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The Whitehead Group of the Novikov Ring

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Abstract. The Bass–Heller–Swan–Farrell–Hsiang–Siebenmann decomposition of the Whitehead group $K_1(A_\rho[z, z^{-1}])$ of a twisted Laurent polynomial extension $A_\rho[z, z^{-1}]$ of a ring A is generalized to a decomposition of the Whitehead group $K_1(A_\rho((z)))$ of a twisted Novikov ring of power series $A_\rho((z)) = A_\rho[[z]][z^{-1}]$. The decomposition involves a summand $W_1(A, \rho)$ which is an Abelian quotient of the multiplicative group $W(A, \rho)$ of Witt vectors $1+a_1z+a_2z^2+\cdots \in A_\rho[[z]]$. An example is constructed to show that in general the natural surjection $W(A, \rho)^{ab} \to W_1(A, \rho)$ is not an isomorphism.

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0. Introduction

We obtain a splitting theorem for the Whitehead group $K_1(A_\rho((z)))$ of the Novikov ring of power series $A_\rho((z))$, which is an analogue of the well-known splitting theorem for the Whitehead group $K_1(A_\rho[z, z^{-1}])$ of the Laurent ring of polynomials $A_\rho[z, z^{-1}]$.

Let A be an associative ring with 1. Given an automorphism $\rho\colon A \to A$ let z be an indeterminate over A such that

$$az = z\rho(a) \quad (a \in A).$$

The *ρ-twisted polynomial extension ring* $A_\rho[z]$ is the ring of polynomials $\sum_{j=0}^{\infty} a_j z^j$ with $a_j = 0 \in A$ for all but a finite number of $j \geqslant 0$. The *ρ-twisted power series ring* $A_\rho[[z]]$ is the ring of power series $\sum_{j=0}^{\infty} a_j z^j$ for arbitrary $a_j \in A$. Inverting z we also obtain the *ρ-twisted Laurent polynomial extension ring* $A_\rho[z, z^{-1}]$ of polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$ with $a_j = 0 \in A$ for all but a finite number of $j \in \mathbf{Z}$, and the *ρ-twisted Novikov formal power series ring* $A_\rho((z))$ of polynomials $\sum_{j=-\infty}^{\infty} a_j z^j$ with $a_j = 0 \in A$ for all but a finite number of $j < 0$.

The Whitehead groups of the polynomial rings split as

$$K_1(A_\rho[z]) = K_1(A) \oplus \widetilde{\mathrm{Nil}}_0(A,\rho),$$
$$K_1(A_\rho[z,z^{-1}]) = K_1(A,\rho) \oplus \widetilde{\mathrm{Nil}}_0(A,\rho) \oplus \widetilde{\mathrm{Nil}}_0(A,\rho^{-1})$$

with $K_1(A,\rho)$ the class group of pairs (P,ϕ) with P a f.g. (finitely generated) projective A-module and $\phi\colon P \to P$ a ρ-twisted automorphism, and $\widetilde{\mathrm{Nil}}_0(A,\rho)$ the reduced class group of pairs (P,ν) with P a f.g. projective A-module and $\nu\colon P \to P$ a nilpotent ρ-twisted endomorphism ([BHS], [B], [FH], [S]).

The augmentation

$$A_\rho[[z]] \to A, \qquad \sum_{j=0}^{\infty} a_j z^j \mapsto a_0$$

induces a split surjection $K_1(A_\rho[[z]]) \to K_1(A)$, with the kernel

$$NK_1(A_\rho[[z]]) = \ker(K_1(A_\rho[[z]]) \to K_1(A))$$

such that

$$K_1(A_\rho[[z]]) = K_1(A) \oplus NK_1(A_\rho[[z]]).$$

Pajitnov [P1] identified $NK_1(A_\rho[[z]])$ with the subgroup $W_1(A,\rho) \subseteq K_1(A_\rho[[z]])$ represented by the Witt vectors, that is the units in $A_\rho[[z]]$ of the type

$$w = 1 + \sum_{j=1}^{\infty} a_j z^j \in A_\rho[[z]]^\bullet.$$

MAIN THEOREM. *The Whitehead group of the Novikov ring splits as*

$$K_1(A_\rho((z))) = K_1(A,\rho) \oplus W_1(A,\rho) \oplus \widetilde{\mathrm{Nil}}_0(A,\rho^{-1}).$$

This splitting was obtained in the untwisted case $\rho = 1$ in Chapter 14 of [R2] by using general results on the algebraic K-theory of localization-completion squares, such as

$$\begin{array}{ccc} A_\rho[z] & \longrightarrow & A_\rho[z,z^{-1}] \\ \downarrow & & \downarrow \\ A_\rho[[z]] & \longrightarrow & A_\rho((z)). \end{array}$$

In principle, the general method also works in the twisted case, using the equivalences of exact categories

{f.g. projective A-modules P with a nilpotent ρ-twisted endomorphism $\nu\colon P \to P$}

$\approx$ {z-primary torsion $A_\rho[z]$-modules of homological dimension 1}

$\approx$ {z-primary torsion $A_\rho[[z]]$-modules of homological dimension 1} but here we prefer to use a direct method. Section 2 of this paper is devoted to a direct proof of the Main Theorem, which follows the direct proof of the splitting theorem for $K_1(A_\rho[z, z^{-1}])$, except for the (Higman) linearization result which does not have an analogue for $K_1(A_\rho((z)))$.

Our Main Theorem is used in the work of the first author [P2] to define a non-Abelian logarithm in the twisted case. Here is the corollary which is used in [P2], and which follows immediately from the proof of the Main Theorem, given in Section 2.5.

COROLLARY 0.1. *The homomorphism* $\widehat{C}_2\colon\ W_1(A,\rho) \to K_1(A_\rho((z)))$ *induced by the inclusion has a left inverse* $\widehat{B}_2\colon\ K_1(A_\rho((z))) \to W_1(A,\rho)$, *which satisfies:*

(1) $\widehat{B}_2$ *vanishes on the image of* $K_1(A)$ *in* $K_1(A_\rho((z)))$,
(2) $\widehat{B}_2(\tau(z)) = 0$, *with* $\tau(z) \in K_1(A_\rho((z)))$ *the torsion of the invertible* 1×1-*matrix* (z).

Write the multiplicative group of Witt vectors in $A_\rho[[z]]$ as

$$W(A,\rho) = 1 + zA_\rho[[z]] \subseteq A_\rho[[z]]^\bullet,$$

with Abelianization

$$W(A,\rho)^{ab} = W(A,\rho)/[W(A,\rho), W(A,\rho)].$$

It was claimed in Proposition 14.6 of [R2] that the natural surjection $W(A,\rho)^{ab} \to W_1(A,\rho)$ is an injection, at least in the untwisted case $\rho = 1$. In Chapter 3 of this paper we correct this, constructing a family of explicit counterexamples, already in the untwisted case $\rho = 1$.

1. Class and Torsion

We recall the definitions of the K-groups K_0, K_1 of additive and exact categories, and also of the less familiar isomorphism torsion group of [R1].

The *class group* $K_0(\mathbb{C})$ of an exact category $\mathbb{C}$ is the Abelian group with one generator $[M]$ for each isomorphism class of objects M in $\mathbb{C}$, and one relation

$$[L] - [M] + [N] = 0$$

for each short exact sequence in $\mathbb{C}$

$$0 \to L \to M \to N \to 0.$$

The *Whitehead group* $K_1(\mathbb{C})$ of an exact category $\mathbb{C}$ has one generator $\tau(f)$ for each automorphism $f\colon\ M \to M$ in $\mathbb{C}$, and relations

(i) $\tau(e) - \tau(f) + \tau(g) = 0$
for each automorphism of a short exact sequence in $\mathbb{C}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle e} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

(ii) $\tau(gf\colon M \to M) = \tau(f\colon M \to M) + \tau(g\colon M \to M)$
for automorphisms $f, g\colon M \to M$ in $\mathbb{C}$.

The algebraic K-groups $K_0(A)$, $K_1(A)$ of a ring A are defined by

$$K_i(A) = K_i(\mathbb{P}(A))$$

with $\mathbb{P}(A)$ the exact category of f.g. projective A-modules; $K_1(A)$ is called the Whitehead group of A.

PROPOSITION 1.1 ([B], p. 397, [R1], Prop. 1.1). *Let $\mathbb{A}$ be an additive category with the split exact structure: a sequence*

$$0 \longrightarrow L \xrightarrow{\;i\;} M \xrightarrow{\;j\;} N \longrightarrow 0$$

is exact if and only if there exists a morphism $u\colon N \to M$ such that

$$(i\ u)\colon L \oplus N \to M$$

is an isomorphism.

(i) *The Whitehead group $K_1(\mathbb{A})$ is (isomorphic to) the Abelian group $K_1'(\mathbb{A})$ with one generator $\tau(f)$ for each automorphism $f\colon M \to M$ in $\mathbb{A}$, and relations*

$$\tau\left(\begin{pmatrix} f & d \\ 0 & f' \end{pmatrix}\colon M \oplus M' \to M \oplus M'\right) = \tau(f\colon M \to M) + \tau(f'\colon M' \to M')$$

for automorphisms $f\colon M \to M$, $f'\colon M' \to M'$ and any morphism $d\colon M' \to M$,

$$\tau(gf\colon M \to M) = \tau(fg\colon N \to N)$$
for isomorphisms $f\colon M \to N$, $g\colon N \to M$ in $\mathbb{A}$.

(ii) *The Whitehead group $K_1(\mathbb{A})$ is (isomorphic to) the Abelian group $K_1''(\mathbb{A})$ with one generator $\tau(f)$ for each automorphism $f\colon M \to M$ in $\mathbb{A}$, and relations*

$$\tau(f \oplus f'\colon M \oplus M' \to M \oplus M') = \tau(f\colon M \to M) + \tau(f'\colon M' \to M')$$
(*for automorphisms $f\colon M \to M$, $f'\colon M' \to M'$*),
$$\tau(gf\colon M \to M) = \tau(f\colon M \to M) + \tau(g\colon M \to M)$$
(*for automorphisms $f\colon M \to M$, $g\colon M \to M$*),
$$\tau(gf\colon M \to M) = \tau(fg\colon N \to N)$$
(*for isomorphisms $f\colon M \to N$, $g\colon N \to M$*).

Proof. Let S be the set of automorphisms in $\mathbb{A}$, and let $R, R', R'' \subset \mathbb{Z}[S]$ be the subgroups defined by

$$\begin{aligned} R = \{&\tau(e) - \tau(f) + \tau(g) \text{ for an exact sequence } 0 \to e \to f \to g \to 0,\\ &\tau(gf) - \tau(f) - \tau(g) \text{ for automorphisms } f, g\},\\ R' = \{&\tau(f) - \tau\begin{pmatrix} f & d \\ 0 & f' \end{pmatrix} + \tau(f') \text{ for automorphisms } f, f',\\ &\tau(gf) - \tau(fg) \text{ for isomorphisms } f, g\},\\ R'' = \{&\tau(f) - \tau(f \oplus f') + \tau(f') \text{ for automorphisms } f, f',\\ &\tau(f) - \tau(gf) + \tau(g) \text{ for automorphisms } f, g,\\ &\tau(gf) - \tau(fg) \text{ for isomorphisms } f, g\}, \end{aligned}$$

so that

$$K_1(\mathbb{A}) = \mathbb{Z}[S]/R, \qquad K_1'(\mathbb{A}) = \mathbb{Z}[S]/R', \qquad K_1''(\mathbb{A}) = \mathbb{Z}[S]/R''.$$

We shall prove that $R = R' = R''$ by first showing that $R \subseteq R' \subseteq R$ and then $R' \subseteq R'' \subseteq R$.

(i) ($R \subseteq R'$) Given an automorphism of a short exact sequence in $\mathbb{A}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{j} & N & \longrightarrow & 0 \\ & & \Big\downarrow e & & \Big\downarrow f & & \Big\downarrow g & & \\ 0 & \longrightarrow & L & \xrightarrow{i} & M & \xrightarrow{e} & N & \longrightarrow & 0 \end{array}$$

there exists a morphism $u \colon N \to M$ such that

$$h = (i\ u) : \ L \oplus N \to M$$

is an isomorphism, with inverse of the form

$$h^{-1} = \begin{pmatrix} v \\ j \end{pmatrix} : \ M \to L \oplus N.$$

Now

$$h^{-1} f h = \begin{pmatrix} e & vfu \\ 0 & g \end{pmatrix} : \ L \oplus N \to L \oplus N.$$

Thus

$$\tau(e) - \tau(f) + \tau(g) = (\tau(e) - \tau(h^{-1}fh) + \tau(g)) + (\tau(h^{-1}fh) - \tau(f)) \in R'$$

and $R \subseteq R'$.

(ii) ($R' \subseteq R$) Every automorphism in $\mathbb{A}$ is of the form

$$\alpha = \begin{pmatrix} f & d \\ 0 & f' \end{pmatrix} \ : \ M \oplus M' \to M \oplus M'$$

and fits into an automorphism of a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & M \oplus M' & \longrightarrow & M' & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow{\scriptstyle f'} & & \\ 0 & \longrightarrow & M & \longrightarrow & M \oplus M' & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

so that

$$\tau(\alpha) - \tau(f) - \tau(f') \in R.$$

(iii) ($R' \subseteq R''$) For any elementary automorphism in $\mathbb{A}$

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} : \ M \oplus M' \to M \oplus M'$$

the automorphisms

$$g \ = \ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \ M \oplus M' \oplus M' \to M \oplus M' \oplus M',$$

$$h \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & e & 1 \end{pmatrix} : \ M \oplus M' \oplus M' \to M \oplus M' \oplus M'$$

are such that

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \oplus 1 = ghg^{-1}h^{-1} : \ M \oplus M' \oplus M' \to M \oplus M' \oplus M'$$

so that

$$\tau \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \in R''.$$

More generally, for any automorphism of the type

$$\begin{pmatrix} f & e \\ 0 & f' \end{pmatrix} : \ M \oplus M' \to M \oplus M'$$

we have

$$\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}^{-1} \begin{pmatrix} f & e \\ 0 & f' \end{pmatrix} = \begin{pmatrix} 1 & f^{-1}e \\ 0 & 1 \end{pmatrix}$$

so that

$$\tau\begin{pmatrix} f & e \\ 0 & f' \end{pmatrix} - \tau\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix} \in R''.$$

(iv) ($R'' \subseteq R'$) For any automorphisms $f\colon\ M \to M$, $g\colon\ M \to M$

$$\begin{pmatrix} gf & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & f \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix}\colon\ M \oplus M \to M \oplus M$$

and for any automorphism $\alpha\colon\ M \to M$ we have the Whitehead lemma identity

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\alpha^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\colon\ M \oplus M \to M \oplus M,$$

so that

$$\tau(f) - \tau(gf) + \tau(g) \in R'.$$

For any isomorphisms $f\colon\ M \to N$, $g\colon\ N \to M$

$$\begin{pmatrix} gf & 0 \\ 0 & (fg)^{-1} \end{pmatrix} = \begin{pmatrix} 0 & g \\ -g^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -f^{-1} \\ f & 0 \end{pmatrix}\colon\ M \oplus N \to M \oplus N,$$

and for any isomorphism $h\colon\ M \to N$ we have

$$\begin{pmatrix} 0 & -h^{-1} \\ h & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\begin{pmatrix} 1 & -h^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}\colon\ M \oplus N \to M \oplus N$$

so that $\tau(gf) - \tau(fg) \in R'$. Thus $R'' \subseteq R'$. □

DEFINITION 1.2 ([R1]). The *isomorphism torsion group* $K_1^{\text{iso}}(\mathbb{A})$ of an additive category $\mathbb{A}$ is the Abelian group with one generator $\tau^{\text{iso}}(f)$ for each isomorphism $f\colon\ M \to N$ in $\mathbb{A}$, and relations

$$\tau^{\text{iso}}(gf\colon\ M \to P) = \tau^{\text{iso}}(f\colon\ M \to N) + \tau^{\text{iso}}(g\colon\ N \to P),$$
$$\tau^{\text{iso}}(f \oplus f'\colon\ M \oplus M' \to N \oplus N') = \tau^{\text{iso}}(f\colon\ M \to N) + \tau^{\text{iso}}(f'\colon\ M' \to N').$$

Every automorphism is an isomorphism, so there is an evident forgetful map

$$K_1(\mathbb{A}) \to K_1^{\text{iso}}(\mathbb{A});\ \tau(f) \mapsto \tau^{\text{iso}}(f).$$

This map is an isomorphism if every isomorphism in $\mathbb{A}$ is an automorphism.

A functor $F\colon\ \mathbb{A} \to \mathbb{B}$ of additive categories induces morphisms of the torsion groups

$$F\colon\ \ K_1(\mathbb{A}) \to K_1(\mathbb{B});\ \tau(f\colon\ M \to M) \mapsto \tau(F(f)\colon\ F(M) \to F(M)),$$
$$F\colon\ K_1^{\text{iso}}(\mathbb{A}) \to K_1^{\text{iso}}(\mathbb{B});\ \tau^{\text{iso}}(f\colon\ M \to N) \mapsto \tau^{\text{iso}}(F(f)\colon\ F(M) \to F(N)).$$

Remark 1.3. (i) There is an essential difference between $K_1(\mathbb{A})$ and $K_1^{\text{iso}}(\mathbb{A})$. An equivalence of additive categories $F\colon \mathbb{A} \to \mathbb{B}$ induces an isomorphism $F\colon K_1(\mathbb{A}) \cong K_1(\mathbb{B})$, but the induced morphism $F\colon K_1^{\text{iso}}(\mathbb{A}) \to K_1^{\text{iso}}(\mathbb{B})$ may not be an isomorphism. See (ii) below for an example.

(ii) For any ring A let $\mathbb{B} = \mathbb{B}(A)$ be the additive category with objects based f.g. free A-modules and A-module morphisms, and let $\mathbb{A} \subset \mathbb{B}$ be the full subcategory with objects A^n with the standard basis $e = \{e_1, e_2, \ldots, e_n\}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. The inclusion $F\colon \mathbb{A} \to \mathbb{B}$ is an equivalence of categories such that $F\colon K_1^{\text{iso}}(\mathbb{A}) = K_1(A) \to K_1^{\text{iso}}(\mathbb{B})$ is not an isomorphism: if $b = \{b_1, b_2, \ldots, b_n\}$ is a non-standard basis for A^n and $f\colon (A^n, e) \to (A^n, b)$ is the isomorphism in $\mathbb{B}$ defined by $f = 1\colon A^n \to A^n$ then $\tau^{\text{iso}}(f) \in K_1^{\text{iso}}(\mathbb{B})$ is not in the image of F.

DEFINITION 1.4. (i) Given a category $\mathbb{A}$ let $\text{Iso}(\mathbb{A})$ be the subcategory with the same objects, but only the isomorphisms as morphisms.

(ii) An *isomorphism torsion structure* F on an additive category $\mathbb{A}$ is a functor

$$F\colon \text{Iso}(\mathbb{A}) \to \text{Iso}(\mathbb{A})$$

which is the identity on objects, and such that

$$\begin{aligned} &F(f \oplus f') = F(f) \oplus F(f')\colon \\ &\qquad F(M \oplus M') = M \oplus M' \to F(N \oplus N') = N \oplus N' \end{aligned}$$

for any isomorphisms $f\colon M \to N$, $f'\colon M' \to N'$.

(iii) The *F-relative torsion* of an isomorphism $f\colon M \to N$ in $\mathbb{A}$ is

$$\tau^F(f) = \tau(F(f)^{-1}f\colon M \to M) \in K_1(\mathbb{A}).$$

(iv) An isomorphism torsion structure F is *idempotent* if $F^2 = F$.

PROPOSITION 1.5. *Let $\mathbb{A}$ be an additive category, and let $F\colon \text{Iso}(\mathbb{A}) \to \text{Iso}(\mathbb{A})$ be an isomorphism torsion structure.*

(i) *The F-relative torsion function defines a morphism*

$$\begin{aligned} &\tau^F\colon K_1^{\text{iso}}(\mathbb{A}) \to K_1(\mathbb{A}); \\ &\quad \tau^{\text{iso}}(f\colon M \to N) \mapsto \tau^F(f) = \tau(F(f)^{-1}f\colon M \to M). \end{aligned}$$

(ii) *F induces an endomorphism of the Whitehead group*

$$F\colon K_1(\mathbb{A}) \to K_1(\mathbb{A}); \tau(f\colon M \to M) \to \tau(F(f)\colon M \to M)$$

such that

$$1 - F\colon K_1(\mathbb{A}) \to K_1^{\text{iso}}(\mathbb{A}) \xrightarrow{\tau^F} K_1(\mathbb{A}).$$

(iii) *If F is idempotent*

$$\begin{aligned}
&F^2 = F\colon\ K_1(\mathbb{A}) \to K_1(\mathbb{A}),\\
&\mathrm{im}(\tau^F\colon\ K_1^{\mathrm{iso}}(\mathbb{A}) \to K_1(\mathbb{A}))\ =\ \mathrm{im}(1 - F\colon\ K_1(\mathbb{A}) \to K_1(\mathbb{A}))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad =\ \ker(F\colon\ K_1(\mathbb{A}) \to K_1(\mathbb{A})),\\
&K_1(\mathbb{A}) = \mathrm{im}(F\colon\ K_1(\mathbb{A}) \to K_1(\mathbb{A})) \oplus \mathrm{im}(\tau^F).
\end{aligned}$$

Proof. Immediate from 1.1. □

PROPOSITION 1.7. Let A be a ring such that the rank of f.g. free A-modules is well defined. The *canonical idempotent isomorphism torsion structure* on the additive category $\mathbb{B}(A)$ of based f.g. free A-modules

$$F^{\mathrm{can}}\colon\ \mathrm{Iso}(\mathbb{B}(A)) \to \mathrm{Iso}(\mathbb{B}(A))$$

sends every isomorphism $f\colon\ M \to N$ to the isomorphism $F^{\mathrm{can}}(f)\colon\ M \to N$ sending the given basis of M to the given basis of N.

PROPOSITION 1.7. (i) *The canonical idempotent isomorphism torsion structure F^{can} on $\mathbb{B}(A)$ induces*

$$F^{\mathrm{can}} = 0\colon\ K_1(\mathbb{B}(A)) = K_1(A) \to K_1(A).$$

(ii) *The F^{can}-relative torsion*

$$\begin{aligned}
&\tau^{F^{\mathrm{can}}}\colon\ K_1^{\mathrm{iso}}(\mathbb{B}(A)) \to K_1(\mathbb{B}(A)) = K_1(A);\\
&\tau^{\mathrm{iso}}(f\colon\ M \to N) \mapsto \tau^{F^{\mathrm{can}}}(f) = \tau(F^{\mathrm{can}}(f)^{-1}f\colon\ M \to M)
\end{aligned}$$

is the standard way of assigning a torsion to an isomorphism of based f.g. free A-modules, defining a surjection splitting the forgetful map $K_1(\mathbb{B}(A)) \to K_1^{\mathrm{iso}}(\mathbb{B}(A))$.

(iii) *The F^{can}-relative torsion of a chain equivalence $f\colon\ D \to E$ of finite chain complexes in $\mathbb{B}(A)$ is the usual torsion $\tau(f) \in K_1(A)$.*

Given a ring morphism $F\colon\ A \to B$ regard B as a (B, A)-bimodule by

$$B \times B \times A \to B;\quad (b, x, a) \mapsto b.x.F(a).$$

As usual, F induces a functor

$$F\colon\ \{A\text{-modules}\} \to \{B\text{-modules}\},\ M \mapsto B \otimes_A M$$

which in turn induces morphisms of the algebraic K-groups

$$F\colon\ K_i(A) \to K_i(B)\quad (i = 0, 1).$$

PROPOSITION 1.8. (i) *A ring endomorphism $F\colon\ A \to A$ determines an isomorphism torsion structure*

$$F\colon\ \mathrm{Iso}(\mathbb{B}(A)) \to \mathrm{Iso}(\mathbb{B}(A))$$

and hence a relative F-torsion

$$\tau^F : K_1^{\text{iso}}(\mathbb{B}(A)) \to K_1(A);$$
$$\tau^{\text{iso}}(f : M \to N) \mapsto \tau^F(f) = \tau(F(f)^{-1}f : M \to M).$$

(ii) *If $F = F^2 : A \to A$ the isomorphism torsion structure is idempotent and*

$$F^2 = F : K_1(A) \to K_1(A),$$
$$\begin{aligned}\text{im}(\tau^F : K_1^{\text{iso}}(\mathbb{B}(A)) \to K_1(A)) &= \text{im}(1 - F : K_1(A) \to K_1(A)) \\ &= \ker(F : K_1(A) \to K_1(A)),\end{aligned}$$
$$K_1(A) = \text{im}(F : K_1(A) \to K_1(A)) \oplus \text{im}(\tau^F).$$

DEFINITION 1.9 ([R1], p. 211). The *isomorphism torsion* of a contractible finite chain complex C in an additive category $\mathbb{A}$ is

$$\tau^{\text{iso}}(C) = \tau^{\text{iso}}(d + \Gamma : C_{\text{odd}} \to C_{\text{even}}) \in K_1^{\text{iso}}(\mathbb{A})$$

with $d + \Gamma$ the isomorphism in $\mathbb{A}$ defined for any chain contraction $\Gamma : 0 \simeq 1 : C \to C$ by

$$d + \Gamma = \begin{pmatrix} d & 0 & 0 & \dots \\ \Gamma & d & 0 & \dots \\ 0 & \Gamma & d & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix} : C_{\text{odd}} = C_1 \oplus C_3 \oplus C_5 \oplus \cdots \to C_{\text{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \cdots .$$

A chain map $f : D \to E$ of finite chain complexes in $\mathbb{A}$ is a chain equivalence if and only if the algebraic mapping cone $C(f)$ is contractible.

Remark 1.10. We refer to Section 4 of [R1] for the precise definition of the isomorphism torsion of a chain equivalence $f : D \to E$ of finite chain complexes in $\mathbb{A}$, which involves the *sign* pairing

$$\epsilon : K_0(\mathbb{A}) \times K_0(\mathbb{A}) \to K_1^{\text{iso}}(\mathbb{A}),$$
$$([M], [N]) \mapsto \epsilon(M, N) = \tau^{\text{iso}}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : M \oplus N \to N \oplus M).$$

The isomorphism torsion of a chain equivalence f is of the form

$$\tau^{\text{iso}}(f) = \tau^{\text{iso}}(C(f)) + \epsilon(M, N) \in K_1^{\text{iso}}(\mathbb{A}) \qquad (*)$$

with M, N objects in $\mathbb{A}$. The isomorphism torsion depends only on the chain homotopy class of f. The sign term in $(*)$ is necessary in order to ensure that the isomorphism torsion of the composite of chain equivalences $f : D \to E$, $g : E \to F$ have the logarithmic property

$$\tau^{\text{iso}}(gf) = \tau^{\text{iso}}(f) + \tau^{\text{iso}}(g) \in K_1^{\text{iso}}(\mathbb{A}).$$

For any isomorphism torsion structure F on $\mathbb{A}$ and any objects M, N in $\mathbb{A}$

$$F\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}:\ M \oplus N \to N \oplus M.$$

Thus the F-torsion of a sign is

$$\tau^F(\epsilon(M, N)) = 0 \in K_1(\mathbb{A})$$

and the sign term in $(*)$ can be ignored for F-torsion purposes.

2. The Main Theorem

The terminology used to deal with polynomial extensions is developed in 2.1. In Sections 2.2, 2.3, 2.4 we recall the splitting theorems for the Whitehead group of a twisted polynomial extension $A_\rho[z]$, the power series ring $A_\rho[[z]]$ and the Laurent ring $A_\rho[z, z^{-1}]$, paving the way to the proof in Section 2.5 of the Main Theorem on the Whitehead group of the Novikov ring $A_\rho((z))$.

2.1. MODULES, MATRICES, POLYNOMIAL EXTENSIONS, ETC.

We shall work with left modules over rings. Let A be a ring with a unit (non-commutative in general). An A-module morphism of finite direct sums of A-modules

$$f:\ M_1 \oplus M_2 \oplus \cdots \oplus M_q \to N_1 \oplus N_2 \oplus \cdots \oplus N_p$$

can be identified with the $p \times q$ matrix $f = (f_{ij})_{1 \leqslant i \leqslant p, 1 \leqslant j \leqslant q}$ with entries A-module morphisms $f_{ij}:\ M_j \to N_i$, such that

$$f(x_1, x_2, \ldots, x_q) = \left(\sum_{j=1}^{q} f_{1j}(x_j), \sum_{j=1}^{q} f_{2j}(x_j), \ldots, \sum_{j=1}^{q} f_{pj}(x_j)\right).$$

An A-module morphism $A \to A$ can be identified with an element of A, using the isomorphism of additive groups

$$A \to \mathrm{Hom}_A(A, A), \qquad a \mapsto (x \mapsto xa).$$

Thus for $M_1 = M_2 = \cdots = M_q = N_1 = N_2 = \cdots = N_p = A$ we can identify a morphism of f.g. free A-modules $f:\ A^q \to A^p$ with a $p \times q$ matrix (f_{ij}) with entries

$$f_{ij} \in \mathrm{Hom}_A(A, A) = A.$$

We shall be working with the twisted polynomial ring $A_\rho[z]$ and its extensions $A_\rho[[z]]$, $A_\rho((z))$, which are defined for an automorphism $\rho:\ A \to A$ of the ring A and an indeterminate z over A such that $az = z\rho(a)$ for $a \in A$, as in the Introduction.

A ρ-*morphism* of A-modules M, N is a morphism $f\colon M \to N$ of the underlying additive groups such that

$$f(ax) = \rho(a)f(x) \quad (a \in A).$$

The composite of ρ-morphisms $f\colon M \to N$, $g\colon N \to P$ is a ρ^2-morphism $gf\colon M \to P$.

For any A-module M and $k \in \mathbb{Z}$ let $z^k M$ be the A-module with elements $z^k x$ $(x \in M)$ and

$$z^k x + z^k y = z^k(x+y),\; a(z^k x) = z^k(\rho^k(a)x) \quad (x, y \in M, a \in A).$$

The function

$$\zeta^k\colon M \to z^k M; x \mapsto z^k x$$

is a ρ^{-k}-isomorphism.

A ρ^k-morphism $f\colon M \to N$ of A-modules determines an A-module morphism

$$z^k f\colon M \to z^k N; x \mapsto z^k f(x)$$

and every A-module morphism $M \to z^k N$ is of this form. Thus there is no essential difference between ρ^k-morphisms $M \to N$ and A-module morphisms $M \to z^k N$.

For any f.g. free A-module A^n there is a defined ρ-isomorphism

$$\begin{aligned}\theta_n \;=\; &\rho \oplus \rho \cdots \oplus \rho\colon A^n \to A^n,\\ &(a_1, a_2, \ldots, a_n) \mapsto (\rho(a_1), \rho(a_2), \ldots, \rho(a_n))\end{aligned}$$

or equivalently an A-module isomorphism

$$z\theta_n\colon A^n \to zA^n; (a_1, a_2, \ldots, a_n) \mapsto z(\rho(a_1), \rho(a_2), \ldots, \rho(a_n)).$$

For any automorphism $\rho\colon A \to A$ and $k \in \mathbb{Z}$ the ring morphism $\rho^k\colon A \to A$ induces a functor

$$(\rho^k)_!\colon \{A\text{-modules}\} \to \{A\text{-modules}\}; M \mapsto (\rho^k)_! M$$

with a natural A-module isomorphism

$$(\rho^k)_! M \to z^k M, (a, x) \mapsto z^k ax.$$

For the inclusion $i\colon A \to A_\rho[z]$ we write the induced $A_\rho[z]$-module as

$$A_\rho[z] \otimes_A M = M_\rho[z] = \sum_{j=0}^{\infty} z^j M.$$

For any A-module M

$$A_\rho[z, z^{-1}] \otimes_A M = M_\rho[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j M$$

and for a f.g. A-module M

$$A_\rho[[z]] \otimes_A M = M_\rho[[z]] = \prod_{j=0}^{\infty} z^j M,$$
$$A_\rho((z)) \otimes_A M = M_\rho((z)) = \sum_{j=-\infty}^{-1} z^j M \oplus \prod_{k=0}^{\infty} z^k M.$$

DEFINITION 2.1. Let R be one of the rings

$A_\rho[z]$, $A_\rho[[z]]$, $A_\rho[z, z^{-1}]$, $A_\rho((z))$.

(i) A f.g. projective R-module is A-*induced* if it is of the form $R \otimes_A P$ for a f.g. projective A-module P.
(ii) Let $\mathbb{P}_A(R)$ be the additive category of A-induced f.g. projective R-modules.

In particular, every f.g. free R-module is A-induced from a f.g. free A-module

$R^n = R \otimes_A A^n$.

If P is a f.g. projective A-module and Q is any A-module it is possible to write every $A_\rho[z]$-module morphism $f\colon P_\rho[z] \to Q_\rho[z]$ as a polynomial

$$f = \sum_{j=0}^{\infty} z^j f_j\colon P_\rho[z] = \sum_{k=0}^{\infty} z^k P \to Q_\rho[z] = \sum_{\ell=0}^{\infty} z^\ell Q;$$
$$\sum_{k=0}^{\infty} z^k x_k \mapsto \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} z^{j+k} f_j(x_k)$$

with coefficients ρ^j-morphisms $f_j\colon P \to Q$ such that $\{j \geqslant 0 \mid f_j \neq 0\}$ is finite. Similarly for $A_\rho[[z]]$, $A_\rho[z, z^{-1}]$, $A_\rho((z))$.

PROPOSITION 2.2. *Let R, $\mathbb{P}_A(R)$ be as in Definition 2.1. The forgetful functor*

$\mathbb{P}_A(R) \to \mathbb{P}(R)$

from the additive category of A-induced f.g. projective R-modules to the additive category of f.g. projective R-modules induces an isomorphism of Whitehead groups

$K_1(\mathbb{P}_A(R)) \cong K_1(\mathbb{P}(R)) = K_1(R)$.

Proof. For any f.g. projective A-modules P, Q

$$\mathrm{Hom}_{\mathbb{P}_A(R)}(R \otimes_A P, R \otimes_A Q) = \mathrm{Hom}_R(R \otimes_A P, R \otimes_A Q). \qquad \square$$

The forgetful functor $\mathbb{P}_A(R) \to \mathbb{P}(R)$ is an equivalence of additive categories, but the induced morphism $K_1^{\mathrm{iso}}(\mathbb{P}_A(R)) \to K_1^{\mathrm{iso}}(\mathbb{P}(R))$ is not an isomorphism (cf. Remark 1.3).

PROPOSITION 2.3. *Let R be one of the rings*

$$A_\rho[z],\ A_\rho[[z]].$$

(i) *The inclusion $A \to R$ is split by the augmentation map*

$$R \to A;\ \sum_{j=0}^{\infty} a_j z^j \mapsto a_0.$$

(ii) *The composite of the augmentation and the inclusion is an idempotent endomorphism*

$$F\colon\ R \to R;\ \sum_{j=0}^{\infty} a_j z^j \mapsto a_0.$$

(iii) *The Whitehead group of R splits as*

$$K_1(R) = K_1(A) \oplus NK_1(R)$$

with

$$\begin{aligned} NK_1(R) &= \ker(K_1(R) \to K_1(A)) \\ &= \operatorname{im}(1 - F\colon\ K_1(R) \to K_1(R)) \\ &= \ker(F\colon\ K_1(R) \to K_1(R)). \end{aligned}$$

(iv) *F determines an idempotent isomorphism structure*

$$F\colon\ \operatorname{Iso}(\mathbb{P}_A(R)) \to \operatorname{Iso}(\mathbb{P}_A(R))$$

with a relative F-torsion

$$\begin{aligned} &\tau^F\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(R)) \to NK_1(R), \\ &\tau^{\mathrm{iso}}(f\colon\ M \to N) \mapsto \tau^F(f) = \tau(F(f)^{-1} f\colon\ M \to M). \end{aligned}$$

Proof. (i), (ii), (iii) Clear.
(iv) Every morphism in $\mathbb{P}_A(R)$ is of the form

$$f = \sum_{j=0}^{\infty} f_j z^j\colon\ M = R \otimes_A P \to N = R \otimes_A Q$$

for some f.g. projective A-modules P, Q and ρ^j-morphisms $f_j\colon\ P \to Q$. If f is an isomorphism then so is

$$F(f) = f_0\colon\ M = R \otimes_A P \to N = R \otimes_A Q.$$

The idempotent isomorphism structure is defined by

$$F\colon\ \operatorname{Iso}(\mathbb{P}_A(R)) \to \operatorname{Iso}(\mathbb{P}_A(R));\ \tau(f\colon\ M \to N) \mapsto \tau(f_0\colon\ M \to N). \qquad \square$$

In the situation of Proposition 2.3 we shall write the natural isomorphism as

$$B = B_1 \oplus B_2 \colon\ K_1(R) \to K_1(A) \oplus NK_1(R)$$

and the inverse isomorphism as

$$B^{-1} = C = C_1 \oplus C_2 \colon\ K_1(A) \oplus NK_1(R) \to K_1(R).$$

The components

$$B_1 \colon\ K_1(R) \to K_1(A), \qquad C_1 \colon\ K_1(A) \to K_1(R)$$

are induced by the augmentation $R \to A; z \mapsto 0$ and the inclusion $A \to R$, with

$$\begin{aligned} B_1C_1 &= 1 \colon\ K_1(A) \to K_1(A), \\ C_1B_1 &= F \colon\ K_1(R) \to K_1(R). \end{aligned}$$

The components

$$B_2 \colon\ K_1(R) \to NK_1(R), \qquad C_2 \colon\ NK_1(R) \to K_1(R)$$

are the natural surjection and injection, with

$$\begin{aligned} B_2C_2 &= 1 \colon\ NK_1(R) \to NK_1(R), \\ C_2B_2 &= 1 - F \colon\ K_1(R) \to K_1(R), \\ B_2C_1 &= 0 \colon\ K_1(A) \to NK_1(R), \\ B_1C_2 &= 0 \colon\ NK_1(R) \to K_1(A). \end{aligned}$$

2.2. THE WHITEHEAD GROUP OF $A_\rho[z]$

By Proposition 2.3 we have

$$K_1(A_\rho[z]) = K_1(A) \oplus NK_1(A_\rho[z])$$

with

$$\begin{aligned} NK_1(A_\rho[z]) &= \ker(K_1(A_\rho[z]) \to K_1(A)) \\ &= \operatorname{im}(1 - F \colon\ K_1(A_\rho[z]) \to K_1(A_\rho[z])) \\ &= \ker(F \colon\ K_1(A_\rho[z]) \to K_1(A_\rho[z])), \end{aligned}$$

where

$$F \colon\ A_\rho[z] \to A_\rho[z]; \sum_{j=0}^{\infty} a_j z^j \mapsto a_0.$$

We shall now recall the identification of $NK_1(A_\rho[z])$ with the reduced ρ-nilpotent class group $\widetilde{\mathrm{Nil}}_0(A, \rho)$.

A ρ-endomorphism $\nu\colon P \to P$ is *nilpotent* if for some $k \geqslant 1$

$$\nu^k = 0\colon P \to P.$$

PROPOSITION 2.4. (i) *A linear morphism of A-induced f.g. projective $A_\rho[z]$-modules*

$$\alpha = \alpha_0 + z\alpha_1\colon Q_\rho[z] \to R_\rho[z]$$

is an isomorphism if and only if $\alpha_0\colon Q \to R$ is an A-module isomorphism and $(\alpha_0)^{-1}\alpha_1\colon Q \to Q$ is a nilpotent ρ-endomorphism.

(ii) (*Higman linearization trick*). *Every element of $K_1(A_\rho[z])$ is the torsion $\tau(\alpha)$ of a linear automorphism of a f.g. free $A_\rho[z]$-module.*

The *nilpotent class group* $\mathrm{Nil}_0(A, \rho)$ is the class group of the exact category $\mathrm{Nil}(A, \rho)$ with objects

$$(P = \text{f.g. projective } A\text{-module}, \nu\colon P \to P \text{ nilpotent } \rho\text{-endomorphism}),$$

that is

$$\mathrm{Nil}_0(A, \rho) = K_0(\mathrm{Nil}(A, \rho)).$$

The *reduced nilpotent class group* $\widetilde{\mathrm{Nil}}_0(A, \rho)$ is defined to be

$$\widetilde{\mathrm{Nil}}_0(A, \rho) = \mathrm{coker}(K_0(A) \to \mathrm{Nil}_0(A, \rho))$$

with

$$K_0(A) \to \mathrm{Nil}_0(A, \rho);\quad [P] \mapsto [P, 0],$$

and is such that

$$\mathrm{Nil}_0(A, \rho) = \widetilde{\mathrm{Nil}}_0(A, \rho) \oplus K_0(A).$$

It is also possible to view $\widetilde{\mathrm{Nil}}_0(A, \rho)$ as

$$\widetilde{\mathrm{Nil}}_0(A, \rho) = \mathrm{coker}(K_0(\mathbb{Z}) \to K_0(\widetilde{\mathrm{Nil}}(A, \rho)))$$

with $\widetilde{\mathrm{Nil}}(A, \rho)$ the full subcategory of $\mathrm{Nil}(A, \rho)$ with objects (F, ν) such that F is a f.g. free A-module, and

$$K_0(\mathbb{Z}) = \mathbb{Z} \to K_0(\widetilde{\mathrm{Nil}}(A, \rho)); [\mathbb{Z}^n] \mapsto [A^n, 0].$$

THEOREM 2.5 ([BHS], [B], [FH]). *The function*

$$\widetilde{\mathrm{Nil}}_0(A, \rho) \to NK_1(A_\rho[z]);\quad [P, \nu] \mapsto \tau(1 - z\nu\colon P_\rho[z] \to P_\rho[z])$$

is an isomorphism.

The following construction of nilpotent classes from isomorphisms of A-induced f.g. projective $A_\rho[z]$-modules will be generalized in Section 2.3 to a construction of Witt vector classes from isomorphisms of A-induced f.g. projective $A_\rho[[z]]$-modules.

PROPOSITION 2.6. (i) *An isomorphism of A-induced f.g. projective $A_\rho[z]$-modules*

$$\alpha = \sum_{j=0}^{k} z^j \alpha_j :\ Q_\rho[z] \to R_\rho[z]$$

determines an object (P, ν) in $\mathrm{Nil}(A, \rho)$, with

$$P = \mathrm{coker}(\alpha| :\ z^{-k-1} Q_\rho[z^{-1}] \to z^{-1} R_\rho[z^{-1}]),$$
$$\nu :\ P \to P, [x] \mapsto [zx].$$

(ii) *For a linear isomorphism*

$$\alpha = \alpha_0 + z\alpha_1 :\ Q_\rho[z] \to R_\rho[z]$$

the object in (i) *is given up to isomorphism by*

$$(P, \nu) = (Q, -(\alpha_0)^{-1}\alpha_1).$$

(iii) *The torsion of an automorphism of an A-induced f.g. projective $A_\rho[z]$-module*

$$\alpha = \sum_{j=0}^{k} z^j \alpha_j :\ Q_\rho[z] \to Q_\rho[z]$$

splits as

$$\tau(\alpha) = \tau(\alpha_0) + \tau(1 - z\nu :\ P_\rho[z] \to P_\rho[z]) \in K_1(A_\rho[z])$$

with (P, ν) as in (i).

(iv) *The isomorphism*

$$B = B_1 \oplus B_2 :\ K_1(A_\rho[z]) \to K_1(A) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho)$$

has components

$$B_1 :\ K_1(A_\rho[z]) \to K_1(A);\quad \tau\left(\sum_{j=0}^{\infty} \alpha_j z^j :\ A_\rho[z]^n \to A_\rho[z]^n\right) \mapsto \tau(\alpha_0 :\ A^n \to A^n),$$

$$B_2 :\ K_1(A_\rho[z]) \to \widetilde{\mathrm{Nil}}_0(A, \rho), \tau\left(\sum_{j=0}^{k} z^j \alpha_j :\ A_\rho[z]^n \to A_\rho[z]^n\right) \mapsto [P, \nu]$$

and the inverse isomorphism

$$B^{-1} = C = C_1 \oplus C_2 \colon\ K_1(A) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho) \to K_1(A_\rho[z])$$

has components

$$\begin{aligned} &C_1\colon\ K_1(A) \to K_1(A_\rho[z]);\ \ \tau(\alpha_0\colon\ A^n \to A^n) \mapsto \tau(\alpha_0\colon\ A_\rho[z]^n \to A_\rho[z]^n),\\ &C_2\colon\ \widetilde{\mathrm{Nil}}_0(A, \rho) \to K_1(A_\rho[z]);\ \ [P, \nu] \mapsto \tau(1 - z\nu\colon\ P_\rho[z] \to P_\rho[z]).\end{aligned}$$

(v) *For* $F\colon\ A_\rho[z] \to A_\rho[z], z \mapsto z$ *the relative* F*-torsion function of 2.3*

$$\tau^F\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[z])) \to NK_1(A_\rho[z]) = \widetilde{\mathrm{Nil}}_0(A, \rho)$$

sends the isomorphism torsion of an isomorphism of A*-induced f.g. projective* $A_\rho[z]$*-modules*

$$\alpha = \sum_{j=0}^{k} z^j \alpha_j\colon\ Q_\rho[z] \to R_\rho[z]$$

to the class of the nilpotent object (P, ν) *in* (i)

$$\tau^F(\alpha) = [P, \nu] \in \widetilde{\mathrm{Nil}}_0(A, \rho).$$

Proof. This is standard except for the isomorphism torsion interpretation in (v), which is just an interpretation of (i)–(iv) in terms of the material in Section 1. See Proposition 2.14 for a detailed proof that the A-module P in (i) is f.g. projective, and that $\nu\colon\ P \to P$ is a nilpotent ρ-morphism. □

DEFINITION 2.7. The *nilpotent torsion* is the group morphism given by the relative F-torsion construction of Proposition 2.6 (v)

$$\nu = \tau^F\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[z])) \to \widetilde{\mathrm{Nil}}_0(A, \rho);\ \ \tau^{\mathrm{iso}}(\alpha) \mapsto [P, \nu].$$

The split surjection B_2 in Proposition 2.6 (iv) is the composite

$$B_2\colon\ K_1(A_\rho[z]) \to K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[z])) \xrightarrow{\ \nu\ } \widetilde{\mathrm{Nil}}_0(A, \rho).$$

Remark 2.8 In view of 1.10, it is also possible to define the nilpotent torsion $\nu(f) \in \widetilde{\mathrm{Nil}}_0(A, \rho)$ of a chain equivalence $f\colon\ D \to E$ of A-induced f.g. projective $A_\rho[z]$-module chain complexes, which depends only on the chain homotopy class of f and has the logarithmic property $\nu(fg) = \nu(f) + \nu(g)$.

2.3. THE WHITEHEAD GROUP OF $A_\rho[[z]]$

The splitting of $K_1(A_\rho[z])$ in Section 2.2 is now extended to $K_1(A_\rho[[z]])$. However, there is an essential difference between the decompositions

$$\begin{aligned}K_1(A_\rho[z]) &= K_1(A) \oplus NK_1(A_\rho[z]),\\ K_1(A_\rho[[z]]) &= K_1(A) \oplus NK_1(A_\rho[[z]])\end{aligned}$$

in that there is no analogue for $K_1(A_\rho[[z]])$ of Higman linearization for $K_1(A_\rho[z])$.

As in Proposition 2.1 we have

$$\begin{aligned}NK_1(A_\rho[[z]]) &= \ker(K_1(A_\rho[[z]]) \to K_1(A))\\ &= \mathrm{im}(1 - F\colon\ K_1(A_\rho[[z]]) \to K_1(A_\rho[[z]]))\\ &= \ker(F\colon\ K_1(A_\rho[[z]]) \to K_1(A_\rho[[z]]))\end{aligned}$$

where

$$F\colon\ A_\rho[[z]] \to A_\rho[[z]];\ \sum_{j=0}^{\infty} a_j z^j \mapsto a_0.$$

PROPOSITION 2.9. *An $A_\rho[[z]]$-module morphism of A-induced f.g. projective $A_\rho[[z]]$-modules*

$$\alpha = \sum_{j=0}^{\infty} z^j \alpha_j\colon\ P_\rho[[z]] \to Q_\rho[[z]]$$

is an isomorphism if and only if $\alpha_0\colon\ P \to Q$ is an A-module isomorphism.

DEFINITION 2.10. (i) A *Witt vector* in $A_\rho[[z]]$ is a unit of the type

$$w = 1 + \sum_{j=1}^{\infty} a_j z^j \in A_\rho[[z]]^\bullet.$$

(ii) Let $W_1(A, \rho) \subseteq K_1(A_\rho[[z]])$ be the subgroup of the torsions $\tau(w)$ of Witt vectors w.

PROPOSITION 2.11 ([P1]).

$$NK_1(A_\rho[[z]]) = W_1(A, \rho) \subseteq K_1(A_\rho[[z]])).$$

Proof. By Proposition 2.3 an endomorphism of a f.g. free $A_\rho[[z]]$-module of rank n

$$\alpha = \sum_{i=0}^{\infty} \alpha_i z^i\colon\ A_\rho[[z]]^n \to A_\rho[[z]]^n$$

is an automorphism if and only if $\alpha_0\colon\ A^n \to A^n$ is an A-module automorphism, in which case the torsion of α is the sum

$$\tau(\alpha) = \tau(\alpha_0) + \tau(\beta) \in K_1(A_\rho[[z]])$$

with

$$\beta = (\alpha_0)^{-1}\alpha = 1 + \sum_{i=1}^{\infty} \beta_i z^i\colon\ A_\rho[[z]]^n \to A_\rho[[z]]^n.$$

The diagonal entries in the matrix $\beta = (\beta_{jk})$ are Witt vectors

$$\beta_{jj} \in 1 + zA_\rho[[z]] \subset A_\rho[[z]]^\bullet\ (1 \leqslant j \leqslant n)$$

so that β can be reduced by elementary row operations to an upper triangular matrix with diagonal entries Witt vectors

$$\gamma = \begin{pmatrix} w_1 & * & \dots & * \\ 0 & w_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_n \end{pmatrix}\colon\ A_\rho[[z]]^n \to A_\rho[[z]]^n.$$

Thus

$$\begin{aligned} \tau(\alpha) &= \tau(\alpha_0) + \tau(\beta) \\ &= \tau(\alpha_0) + \tau(\gamma) = \tau(\alpha_0) + \sum_{j=1}^{n} \tau(w_j) \\ &\in K_1(A_\rho[[z]]). \end{aligned}$$

In this terminology

$$\begin{aligned} B_1(\tau(\alpha)) &= \tau(\alpha_0) \in K_1(A), \\ B_2(\tau(\alpha)) &= \tau((\alpha_0)^{-1}\alpha) = \sum_{j=1}^{n} \tau(w_j) \in W_1(A,\rho). \end{aligned}$$ □

The proof of the Main Theorem will make use of the following construction of elements in $W_1(A,\rho)$, analogous to the construction 2.6 of elements in $\widetilde{\mathrm{Nil}}_0(A,\rho)$.

DEFINITION 2.12. The *Witt vector torsion* is the group morphism given by

$$w\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[[z]])) \to W_1(A,\rho);\ \tau^{\mathrm{iso}}(\alpha) \mapsto w(\alpha) = \tau((\alpha_0)^{-1}\alpha).$$

Equivalently, this is the relative F-torsion of Proposition 2.3

$$\tau^F\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[[z]])) \to NK_1(A_\rho[[z]])) = W_1(A,\rho)$$

with $F\colon\ A_\rho[[z]] \to A_\rho[[z]];\ z \mapsto 0.$

Remark 2.13. In view of 1.10, it is also possible to define the Witt vector torsion $w(f) \in W_1(A, \rho)$ of a chain equivalence $f\colon\ D \to E$ of A-induced f.g. projective $A_\rho[[z]]$-module chain complexes, which depends only on the chain homotopy class of f and has the logarithmic property $w(fg) = w(f) + w(g)$.

2.4. THE WHITEHEAD GROUP OF $A_\rho[z, z^{-1}]$

The aim of this section is to describe the decomposition of $K_1(A_\rho[z, z^{-1}])$ in a way which will serve as a model for our main result on the decomposition of $K_1(A_\rho((z)))$.

PROPOSITION 2.14. (i) *An isomorphism of A-induced f.g. projective $A_\rho[z, z^{-1}]$-modules*

$$\alpha = \sum_{j=-k_+}^{k_-} z^j \alpha_j\colon\ Q_\rho[z, z^{-1}] \to R_\rho[z, z^{-1}]$$

determines an object (P_+, ν_+) in $\mathrm{Nil}(A, \rho^{-1})$ *and an object (P_-, ν_-) in* $\mathrm{Nil}(A, \rho)$*, with*

$$\begin{aligned}
P_+ &= \mathrm{coker}(\alpha_+\colon\ z^{k_+} Q_\rho[z] \to R_\rho[z]),\\
\nu_+&\colon\ P_+ \to P_+;\quad [x] \mapsto [z^{-1}x],\\
P_- &= \mathrm{coker}(\alpha_-\colon\ z^{-k_- -1} Q_\rho[z^{-1}] \to z^{-1} R_\rho[z^{-1}]),\\
\nu_-&\colon\ P_- \to P_-;\quad [x] \mapsto [zx],
\end{aligned}$$

with α_+, α_- restrictions of α.

(ii) *The constructions of* (i) *define group morphisms*

$$\begin{aligned}
\nu_+&\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[z, z^{-1}])) \to \widetilde{\mathrm{Nil}}_0(A, \rho^{-1});\quad \tau^{\mathrm{iso}}(\alpha) \mapsto [P_+, \nu_+],\\
\nu_-&\colon\ K_1^{\mathrm{iso}}(\mathbb{P}_A(A_\rho[z, z^{-1}])) \to \widetilde{\mathrm{Nil}}_0(A, \rho);\quad \tau^{\mathrm{iso}}(\alpha) \mapsto [P_-, \nu_-].
\end{aligned}$$

Proof. (i) It is clear that P_+ is a f.g. $A_\rho[z]$-module, and that there is defined an A-induced f.g. projective $A_\rho[z]$-module resolution

$$0 \longrightarrow z^{k_+} Q_\rho[z] \xrightarrow{\alpha_+} R_\rho[z] \xrightarrow{\pi_+} P_+ \longrightarrow 0$$

with

$$\pi_+ = \text{projection}\colon\ R_\rho[z] \to P_+,$$

and similarly for the $A_\rho[z^{-1}]$-module P_- with a resolution

$$0 \longrightarrow z^{-k_- -1} Q_\rho[z^{-1}] \xrightarrow{\alpha_-} z^{-1} R_\rho[z^{-1}] \xrightarrow{\pi_-} P_- \longrightarrow 0.$$

It follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & z^{k_+}Q_\rho[z]\oplus z^{-k_--1}Q_\rho[z^{-1}] & \longrightarrow & Q_\rho[z,z^{-1}] & \longrightarrow & \sum\limits_{j=-k_-}^{k_+-1} z^jQ & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \alpha} & & \Big\downarrow & & \\ 0 & \longrightarrow & z^{k_+}Q_\rho[z]\oplus z^{-k_--1}Q_\rho[z^{-1}] & \xrightarrow{\alpha_+\oplus\alpha_-} & R_\rho[z,z^{-1}] & \longrightarrow & P_+\oplus P_- & \longrightarrow & 0 \end{array}$$

that there is defined an A-module isomorphism

$$\sum_{j=-k_-}^{k_+-1} z^jQ \;\cong\; P_+\oplus P_-,$$

so that P_+, P_- are f.g. projective A-modules. The ρ^{-1}-endomorphism

$$z\colon\ R_\rho[z]\to R_\rho[z];\, x\mapsto zx$$

fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & z^{k_+}Q_\rho[z] & \xrightarrow{\alpha_+} & R_\rho[z] & \xrightarrow{\pi_+} & P_+ & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle z} & & \Big\downarrow{\scriptstyle z} & & \Big\downarrow{\scriptstyle \nu_+} & & \\ 0 & \longrightarrow & z^{k_+}Q_\rho[z] & \xrightarrow{\alpha_+} & R_\rho[z] & \xrightarrow{\pi_+} & P_+ & \longrightarrow & 0 \end{array}$$

Write the inverse of α as

$$\alpha^{-1}=\beta=\sum_{j=-\ell_+}^{\ell_-} z^j\beta_j\colon\ R_\rho[z,z^{-1}]\to\ Q_\rho[z,z^{-1}]$$

for some $\ell_+,\ell_-\geqslant 0$. For any $x\in R_\rho[z]$

$$z^{k_++\ell_+}x=\alpha\beta(z^{k_++\ell_+}x)=\alpha_+(z^{k_+}\beta(z^{\ell_+}x))\in \mathrm{im}(\alpha_+\colon\ z^{k_+}Q_\rho[z]\to R_\rho[z])$$

so that

$$(\nu_+)^{k_++\ell_+}\pi_+(x)=0\in P_+.$$

Thus $\nu_+\colon\ P_+\to P_+$ is nilpotent, with

$$(\nu_+)^{k_++\ell_+}=0\colon\ P_+\to P_+.$$

Similarly, the ρ-endomorphism

$$z^{-1}\colon\ z^{-1}R_\rho[z^{-1}]\to z^{-1}R_\rho[z^{-1}];\quad x\mapsto z^{-1}x$$

fits into the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & z^{-k_--1}Q_\rho[z^{-1}] & \xrightarrow{\alpha_-} & z^{-1}R_\rho[z^{-1}] & \xrightarrow{\pi_-} & P_- & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z^{-1}} & & \downarrow{\scriptstyle z^{-1}} & & \downarrow{\scriptstyle \nu_-} & & \\
0 & \longrightarrow & z^{-k_--1}Q_\rho[z^{-1}] & \xrightarrow{\alpha_-} & z^{-1}R_\rho[z^{-1}] & \xrightarrow{\pi_-} & P_- & \longrightarrow & 0
\end{array}$$

and

$$(\nu_-)^{k_-+\ell_-} = 0\colon\ P_- \to P_-.$$

(ii) The composite of isomorphisms of A-induced f.g. projective $A_\rho[z]$-modules

$$\alpha = \sum_{j=-k_-}^{k_+} z^j\alpha_j\colon\ Q_\rho[z,z^{-1}] \to R_\rho[z,z^{-1}],$$

$$\alpha' = \sum_{j=-k'_-}^{k'_+} z^j\alpha'_j\colon\ R_\rho[z,z^{-1}] \to S_\rho[z,z^{-1}]$$

is an isomorphism of A-induced f.g. projective $A_\rho[z,z^{-1}]$-modules

$$\alpha'' = \alpha'\alpha = \sum_{j=-k_--k'_-}^{k_++k'_+} z^j\alpha''_j\colon\ Q_\rho[z,z^{-1}] \to S_\rho[z,z^{-1}]$$

such that the corresponding nilpotent objects fit into exact sequences

$$0 \to (P_+,\nu_+) \to (P''_+,\nu''_+) \to (P'_+,\nu'_+) \to 0,$$
$$0 \to (P_-,\nu_-) \to (P''_-,\nu''_-) \to (P'_-,\nu'_-) \to 0. \qquad \square$$

DEFINITION 2.15. (i) The *automorphism class group* of A,ρ is the class group

$$\mathrm{Aut}_0(A,\rho) = K_0(\mathrm{Aut}(A,\rho))$$

of the exact category $\mathrm{Aut}(A,\rho)$ of pairs (P,ϕ) with P a f.g. projective A-module and $\phi\colon\ P \to P$ a ρ-isomorphism.
(ii) ([S]). The *class-torsion group* $K_1(A,\rho)$ is the quotient of $\mathrm{Aut}_0(A,\rho)$ by the subgroup generated by the differences $(P,\phi)-(P',\phi')$ for which there exists an isomorphism $h\colon\ P \to P'$ such that

$$\tau(h^{-1}\phi'^{-1}(zh)\phi\colon\ P \to zP \to zP' \to P' \to P) = 0 \in K_1(A).$$

The group $K_1(A, \rho)$ fits into the long exact sequence

$$K_1(A) \xrightarrow{1-\rho} K_1(A) \xrightarrow{i} K_1(A, \rho) \xrightarrow{j} K_0(A) \xrightarrow{1-\rho} K_0(A)$$

with

$$i\colon\ K_1(A) \to K_1(A, \rho);\quad \tau(\alpha\colon\ A^n \to A^n) \mapsto [A^n, \theta_n \alpha] - [A^n, \theta_n],$$
$$j\colon\ K_1(A, \rho) \to K_0(A);\quad [P, \phi] \mapsto [P].$$

THEOREM 2.16 ([B], [FH], [S]). *There is a natural decomposition*

$$K_1(A_\rho[z, z^{-1}]) = K_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1})$$

with the map

$$C = C_1 \oplus C_2 \oplus C_3\colon$$
$$K_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}) \to K_1(A_\rho[z, z^{-1}])$$

defined by

$$C_1\colon\ K_1(A, \rho) \to K_1(A_\rho[z, z^{-1}]);$$
$$(P, \phi) \mapsto \tau(z\phi\colon\ P_\rho[z, z^{-1}] \to P_\rho[z, z^{-1}]),$$
$$C_2\colon\ \widetilde{\mathrm{Nil}}_0(A, \rho) \to K_1(A_\rho[z, z^{-1}]);$$
$$(P_+, \nu_+,) \mapsto \tau(1 - z\nu_+\colon\ (P_+)_\rho[z, z^{-1}] \to (P_+)_\rho[z, z^{-1}]),$$
$$C_3\colon\ \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}) \to K_1(A_\rho[z, z^{-1}]);$$
$$(P_-, \nu_-) \mapsto \tau(1 - z^{-1}\nu_-\colon\ (P_-)_\rho[z, z^{-1}] \to (P_-)_\rho[z, z^{-1}])$$

an isomorphism.

The inverse isomorphism

$$C^{-1} = B = B_1 \oplus B_2 \oplus B_3\colon\ K_1(A_\rho[z, z^{-1}])$$
$$\to K_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1})$$

is constructed as follows:

LEMMA 2.17. (i) *The following sequence is an A-induced f.g. projective $A_\rho[z]$-module resolution of P_+:*

$$0 \longrightarrow (zP_+)_\rho[z] \xrightarrow{z\zeta^{-1}-\nu_+} (P_+)_\rho[z] \xrightarrow{\pi_+} P_+ \longrightarrow 0$$

with

$$\zeta^{-1}\colon\ zP_+ \to P_+;\quad zx \mapsto x \quad (a\ \rho\text{-isomorphism}),$$

$$z\zeta^{-1} - \nu_+\colon\ (zP_+)_\rho[z] \to (P_+)_\rho[z];\quad \sum_{j=1}^{\infty} z^j x_j \mapsto \sum_{j=1}^{\infty} z^j x_j - \sum_{j=0}^{\infty} z^j \nu_+(x_{j+1}),$$

$$\pi_+\colon\ (P_+)_\rho[z] \to P_+;\quad \sum_{j=0}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} (\nu_+)^j (x_j).$$

(ii) *The two $A_\rho[z]$-module resolutions for P_+ are related by a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & z^{k_+}A_\rho[z]^n & \xrightarrow{\alpha_+} & A_\rho[z]^n & \xrightarrow{\pi_+} & P_+ & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle g_+} & & \Big\downarrow{\scriptstyle f_+} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & (zP_+)_\rho[z] & \xrightarrow{z\zeta^{-1}-\nu_+} & (P_+)_\rho[z] & \xrightarrow{\widetilde{\pi}_+} & P_+ & \longrightarrow & 0
\end{array}$$

with

$$\widetilde{\pi}_+ :\ (P_+)_\rho[z] \to P_+;\ \ \sum_{j=0}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} (\nu_+)^j(x_j),$$

$$f_+ :\ A_\rho[z]^n \to (P_+)_\rho[z];\ \ \sum_{j=0}^{\infty} z^j a_j \mapsto \sum_{j=0}^{\infty} z^j \pi_+(a_j),$$

$$g_+ :\ z^{k_+}A_\rho[z]^n \to (zP_+)_\rho[z];\ \ \sum_{j=k_+}^{\infty} z^j b_j \mapsto \sum_{j=1}^{\infty}\sum_{i=1}^{j} z^i(\nu_+)^{j-i}(y_j)$$

$$\left(f_+\alpha_+ \left(\sum_{j=k_+}^{\infty} z^j b_j \right) = \sum_{j=0}^{\infty} z^j y_j \right).$$

Regarding the commutative diagram in 2.17 (ii) as a chain equivalence of one-dimensional A-induced f.g. projective $A_\rho[z]$-module chain complexes

$$\begin{aligned}
&(f_+, g_+) :\ C(\alpha_+ :\ z^{k_+}A_\rho[z]^n \to A_\rho[z]^n) \\
&\quad \to C(z\zeta^{-1} - \nu_+ :\ (zP_+)_\rho[z] \to (P_+)_\rho[z])
\end{aligned}$$

we have that the algebraic mapping cone is a short exact sequence of A-induced f.g. projective $A_\rho[z]$-modules

$$C(f_+, g_+) :\ 0 \to (z^{k_+}A^n)_\rho[z] \to (zP_+ \oplus A^n)_\rho[z] \to (P_+)_\rho[z] \to 0.$$

Choosing a splitting there is obtained an $A_\rho[z]$-module isomorphism

$$h = \sum_{j=0}^{\infty} z^j h_j :\ (P_+ \oplus z^{k_+}A^n)_\rho[z] \xrightarrow{\ \cong\ } (zP_+ \oplus A^n)_\rho[z].$$

The components are ρ^j-morphisms

$$h_j :\ P_+ \oplus z^{k_+}A^n \to zP_+ \oplus A^n \qquad (j \geqslant 0)$$

with $h_0 :\ P_+ \oplus z^{k_+}A^n \to zP_+ \oplus A^n$ an A-module isomorphism. Use the ρ-isomorphism

$$z^{-1}h_0 :\ P_+ \oplus z^{k_+}A^n \to P_+ \oplus z^{-1}A^n;\ \ x \mapsto z^{-1}(h_0(x))$$

and the A-module isomorphisms

$$\begin{aligned} & z^{k_+}(\theta_n)^{k_+}\colon\ A^n \to z^{k_+}A^n;\\ &\quad (a_1, a_2, \dots, a_n) \mapsto z^{k_+}(\rho^{k_+}(a_1), \rho^{k_+}(a_2), \dots, \rho^{k_+}(a_n))\\ &\quad z\theta_n\colon\ z^{-1}A^n \to A^n;\, z^{-1}(a_1, a_2, \dots, a_n) \mapsto (\rho(a_1), \rho(a_2), \dots, \rho(a_n)) \end{aligned}$$

to define a ρ-isomorphism

$$\phi = (1 \oplus z\theta_n)(z^{-1}h_0)(1 \oplus z^{k_+}(\theta_n)^k)\colon\ P_+ \oplus A^n \to P_+ \oplus A^n.$$

The components B_i of $C = B^{-1}$ are given by

$$\begin{aligned} & B_1\colon\ K_1(A_\rho[z, z^{-1}]) \to K_1(A, \rho);\quad \tau(\alpha) \mapsto [P_+ \oplus A^n, \phi] - [A^n, \theta_n],\\ & B_2\colon\ K_1(A_\rho[z, z^{-1}]) \to \widetilde{\mathrm{Nil}}_0(A, \rho);\quad \tau(\alpha) \mapsto [P_-, \nu_-],\\ & B_3\colon\ K_1(A_\rho[z, z^{-1}]) \to \widetilde{\mathrm{Nil}}_0(A, \rho^{-1});\quad \tau(\alpha) \mapsto [P_+, \nu_+]. \end{aligned}$$

2.5. THE WHITEHEAD GROUP OF $A_\rho((z))$

The splitting of $K_1(A_\rho[z, z^{-1}])$ in Section 2.4 is now extended to $K_1(A_\rho((z)))$. However, as for $K_1(A_\rho[z])$, $K_1(A_\rho[[z]])$ there is an essential difference between the proofs of these results: there is no analogue for $K_1(A_\rho((z)))$ of the Higman linearization trick by which every element of $K_1(A_\rho[z, z^{-1}])$ can be represented by a linear automorphism

$$\alpha = \alpha_0 + z\alpha_1\colon\ A_\rho[z, z^{-1}]^n \to A_\rho[z, z^{-1}]^n.$$

We shall show that the map

$$\widehat{C} = \widehat{C}_1 \oplus \widehat{C}_2 \oplus \widehat{C}_3\colon\ K_1(A, \rho) \oplus W_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}) \to K_1(A_\rho((z)))$$

defined by

$$\begin{aligned} & \widehat{C}_1\colon\ K_1(A, \rho) \to K_1(A_\rho((z)));\quad (P, \phi) \mapsto \tau(z\phi\colon\ P_\rho((z)) \to P_\rho((z))),\\ & \widehat{C}_2\colon\ W_1(A, \rho) \to K_1(A_\rho((z)));\quad w \mapsto w,\\ & \widehat{C}_3\colon\ \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}) \to K_1(A_\rho((z)));\\ &\quad (P, \nu) \mapsto \tau(1 - z^{-1}\nu\colon\ P_\rho((z)) \to P_\rho((z))) \end{aligned}$$

is an isomorphism by constructing an explicit inverse

$$\begin{aligned} \widehat{C}^{-1} \ =\ \widehat{B} = \widehat{B}_1 \oplus \widehat{B}_2 \oplus \widehat{B}_3\colon\ & K_1(A_\rho((z)))\\ & \to K_1(A, \rho) \oplus W_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}). \end{aligned}$$

The definition of $\widehat{B}$ is based on the constructions of several auxiliary objects, to which we now proceed.

An element $\alpha \in K_1(A_\rho((z)))$ is represented by an automorphism of a f.g. free $A_\rho((z))$-module

$$\alpha = \sum_{j=-k}^{\infty} z^j \alpha_j : \ A_\rho((z))^n \to A_\rho((z))^n$$

with coefficients ρ^j-morphisms $\alpha_j : \ A^n \to A^n$, for some $k \geqslant 0$. Write the inverse of α as

$$\alpha^{-1} = \beta = \sum_{j=-\ell}^{\infty} z^j \beta_j : \ A_\rho((z))^n \to A_\rho((z))^n$$

for some $\ell \geqslant 0$.

Now we can define the object which plays the most important role in our construction: the A-module

$$P = \operatorname{coker}(\widetilde{\alpha} : \ z^k A_\rho[[z]]^n \to A_\rho[[z]]^n)$$

with $\widetilde{\alpha}$ the restriction of α.

LEMMA 2.18. (i) *P is a f.g. projective A-module.*
(ii) *The ρ^{-1}-endomorphism*

$$z : \ P \to P ; x \mapsto zx$$

is nilpotent.

Proof. (i) It is clear that P is a f.g. $A_\rho[[z]]$-module, and that there is defined a f.g. free $A_\rho[[z]]$-module resolution

$$0 \longrightarrow z^k A_\rho[[z]]^n \xrightarrow{\widetilde{\alpha}} A_\rho[[z]]^n \xrightarrow{\pi} P \longrightarrow 0$$

with

$$\pi = \text{projection} : \ A_\rho[[z]]^n \to P.$$

To show that P is a f.g. projective A-module, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & z^k A_\rho[[z]]^n & \xrightarrow{\widetilde{\alpha}} & A_\rho[[z]]^n & \xrightarrow{\pi} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \widetilde{\beta}} & & \downarrow & & \\
0 & \longrightarrow & z^k A_\rho[[z]]^n & \longrightarrow & z^{-\ell} A_\rho[[z]]^n & \longrightarrow & \sum\limits_{j=-\ell}^{k-1} z^j A^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \gamma} & & \downarrow & & \\
0 & \longrightarrow & z^k A_\rho[[z]]^n & \xrightarrow{\widetilde{\alpha}} & A_\rho[[z]]^n & \xrightarrow{\pi} & P & \longrightarrow & 0
\end{array}$$

with $\widetilde{\beta}$ the restriction of β, and γ the A-module morphism defined by

$$\gamma\colon\ z^{-\ell}A_\rho[[z]]^n \to A_\rho[[z]]^n;\quad \sum_{j=-\ell}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} z^j y_j$$
$$\left(\alpha\left(\sum_{j=-\ell}^{\infty} z^j x_j\right) = \sum_{j=-k-\ell}^{\infty} z^j y_j\right).$$

It follows from the identity

$$\gamma\widetilde{\beta} = (\alpha\beta)|_{A_\rho[[z]]^n} = 1\colon\ A_\rho[[z]]^n \to A_\rho[[z]]^n$$

that P is a direct summand of the f.g. free A-module $\sum_{j=-\ell}^{k-1} z^j A^n$, and hence a f.g. projective A-module. In fact, the A-module defined by

$$Q = \operatorname{coker}(\widetilde{\beta}\colon\ A_\rho[[z]]^n \to z^{-\ell}A_\rho[[z]]^n)$$

is such that there exists an A-module isomorphism

$$P \oplus Q \cong \sum_{j=-\ell}^{k-1} z^j A^n.$$

(ii) Recall that P is not only a left A-module, but also a left $A_\rho[[z]]$-module. The left multiplication by the element $z \in A_\rho[[z]]$ defines a ρ^{-1}-endomorphism of the A-module $A_\rho[[z]]^n$

$$z\colon\ A_\rho[[z]]^n \to A_\rho[[z]]^n;\ x \mapsto zx$$

which induces a ρ^{-1}-endomorphism of the A-module P

$$\nu\colon\ P \to P;\ x \mapsto zx$$

with a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & z^k A_\rho[[z]]^n & \xrightarrow{\widetilde{\alpha}} & A_\rho[[z]]^n & \xrightarrow{\pi} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle \nu} & & \\
0 & \longrightarrow & z^k A_\rho[[z]]^n & \xrightarrow{\widetilde{\alpha}} & A_\rho[[z]]^n & \xrightarrow{\pi} & P & \longrightarrow & 0
\end{array}$$

For any $x \in A_\rho[[z]]^n$

$$z^{k+\ell}x = \alpha\beta(z^{k+\ell}x) = \widetilde{\alpha}(z^k\beta(z^\ell x)) \in \operatorname{im}(\widetilde{\alpha}\colon\ z^k A_\rho[[z]]^n \to A_\rho[[z]]^n)$$

so that

$$\nu^{k+\ell}\pi(x) = 0 \in P$$

and

$$\nu^{k+\ell} = 0\colon\ P \to P. \qquad \square$$

We thus obtain an element

$$(P, \nu) \in \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}).$$

Define an $A_\rho[[z]]$-module morphism

$$\pi\colon\ P_\rho[[z]] \to P;\ \ \sum_{j=0}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} \nu^j(x_j).$$

Note that the right hand side of the formula is well defined since ν is nilpotent, and therefore the sum contains only a finite number of terms.

Identify

$$zP_\rho[[z]] = \prod_{j=1}^{\infty} z^j P, \qquad P_\rho[[z]] = \prod_{j=0}^{\infty} z^j P.$$

By analogy with 2.17:

LEMMA 2.19. (i) *The following sequence is an A-induced f.g. projective $A_\rho[[z]]$-module resolution of P:*

$$0 \longrightarrow zP_\rho[[z]] \xrightarrow{z\zeta^{-1}-\nu} P_\rho[[z]] \xrightarrow{\widetilde{\pi}} P \longrightarrow 0$$

with

$$\zeta^{-1}\colon\ zP \to P;\, zx \mapsto x,$$

$$z\zeta^{-1} - \nu\colon\ zP_\rho[[z]] \to P_\rho[[z]];\ \ \sum_{j=1}^{\infty} z^j x_j \mapsto \sum_{j=1}^{\infty} z^j x_j - \sum_{j=0}^{\infty} z^j \nu(x_{j+1}),$$

$$\widetilde{\pi}\colon\ P_\rho[[z]] \to P;\ \ \sum_{j=0}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} \nu^j(x_j).$$

(ii) *The two $A_\rho[[z]]$-module resolutions for P are related by a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & z^k A_\rho[[z]]^n & \xrightarrow{\widetilde{\alpha}} & A_\rho[[z]]^n & \xrightarrow{\pi} & P & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & zP_\rho[[z]] & \xrightarrow{z\zeta^{-1}-\nu} & P_\rho[[z]] & \xrightarrow{\widetilde{\pi}} & P & \longrightarrow & 0
\end{array}$$

with

$$f\colon\ A_\rho[[z]]^n \to P_\rho[[z]];\ \ \sum_{j=0}^{\infty} z^j a_j \mapsto \sum_{j=0}^{\infty} z^j \pi(a_j),$$

$$g\colon\ z^k A_\rho[[z]]^n \to zP_\rho[[z]];\ \ \sum_{j=k}^{\infty} z^j b_j \mapsto \sum_{j=1}^{\infty}\sum_{i=1}^{j} z^i \nu^{j-i}(y_j)$$

$$\left(f\widetilde{\alpha}\left(\sum_{j=k}^{\infty} z^j b_j\right) = \sum_{j=0}^{\infty} z^j y_j \right).$$

Proof. (i) The sequence is part of a direct sum system of A-modules

$$zP_\rho[[z]] \underset{\tau}{\overset{z\zeta^{-1}-\nu}{\rightleftarrows}} P_\rho[[z]] \underset{\sigma}{\overset{\widetilde{\pi}}{\rightleftarrows}} P,$$

with the A-module morphisms defined by

$$\sigma\colon\ P \to P_\rho[[z]];\ x \mapsto x,$$

$$\tau\colon\ P_\rho[[z]] \to zP_\rho[[z]];\ \ \sum_{j=0}^{\infty} z^j x_j \mapsto \sum_{j=0}^{\infty} z^{j+1}\left(\sum_{k=0}^{\infty} \nu^k(x_{j+k+1})\right)$$

such that

$$\widetilde{\pi}\circ\sigma = \mathrm{id}\colon\ P \to P,$$

$$\tau\circ(z\zeta^{-1}-\nu) = \mathrm{id}\colon\ zP_\rho[[z]] \to zP_\rho[[z]],$$

$$\sigma\circ\widetilde{\pi} + (z\zeta^{-1}-\nu)\circ\tau = \mathrm{id}\colon\ P_\rho[[z]] \to P_\rho[[z]].$$

(ii) By construction. □

Regard the commutative diagram of Lemma 2.19 (ii) as a chain equivalence of one-dimensional A-induced f.g. projective $A_\rho[[z]]$-module chain complexes

$$(f, g, k)\colon\ C(\widetilde{\alpha}\colon\ z^k A_\rho[[z]]^n \to A_\rho[[z]]^n)$$
$$\to C(z\zeta^{-1}-\nu\colon\ zP_\rho[[z]] \to P_\rho[[z]]).$$

By Definition 2.3 and Remark 2.3 we now have a Witt vector torsion

$$w(f, g, k) \in W_1(A, \rho).$$

More precisely, the algebraic mapping cone is a short exact sequence of A-induced f.g. projective $A_\rho[[z]]$-modules

$$C(f, g, k)\colon\ 0 \to (z^k A^n)_\rho[[z]] \to (zP \oplus A^n)_\rho[[z]] \to P_\rho[[z]] \to 0.$$

Choosing a splitting there is obtained an $A_\rho[[z]]$-module isomorphism

$$h = \sum_{j=0}^{\infty} z^j h_j \colon\ (P \oplus z^k A^n)_\rho[[z]] \xrightarrow{\cong} (zP \oplus A^n)_\rho[[z]]$$

such that the components are ρ^j-morphisms

$$h_j \colon\ P \oplus z^k A^n \to zP \oplus A^n \qquad (j \geqslant 0)$$

with $h_0 \colon\ P \oplus z^k A^n \to zP \oplus A^n$ an A-module isomorphism, and

$$(h_0)^{-1}h = 1 + \sum_{j=1}^{\infty} z^j (h_0)^{-1} h_j \colon\ (P \oplus z^k A^n)_\rho[[z]] \to (P \oplus z^k A^n)_\rho[[z]]$$

an automorphism of an A-induced f.g. projective $A_\rho[[z]]$-module, such that

$$\begin{aligned}
&\tau^{\mathrm{iso}}(f, g, k) = \tau^{\mathrm{iso}}(h) \in K_1^{\mathrm{iso}}(\mathbb{P}(A_\rho[[z]]))\\
&w(f, g, k) = \tau^F(f, g, k) = \tau^F(h) = \tau((h_0)^{-1}h)\\
&\qquad \in W_1(A, \rho) = \ker(K_1(A_\rho[[z]]) \to K_1(A))
\end{aligned}$$

with $F \colon\ A_\rho[[z]] \to A_\rho[[z]];\ z \mapsto 0$.

Continuing with our preparation for the construction of $\widehat{B}^{-1} = \widehat{C}$, use the ρ-isomorphism

$$z^{-1}h_0 \colon\ P \oplus z^k A^n \to P \oplus z^{-1}A^n;\ x \mapsto z^{-1}(h_0(x))$$

and the A-module isomorphisms

$$\begin{aligned}
&z^k(\theta_n)^k \colon\ A^n \to z^k A^n;\quad (a_1, a_2, \dots, a_n) \mapsto z^k(\rho^k(a_1), \rho^k(a_2), \dots, \rho^k(a_n))\\
&z\theta_n \colon\ z^{-1}A^n \to A^n;\quad z^{-1}(a_1, a_2, \dots, a_n) \mapsto (\rho(a_1), \rho(a_2), \dots, \rho(a_n))
\end{aligned}$$

to define a ρ-isomorphism

$$\phi = (1 \oplus z\theta_n)(z^{-1}h_0)(1 \oplus z^k(\theta_n)^k) \colon\ P \oplus A^n \to P \oplus A^n.$$

We define an inverse for

$$\widehat{C} = \widehat{C}_1 \oplus \widehat{C}_2 \oplus \widehat{C}_3 \colon\ K_1(A, \rho) \oplus W_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1}) \to K_1(A_\rho((z)))$$

by setting

$$\widehat{B} = \widehat{B}_1 \oplus \widehat{B}_2 \oplus \widehat{B}_3 \colon\ K_1(A_\rho((z))) \to K_1(A, \rho) \oplus W_1(A, \rho) \oplus \widetilde{\mathrm{Nil}}_0(A, \rho^{-1})$$

with

$$\begin{aligned}
&\widehat{B}_1 \colon\ K_1(A_\rho((z))) \to K_1(A, \rho);\\
&\qquad \tau(\alpha \colon\ A_\rho((z))^n \to A_\rho((z))^n) \mapsto [P \oplus A^n, \phi] - [A^n, \theta_n],\\
&\widehat{B}_2 \colon\ K_1(A_\rho((z))) \to W_1(A, \rho);\\
&\qquad \tau(\alpha \colon\ A_\rho((z))^n \to A_\rho((z))^n) \mapsto w(f, g, k) = \tau((h_0)^{-1}h),\\
&\widehat{B}_3 \colon\ K_1(A_\rho((z))) \to \widetilde{\mathrm{Nil}}_0(A, \rho^{-1});\\
&\qquad \tau(\alpha \colon\ A_\rho((z))^n \to A_\rho((z))^n) \mapsto [P, \nu].
\end{aligned}$$

We have to verify that the maps $\widehat{B}_i$ are well defined, that

$$\widehat{B}_i\widehat{C}_j = \begin{cases} \text{Id} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and that

$$\widehat{C}_1\widehat{B}_1 + \widehat{C}_2\widehat{B}_2 + \widehat{C}_3\widehat{B}_3 = \text{Id}\colon\ K_1(A_\rho((z))) \to K_1(A_\rho((z))).$$

The identities

$$\widehat{B}_i\widehat{C}_j = 0 \text{ if } i \neq j, \qquad \widehat{B}_1\widehat{C}_1 = \text{Id}_{K_1(A_\rho((z)))}, \qquad \widehat{B}_3\widehat{C}_3 = \text{Id}_{\widetilde{\text{Nil}}_0(A,\rho^{-1})}$$

are easily reduced to the corresponding identites in the K-theory of Laurent polynomial ring (Subsection 2.4). The identity $\widehat{B}_2\widehat{C}_2 = \text{Id}_{W_1(A,\rho)}$ is a matter of a trivial computation, once we have proved that $\widehat{B}_2$ is well defined. This is proved below just after the next lemma, which proves $\widehat{C}\widehat{B} = \text{Id}$.

LEMMA 2.20. *For every $A_\rho((z))$-module automorphism $\alpha\colon A_\rho((z))^n \to A_\rho((z))^n$*

$$\tau(\alpha) = (\widehat{C}_1\widehat{B}_1 + \widehat{C}_2\widehat{B}_2 + \widehat{C}_3\widehat{B}_3)\tau(\alpha) \in K_1(A_\rho((z))).$$

Proof. Apply $A_\rho((z)) \otimes_{A_\rho[[z]]} -$ to the chain equivalence of one-dimensional A-induced f.g. projective $A_\rho[[z]]$-module chain complexes

$$\begin{aligned}(f, g, k)\colon\ & C(\widetilde{\alpha}\colon\ z^k A_\rho[[z]]^n \to A_\rho[[z]]^n) \\ & \to C(z\zeta^{-1} - \nu\colon\ zP_\rho[[z]] \to P_\rho[[z]])\end{aligned}$$

to obtain a chain equivalence of contractible one-dimensional A-induced f.g. projective $A_\rho((z))$-module chain complexes

$$\begin{aligned}1 \otimes (f, g, k)\colon\ & C(\widetilde{\alpha}\colon\ z^k A_\rho((z))^n \to A_\rho((z))^n) \\ & \to C(z\zeta^{-1} - \nu\colon\ zP_\rho((z)) \to P_\rho((z))).\end{aligned}$$

Use the $A_\rho((z))$-module isomorphisms

$$\begin{aligned}& z^{-k}\colon\ z^k A_\rho((z))^n \to A_\rho((z))^n, \\ & z^{-1}\colon\ zP_\rho((z)) \to P_\rho((z))\end{aligned}$$

to define $A_\rho((z))$-module isomorphisms

$$\begin{aligned}& C(\widetilde{\alpha}\colon\ z^k A_\rho((z))^n \to A_\rho((z))^n) \cong C(\alpha\colon\ A_\rho((z))^n \to A_\rho((z))^n), \\ & C(z\zeta^{-1} - \nu\colon\ zP_\rho((z)) \to P_\rho((z))) \cong C(1 - z^{-1}\nu\colon\ P_\rho((z)) \to P_\rho((z)))\end{aligned}$$

with

$$\widehat{C}_3\widehat{B}_3\tau(\alpha) = \tau(1 - z^{-1}\nu\colon\ P_\rho((z)) \to P_\rho((z))) \in K_1(A_\rho((z))).$$

It now follows from the sum formula for torsion that the morphism

$$\widehat{C}_2\colon\ W_1(A,\rho) \to K_1(A_\rho((z)));\ w \mapsto w$$

sends $\widehat{B}_2\tau(\alpha) = w(f,g,k)$ to

$$\widehat{C}_2\widehat{B}_2\tau(\alpha) = \tau(\alpha) - \widehat{C}_1\widehat{B}_1\tau(\alpha) - \widehat{C}_3\widehat{B}_3\tau(\alpha) \in K_1(A_\rho((z))). \qquad \square$$

From now on, we shall write $\widehat{B}_2(\tau(\alpha))$ as $\widehat{B}_2(\alpha)$. We have to check that $\widehat{B}_2$ is well defined, that is

(i) $\widehat{B}_2(\beta \circ \alpha) = \widehat{B}_2(\beta) + \widehat{B}_2(\alpha) \in W_1(A,\rho)$,

(ii) $\widehat{B}_2(\alpha) = w(f,g,k) \in W_1(A,\rho)$ does not depend on the choice of k.

The general observation is that the composite of two automorphisms of a f.g. free $A_\rho((z))$-module

$$\alpha = \sum_{j=-k}^{\infty} z^j\alpha_j, \qquad \alpha' = \sum_{j=-k'}^{\infty} z^j\alpha'_j\colon\ A_\rho((z))^n \to A_\rho((z))^n$$

is an automorphism

$$\alpha'' = \alpha'\alpha = \sum_{j=-k''}^{\infty} z^j\alpha''_j\colon\ A_\rho((z))^n \to A_\rho((z))^n\ (k'' = k + k')$$

such that the cokernel $A_\rho[[z]]$-modules

$$\begin{aligned} P &= \mathrm{coker}(\widetilde{\alpha}\colon\ z^k A_\rho[[z]]^n \to A_\rho[[z]]^n),\\ P' &= \mathrm{coker}(\widetilde{\alpha}'\colon\ z^{k'} A_\rho[[z]]^n \to A_\rho[[z]]^n),\\ P'' &= \mathrm{coker}(\widetilde{\alpha}''\colon\ z^{k''} A_\rho[[z]]^n \to A_\rho[[z]]^n) \end{aligned}$$

fit into a short exact sequence

$$0 \to z^{k'}P \to P'' \to P' \to 0$$

which in principle gives (i). The case

$$\alpha' = 1\colon\ A_\rho((z))^n \to A_\rho((z))^n, \qquad k' = 1$$

in principle gives (ii). However, for the sake of precision we shall now verify (i) and (ii) in detail.

We first make another general remark. Let P be an $A_\rho[[z]]$-module which admits an A-induced f.g. projective $A_\rho[[z]]$-module resolution

$$R\colon\ 0 \longrightarrow M_\rho[[z]] \longrightarrow N_\rho[[z]] \longrightarrow P \longrightarrow 0.$$

Any two such resolutions R_1, R_2 are related by a chain equivalence $R_1 \to R_2$ with an isomorphism torsion $\tau^{\text{iso}}(R_1 \to R_2) \in K_1^{\text{iso}}(A_\rho[[z]])$. Denote the Witt vector torsion class (Definition 2.3) by

$$\sigma(R_1, R_2) = w(\tau^{\text{iso}}(R_1 \to R_2)) \in W_1(A, \rho).$$

For any three such resolutions R_1, R_2, R_3 we have the sum formula

$$\sigma(R_1, R_3) = \sigma(R_1, R_2) + \sigma(R_2, R_3).$$

Thus if $\alpha\colon\ A_\rho((z))^n \to A_\rho((z))^n$ is an automorphism

$$w(f, g, k) = \sigma(\mu(\alpha, k), \theta(\alpha, k)) \in W_1(A, \rho)$$

where $\mu(\alpha, k)$ and $\theta(\alpha, k)$ are the following resolutions of the module $P = P(k, \alpha)$:

$$\mu(\alpha, k)\colon\ 0 \longrightarrow z^k A_\rho[[z]]^n \xrightarrow{\widetilde{\alpha}} A_\rho[[z]]^n \xrightarrow{\pi} P \longrightarrow 0 \tag{1}$$

and

$$\theta(\alpha, k)\colon\ 0 \longrightarrow zP_\rho[[z]] \xrightarrow{z\zeta^{-1}-\nu} P_\rho[[z]] \xrightarrow{\widetilde{\pi}} P \longrightarrow 0. \tag{2}$$

PROPOSITION 2.21. *The element*

$$\sigma(\mu(\alpha, k), \theta(\alpha, k)) \in W_1(A, \rho)$$

does not depend on k.

Proof. It suffices to check that the passage from k to $k+1$ does not change the invariant. Let us consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & z^{k+1}A_\rho[[z]]^n & \xrightarrow{\text{id}} & z^{k+1}A_\rho[[z]]^n & \hookrightarrow & z^k A_\rho[[z]]^n & & \\
 & & \cap\downarrow & \boxed{\mu} & \downarrow\alpha & \boxed{\mu\mu} & \downarrow\alpha & & \\
 & & z^k A_\rho[[z]]^n & \hookrightarrow & A_\rho[[z]]^n & \xrightarrow{\text{id}} & A_\rho[[z]]^n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{i} & P(k+1, \alpha) & \xrightarrow{p} & P(k, \alpha) & \longrightarrow & 0
\end{array} \tag{3}$$

The two columns on the right are exactly $\mu(\alpha, k+1)$ and $\mu(\alpha, k)$. The squares $\boxed{\mu}$ and $\boxed{\mu\mu}$ are obviously homotopy commutative, and this implies the existence of the bottom line. Recall that the A-modules $P(k, \alpha)$, $P(k+1, \alpha)$ come equipped with nilpotent ρ^{-1}-homomorphisms ν_k and, respectively, ν_{k+1}. Endow the A-module $P' \cong z^k A$ with the zero nilpotent endomorphism, then the bottom line becomes an exact sequence in the category of modules and their nilpotent ρ^{-1}-endomorphisms. This exact sequence induces the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & zP'_\rho[[z]] & \xrightarrow{\mathrm{id}} & zP(k+1,\alpha)_\rho[[z]] & \longrightarrow & zP(k,\alpha)_\rho[[z]] & & \\
 & & \cap\downarrow & \boxed{\vartheta} & \downarrow & \boxed{\vartheta\vartheta} & \downarrow & & \\
 & & P'_\rho[[z]] & \xrightarrow{\mathrm{id}} & P(k+1,\alpha)_\rho[[z]] & \longrightarrow & P(k,\alpha)_\rho[[z]] & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P' & \xrightarrow{i} & P(k+1,\alpha) & \xrightarrow{p} & P(k,\alpha) & \longrightarrow & 0
\end{array}
\tag{4}
$$

where all the rows are exact sequences, and the two columns on the right are the resolutions $\theta(k+1, \alpha)$ and resp. $\theta(k, \alpha)$. The computation of $\sigma(\mu(\alpha, k+1), \theta(\alpha, k+1))$ is reduced to the comparison of the resolutions represented by the central columns of the both diagrams.

Consider the square $\boxed{\mu}$ as a map of two resolutions corresponding to modules P' and $P(k+1, \alpha)$. Using the diagram (3) it is easy to prove:

LEMMA 2.22. *There is an epimorphic chain homotopy equivalence*

$$C\left(\boxed{\mu}\right) \to \mu(\alpha, k)$$

with kernel the chain complex

$$\{0 \longrightarrow z^{k+1}A_\rho[[z]]^n \xrightarrow{\mathrm{id}} z^{k+1}A_\rho[[z]]^n \longrightarrow 0\}. \tag{5}$$

Sketch of the proof. The composition of two maps of resolutions, – the first corresponding to $\boxed{\mu}$, the second to $\boxed{\mu\mu}$ is homotopic to zero, being a lifting of the zero map $P' \to P(k, \alpha)$. This implies a direct construction of the map $C\left(\boxed{\mu}\right) \to \mu(\alpha, k)$, and checking through this construction leads to the proof of the lemma. □

Let

$$\xi = \{0 \longrightarrow zP'[[z]] \longrightarrow P'[[z]] \longrightarrow 0\}. \tag{6}$$

Similar reasoning gives the following:

LEMMA 2.23. *There is an epimorphic A-induced chain homotopy equivalence* $\pi\colon\ C\left(\boxed{\vartheta}\right) \to \theta(\alpha, k)$ *with kernel*

$$\ker(\pi) = C(\mathrm{id}\colon\ \xi \to \xi).$$

In particular $C\left(\boxed{\vartheta}\right)$ is a resolution for $S(k, \alpha)$. It follows from the preceding lemmas that

$$\sigma\left(C\left(\boxed{\vartheta}\right), \theta(\alpha, k)\right) = 0 = \sigma\left(C\left(\boxed{\mu}\right), \mu(\alpha, k)\right),$$

and therefore

$$\sigma(\mu(\alpha, k), \theta(\alpha, k)) = \sigma\left(C\left(\boxed{\mu}\right), C\left(\boxed{\vartheta}\right)\right). \tag{7}$$

Let

$$\eta = \{0 \longrightarrow z^{k+1}A_\rho[[z]]^n \longrightarrow z^k A_\rho[[z]]^n \longrightarrow 0\}. \tag{8}$$

PROPOSITION 2.24. *There is a commutative diagram of $A_\rho[[z]]$-module chain complexes and chain maps*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu(\alpha, k+1) & \longrightarrow & C\left(\boxed{\mu}\right) & \longrightarrow & \Sigma(\eta) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \theta(\alpha, k+1) & \longrightarrow & C\left(\boxed{\vartheta}\right) & \longrightarrow & \Sigma(\xi) & \longrightarrow & 0 \end{array} \tag{9}$$

(here Σ stands for suspension in the category of chain complexes, and all the horizontal arrows are A-induced homomorphisms).

Proof. Consider the following square of chain complexes.

$$\begin{array}{ccc} \eta & \longrightarrow & \mu(k+1, \alpha) \\ \big\downarrow & & \big\downarrow \\ \xi & \longrightarrow & \theta(k+1, \alpha) \end{array} \tag{10}$$

(Here the upper horizontal arrow comes from the square $\boxed{\mu}$, the lower comes from $\boxed{\vartheta}$, the vertical arrows are the maps of resolutions, induced by the identity maps of the corresponding complexes). The square (10) is homotopy commutative (indeed, all these complexes are actually resolutions of the corresponding $A_\rho[[z]]$-modules, and to check the homotopy commutativity it suffices to check the commutativity on the level of the modules themselves, which is obvious). By the basic properties of the algebraic mapping cone construction this implies the existence of the left square (strictly commutative) of (9) and the rest of this diagram is now obvious. □

Therefore,

$$\begin{aligned}&\sigma\left(C\left(\boxed{\mu}\right), C\left(\boxed{\theta}\right)\right)\\ &\quad = \sigma(\mu(\alpha, k+1), \theta(\alpha, k+1)) + \sigma(\Sigma(\eta), \Sigma(\xi))\end{aligned} \tag{11}$$

and the proof of Proposition 2.5 is concluded. □

We can now identify

$$\widehat{B}_2(\alpha) = \sigma(\mu(\alpha, k), \theta(\alpha, k)) \in W_1(A, \rho)$$

and proceed to the next step of the verification. It is quite obvious that

$$\widehat{B}_2(\alpha \oplus \beta) = \widehat{B}_2(\alpha) + \widehat{B}_2(\beta),\ \widehat{B}_2(\mathrm{id}) = 0 \in W_1(A, \rho),$$

so $\widehat{B}_2$ is a well-defined map $K_1(A_\rho((z))) \to W_1(A, \rho)$. It remains to check that this map is a homomorphism.

PROPOSITION 2.25. $\widehat{B}_2(\beta \circ \alpha) = \widehat{B}_2(\beta) + \widehat{B}_2(\alpha) \in W_1(A, \rho)$.

Proof. Note first of all that it suffices to prove the proposition for the particular case when $\beta(A_\rho[[z]]^n) \subset A_\rho[[z]]^n$. (Indeed, the general case is easily reduced to this particular one considering $\beta' = \zeta^N\beta$ where $\zeta\colon A_\rho[[z]]^n \to A_\rho[[z]]^n$ is defined by $\zeta(x_1, \dots, x_n) = (z\rho(x_1), \dots, z\rho(x_n))$, and N is sufficiently large).

For this particular case the proof goes on the same lines as the proof for Proposition 2.5. Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & z^kA_\rho[[z]]^n & \xrightarrow{\mathrm{id}} & z^kA_\rho[[z]]^n & \xrightarrow{\alpha} & z^kA_\rho[[z]]^n & & \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta\alpha} & & \downarrow{\scriptstyle\beta} & & \\
 & & A_\rho[[z]]^n & \xrightarrow{\beta} & A_\rho[[z]]^n & \xrightarrow{\mathrm{id}} & A_\rho[[z]]^n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P(k, \alpha) & \xrightarrow{i} & P(k, \beta\alpha) & \xrightarrow{p} & P(k, \beta) & \longrightarrow & 0
\end{array} \tag{12}$$

where the squares in the middle are obvious, and the bottom line is obtained from these two squares. Form the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & zP(k,\alpha)[[t]] & \longrightarrow & zP(k,\beta\alpha)[[t]] & \longrightarrow & zP(k,\beta)[[t]] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P(k,\alpha)[[z]] & \longrightarrow & P(k,\beta\alpha)[[z]] & \longrightarrow & P(k,\beta)[[z]] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P(k,\alpha) & \xrightarrow{i} & P(k,\beta\alpha) & \xrightarrow{p} & P(k,\beta) & \longrightarrow & 0 \end{array} \tag{13}$$

The rest of the proof repeats step by step the proof of Proposition 2.5 with the corresponding changes. □

3. On the Image of the Witt Vectors in the Whitehead Group

The Main Theorem reduces the computation of the group $K_1(A_\rho((z)))$ to the computation of three groups. Two of these three summands are classical algebraic K-theoretic groups: $K_1(A,\rho)$ and $\widetilde{\mathrm{Nil}}_0(A,\rho^{-1})$, which have already appeared in the computation of $K_1(A_\rho[z,z^{-1}])$. Much less is known about the third summand $W_1(A,\rho)$. By definition

$$W_1(A,\rho) = \mathrm{im}(W(A,\rho) \longrightarrow K_1(A_\rho((z))))$$

with

$$W(A,\rho) = \{1 + a_1 z + a_2 z^2 + \cdots \mid a_i \in A\} \subseteq \mathfrak{A}$$

the subgroup of the Witt vectors in the multiplicative group of units $\mathfrak{A} = A_\rho[[z]]^\bullet$. The aim of this section is to obtain some partial information about the Abelian group $W_1(A,\rho)$.

For any group G we write the commutators in the usual fashion as

$$[g,g'] = gg'g^{-1}g'^{-1} \in G \; (g,g' \in G),$$

and denote by G^{ab} the Abelian quotient of G by the normal subgroup $[G,G] \subseteq G$ of the elements $x \in G$ which can be expressed as products of commutators

$$x = \prod_{r=1}^{n} [g_r, g'_r] \in G.$$

There is a natural surjection

$$J:\ W(A,\rho)^{ab} \to W_1(A,\rho) \tag{14}$$

so that $W(A,\rho)^{ab}$ is in a sense a first approximation to $W_1(A,\rho)$, and the problem of computing of $W_1(A,\rho)$ may be viewed as the problem of computing the group $\ker J$. The next step of approximation can be obtained via $\mathfrak{A}$ and the factorization

$$\begin{array}{ccc} W(A,\rho)^{ab} & \xrightarrow{J'} & \mathfrak{A}^{ab} \\ & \searrow^{J} & \downarrow \\ & & K_1(A_\rho((z))) \end{array}$$

where the vertical arrow is induced by the natural inclusion. In particular, $\ker J' \subset \ker J$. It turns out that $\ker J'$ is easily described in terms of the group $W(A,\rho)^{ab}$ and the conjugation action of $\mathfrak{A}$ on this group. This can be done in a more general framework of semidirect product of groups, and this is the aim of the next section.

3.1. ON SEMIDIRECT PRODUCTS OF GROUPS

The semidirect product of groups H, K twisted by a homomorphism $\xi:\ K \to \mathrm{Aut}(H)$ is the group

$$G = H \times_\xi K$$

in which every element $x \in G$ has a unique expression as a product

$$x = hk \in G \quad (h \in H, k \in K)$$

with

$$(h_1k_1)(h_2k_2) = h_1\xi(k_1)(h_2)k_1k_2 \in G, \qquad \xi(k)(h) = khk^{-1} \in H$$

and there is defined a short exact sequence of groups

$$1 \longrightarrow H \xrightarrow{i} G \xrightarrow{j} K \longrightarrow 1 \tag{15}$$

with

$$i:\ H \to G; h \mapsto h, \qquad j:\ G \to K;\ \ hk \mapsto k.$$

The Abelianization H^{ab} is a $\mathbf{Z}[K]$-module via

$$\mathbf{Z}[K] \times H^{ab} \to H^{ab}; (nk, h) \mapsto \xi(k)(h^n) = kh^nk^{-1}.$$

In our applications we shall have

$$G = A_\rho[[z]]^\bullet = \mathfrak{A} = H \times_\xi K$$

with

$$H = W(A,\rho),\ K = A^{\bullet}, \qquad \xi :\ K \to \mathrm{Aut}(H);\, k \mapsto (h \mapsto khk^{-1}).$$

PROPOSITION 3.1. *For any semidirect product $G = H \times_{\xi} K$ the kernel of the induced morphism of Abelian groups $i^{ab}:\ H^{ab} \to G^{ab}$ is given by*

$$\ker(i^{ab}) = \ker(\epsilon)H^{ab} \subseteq H^{ab}$$

with $\epsilon:\ \mathbf{Z}[K] \to \mathbf{Z};\, k \mapsto 0$ the usual augmentation.

Proof. (i) We prove first that $\ker i^{ab} \supset (\ker\epsilon)(H^{ab})$. It suffices to observe that for any $h \in H$, $k \in K$

$$i(\xi(k)(h) - h) = [k,h] \in G,$$

so that

$$i^{ab}(\xi(k)(h) - h) = 0 \in G^{ab}.$$

(ii) Now for the reverse inclusion: we have to show that if $x \in H$ is such that $i(x) \in [G,G]$ then $[x] = y(h) \in H^{ab}$ for some $y \in \ker(\epsilon)$, $h \in H$. For any $hk, h'k' \in G$ we have

$$[hk, h'k'] = \xi(k)(h)\xi(k'k^{-1})(hh')\xi(k'^{-1})(hh'h^{-1})[h,h'][k,k'] \in [G,G].$$

If $x \in H$ is such that $i(x) \in [G,G]$ then

$$x = \prod_{r=1}^{n}[h_r k_r, h'_r k'_r] \in G$$

for some $h_r \in H$, $k_r \in K$ with

$$\prod_{r=1}^{n}[k_r, k'_r] = 1 \in K$$

and

$$\begin{aligned} i(x) &= \sum_{r=1}^{n}\xi(k)(h)\xi(k'_r k_r^{-1})(h_r h'_r)\xi(k_r'^{-1})(h_r h'_r h_r^{-1}) \\ &= \sum_{r=1}^{n}((\xi(k_r) - \xi(k_r'^{-1}))(h_r) + (\xi(k'_r k_r^{-1}) - \xi(k_r'^{-1}))(h_r h'_r) \\ &\in \ker(\epsilon)H^{ab} \subseteq H^{ab}. \end{aligned}$$

□

3.2. EXPLICIT EXAMPLES

Now we return to Witt vectors. The direct application of Proposition 3.1 gives

$$\ker J' = (\ker \epsilon)(W(A,\rho)^{ab}) \tag{16}$$

where $\epsilon\colon \mathbf{Z}\mathfrak{A} \to \mathbf{Z}$ is the augmentation in the group ring of the group $\mathfrak{A}$. This result leads quickly to the construction of non-trivial elements in $\ker J'$. Indeed, any element of type $y = \xi(\alpha)(x) - x$ where $\alpha \in \mathfrak{A}$, $x \in W(A,\rho)$ is in $\ker J'$. If

$$x = 1 + \beta z \; (\beta \in A)$$

then y has a following representative:

$$\bar{y} = \alpha(1+\beta z)\alpha^{-1}(1+\beta z)^{-1} \in W(A,\rho). \tag{17}$$

Assume now that $\rho = \mathrm{id}$, and $\alpha\beta\alpha^{-1} \neq \beta$. The element $\bar{y}$ is of the form

$$1 + (\alpha\beta\alpha^{-1} - \beta)z + \mathrm{O}(z^2) \tag{18}$$

and it does not belong to $[W(A,\rho), W(A,\rho)]$ since every element

$$1 + a_1 z + a_2 z^2 + \cdots \in [W(A,\rho), W(A,\rho)]$$

obviously has $a_1 = 0$.

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