V_{n+1}}$; so $\partial U_n = \partial V_n \cup \partial V_{n+1}$.

$\partial V_n \subset U_n$, $\partial V_{n+1} \subset U_n$ induce fundamental group isomorphisms (because $V_i$ is 1-nhd).

$$\begin{array}{ccccc} & & \pi_1(V_{n+1}) & & \\ & \overset{\cong}{\nearrow} & & \overset{\cong}{\searrow} & \\ \pi_1(\partial V_{n+1}) & & & & \pi_1(V_n) \\ & \searrow & & \nearrow & \\ & & \pi_1(U_n) & & \end{array}$$

Van Kampen's Th$^m$ $\Rightarrow$ this is a pushout diagram.

$\therefore$ All isos; we know that $\pi_1(\partial V_{n+1}) \cong \pi_1(V_n)$ and since $V_n, V_{n+1}$ are 1-nhds, $\pi_1(V_{n+1}) \to \pi_1(V_n)$ iso.

$\therefore$ Since diagram is a pushout, $\pi_1(U_n) \to \pi_1(V_n)$ & $\pi_1(\partial V_{n+1}) \to \pi_1(U_n)$ are isos.

Similarly $\partial V_n \subset U_n$ induces $\pi_1$ iso.

$\exists$ handle decomposition of $U_i$ on $\partial V_i$ without handles of index $0, 1, n-1, n$.

$\therefore$ $V$ can be obtained from $\partial V$ by attaching handles of index $\leq n-2$.

$\therefore$ $V \simeq$ CW complex $K$ with $\partial V$ as a subcomplex
and with all cells of $K - \partial V$ of dimension $\le n-2$.

[ Attach handles of $V - \partial V$ one at a time, giving
$\partial V = X_0 \subset X_1 \subset \ldots$ with $\cup X_i = V$ and $X_i$
obtained from $X_{i-1}$ by attaching $r$-handle, $r \le n-2$.
Suppose inductively $X_{i-1} \simeq$ complex $K_{i-1}$ of required
form. Then $X_i \simeq X_{i-1} \cup r\text{-handle}$

$$\simeq X_{i-1} \cup r\text{-cell}$$
$$\simeq K_{i-1} \cup e^r$$

Replace attaching map of $e^r$ by a homotopic
cellular map. $K_i = K_{i-1} \cup e^r$, and $X_i \simeq K_i$.
Put $K = \cup K_i$; then $V \simeq K$. ]

$C_*(\tilde V, \partial \tilde V)$ is equivalent to a (not nec. f.g.) free complex
of dim $\le n-2$. But $C_*(\tilde V, \partial \tilde V)$ is equiv. to a f.g.
proj. complex.

Lm 2.3 (second half of proof) shows $C_*(\tilde V, \partial \tilde V) \simeq$ f.g.
proj complex $E_*$ of dimension $\le n-2$.
We have exact sequence

$$0 \to H_{n-2}(E_*) \to E_{n-2} \to E_{n-3} \to \cdots \to E_0 \to 0.$$

(since $V$ is an $(n-3)$-nhd).

$\therefore$ $H_{n-2}(E_*)$ is f.g. proj.
Moreover, $H_{n-2}(E_*)$ represents $(-1)^n \sigma(E_*)$
$$= (-1)^n \sigma(C_*(\tilde V, \partial \tilde V)).$$

$H_i(\tilde V, \partial \tilde V) = 0$ if $i > n-2$.

Corollary 5.14 Let $\mathcal{E}$ be an end of manifold $W$
of dimension $\geq 6$. Then $\mathcal{E}$ has a collar iff $\mathcal{E}$
has arbitrarily small $(n-2)$-nhds.

Proof. Necessity clear

Let $V$ be an $(n-2)$-nhd of $\mathcal{E}$, let $V'$ be another $(n-2)$ nhd, $V' \subset \operatorname{int} V$.

$\mathcal{U} = \overline{V - V'}$ is an $h$-cobordism from $\partial V$ to $\partial V'$

$$\begin{array}{ccccc} H_r(\tilde{V}, \tilde{\mathcal{U}}) & \longrightarrow & H_{r-1}(\tilde{\mathcal{U}}, \partial\tilde{V}) & \longrightarrow & H_{r-1}(\tilde{V}, \partial\tilde{V}) \\ \| & & & & \| \\ H_r(\tilde{V}', \partial\tilde{V}') & & & & 0 \\ \| & & & & \\ 0 & & & & \end{array}$$

Let $\partial V \subset \mathcal{U}$ have torsion $\tau$.

Let $\mathcal{U}' = \mathcal{U} \cup (1\text{-handles}) \cup (2\text{-handles})$

where the 1-handles & 2-handles are contained in $V'$, and are chosen so that $\mathcal{U} \to \mathcal{U}'$ has torsion $-\tau$.

Let $V'' = \overline{V - \mathcal{U}'}$ ; $V''$ is a nhd of $\mathcal{E}$ contained in $V'$.

$\mathcal{U}'$ is an $h$-cobordism with torsion $0$; ie an $s$-cobordism, dim $\geq 6$.

$\therefore$ $\mathcal{U}' \cong \partial V \times I$.

$\therefore$ $\exists$ arbitrarily small nhds $V''$ of $\mathcal{E}$, s.t. $V'' \subset \operatorname{int} V$ and $\overline{V - V''} \cong \partial V \times I$.

Now easy to show that $V \cong \partial V \times [0, \infty)$, so $V$ is a collar.

**Theorem 5.15** Let $\mathcal{E}$ be an end of a manifold $W$, of dimension $n \geq 6$. Then $\mathcal{E}$ has a collar iff $\mathcal{E}$ is tame and $\sigma(\mathcal{E}) = 0$. $[\sigma(\mathcal{E}) \in \tilde{K}_0(\mathbb{Z}[\pi])]$

Proof Necessity clear (take collar as $(n-3)$ nhd to calculate $\sigma(\mathcal{E})$.)

**Conversely**: let $V$ be an $(n-3)$-nhd of $\mathcal{E}$, so $H_{n-2}(\tilde{V}, \partial\tilde{V})$ is stably free (since $\sigma(\mathcal{E}) = 0$)

(ie $H_{n-2}(\tilde V, \partial \tilde V) \oplus F \cong G$ for f.g. free $F, G$).

Wlog assume $H_{n-2}(\tilde V, \partial\tilde V)$ is actually free:
(for we can add $\mathbb{Z}[\pi]$ to $H_{n-2}(\tilde V,\partial\tilde V)$ by swapping
a trivial $(n-3)$-disc across $\partial V$)

Since ~~(n-2)~~ $H_{n-2}(\tilde V,\partial\tilde V)$ is f.g., $\exists$ $(n-3)$-nhd $V' \subset \operatorname{int} V$
s.t. if $U = \overline{V - V'}$, then $H_{n-2}(\tilde U, \partial\tilde V) \to H_{n-2}(\tilde V, \partial\tilde V)$
is onto. Exact sequence of $(\tilde V, \tilde U, \partial \tilde V)$

$$0 \to H_{n-2}(\tilde U, \partial\tilde V) \underset{\cong}{\twoheadrightarrow} H_{n-2}(\tilde V, \partial \tilde V) \xrightarrow{0} H_{n-2}(\tilde V', \partial\tilde V') \underset{\cong}{\to} H_{n-3}(\tilde U, \partial\tilde V) \to 0$$

So $H_{n-2}(\tilde U, \partial\tilde V) \cong H_{n-2}(\tilde V, \partial\tilde V)$

$H_{n-2}(\tilde V', \partial\tilde V') \cong H_{n-3}(\tilde U, \partial\tilde V)$

$V'$ is an $(n-3)$-nhd, wlog $H_{n-2}(\tilde V', \partial\tilde V')$ is f.g. free.

Let $x_1, \dots, x_k$ be free basis for $H_{n-3}(\tilde U, \partial\tilde V)$.

By Lemma 5.12, can represent $x_1, \dots, x_k$ by disjoint embedded $D^{n-3}$'s. (Embed discs one at a time;
embed $D_i^{n-3}$ in complement of regular nhd of $D_1^{n-3} \cup \dots \cup D_{i-1}^{n-3}$.)

Swap these discs across $\partial V$, giving $V^*$, $U^*$.

Then $H_{n-2}(\tilde U^*, \partial\tilde V^*) \to H_{n-2}(\tilde V^*, \partial\tilde V^*)$ is still onto.

Enough to check that $H_{n-2}(\tilde V, \partial\tilde V) \twoheadrightarrow H_{n-2}(\tilde V^*, \partial\tilde V^*)$

Exact sequence

$$H_{n-2}(\tilde V, \partial\tilde V) \twoheadrightarrow H_{n-2}(\tilde V, \tilde H) \xrightarrow{0} H_{n-3}(\tilde H, \partial\tilde V) \rightarrowtail H_{n-3}(\tilde V, \partial\tilde V)$$

Replace $V$ by $V^*$, $U$ by $U^*$; now $H_{n-2}(\tilde U, \partial\tilde V) = 0$.

$\partial V \subset U$ induces $\pi_1$ isomorphism, $\partial\tilde V \subset \tilde U$ induces
~~so~~ homology isomorphisms in dimension $\leq n-4$.

$\therefore$ $H_i(\tilde U, \partial\tilde V) = 0$ for $i \neq n-3, n-2$.

$U$ has a handle decomposition on $\partial V$, handles of
dimension $n-3$, $n-2$ only.

Let $X$ = reg nhd of $\partial V \cup (n-3)$-handles, $Y = \overline{U - X}$, $Z = X \cap Y$.

Let $C_{n-2} = H_{n-2}(\tilde{Y}, \tilde{Z})$ , $C_{n-3} = H_{n-3}(\tilde{X}, \partial\tilde{V})$,
bases $c_{n-2}$, $c_{n-3}$ given by handles. Chain complex

$$0 \longrightarrow C_{n-2} \xrightarrow{\partial} C_{n-3} \longrightarrow 0$$ with homology groups

$H_{n-2}(\tilde{U}, \partial\tilde{V})$ , $H_{n-3}(\tilde{U}, \partial\tilde{V})$ . from exact sequence of
$(\tilde{U}, \tilde{X}, \partial\tilde{V})$.

Let $B_{n-3}$ be the boundary group $\partial(C_{n-2})$.

If we put in extra $(n-3)$-handle into $X$, and complementary $(n-2)$-handle into $Y$, then we add $\mathbb{Z}[\pi]$ to $C_{n-2}$, $C_{n-3}$, $B_{n-3}$, and do not affect the homology groups.

$B_{n-3}$ is stably free $(0 \to B_{n-3} \to C_{n-3} \to H_{n-3}(C_*) \to 0)$
by adding enough complementary pairs of handles, we can make $B_{n-3}$ free.

Choose basis ~~$c_{n-2}$ of $C_{n-2}$~~ of $H_{n-2}(C_*)$, and extend to a basis of $C_{n-2}$, say $c'_{n-2}$, using exact sequence $0 \to H_{n-2}(C_*) \to C_{n-2} \to B_{n-2} \to 0$.

Let $M \in GL(k, \mathbb{Z}[\pi])$ ( $k$ = dimension of $C_{n-2}$ ) be such that $c'_{n-2} = M c_{n-2}$. Let $D$ = free module $(\mathbb{Z}[\pi])^k$, standard basis $d$. Put in extra handles as above to replace $C_{n-2}$ by $C_{n-2} \oplus D$, and $c_{n-2}$ by $c_{n-2} \oplus d$. Replace $c'_{n-2}$ by $c_{n-2} \oplus M^{-1}d$; then $c'_{n-2} = L c_{n-2}$ where $L \in GL(2k, \mathbb{Z}[\pi])$ is a product of elementary matrices.

By handle addition theorem, we can change $(n-2)$-handles so that they give the basis $c'_{n-2}$. Then $H_{n-2}(C_*)$ is generated by handles $h_1^{n-2}, \ldots, h_r^{n-2}$, which form a free basis of $H_{n-2}(C_*)$. Since $\partial h_i^{n-2}$ represents $0$ in $C_{n-3} = H_{n-3}(\tilde{X}, \partial\tilde{V})$, we can apply

the Whitney process to isotop $h_i^{n-2}$ off the $(n-3)$-handles in $X$ (as in 5.12, we need $n \geq 6$).
We finish with embedded discs $D_1^{n-2}, \dots, D_r^{n-2}$ with $\partial D_i^{n-2} \subset \partial V$ representing a basis of $H_{n-2}(\tilde{U}, \partial\tilde{V}) \cong H_{n-2}(\tilde{V}, \partial\tilde{V})$. Swap $D_1^{n-2}, \dots, D_r^{n-2}$ across $\partial V$, obtaining a nbd $V_1$ of $\mathcal{E}$ : Claim this is an $(n-2)$-nbd.

<u>1-nbd</u>: Let $U_1 = U \cap V_1$ : $U_1$ has a handle decomp$^n$ on $\partial V_1$ with $(n-3)$ & $(n-2)$-handles only.
$n-3 \geq 3$, so $\pi_1(\partial V_1) \to \pi_1(U_1)$ is iso.
$U_1$ has handle decomposition on $\partial V'$, with 2-handles + 3-handles only.
$\therefore$ $\pi_1(\partial V') \longrightarrow \pi_1(U_1)$ is onto.
But we have $\pi_1(\partial V') \to \pi_1(U_1) \to \pi_1(U)$ an isomorphism; so $\pi_1(\partial V') \xrightarrow{\cong} \pi_1(U_1)$
Van Kampen for $\pi_1(V_1)$ $\quad (V_1 = U_1 \cup V')$
$\therefore \pi_1(V_1) \cong \pi_1(U_1) \cong \pi_1(\partial V') \cong \pi_1(V') \cong \pi_1(\mathcal{E})$
and $\pi_1(\partial V_1) \cong \pi_1(U_1) \cong \pi_1(V_1)$
$\therefore$ $V_1$ is a 1-nbd.
Let $H = \overline{U - U_1}$ = union of handles swapped.
Exact sequence of $(\tilde{V}, \widetilde{\partial V \cup H}, \partial\tilde{V})$ gives

$$\to H_{n-2}(\widetilde{\partial V \cup H}, \partial\tilde{V}) \xrightarrow{i_*} H_{n-2}(\tilde{V}, \partial\tilde{V}) \to H_{n-2}(\tilde{V}_1, \partial\tilde{V}_1) \to 0$$

$i_*$ is mono because $D_1^{n-2} \cup \dots \cup D_r^{n-2}$ is free basis for $H_{n-2}(\tilde{V}, \partial\tilde{V})$.
In any case, $H_{n-2}(\tilde{V}_1, \partial\tilde{V}_1) = 0$, similarly $H_i(\tilde{V}_1, \partial\tilde{V}_1) = 0$ for $i < n-2$, so $V_1$ is $(n-2)$-nbd of $\mathcal{E}$.
$\therefore$ By Cor 5.14, $\mathcal{E}$ has a collar, as required.

Remarks.

i) $\exists$ ends $\mathcal{E}$ which are tame but $\tau(\mathcal{E}) \neq 0$.

ii) $X$ finite CW complex s.t. $X \times S^1 \simeq$ closed man. $M$.

$\tilde{M}$ = covering corr to $\pi_1(X) \subset \pi_1(X \times S^1)$

Then $\tilde{M}$ has just two ends $\mathcal{E}_1, \mathcal{E}_2$, both tame, and both have nbds, $\tilde{M}$, which is $\simeq$ finite complex $X$.

Can happen that $\sigma(\mathcal{E}_i) \neq 0$.