REPRESENTATIONS OF ALGEBRAS AS UNIVERSAL LOCALIZATIONS

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Contents

Introduction

Given a presentation of a finitely presented group, there is a natural way to represent the group as the fundamental group of a 2-complex. The first part of this paper demonstrates one possible way to represent a finitely presented algebra S in a similarly compact form. From a presentation of the algebra, we construct a quiver with relations whose path algebra is finite dimensional. When we adjoin inverses to some of the arrows in the quiver, we show that the path algebra of the new quiver with relations is $M_n(S)$ where n is the number of vertices in our quiver. The slogan would be that every finitely presented algebra is Morita equivalent to a universal localization of a finite dimensional algebra.

Two applications of this are then considered. Firstly, given a ring homomorphism $A \to B$, we say that B is *stably flat* over A if and only if $\mathrm{Tor}_i^A(B,B) = 0$ for all $i > 0$. In a recent paper [2], the first two authors show that there is a long exact sequence in algebraic K-theory associated to a universal localization provided the localization is stably flat. Given a finitely presented algebra S we construct in Section 1 a finite dimensional algebra A with a universal localization $\sigma^{-1}A = M_n(S)$ (Theorem 1.1), such that A has global dimension $\leqslant 2$ (Proposition 1.2). For a stably flat universal localization $A \to \sigma^{-1}A$ the global dimensions are such that $\mathrm{g.d.}(\sigma^{-1}A) \leqslant \mathrm{g.d.}(A)$

Date: May 31, 2002.

Key words and phrases. representations of algebras, noncommutative localization

A.R. acknowledges the support of the Leverhulme Trust.

(Lemma 1.4). Thus any finitely presented algebra S of global dimension $\geqslant 3$ provides an example of a universal localization which is not stably flat.

Secondly, the Malcolmson normal form states that every element of the localised ring can be written in the form $as^{-1}b$ where $s\colon P \to Q$ lies in the upper triangular closure of σ, $a\colon A \to Q$ and $b\colon P \to A$ are maps in the category of finitely generated projective modules over the original ring A, and gives an equivalence condition on such elements which determines when they define the same element of the localised ring. This equivalence condition depends on the existence of certain maps in σ and the category of finitely generated projective modules. One might reasonably ask if such an equation could be constructed algorithmically. We show that this cannot be done.

In Section 2, we consider a related construction of a ring by universal localization where we calculate explicitly the values of $\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A)$. For any $n \geqslant 3$ we obtain an injective universal localization $A \to \sigma^{-1}A$ with $\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) = 0$ for $1 \leqslant i \leqslant n-2$ and $\neq 0$ for $i = n-1$.

1. Algebras

An *algebra* over a field k is a ring A with a homomorphism from k to the centre of R. By definition, the algebra A is *finite dimensional* if it is a finite dimensional vector space over k. By definition, an algebra S is *finitely presented* if it has a finite number of generators and relations, so that it has the form

$$S \ = \ k\langle x_1, x_2, \dots, x_a\rangle/\langle y_1, y_2, \dots, y_b\rangle \ .$$

A finite dimensional algebra S is finitely presented, since for any basis $e_1, e_2, \dots, e_a$ the coefficients $c_{pqr} \in k$ in

$$e_p e_q \ = \ \sum_r c_{pqr} e_r \in S \ (1 \leqslant p, q, r \leqslant a)$$

are such that

$$S \ = \ k\langle x_1, x_2, \dots, x_a\rangle/\langle x_p x_q - \sum_r c_{pqr} x_r\rangle \ .$$

For any finitely presented algebra S over k we shall exhibit the matrix algebra $M_n(S)$ for some integer n as the universal localization $\sigma^{-1}A$ of a finite dimensional algebra A over k inverting a finite set σ of maps between finitely generated projective A-modules. We shall construct A as the path algebra of a quiver with relations, and it will be clear from the construction that A is of global dimension 2, but the natural map $A \to \sigma^{-1}A = M_n(S)$ may not be an injection. Then a variation of the construction allows us to ensure that $A \to \sigma^{-1}A$ is injective and A has finite global dimension. From this it is fairly clear that for suitable choice of S, for example, of infinite global dimension, the $\mathrm{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A)$ cannot all vanish. We present examples to show variations on these techniques.

First of all, recall the language of quivers with relations.

A *quiver* Q has a finite *vertex set* $V_Q = \{v, w, \dots\}$ and finite *arrow set* $A_Q = \{a, b, \dots\}$. Each arrow $a \in A$ has a *tail* $ta \in V_Q$ and *head* $ha \in V_Q$. A *path of length* i is a formal word in the arrows $a_1, \dots, a_i$ such that for $1 \leqslant j < i$, $ha_j = ta_{j+1}$. Its tail is $ta_1 = v$ and its head is $ha_i = w$ and we say that it is a path from v to w. For each vertex $v \in V_Q$ we have a path $f_v \in A_Q$ of length 0 at v whose head and tail are both v. For vertices v and w, we define $[v, w]$ to be the vector space with basis the set of paths from v to w. The *path algebra* of Q is the vector space

$$\Lambda(Q) \;=\; \bigoplus_{v,w \in V_Q} [v, w]$$

with the product given by the composition of arrows, which makes it into an associative algebra with $1 = \sum_v f_v$. Note that this composition gives an injective linear map from $[u, v] \otimes [v, w]$ to $[u, w]$.

A *quiver with relations* (Q, R) is a quiver Q together with a set of relations $R = \{r_i\}$ where each r_i is an element of $\cup_{v,w}[v, w]$. In the examples we shall be discussing R is a finite set. Each element r of R has a well-defined head and tail which we shall write as tr and hr. For vertices v, w, define $R[v, w]$ to be the linear subspace of $[v, w]$ of the form $\sum_{r \in R}[v, tr]r[hr, w]$. Then $\oplus_{v,w} R[v, w]$ is an ideal in $\Lambda(Q)$ and the factor algebra

$$\Lambda(Q, R) \;=\; \Lambda(Q)/ \oplus_{v,w} R[v, w]$$

is called the *path algebra* of the quiver with relations (Q, R). We define

$$(v, w) \;=\; [v, w]/R[v, w] \;,$$

so

$$\Lambda(Q, R) \;=\; \bigoplus_{v,w \in V_Q} (v, w) \;.$$

We begin with notation. Let

$$S \;=\; k\langle X : Y\rangle$$

where $X = \{x_i : 1 \leqslant i \leqslant a\}$ and $Y = \{y_j : 1 \leqslant j \leqslant b\}$ is a finite subset of $k\langle X\rangle$. In turn, each element of Y can be written in a unique way as a linear combination of words in the set X. Thus

$$y_j \;=\; \sum_{\ell=1}^{c_j} \lambda_{j\ell} w_{j\ell}$$

for suitable elements $\lambda_{j\ell} \in k$ and words $w_{j\ell}$. Let $n-1$ be the maximal length of a word $w_{j\ell}$.

We consider the quiver Q with vertex and arrow sets

$$V_Q = \{1, \ldots, n\} \ , \ A_Q = \{e_1, \ldots, e_{n-1}\} \cup \{1, \ldots, n-1\} \times X$$

where e_m is an arrow from m to $m+1$ and $a_{mi} = (m, x_i)$ is also an arrow from m to $m+1$. Eventually we are going to invert the arrows e_m and then they and their inverses will generate a copy of $M_n(k)$. With this in mind and for convenience of notation we define for $1 \leqslant s < t \leqslant n$, $e_{s,t} = e_s \ldots e_{t-1}$. Thus $e_{s,t}$ is the unique path using the arrows e_m from the vertex s to the vertex t. We also define $e_{m,m}$ to be the empty path from m to m.

We construct a set of relations on this quiver. Our first set of relations is

$$T = \{t_{mi} : 1 < m < n, 1 \leqslant i \leqslant a\}$$

where

$$t_{mi} = a_{1i}e_{2,n} - e_{1,m}a_{mi}e_{m+1,n} \ .$$

These, in a sense which will become clear soon, ensure that a_{mi} for fixed i all represent the element x_i. Now let $w = x_{i_1} \ldots x_{i_u}$ be a word of length less than n. We define $w' = a_{1,i_1} \ldots a_{u,i_u}e_{u+1,n}$, a path in the quiver Q from 1 to n. We define

$$Y' = \{y'_j : 1 \leqslant j \leqslant b\}$$

where

$$y'_j = \sum_{\ell=1}^{c_j} \lambda_{j\ell} w'_{j\ell} \ .$$

Our relations on the quiver are $T \cup Y'$. Its path algebra A is evidently finite dimensional and it is a simple matter as we shall see to check that A has global dimension 2.

For each vertex m, let P_m be the corresponding projective representation of the quiver Q. Given a path p in the quiver Q from s to t, there is a corresponding map $\hat{p}\colon P_t \to P_s$. We shall abuse notation by writing P_m for the corresponding projective module for A and $\hat{p}$ for the corresponding homomorphism of projective A modules. Let $\sigma = \{\hat{e}_1, \ldots, \hat{e}_{n-1}\}$.

Theorem 1.1. *For S, A, σ and n defined as above there is an isomorphism of rings*

$$\sigma^{-1}A \ \cong \ M_n(S) \ .$$

Proof. We continue to use the notation from the preceding discussion. In the special case when $S = k\langle x_1, x_2, \ldots, x_a\rangle$ has no relations, it is well-known that $M_2(S) = \sigma^{-1}A$ is the universal localization inverting the top arrow in the path algebra A of the quiver Q with two vertices and $a+1$ arrows.

From now on, we assume that $S = k\langle X : Y\rangle$ has at least one relation, i.e. $b \geqslant 1$. We enlarge our quiver with relations Q to a quiver with relations Q' by adjoining arrows f_m from the vertex $m+1$ to the vertex m together with relations $e_m f_m = e_{m,m}$ and $f_m e_m = e_{m+1,m+1}$ for $1 \leqslant m \leqslant n-1$. The path algebra A' of the quiver with relations Q' is just $\sigma^{-1}A$. If we consider the subquiver with relations with the same vertex set and arrows e_m, f_m for $1 \leqslant m \leqslant n-1$, it is clear that there is a unique path $e_{s,t}$ from vertex s to vertex t that involves no subpath of type $e_m f_m$ or of type $f_m e_m$ and that $e_{s,t}e_{t,u} = e_{s,u}$ for any s, t, u. It follows that the path algebra of this subquiver with relations is just $M_n(k)$.

Since the arrows of Q' generate A' over the subring $\times_{i=1}^{n} k$, the set of paths, $\{x_{m,i} = e_{1,m}a_{m,i}e_{m+1,1}\}$ for all m and i, generate A' over the subring $M_n(k)$ given by the paths e_m, f_m. They differ from the elements $a_{m,i}$ by multiplication by invertible paths and we can rewrite our relations between the elements $a_{m,i}$ and e_m as equivalent relations between the elements $x_{m,i}$. Moreover they generate the ring $B = e_{1,1}A'e_{1,1}$ and $A' \cong M_n(B)$. After noting that $x_{m,i} = e_{1,m}a_{m,i}e_{m+1,n}e_{n,1}$, we see that the relations in the set T can be rewritten in terms of the elements $x_{m,i}$ as $x_{m,i} - x_{1,i}$; therefore, we write $x_i = x_{m,i}$ and find the relations between these elements induced by the relations in Y'. We note that a word $a_{1,i_1} \dots a_{u,i_u} = x_{i_1} \dots x_{i_u} e_{1,u+1}$ and so taking a relation y'_j in Y', we see that the corresponding relation between the elements x_i is y_j. Thus B is isomorphic to S as required. □

Proposition 1.2. *The path algebra A in Theorem 1.1 has global dimension $\leqslant 2$.*

Proof. We consider the homological dimension of the simple representations of the quiver. There is one simple S_i for each vertex of the quiver; this is the representation which assigns the field k to the vertex i and 0 to every other vertex and where each arrow gives the zero map. The simple representation S_n is also the projective representation P_n. Because there are no relations on the full subquiver on the vertices $\{2, \dots, n\}$, the simple representations S_m for $m = 2$ to $n-1$ are of homological dimension 1; in fact, for $m = 2$ to $n-1$ we have a short exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{a} P_{m+1} \xrightarrow{\ \phi\ } P_m \longrightarrow S_m \longrightarrow 0$$

where the 0th component of ϕ is $\hat{e}_m$ and the ith component is $\hat{a}_{mi}$ for $i > 0$. Now it is clear that the simple S_1 has homological dimension 2 since the kernel of the homomorphism from P_1 to S_1 has only simples of the form S_m for $m > 1$ as composition factors so that it has homological dimension at most 1 and it is not projective since it is a factor of $\oplus_{i=0}^{a} P_2$ by a semisimple subrepresentation. □

Of course, there is no reason to suppose that A is a subalgebra of $M_n(S)$. Any relation in S between the elements x_i such that the longest monomial has length less than $n-1$ will give nonzero elements of A whose image is 0 in $M_n(S)$. However, the image, $\bar{A}$ of A in $M_n(S)$ is the path algebra of a quiver with relations on the same vertex and arrow set so the quiver is directed and $\bar{A}$ must have finite global dimension. Moreover, it is clear that $\bar{A}_\sigma$ is isomorphic to $\sigma^{-1}A$. In fact, it is a fairly simple matter to describe $\bar{A}$. We consider the filtration of S induced by saying the generators have degree 1; that is, $S_0 = k$ and S_i is the finite dimensional vector space spanned by the monomials in the generators of length at most i. Then $\bar{A}$ is the upper triangular subalgebra of $M_n(S)$ whose elements have entries from S_i in the ith diagonal where the main diagonal is taken to be the 0th.

We summarise this in the following theorem.

Theorem 1.3. *Let S be a finitely presented algebra. Let the largest degree of a relation be $n-1$. Then there is an upper triangular finite dimensional subalgebra C of $M_n(S)$ of which $M_n(S)$ is a universal localization. In particular, C has finite global dimension.* □

In order to see that the examples in the last lemma usually give us examples of universal localizations that are not stably flat we note the following lemma.

Lemma 1.4. *If $\phi\colon R \to S$ is a stably flat epimorphism of rings then*

$$\text{global dimension}(S) \leqslant \text{global dimension}(R)\ .$$

Proof. That ϕ is an epimorphism of rings is equivalent to the condition that the multiplication map from $S \otimes_R S$ to S is an isomorphism ([1]). Therefore, by Lemma 3.30 of [2], $\mathrm{Tor}_i^R(S, M) = 0$ for any S module M and $S \otimes_R M = M$. Therefore, we can construct a projective resolution of M by applying $S \otimes_R \_$ to a projective resolution of M as R module. It follows that the homological dimension of M as S module is bounded by its homological dimension as R module. □

There are many possible variations on this method for representing algebras as universal localizations of finite dimensional algebras. We give two examples to illustrate possible changes. Let Q be the quiver with relations having vertices $1, 2, 3, 4$ and arrows e_1, x_1 from 1 to 2, e_2, y_2 from 2 to 3 and e_3, x_3 from 3 to 4 together with relations $x_1e_2e_3 - e_1e_2x_3$ and $x_1y_2e_3 - e_1y_2x_3 - e_1e_2e_3$. On inverting the arrows e_1, e_2, e_3 the path algebra we obtain is $M_4(R_1)$ where R_1 is the first Weyl algebra.

Let Q be the quiver with relations having vertices $1, 2, 3, 4$ and arrows e_1, x_1 from 1 to 3, e_2, y_2 from 2 to 3 and e_3, x_3 from 3 to 4 together with relations $e_1x_3 - x_1e_3$, x_1x_3 and y_2x_3. On inverting the arrows e_1, e_2, e_3 the path algebra we obtain is $M_4(k\langle x, y : x^2, yx\rangle)$. The important point in this example is that the set of arrows we invert can be simply a maximal subtree of the quiver, and there may occasionally be an advantage to doing this if the relations we are interested in can be described compactly on a tree.

At this point, we return to the question of whether there can be an algorithm to determine the equality of elements in a universal localization. Thus let A be a ring and σ a set of maps between finitely generated projective modules over A. The Malcolmson normal form states that every element of the localised ring $\sigma^{-1}A$ can be written in the form $as^{-1}b$ where $s\colon P \to Q$ lies in the upper triangular closure of σ, $a\colon A \to Q$ and $b\colon P \to A$ are maps in the category of finitely generated projective modules over the original ring A and gives an equivalence condition on such elements which determines when they define the same element of the localised ring. This equivalence condition depends on the existence of certain maps in σ and the category of finitely generated projective modules. One might reasonably ask if such an equation could be constructed algorithmically. In order to show that this is not possible we do not need to know the exact nature of the equivalence relation defined by Malcolmson since it is simply important to be able to demonstrate that there can be no algorithm to determine the equality of two such elements. We say that the *equality problem* for (A, σ) is *solvable* if there is an algorithm to determine the equality of two such elements in $\sigma^{-1}A$.

Our proof that the equality problem is not always solvable comes from the fact that the word problem for groups is not always solvable. Thus let G be a finitely presented group with generators $\{x_i : 1 \leqslant i \leqslant c\}$ and relations $\{r_j : 1 \leqslant j \leqslant d\}$. We obtain a finite presentation of its group algebra kG by taking as generators $\{x_i, \bar{x}_i : 1 \leqslant i \leqslant c\}$ and as relations $\{x_i\bar{x}_i - 1 : 1 \leqslant i \leqslant c\} \cup \{\bar{x}_ix_i - 1 : 1 \leqslant i \leqslant c\} \cup \{s_j - 1 : 1 \leqslant j \leqslant d\}$ where s_j is obtained from r_j by replacing each occurrence of each x_i^{-1} by $\bar{x}_i$. Let A be the finite dimensional algebra we produce by the method considered in and preceding theorem 1.1 and let σ be the set of maps between finitely generated projective modules over A considered there so that $\sigma^{-1}A$ is isomorphic to $M_n(kG)$ for a suitable integer n.

Theorem 1.5. *Let G be a finitely presented group for which the word problem is not solvable. Let A be the finite dimensional algebra and let σ be the set of maps between finitely generated projective modules considered in the previous paragraph. Then the equality problem for (A, σ) is not solvable.*

Proof. We use the notation developed before theorem 1.1. The group ring occurs as the endomorphism ring of $P_1 \otimes M_n(kG)$. The generators of the group and their inverses occur as elements of the form $x_{m1}e_1^{-1}$. Therefore words in these elements can be written algorithmically in the form $as^{-1}b$ for suitable maps a and b between finitely generated projective modules and s in the upper triangular closure of σ. If there were an algorithm to determine whether such an element were equal to the identity map on P_1 then we would be able to solve the word problem for the group G. Since we cannot solve the word problem, there can be no such algorithm. $\square$

2. An explicit computation

Notation 2.1. In this section, let k be a ring and S a k-ring, i.e. a ring homomorphism $k \to S$. We will assume throughout that S is flat as a left k-module. $\square$

We define a functor from the category of left S-modules to itself.

Definition 2.2. Recall the short exact sequence of S bimodules

$$0 \longrightarrow \Omega_k(S) \longrightarrow S \otimes_k S \xrightarrow{m} S \longrightarrow 0$$

where $\Omega_k(S)$ is the universal bimodule of derivations of S over k and m is the multiplication map. It is split considered as a sequence of left or right S modules since S is a projective module but not as a sequence of bimodules.

Given a left S-module M, we define $\mathcal{K}(M)$ from the exact sequence obtained by tensoring with M on the right

$$0 \longrightarrow \mathcal{K}(M) = \Omega_k(S) \otimes_S M \longrightarrow S \otimes_k M \xrightarrow{\mu_M} M \longrightarrow 0$$

with $\mu_M = m \otimes_S 1_M$ the multiplication map. Thus $\mathcal{K}(M)$ is the kernel of the multiplication map and is isomorphic to $\Omega_k(S) \otimes_S M$.

$\square$

Lemma 2.3. *As in Definition 2.2, let M be a left S-module. If M is flat as a (left) k-module, then so is $\mathcal{K}(M)$.*

Proof. By hypothesis, both M and S are flat as left k-modules. It follows that $S \otimes_k M$ is also a flat left k-module. In the exact sequence

$$0 \longrightarrow \mathcal{K}(M) \longrightarrow S \otimes_k M \xrightarrow{\mu_M} M \longrightarrow 0$$

we now know that both M and $S \otimes_k M$ are flat as left k-modules. From the exact sequence for Tor it now follows that so is $\mathcal{K}(M)$. $\square$

Let M be a left S-module, flat over k. The above produces for us exact sequences of left S-modules, all flat over k

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}(M) & \xrightarrow{i_M} & S \otimes_k M & \xrightarrow{\mu_M} & M & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{K}^2(M) & \xrightarrow{i_{\mathcal{K}(M)}} & S \otimes_k \mathcal{K}(M) & \xrightarrow{\mu_{\mathcal{K}(M)}} & \mathcal{K}(M) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{K}^3(M) & \xrightarrow{i_{\mathcal{K}^2(M)}} & S \otimes_k \mathcal{K}^2(M) & \xrightarrow{\mu_{\mathcal{K}^2(M)}} & \mathcal{K}^2(M) & \longrightarrow & 0
\end{array}$$

Splicing these short exact sequences, we deduce

Lemma 2.4. *Let M be a left S-module, flat over k. To make the notation work nicely, define $\mathcal{K}^0(M) = M$. For $n \geqslant 1$ we have defined $\mathcal{K}^n(M)$ above. For each $j \geqslant 1$ there is an exact sequence of left S-modules, all flat over k*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}^j(M) & \xrightarrow{i_{\mathcal{K}^{j-1}(M)}} & S \otimes_k \mathcal{K}^{j-1}(M) & \longrightarrow & \cdots & & \\
\cdots & \longrightarrow & S \otimes_k \mathcal{K}(M) & \xrightarrow{i_M \mu_{\mathcal{K}(M)}} & S \otimes_k \mathcal{K}^0(M) & \xrightarrow{\mu_M} & M & \longrightarrow & 0.
\end{array}$$

□

The case of most interest to us is where $M = S$. We can assemble the first n of these exact sequences in vector form.

Lemma 2.5. *We have an exact sequence*

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ k \end{pmatrix} \otimes_k \mathcal{K}^{n-1}(S) \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ k \\ S \end{pmatrix} \otimes_k \mathcal{K}^{n-2}(S) \longrightarrow \cdots$$

$$\cdots \longrightarrow \begin{pmatrix} k \\ S \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix} \otimes_k \mathcal{K}^{0}(S) \longrightarrow \begin{pmatrix} S \\ S \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

□

Definition 2.6. Let A be the ring of $n \times n$ lower triangular matrices

$$A = \begin{pmatrix} k & 0 & 0 & \cdots & 0 & 0 & 0 \\ S & k & 0 & \cdots & 0 & 0 & 0 \\ S & S & k & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ S & S & S & \cdots & k & 0 & 0 \\ S & S & S & \cdots & S & k & 0 \\ S & S & S & \cdots & S & S & k \end{pmatrix}$$

That is, the terms above the diagonal vanish, the diagonal terms lie in k, while the terms below the diagonal may be any elements of S. $\square$

The columns of the matrix ring A are left A-modules. We denote them

$$P_1 = \begin{pmatrix} k \\ S \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ k \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix}, \quad \cdots \quad P_{n-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ k \\ S \end{pmatrix}, \quad P_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ k \end{pmatrix}$$

and as a left A-module

$$A = P_1 \oplus P_2 \oplus \cdots \oplus P_n .$$

Then Lemma 2.5 says that we have an exact sequence

$$0 \longrightarrow P_n \otimes_k \mathcal{K}^{n-1}(S) \longrightarrow P_{n-1} \otimes_k \mathcal{K}^{n-2}(S) \longrightarrow \cdots$$

$$\cdots \longrightarrow P_1 \otimes_k \mathcal{K}^0(S) \longrightarrow \begin{pmatrix} S \\ S \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is clearly a resolution of left A-modules. The modules P_i are all direct summands of A, hence they are projective left A-modules. Being projective, they are certainly flat left A-modules. The modules $\mathcal{K}^{i-1}(S)$ are flat left k-modules. It follows that $P_i \otimes_k \mathcal{K}^{i-1}(S)$ are all flat left A-modules. Summarizing the above, we have

Lemma 2.7. *The left A-module*

$$N = \begin{pmatrix} S \\ S \\ S \\ \vdots \\ S \\ S \\ S \end{pmatrix}$$

has a flat resolution

$$0 \longrightarrow P_n \otimes_k \mathcal{K}^{n-1}(S) \longrightarrow \dots \longrightarrow P_1 \otimes_k \mathcal{K}^0(S) \longrightarrow N \longrightarrow 0 .$$

□

Define also the right A-module

$$M = \begin{pmatrix} S & S & S & \cdots & S & S & S \end{pmatrix} .$$

Lemma 2.8. *We have*

$$M \otimes_A P_i = S \ (1 \leqslant i \leqslant n)$$

and

$$M_n(S) \otimes_A P_i = N \ (1 \leqslant i \leqslant n) .$$

Proof. We begin with $M \otimes_A P_i = S$. There are obvious maps

$$S \xrightarrow{\alpha_i} M \otimes_A P_i \xrightarrow{\beta_i} S$$

defined by

$$\alpha_i(s) = (0, \dots, 0, s) \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} ,$$

$$\beta_i\left((s_1, s_2, \dots, s_n) \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_i \\ \vdots \\ x_n \end{pmatrix}\right) = \sum_{j=i}^{n} s_j x_j .$$

It is clear that the composite $\beta_i \alpha_i$ is the identity. It suffices to show that α_i is surjective, which we leave to the reader.

The identity $M_n(S) \otimes_A P_i = N$ reduces to the above, after observing that $M_n(S) = \bigoplus_{i=1}^n M$ as a right A-module. □

Proposition 2.9. *The* Tor*-groups are*

$$\mathrm{Tor}_i^A(M,N) \ = \ \begin{cases} S & if\ i=0 \\ \mathcal{K}^n(S) & if\ i=n-1 \\ 0 & otherwise\ . \end{cases}$$

Consequently

$$\mathrm{Tor}_i^A(M_n(S),M_n(S)) = M_n(\mathrm{Tor}_i^A(M,N)) = \begin{cases} M_n(S) & if\ i=0 \\ M_n(\mathcal{K}^n(S)) & if\ i=n-1 \\ 0 & otherwise\ . \end{cases}$$

Proof. By definition, $\mathrm{Tor}_i^A(M,N)$ is the ith homology of the complex obtained from any flat resolution of N by tensoring over A with M. We use the resolution provided by Lemma 2.7. Lemma 2.8 allows us to identify $\mathrm{Tor}_i^A(M,N)$ with the ith homology of

$$S\otimes_k \mathcal{K}^{n-1}(S) \longrightarrow S\otimes_k \mathcal{K}^{n-2}(S) \longrightarrow \cdots \longrightarrow S\otimes_k \mathcal{K}^{1}(S) \longrightarrow S\otimes_k \mathcal{K}^{0}(S)\ .$$

□

Definition 2.10. Let $\phi : k \longrightarrow S$ be the ring homomorphism giving S the structure of an A-ring. Define σ to be the set of maps $s_i : P_n \longrightarrow P_i$ given by the matrices

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \phi \end{pmatrix} \quad : \quad \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ k \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ k \\ S \\ \vdots \\ S \\ S \end{pmatrix}$$

□

Lemma 2.11. *The ring homomorphism $A \to M_n(S)$ is σ-inverting.*

Proof. By Lemma 2.8

$$1\otimes s_i \ : \ M_n(S)\otimes_A P_n \to M_n(S)\otimes_A P_i$$

can be identified with $1 : N \to N$. □

Theorem 2.12. *For $n \geqslant 3$, $A \to M_n(S)$ is universally σ-inverting,*

$$\sigma^{-1}A \ = \ M_n(S)\ .$$

Proof. Let T be a σ-inverting A-ring. We need to exhibit a unique factorization

$$A \to M_n(S) \to T \ .$$

It follows from $A = \bigoplus_{i=1}^{n} P_i$ that

$$T \ = \ \bigoplus_{i=1}^{n} T \otimes_A P_i$$

with the $T \otimes_A P_i$'s isomorphic f.g. projective T-modules. Also,

$$T \ = \ \mathrm{End}_T(T) \ = \ M_n(\mathrm{End}_T(T \otimes_A P_1)) \ .$$

It therefore suffices to produce a homomorphism

$$S \to \mathrm{End}_T(T \otimes_A P_1) \ .$$

For $x \in S$ define the A-module morphisms

$$r_x \ : \ P_i \ = \ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ k \\ \vdots \\ S \end{pmatrix} \longrightarrow P_j \ = \ \begin{pmatrix} 0 \\ \vdots \\ k \\ \vdots \\ S \\ \vdots \\ S \end{pmatrix}$$

with components right multiplication by x. Define

$$S \longrightarrow \mathrm{End}_T(T \otimes_A P_1)$$

by sending $x \in S$ to

$$P_1 \xrightarrow{(r_1)^{-1}} P_2 \xrightarrow{r_x} P_1 \ .$$

Because $r_{x+y} = r_x + r_y$ this is a homomorphism of abelian groups. The multiplicative identity $r_{xy} = r_x r_y$ follows from the commutative diagram

$$\begin{tikzcd}
P_n \arrow[r, "r_1"] \arrow[d, "r_y"'] & P_{n-1} \arrow[r, "r_1"] \arrow[d, "r_y"] & P_1 \\
P_{n-1} \arrow[r, "r_1"] \arrow[d, "r_x"'] & P_1 & \\
P_1 & &
\end{tikzcd}$$

(This diagram only makes sense if $n > n-1 > 1$, i.e. $n \geqslant 3$). $\square$

Remark 2.13. Suppose k is a field. For any finite-dimensional k-algebra S let $d = \dim_k(S)$. It follows from the exact sequence

$$0 \longrightarrow \mathcal{K}(M) \longrightarrow S \otimes_k M \longrightarrow M \longrightarrow 0$$

that for any f.g. S-module M

$$\dim_k \mathcal{K}(M) \ = \ (d-1)\dim_k(M) \ .$$

By induction, for $M = S$ and $n \geqslant 1$

$$\dim_k \mathcal{K}^n(S) \ = \ (d-1)^n d \ .$$

Thus if $n \geqslant 3$ and $d > 1$ then

$$\mathrm{Tor}^A_{n-1}(\sigma^{-1}A, \sigma^{-1}A) \ = \ M_n(\mathcal{K}^n(S)) \neq 0 \ .$$

In particular, for $S = k[\varepsilon]/(\varepsilon^2)$ $(d = 2)$ the ring

$$A \ = \ \begin{pmatrix} k & 0 & 0 \\ S & k & 0 \\ S & S & k \end{pmatrix}$$

is the path algebra of the quiver with relations (Q, R) constructed as in section 1, with $\sigma^{-1}A = M_3(S)$, $\mathrm{Tor}^A_2(\sigma^{-1}A, \sigma^{-1}A) = M_3(S)$. $\square$

References

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