ALGEBRAIC L-THEORY AND TRIANGULAR WITT GROUPS

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Abstract. The classical Witt groups of quadratic forms have been generalized in algebraic topology, with Wall's and Ranicki's L-groups for surgery theory, and in algebraic geometry, with Knebusch's Witt groups of schemes and Balmer's Witt groups of triangulated categories with duality.

We introduce algebraic L-theory groups for so-called pre-triangulated differential graded categories with duality, generalizing and unifying the two aforementioned approaches.

Contents

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1. Introduction

In this paper we introduce algebraic L-theory groups for pre-triangulated differential graded categories with duality, somewhat unifying and generalizing Andrew Ranicki's L-groups on the one hand and Paul Balmer's triangular Witt groups on the other.

The common ancestor of all these theories is the classical Witt group of quadratic forms, introduced by Ernst Witt in the 1930's [Wit36]. It classifies non-degenerate quadratic forms defined on vector spaces over a fixed field, for simplicity say of characteristic different from 2, modulo stabilization with "trivial" forms: the ones coming from hyperbolic spaces. It is not difficult to generalize this definition to fields of characteristic 2, where one first encounters the difference between quadratic forms and symmetric bilinear forms, and then also to commutative rings or more generally to rings with involution. (In these more general settings hyperbolic spaces are replaced by so-called metabolic spaces, of which hyperbolics are special examples.)

In the 1960's C.T.C. Wall [Wal99] introduced the 4-periodic (quadratic) L-groups of rings with involution, that arise as obstruction groups in surgery theory, which aims to classify manifolds up to automorphisms within a given homotopy type. Wall's even L-groups are nothing but Witt groups of symmetric or skew-symmetric quadratic forms, whereas the odd L-groups are defined in terms of automorphisms of hyperbolic forms, or equivalently in terms of so-called formations. Answering a question posed by Wall and generalizing an early attempt by Miščenko, Andrew Ranicki developed in the 1970's an algebraic L-theory based on chain complexes of modules (with suitable finiteness conditions) over a ring with involution. His theory provides a unified definition of the even and the odd L-groups, as well as parallel symmetric and quadratic (and hyperquadratic) versions. In addition to these abstract conceptual benefits, his framework has proved itself flexible and powerful for both the algebraic and the topological applications: on the one hand, it can be used for example to produce long exact localization sequences, product structures on the various L-groups; on the other hand, it allows for a direct description of the symmetric signature and of the surgery obstruction of a degree one normal map without the need of performing first surgeries below the middle dimension, and also a purely algebraic description of the surgery long exact sequence for topological manifolds. In Ranicki's more general and recent formulations [Ran92a], L-groups are defined for categories of chain complexes in additive categories equipped with a so-called chain duality.

On the other hand, in the realm of algebraic geometry, Manfred Knebusch [Kne77] introduced in the 1970's Witt groups of symmetric bilinear forms defined on vector bundles over algebraic varieties—or more generally over schemes—, which of course agree with the classical Witt groups in the affine case. His definition easily generalizes to arbitrary small exact categories with duality. Around the turn of the century Paul Balmer [Bal99, Bal00, Bal01a] developed a theory of 4-periodic shifted Witt groups for small triangulated categories with duality, and proved that his new zeroth Witt group of the derived category of an exact category with duality where 2 is invertible agrees with the Witt group of said exact category. (One says that 2 is invertible in a linear category if every homomorphism is uniquely 2-divisible.) In particular, Knebusch's Witt group of a scheme for which 2 is invertible in the structure sheaf can be recovered as the zeroth Witt group of the associated derived category of vector bundles. When restricted to a category of suitably well-behaved schemes, and always under the assumption that 2 is invertible, Balmer's triangular Witt groups yield a generalized cohomology theory. The key ingredient is Balmer's localization theorem, which produces long exact sequences of Witt groups associated to localization (or short exact) sequences of triangulated categories with duality where 2 is invertible. This in particular applies to the restriction map from a scheme where 2 is invertible to an open subscheme and allows one to describe the induced map on the Witt groups in terms of shifted Witt groups of related triangulated categories, thus answering a long-standing and fundamental open question.

Although originating from and motivated by distinct fields of research, and although developed using a somewhat different language, the similarities between Balmer's approach and Ranicki's algebraic L-theory were surely evident to all the experts. Our initial motivation and main goal was to make this precise, and secondly to try and extend Balmer's theory beyond the assumption on the invertibility of 2. It seems though that it is impossible to carry out this program in the great generality of triangulated categories with duality. But the good news is that all triangulated categories encountered in real life arise naturally as homotopy categories of categories with some richer structure, or quotients (localizations) thereof. In particular, all triangulated categories arising from schemes on the one hand and from rings with involution on the other can be described as homotopy categories of so-called differential graded categories. A differential graded category is, in short, a category where between objects there are chain complexes of morphisms (instead of just sets, or abelian groups), and where all the operations describing the structure of a category—i.e., compositions and units—are compatible with this extra structure. In other and more precise words, differential graded categories, henceforth abbreviated dg categories, are categories enriched over the category of chain complexes over a fixed commutative ground ring. The main examples of dg categories are provided by categories of chain complexes in some additive category. In these examples, the associated homotopy categories are naturally triangulated, but all the relevant constructions (i.e., suspensions and cones) that endow the homotopy category with the structure of a triangulated category are already present before passing to the homotopy category. As it turns out, one can formulate the definition of suspensions and cones in an arbitrary dg category, and those dg categories where suspensions and cones always exist are called pre-triangulated. This notion has been introduced and studied independently by Bondal and Kapranov [BK90] (under the name of framed, or enhanced, triangulated categories) and by Keller [Kel99] (who used the term exact dg categories), and provides the abstract setting in which we work.

Our main observation is that Ranicki's approach to L-theory can be set up to work also in the more general and abstract setting of pre-triangulated dg categories with duality. In a nutshell, the moral is that we can "do algebraic L-theory" for categories with duality where between objects we are given chain complexes of homomorphisms, regardless of the fact that the objects themselves are chain complexes—and where we have notions of suspensions and cones inducing a triangulated structure on the homotopy category. And these are precisely pre-triangulated dg categories (with duality). More precisely, the enrichment over chain complexes allows to define the correct analogs of symmetric and quadratic spaces, whereas the pre-triangulated structure is used to identify which spaces are metabolic.

Following these ideas we are able to: define all variants of L-groups for arbitrary pre-triangulated dg categories with duality, that of course agree with Ranicki's L-groups in the case of additive categories with duality; prove that they enjoy the same basic properties as Ranicki's L-groups; and finally show that when both are available, our L-groups coincide with Balmer's Witt groups of the associated triangulated categories. Our main results are summarized in the following theorem, which is proved in section 3.3.

Theorem. *Let $\mathcal{C}$ be any pre-triangulated dg category with duality. Then for any integer $n \in \mathbb{Z}$ there are functorially defined*

$$\text{quadratic L-groups } L_n(\mathcal{C}) \ , \quad \text{symmetric L-groups } L^n(\mathcal{C}) \ ,$$

$$\textit{and } \text{hyperquadratic L-groups } \widehat{L}^n(\mathcal{C})$$

satisfying the following properties (for all $n \in \mathbb{Z}$):

(i) All L-groups are 4-periodic, i.e., one has

$$L_n(\mathcal{C}) \cong L_{n+4}(\mathcal{C}) \ , \quad L^n(\mathcal{C}) \cong L^{n+4}(\mathcal{C}) \ , \quad \textit{and} \quad \widehat{L}^n(\mathcal{C}) \cong \widehat{L}^{n+4}(\mathcal{C}) \ ;$$

(ii) There are natural long exact sequences

$$\ldots \to L_n(\mathcal{C}) \to L^n(\mathcal{C}) \to \widehat{L}^n(\mathcal{C}) \to L_{n-1}(\mathcal{C}) \to \ldots \ ;$$

(iii) There are natural isomorphisms

$$L_n(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{2}\big] \cong L^n(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{2}\big] \quad \textit{and} \quad \widehat{L}^n(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{Z}\big[\tfrac{1}{2}\big] \cong 0 \ ;$$

(iv) Let $\mathcal{A}$ be an additive category with chain duality. Define $\mathcal{C} = \mathsf{ch}^{\mathsf{b}}\mathcal{A}$ as the corresponding pre-triangulated dg category with duality of bounded chain complexes in $\mathcal{A}$. Then

$$L_n(\mathcal{C}) \cong RL_n(\mathcal{A}) \ , \quad L^n(\mathcal{C}) \cong RL^n(\mathcal{A}) \ , \quad \textit{and} \quad \widehat{L}^n(\mathcal{C}) \cong \widehat{RL}^n(\mathcal{A})$$

where RL denote Ranicki's L-groups;

(v) If 2 is invertible in $\mathcal{C}$ then there are isomorphisms

$$L_n(\mathcal{C}) \cong L^n(\mathcal{C}) \quad \textit{and} \quad \widehat{L}^n(\mathcal{C}) \cong 0 \ ,$$

and

$$L^n(\mathcal{C}) \cong W^n(\mathrm{ho}(\mathcal{C}))$$

where $\mathrm{ho}(\mathcal{C})$ is the associated homotopy category of $\mathcal{C}$, a triangulated category with duality, and W^n denote Balmer's triangular Witt groups;

(vi) Any morphism of pre-triangulated dg categories with duality that induces an equivalence of the associated triangulated categories, induces isomorphisms in all L-theory groups.

To summarize we can say that we are able to define a well-behaved algebraic L-theory for arbitrary pre-triangulated dg categories with duality, extending Ranicki's definition for the case of additive categories, and coinciding with Balmer's triangular Witt groups when both are defined. Balmer's Witt groups are available for any triangulated category with duality where 2 is invertible, whereas our L-groups require the existence of a dg model (that in the cases of interest is always present) but dispense with the assumption about the invertibility of 2.

1.1. **Structure of the article.** In chapter 2 *Categorical setup*, we introduce our language and examples. The first section 2.1 deals with the abstract and general notion of categories with duality, that in subsequent sections of the chapter are "dressed up" with some compatible additional structure, such as exact, differential graded, or triangulated structures. In section 2.2 we recall the definition of exact categories, and of exact categories with duality, and present the main examples coming from finitely generated projective modules over a ring with involution and from vector bundles over a scheme. Section 2.3 fixes our basic notations and conventions, especially in regard to signs, about chain complexes in additive categories. The following sections set up the central definitions of differential graded (dg) categories with duality (section 2.4), and of pre-triangulated dg categories (section 2.5). We then review the definition of triangulated categories with duality in section 2.6. And finally section 2.7 deals with Drinfeld's construction of quotients of pre-triangulated dg categories, extending his results to the cases where a duality is present. This last section, together with the ones about pre-triangulated differential categories, contain the least well-known material.

The main chapter 3 treats *Witt groups and L-theory* for all the types of categories with duality introduced in chapter 2. In the first section 3.1 the "classical" definition of Witt groups for exact categories with duality is recalled, along with their many properties and some examples. Section 3.2 is dedicated to Balmer's Witt groups of triangulated categories with duality. And finally section 3.3 contains our new definition of algebraic L-theory for pre-triangulated dg categories, and states and proves our main results. We strive to treat all these three cases in a similar way, hoping that this highlights the analogies and helps to motivate the relevant definitions.

The expert reader can probably start to read this article from the bottom, jumping back to previous sections in order to look up the relevant definitions, in particular those contained in sections 2.4 and 2.5.

1.2. **Conventions.** We fix once and for all a ground ring $\mathfrak{K}$, assumed to be commutative, associative, and with unit—and we implicitly work "over $\mathfrak{K}$"; for example, the words chain complex of modules refer to a chain complex of modules over $\mathfrak{K}$, and we say dg category for a $\mathfrak{K}$-dg category, and so on....

In some cases our notations diverge a little from those that can be found elsewhere in the literature, in particular of Balmer and Ranicki. For us symmetric or quadratic spaces are not necessarily assumed to be non-degenerate, and the so-called Q-modules of symmetric and quadratic forms on objects of a category with duality are contravariant: Ranicki's $Q(X)$ coincide with our $Q(X^\vee)$, where $X^\vee$ denotes the dual of X. For triangulated categories, we consider dualities in the graded sense, instead of speaking of shifted dualities: An n-dimensional symmetric space for us is a symmetric space for the $(-n)$-shifted duality in the sense of Balmer.

We use the language of categories enriched over closed symmetric monoidal categories without further explanation; all that is needed can be found in the standard reference [ML98, section VII.7, pages 180–181]. In a nutshell, given a closed symmetric monoidal category $\mathcal{M}$, a category enriched over $\mathcal{M}$ differs from an ordinary category only in that the morphisms between two objects are not a set but an object of $\mathcal{M}$, and units and compositions are expressed in terms of the symmetric monoidal structure in $\mathcal{M}$. In this language, an ordinary category is a category enriched over the closed symmetric monoidal category of sets.

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2. Categorical setup

2.1. **Categories with duality.** The notion of duality is fundamental for this work. In this first section we introduce the very general and abstract definition of a category with duality. Although this extreme level of generality is not of interest in itself, it sets the stage for all what follows: In the next sections of this chapter categories with some additional structure (e.g., exact categories, triangulated categories, differential graded categories) and with duality are introduced, simply by requiring that the obvious compatibility conditions between said additional structure and the duality are satisfied.

Definition 2.1.1. A *category with duality* is a category $\mathcal{C}$ together with a functor ${}^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ and a natural isomorphism $\mu\colon \mathrm{id}_{\mathcal{C}} \xrightarrow{\cong} {}^\vee \circ {}^\vee$ such that for every object X of $\mathcal{C}$

$$\mathrm{id}_{X^\vee} = (\mu_X)^\vee\, \mu_{(X^\vee)}\colon X^\vee \xrightarrow{\mu_{(X^\vee)}} X^{\vee\vee\vee} \xrightarrow{(\mu_X)^\vee} X^\vee \ , \tag{2.1.2}$$

where as customary we write $X^\vee$ instead of ${}^\vee(X)$ (and call it the *dual* of X), and similarly for homomorphisms.

(In other words, the functor ${}^\vee$ is an involutive equivalence of categories, and μ is a unit of the adjunction.)

We stress that the choice of a natural identification of every object with its double dual is part of the definition of a duality: A category with duality is indeed a triple $(\mathcal{C}, {}^\vee, \mu)$. Nonetheless, when this structure is understood, we sometimes write simply $\mathcal{C}$ instead of $(\mathcal{C}, {}^\vee, \mu)$ in order to alleviate the notation.

Given a category with duality $(\mathcal{C}, {}^\vee, \mu)$, the choice of an element ϵ in the automorphism group of the identity functor $\mathrm{id}_{\mathcal{C}}$ such that $(\epsilon_{X^\vee})^{-1} = (\epsilon_X)^\vee$ yields another category with duality $(\mathcal{C}, {}^\vee, \epsilon\mu)$.

For every pair of objects X and Y of $\mathcal{C}$ the duality induces a natural isomorphism $\tau_{X,Y}\colon \hom_{\mathcal{C}}(X, Y^\vee) \to \hom_{\mathcal{C}}(Y, X^\vee)$, defined as the composition

$$\begin{aligned}\tau_{X,Y}\colon \hom_{\mathcal{C}}(X, Y^\vee) &\xrightarrow{\vee} \hom_{\mathcal{C}}(Y^{\vee\vee}, X^\vee) \xrightarrow{\hom_{\mathcal{C}}(\mu_Y, \mathrm{id}_{X^\vee})} \hom_{\mathcal{C}}(Y, X^\vee) \ , \\ f &\mapsto f^\vee \mu_Y \ ,\end{aligned}$$

with ${\tau_{X,Y}}^{-1} = \tau_{Y,X}$ (here one uses (2.1.2)).

In particular, for every object X in $\mathcal{C}$, one has a natural action of the cyclic group of order two $\mathrm{C}_2 = \langle\, \tau \mid \tau^2 = 1 \,\rangle$ on $\hom_{\mathcal{C}}(X, X^\vee)$, with $\tau(f) = \tau_{X,X}(f) = f^\vee \mu_X$.

We close this section by formalizing the idea of "functors that respect the duality".

Definition 2.1.3. A *morphism* $(\mathcal{C}, {}^\vee, \mu^{\mathcal{C}}) \to (\mathcal{D}, {}^\#, \mu^{\mathcal{D}})$ of categories with duality is a functor $F\colon \mathcal{C} \to \mathcal{D}$ together with a natural isomorphism $\eta\colon F \circ {}^\vee \xrightarrow{\cong} {}^\# \circ F$ such that for every object X of $\mathcal{C}$ the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(\mu^{\mathcal{C}}_X)} & F(X^{\vee\vee}) \\ {\scriptstyle \mu^{\mathcal{D}}_{F(X)}}\downarrow & & \downarrow{\scriptstyle \eta_{(X^\vee)}} \\ F(X)^{\#\#} & \xrightarrow[(\eta_X)^\#]{} & F(X^\vee)^\# \end{array}$$

commutes.

Notice that given a morphism of categories with duality as in the definition above, then for every pair of objects X and Y in $\mathcal{C}$ there is an induced map $\hom_{\mathcal{C}}(X, Y^\vee) \to \hom_{\mathcal{D}}(F(X), F(Y)^\#)$, defined as the composition

$$\hom_{\mathcal{C}}(X, Y^\vee) \xrightarrow{F} \hom_{\mathcal{D}}(F(X), F(Y^\vee)) \xrightarrow{\hom_{\mathcal{D}}(\mathrm{id}_{F(X)}, \eta_Y)} \hom_{\mathcal{D}}(F(X), F(Y)^\#)\,,$$

and the diagram

$$\begin{array}{ccc} \hom_{\mathcal{C}}(X, Y^\vee) & \xrightarrow{\tau^{\mathcal{C}}_{X,Y}} & \hom_{\mathcal{C}}(Y, X^\vee) \\ \downarrow & & \downarrow \\ \hom_{\mathcal{D}}(F(X), F(Y)^\#) & \xrightarrow{\tau^{\mathcal{D}}_{X,Y}} & \hom_{\mathcal{D}}(F(Y), F(X)^\#) \end{array}$$

commutes, since for every $f\colon X \to Y^\vee$ the diagram

$$\begin{array}{ccccc} F(Y) & \xrightarrow{F(\mu^{\mathcal{C}}_Y)} & F(Y^{\vee\vee}) & \xrightarrow{F(f^\vee)} & F(X^\vee) \\ {\scriptstyle \mu^{\mathcal{D}}_{F(Y)}}\downarrow & & \downarrow{\scriptstyle \eta_{(Y^\vee)}} & & \downarrow{\scriptstyle \eta_X} \\ F(Y)^{\#\#} & \xrightarrow[(\eta_Y)^\#]{} & F(Y^\vee)^\# & \xrightarrow[F(f)^\#]{} & F(X)^\# \end{array}$$

commutes.

In particular, for every object X in $\mathcal{C}$ the induced map $\hom_{\mathcal{C}}(X, X^\vee) \to \hom_{\mathcal{D}}(F(X), F(X)^\#)$ is equivariant with respect to the natural actions of C_2 defined above.

2.2. **Exact categories with duality.** Let us fix a commutative associative unital ring $\mathfrak{K}$ (usually $\mathfrak{K} = \mathbb{Z}$). We denote by $\mathtt{mod}/\mathfrak{K}$ the category of modules over $\mathfrak{K}$, which is a closed symmetric monoidal category with respect to $\otimes_{\mathfrak{K}}$, the tensor product over $\mathfrak{K}$. (For closed symmetric monoidal categories and enriched categories—a concept that is frequently used below—see for example [ML98, section VII.7, pages 180–181].)

Throughout this work "everything" is implicitly understood as being "over $\mathfrak{K}$", e.g., the words module and homomorphism refer to $\mathfrak{K}$-modules and $\mathfrak{K}$-linear homomorphisms, we say dg category instead of $\mathfrak{K}$-dg category (see section 2.4 below), and so on.

A $\mathfrak{K}$-*linear category* is a category enriched over $\mathtt{mod}/\mathfrak{K}$. Since this is the minimal structure possessed by all the categories we consider, let us agree that from now on the word category is short for $\mathfrak{K}$-linear category.

An additive category is a ($\mathfrak{K}$-linear) category with all finite sums (coproducts), and remember that this already implies that finite sums and finite products agree.

Now we recall the definition of an exact category, using Quillen's original terminology [Qui73, section 2] but following Keller's equivalent (and shorter) axiomatization [Kel90, appendix A].

Definition 2.2.1 (exact category)**.** Let $\mathcal{E}$ be an additive category. A *short sequence* in $\mathcal{E}$ is a pair of composable homomorphisms $L \to M \to N$ such that $L \to M$ is a kernel for $M \to N$ and $M \to N$ is a cokernel for $L \to$

M. Homomorphisms of short sequences are defined in the obvious way as commutative diagrams.

An *exact category* is an additive category $\mathcal{E}$ together with a choice of a class of short sequences, called *short exact sequences*, closed under isomorphisms and satisfying the axioms below. A short exact sequence is displayed as $L \rightarrowtail M \twoheadrightarrow N$ and in a short exact sequence the homomorphism $L \rightarrowtail M$ is called *admissible monomorphism*, the homomorphism $M \twoheadrightarrow N$ is called *admissible epimorphism*. Here come the axioms:

(i) The identity of the zero object is an admissible monomorphism and an admissible epimorphism.

(ii) The class of admissible monomorphisms is closed under composition and "cobase changes" by pushouts along arbitrary homomorphisms, i.e., given any admissible monomorphism $L \rightarrowtail M$ and any homomorphism $L \to L'$ their pushout M' exists and the induced homomorphism from L' to M' is again an admissible monomorphism.

$$\begin{array}{ccc} L & \rightarrowtail & M \\ \downarrow & & \downarrow \\ L' & \rightarrowtail & M' \end{array}$$

(iii) Dually, the class of admissible epimorphisms is closed under composition and "base changes" by pullbacks along arbitrary homomorphisms, i.e., given any admissible epimorphism $M \twoheadrightarrow N$ and any homomorphism $N' \to N$ their pullback M' exists and the induced homomorphism from M' to N' is again an admissible epimorphism.

$$\begin{array}{ccc} M' & \dashrightarrow & M \\ \downarrow & & \downarrow \\ N' & \longrightarrow & N \end{array}$$

A ($\mathfrak{K}$-linear) functor $\mathcal{E} \to \mathcal{F}$ between exact categories is called *exact* if it preserves short exact sequences, i.e., if it sends short exact sequences in $\mathcal{E}$ to short exact sequences in $\mathcal{F}$. An exact functor is said to *reflect* exactness if it detects short exact sequences, i.e., if any short sequence in $\mathcal{E}$ which is sent to a short exact sequence in $\mathcal{F}$ is itself exact.

Keller [Kel90, appendix A] proved that this set of axioms implies all other axioms in Quillen's original definition [Qui73, section 2]. In particular, any *split* sequence, i.e., any short sequence isomorphic to one of the form

$$L \xrightarrow{\binom{1}{0}} L \oplus M \xrightarrow{(0\ 1)} M \tag{2.2.2}$$

must be exact. Therefore exact functors are automatically additive.

There is a minimal choice of short exact sequences for any additive category $\mathcal{E}$. Indeed, the class of short sequences isomorphic to those of the form (2.2.2) satisfies the axioms above and hence defines on any additive category $\mathcal{E}$ a structure of an exact category. We call this a *split exact category*.

Notice that the definition of exact category is self-dual, in the sense that an additive category $\mathcal{E}$ is exact if and only if its opposite category $\mathcal{E}^{\mathrm{op}}$ is exact, where $L \to M \to N$ is a short exact sequence in $\mathcal{E}$ if and only if $L^{\mathrm{op}} \leftarrow M^{\mathrm{op}} \leftarrow N^{\mathrm{op}}$ is a short exact sequence in $\mathcal{E}^{\mathrm{op}}$.

Any abelian category with the usual notion of short exact sequences can be seen as an exact category. If $\mathcal{E}$ is a full subcategory of an exact category $\mathcal{F}$ we say that $\mathcal{E}$ is closed under extensions in $\mathcal{F}$ if for all short exact sequences $L \rightarrowtail M \twoheadrightarrow N$ in $\mathcal{F}$ with L and N in $\mathcal{E}$, then M is isomorphic to an object of $\mathcal{E}$. Let $\mathcal{E}$ be a full subcategory of an abelian category $\mathcal{A}$, and suppose that $\mathcal{E}$ is closed under extensions in $\mathcal{A}$. Then $\mathcal{E}$ inherits from $\mathcal{A}$ a structure of an exact category. Conversely, the Gabriel-Quillen embedding theorem [Gab62, II section 2; Qui73, section 2; TT90, appendix A] states that given any small exact category $\mathcal{E}$ there is an abelian category $\mathcal{A}$ and a fully faithful exact functor $\mathcal{E} \to \mathcal{A}$ that reflects exactness, and such that $\mathcal{E}$ is closed under extension in $\mathcal{A}$. A canonical construction is provided by the Yoneda embedding h of $\mathcal{E}$ into the abelian category of left-exact functors $\mathcal{E}^{\mathrm{op}} \to \mathtt{mod}/\mathfrak{K}$, defined by sending an object M in $\mathcal{E}$ to the representable functor $h(M) = \hom_{\mathcal{E}}(?, M)$.

Now we introduce our two main examples; here $\mathfrak{K} = \mathbb{Z}$ for simplicity.

Example 2.2.3. Let R be an associative unital ring. We denote by $\mathtt{fgp}/R$ the additive category of finitely generated and projective left R-modules, and view it as a split exact category.

Example 2.2.4. Let S be a scheme. Let $\mathtt{vect}/S$ be the category of vector bundles over S, i.e., the category of locally free $\mathcal{O}_S$-modules of finite rank—see for example [Har77, exercise II.5.18 on page 128-129]. It is an extension closed full subcategory of $\mathtt{qcoh}/S$, the abelian category of quasi coherent $\mathcal{O}_S$-modules, hence $\mathtt{vect}/S$ inherits a structure of an exact category.

If R is a commutative ring then vector bundles over the affine scheme $\mathrm{Spec}(R)$ are nothing but finitely generated projective modules over R, i.e., one has an equivalence of exact categories $\mathtt{vect}/\mathrm{Spec}(R) \simeq \mathtt{fgp}/R$. But notice that for general, non affine schemes S the exact category $\mathtt{vect}/S$ is not split.

We close this section with the definition and main examples of exact categories with duality.

Definition 2.2.5. An *exact category with duality* is a category with duality $(\mathcal{E}, {}^\vee, \mu)$ in the sense of definition 2.1.1 such that $\mathcal{E}$ is an exact category and the functor ${}^\vee \colon \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is exact.

A *morphism* of exact categories with duality is a morphism (F, η) of the underlying categories with duality in the sense of definition 2.1.3 such that the functor F is exact.

Recall that exact functors are in particular additive, and natural transformations between additive functors are automatically additive. For this reason there is no need to impose compatibility conditions on the natural transformations in the definition above. In particular, for every pair of objects L and M in $\mathcal{E}$ one automatically has that $\mu_{L \oplus M} = \mu_L \oplus \mu_M$.

All the remarks following definition 2.1.1 apply also here. For example, given an exact category with duality $(\mathcal{E}, {}^\vee, \mu)$, we obtain a new exact category with duality $(\mathcal{E}, {}^\vee, -\mu)$ by changing the sign of the identification with the double dual. This allows us in section 3.1 to treat symmetric and skew-symmetric forms simultaneously, skew-symmetric forms simply being symmetric forms in the category with duality $(\mathcal{E}, {}^\vee, -\mu)$.

Furthermore, for every object M in $\mathcal{E}$, the module $\hom_{\mathcal{E}}(M, M^\vee)$ inherits a natural linear action of the cyclic group of order two C_2.

Example 2.2.6. Let R be a *ring with involution*, i.e., a ring R together with a homomorphism $^-\colon R^{\mathrm{op}} \to R$ such that $\overline{\overline{r}} = r$ for all $r \in R$ (then automatically $\overline{1} = 1$).

Examples of rings with involution are commutative rings with the identity as involution (notice that the identity defines an involution if and only if the ring is commutative); the field of complex numbers with conjugation; or group algebras $\mathfrak{K}^\omega G$ of groups G together with a group homomorphism $\omega\colon G \to \{\pm 1\}$ to the cyclic group of order two (for example, the trivial homomorphism), with involution given by extending linearly the group inverse and twisting with ω: explicitly $\overline{\sum \lambda_g g} = \sum \omega(g)\lambda_g g^{-1}$.

If R is a ring with involution, then the involution defines an isomorphism between the categories of left and right R-modules. In particular, if M is a left-module over R, then its dual $M^\vee = \hom_R(M, R)$ is naturally a right R-module and we use the involution to consider it as a left R-module again.

Let $\mathtt{fgp}/R$ be the split exact category of finitely generated projective left modules over R, see example 2.2.3. Then the dual $M^\vee = \hom_R(M, R)$ of any finitely generated projective module M is again finitely generated and projective. The canonical homomorphism $\mu_M\colon M \to M^{\vee\vee}$, that sends an element $m \in M$ to the homomorphism $\mathrm{ev}_m\colon M^\vee \to R$ defined by $\mathrm{ev}_m(f) = \overline{f(m)}$, is an isomorphism for any finitely generated projective module M. This structure turns $\mathtt{fgp}/R$ into a split exact category with duality.

Example 2.2.7. Let S be a scheme and $\mathtt{vect}/S$ the exact category of vector bundles on S, see example 2.2.4. Fix a line bundle $\mathcal{L}$ over S, for example $\mathcal{L} = \mathcal{O}_S$, the trivial line bundle. For every vector bundle M in $\mathtt{vect}/S$ define $M^\vee = \hom_{\mathcal{O}_S}(M, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{L}$. The natural identification $\mu_M\colon M \to M^{\vee\vee}$ is defined as usual, using the canonical isomorphism $\hom_{\mathcal{O}_S}(\mathcal{L}, \mathcal{O}_S) \otimes_{\mathcal{O}_S} \mathcal{L} \cong \mathcal{O}_S$. This structure turns $\mathtt{vect}/S$ into an exact category with duality.

2.3. **Categories of chain complexes.** Here we recall some basic definitions about chain complexes. If we go into considerable detail in spite of the well-known nature of the material, it is in order to fix once and for all and explicitly our conventions, especially about signs.

Let $\mathcal{E}$ be a ($\mathfrak{K}$-linear) category. We denote by $\mathtt{ch}\,\mathcal{E}$ the category whose objects are chain complexes in $\mathcal{E}$; for us chain complexes are always understood to be indexed over the integers $\mathbb{Z}$, and we follow the customs of the tribe of topologists, for whom the differentials in a complex X decrease degree, $\partial^X_j\colon X_j \to X_{j-1}$. We denote by $\mathrm{chhom}_{\mathcal{E}}(X, Y)$ the module of (degree preserving) chain homomorphisms between two chain complexes X and Y. Let $\mathtt{ch}^{\mathrm{b}}\mathcal{E}$ be the full subcategory consisting of the bounded chain complexes, i.e., those chain complexes X such that $X_j \neq 0$ only for a finite number of j's.

Now let $\mathcal{E}$ be an additive category. Notice that $\mathtt{ch}\,\mathcal{E}$ obviously inherits the structure of an additive category, with $\mathtt{ch}^{\mathrm{b}}\mathcal{E}$ an additive subcategory.

For every $n \in \mathbb{Z}$ there are two functors $S^n, D^n : \mathcal{E} \to \mathsf{ch}^{\mathrm{b}}\mathcal{E} \subset \mathsf{ch}\,\mathcal{E}$ defined by setting

$$\begin{array}{llccccccccccc} S^n M & : & \ldots \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \ldots & , \\ D^n M & : & \ldots \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{\mathrm{id}} & M & \longrightarrow & 0 & \longrightarrow \ldots & , \\ & & & {\scriptstyle j=n+1} & & {\scriptstyle j=n} & & {\scriptstyle j=n-1} & & {\scriptstyle j=n-2} & & \end{array}$$

for every object M of $\mathcal{E}$, and analogously for homomorphisms.

Notice that if $\mathcal{E}$ has kernels and cokernels, in particular if $\mathcal{E}$ is an abelian category like for example $\mathsf{mod}/\mathfrak{K}$, then one has the following adjunctions:

$$\begin{aligned} \mathrm{chhom}_{\mathcal{E}}(S^n M, X) &\cong \hom_{\mathcal{E}}(M, \ker \partial_n^X) \ , \\ \mathrm{chhom}_{\mathcal{E}}(X, S^n M) &\cong \hom_{\mathcal{E}}(\mathrm{coker}\, \partial_{n+1}^X, M) \ , \end{aligned}$$

and

$$\begin{aligned} \mathrm{chhom}_{\mathcal{E}}(D^n M, X) &\cong \hom_{\mathcal{E}}(M, X_n) \ , \\ \mathrm{chhom}_{\mathcal{E}}(X, D^n M) &\cong \hom_{\mathcal{E}}(X_{n-1}, M) \ . \end{aligned}$$

As usual we abbreviate $Z_n X = \ker \partial_n^X$.

Now suppose that $(\mathcal{E}, \otimes)$ is an additive symmetric monoidal category, e.g., $(\mathsf{mod}/\mathfrak{K}, \otimes_{\mathfrak{K}})$. Then the product of two chain complexes X and Y is the chain complex $X \otimes Y$ defined by

$$(X \otimes Y)_j = \bigoplus_{\substack{s,t \in \mathbb{Z} \\ s+t=j}} X_s \otimes Y_t \ , \tag{2.3.1}$$

$$\begin{aligned} \partial_j^{X \otimes Y}|_{X_s \otimes Y_t} : X_s \otimes Y_t \xrightarrow{\begin{pmatrix} \partial_s^X \otimes \mathrm{id}_{Y_t} \\ (-1)^{-s}\, \mathrm{id}_{X_s} \otimes \partial_t^Y \end{pmatrix}} & X_{s-1} \otimes Y_t \oplus X_s \otimes Y_{t-1} \\ & \subset (X \otimes Y)_{j-1} \ . \end{aligned}$$

In the formula for the differential, some signs need to be introduced in order to ensure that $\partial^2 = 0$. Our choice agrees well with certain geometric constructions, and follows the general convention that if we pass something of degree i through something of degree j, we should multiply by the sign $(-1)^{ij}$, and we regard ∂ as being of degree -1. Notice that $S^0 E \otimes X \cong X \cong X \otimes S^0 E$ for every chain complex X, where E is the neutral element of $\mathcal{E}$ with respect to the symmetric monoidal product $\otimes$.

The associativity isomorphisms in $\mathcal{E}$ applied degree-wise and to each summand $X_s \otimes Y_t$ in (2.3.1) induce associativity isomorphisms in $\mathsf{ch}\,\mathcal{E}$, whereas the commutativity isomorphism in $\mathsf{ch}\,\mathcal{E}$ are defined by applying the commutativity isomorphism in $\mathcal{E}$ to each summand $X_s \otimes Y_t$ in (2.3.1) with sign changed to $(-1)^{st}$.

The category $\mathsf{ch}\mathcal{E}$ thus inherits the structure of a symmetric monoidal category. If furthermore the symmetric monoidal category $\mathcal{E}$ is closed, then the same is true for $\mathsf{ch}\,\mathcal{E}$. For example, consider the closed symmetric monoidal category $\mathsf{mod}/\mathfrak{K}$. Then given two chain complexes X and Y we define a chain complex $\hom_{\mathsf{ch}\,\mathsf{mod}/\mathfrak{K}}(X, Y)$ precisely in such a way that $\hom_{\mathsf{ch}\,\mathsf{mod}/\mathfrak{K}}(X, -)$ is right adjoint to $- \otimes_{\mathfrak{K}} X$, i.e.,

$$\mathrm{chhom}_{\mathsf{mod}/\mathfrak{K}}(W \otimes_{\mathfrak{K}} X, Y) \cong \mathrm{chhom}_{\mathsf{mod}/\mathfrak{K}}(W, \hom_{\mathsf{ch}\,\mathsf{mod}/\mathfrak{K}}(X, Y)) \ .$$

This yields the following definition:

$$\hom_{\mathsf{ch\,mod}/\mathfrak{K}}(X,Y)_j = \prod_{s\in\mathbb{Z}} \hom_{\mathfrak{K}}(X_s, Y_{s+j}) \quad \ni \quad f = (f_s\colon X_s \to Y_{s+j})_{s\in\mathbb{Z}},$$

$$\partial_j f = (\partial^Y_{s+j} f_s - (-1)^{-j} f_{s-1}\partial^X_s)_{s\in\mathbb{Z}}, \tag{2.3.2}$$

or in other words, $\partial^Y_{s+j}(f(x_s)) = \partial_{s+j}(f)(x_s) + (-1)^{-j} f(\partial^X_s x_s)$. Notice that for $X = Y$ the differential of $\hom_{\mathsf{ch\,mod}/\mathfrak{K}}(X,X)$ is equal to $[\partial^X, -]$, the graded commutator with $\partial^X \in \hom_{\mathsf{ch\,mod}/\mathfrak{K}}(X,X)_{-1}$.

Now let $\mathcal{E}$ be any (additive) category. Notice that the definition (2.3.2) makes sense for chain complexes in any category $\mathcal{E}$, yielding a chain complex $\hom_{\mathsf{ch}\,\mathcal{E}}(X,Y)$ of modules (over $\mathfrak{K}$) for every pair of chain complexes X and Y in $\mathcal{E}$. These make $\mathsf{ch}\,\mathcal{E}$ into a dg category, see the next section 2.4.

For all $n \in \mathbb{Z}$ we define the *n-th suspension* $\Sigma^n X$ of a chain complex X by setting

$$(\Sigma^n X)_j = X_{j-n}\ , \qquad \partial^{\Sigma^n X}_j = (-1)^n \partial^X_{j-n}\ . \tag{2.3.3}$$

Notice that the identity on X yields an isomorphism of degree n in $Z_n \hom_{\mathsf{ch}\,\mathcal{E}}(X, \Sigma^n X)$.

We make suspensions into functors (actually, self-isomorphisms of $\mathsf{ch}\mathcal{E}$) by just shifting indices on chain homomorphisms (i.e., no signs added). Notice that $\Sigma^0 = \mathrm{id}$ and $\Sigma^n\Sigma^m = \Sigma^{m+n}$ for all $m,n \in \mathbb{Z}$, and notice that if $\mathcal{E}$ is a symmetric monoidal category then $\Sigma^n X \cong S^n E \otimes_{\mathfrak{K}} X$ (this also explains the signs appearing in the definition of the suspensions).

For every $n \in \mathbb{Z}$ one has

$$Z_n(\hom_{\mathsf{ch}\,\mathcal{E}}(X,Y)) = \mathrm{chhom}_{\mathcal{E}}(\Sigma^n X, Y)\ ,$$

and $H_n(\hom_{\mathsf{ch}\,\mathcal{E}}(X,Y))$ equals the module of chain homotopy classes of chain homomorphisms $\Sigma^n X \to Y$; moreover

$$\hom_{\mathsf{ch}\,\mathcal{E}}(\Sigma^{-n}X, Y) = \Sigma^n \hom_{\mathsf{ch}\,\mathcal{E}}(X,Y) = \hom_{\mathsf{ch}\,\mathcal{E}}(X, \Sigma^n Y)\ .$$

For a chain homomorphism $f\colon X \to Y$ define

$$\mathrm{cone}(f)_j = Y_j \oplus X_{j-1}\ , \qquad \partial_j = \begin{pmatrix} \partial^Y_j & f_{j-1} \\ 0 & -\partial^X_{j-1} \end{pmatrix} = \begin{pmatrix} \partial^Y_j & f_{j-1} \\ 0 & \partial^{\Sigma X}_j \end{pmatrix}\ ,$$

i.e., $\partial_j(y,x) = (\partial^Y_j y + f_{j-1}x, -\partial^X_{j-1}x)$. Notice that $\Sigma X \cong \mathrm{cone}(X \to 0)$ and $X \oplus Y = \mathrm{cone}(\Sigma^{-1}Y \xrightarrow{0} X)$. There are obvious homomorphisms of degree zero

$$\Sigma X \xrightarrow{i} \mathrm{cone}(f) \xrightarrow{p} \Sigma X\ , \quad Y \xrightarrow{j} \mathrm{cone}(f) \xrightarrow{s} Y\ , \tag{2.3.4}$$

satisfying

$$pi = \mathrm{id}_{\Sigma X}\ , \quad sj = \mathrm{id}_Y\ , \quad si = 0\ , \quad pj = 0\ , \quad ip \oplus js = \mathrm{id}_{Y\oplus\Sigma X}\ ,$$

and

$$\partial j = 0\ , \quad \partial p = 0\ , \quad \partial i = jf\ , \quad \partial s = -fp\ .$$

In particular, in the category of $\mathbb{Z}$-graded objects in $\mathcal{E}$, $\mathrm{cone}(f)$ is just the direct sum of Y and ΣX, and there are chain homomorphisms

$$X \xrightarrow{f} Y \xrightarrow{j} \mathrm{cone}(f) \xrightarrow{p} \Sigma X$$

such that $pj = 0$ and the other two compositions jf and $(\Sigma f)p$ are null-homotopic.

The cone has the following properties. Let $g\colon Y \to Z$ be a chain homomorphism such that gf is null-homotopic. Then the choice of a null-homotopy $h \in \hom_{\mathfrak{K}}(X,Z)_1$, $\partial h = gf$, defines a chain homomorphism $\operatorname{cone}(f) \to Z$, $(y,x) \mapsto gy + hx$, extending g over $\operatorname{cone}(f)$. On the other hand, if $g\colon W \to X$ is a chain homomorphism such that fg is null-homotopic, then the choice of a null-homotopy $h \in \hom_{\mathfrak{K}}(W,Y)_1$, $\partial h = fg$, defines a chain homomorphism $W \to \Sigma^{-1}\operatorname{cone}(f)$, $w \mapsto (-hw, gw)$, lifting g to $\Sigma^{-1}\operatorname{cone}(f)$.

From the definition of the cone (see also section 2.5) it follows immediately that if $f\colon X \to Y$ is a chain homomorphism and Z any chain complex, then

$$\hom_{\mathsf{ch}\,\mathcal{E}}(Z, \operatorname{cone}(f)) \cong \operatorname{cone}(\hom_{\mathsf{ch}\,\mathcal{E}}(Z,X) \to \hom_{\mathsf{ch}\,\mathcal{E}}(Z,Y)) \ ,$$

and dually

$$\hom_{\mathsf{ch}\,\mathcal{E}}(\operatorname{cone}(f), Z) \cong \operatorname{cone}(\hom_{\mathsf{ch}\,\mathcal{E}}(Y,Z) \to \hom_{\mathsf{ch}\,\mathcal{E}}(X,Z)) \ .$$

Now let $\mathcal{E}$ be an exact category. Recall that a chain complex X in $\mathsf{ch}\,\mathcal{E}$ is called *acyclic* if for every $n \in \mathbb{Z}$ there exist short exact sequences

$$W_{n+1} \overset{m_{n+1}}{\rightarrowtail} X_n \overset{e_n}{\twoheadrightarrow} W_n$$

such that each differential $\partial_n\colon X_n \to X_{n-1}$ decomposes as

$$\partial_n = m_n e_n \colon X_n \overset{e_n}{\twoheadrightarrow} W_n \overset{m_n}{\rightarrowtail} X_{n-1} \ ,$$

i.e., as in the following commutative diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{n+1} & \xrightarrow{\partial_{n+1}} & X_n & \xrightarrow{\partial_n} & X_{n-1} & \longrightarrow & \cdots \\
 & \nearrow & & \searrow^{e_{n+1}} \quad \nearrow_{m_{n+1}} & & \searrow^{e_n} \quad \nearrow_{m_n} & & \searrow & \\
\cdots & & & W_{n+1} & & W_n & & & \cdots
\end{array} .$$

We denote by $\mathsf{ach}\mathcal{E}$ the full subcategory of $\mathsf{ch}\mathcal{E}$ consisting of the acyclic chain complexes, and by $\mathsf{ach}^{\mathsf{b}}\mathcal{E}$ its full subcategory consisting of the bounded ones. It is easy to check that if $f\colon X \to Y$ is a chain homomorphism between acyclic chain complexes, then $\operatorname{cone}(f)$ is again acyclic (see for example [Nee90, lemma 1.1 on page 398]). More on this is to be found in section 2.7 below, where we recall the definition of the derived category.

Finally, let $\mathcal{E}$ be a category with duality. Then for every chain complex X in $\mathsf{ch}\,\mathcal{E}$ we define $X^\vee$ by putting

$$(X^\vee)_j = (X_{-j})^\vee \quad \text{and} \quad \partial_j^{(X^\vee)} = (-1)^{j+1}(\partial_{j-1}^X)^\vee \ .$$

The signs are chosen in such a way that if $\mathcal{E} = \mathsf{ch}\,\mathsf{mod}/R$ for a ring with involution R (see example 2.2.6) then $X^\vee = \hom_{\mathsf{ch}\,\mathsf{mod}/R}(X, S^0R)$, and also if $\mathcal{E} = \mathsf{vect}/S$ for a scheme S (see example 2.2.7) then $X^\vee = \hom_{\mathsf{ch}\,\mathsf{vect}/S}(X, S^0\mathcal{O}_S) \otimes_{\mathcal{O}_S} S^0\mathcal{L}$. Notice that taking duals of homomorphisms (no sign changes) defines a chain homomorphism $\hom_{\mathsf{ch}\,\mathcal{E}}(X,Y) \to \hom_{\mathsf{ch}\,\mathcal{E}}(Y^\vee, X^\vee)$. Applying the natural isomorphism μ degree-wise one gets a chain isomorphism $\mu\colon X \xrightarrow{\cong} X^{\vee\vee}$. Therefore $\mathsf{ch}\mathcal{E}$ is again a category with duality (even better, it is a dg category with duality, see section 2.4 below).

Notice that for every $n \in \mathbb{Z}$ and for every chain complex X one has that $(\Sigma^n X)^\vee = \Sigma^{-n}(X^\vee)$, and for every chain homomorphism $f\colon X \to Y$ one has $(\operatorname{cone}(f))^\vee = \Sigma^{-1}\operatorname{cone}(f^\vee)$.

Finally notice that if $\mathcal{E}$ is an exact category with duality, then from the definition of acyclicity it follows that the dual $X^\vee$ of every acyclic chain complex X is again acyclic.

2.4. **Differential graded (dg) categories with duality.**

Definition 2.4.1. A *differential graded category*, or *dg category*, is a category enriched over $\mathsf{ch}\,\mathsf{mod}/\mathfrak{K}$; dg functors and dg subcategories are just enriched ones. A graded category $\mathcal{C}$ (explicitly, a $\mathbb{Z}$-graded $\mathfrak{K}$-linear category) can be thought of as a dg category where the differentials in the $\hom_{\mathcal{C}}$-chain complexes are all zero.

By definition any dg category $\mathcal{C}$ has an underlying graded category, denoted by $\mathcal{C}^{\mathrm{gr}}$, which is obtained by forgetting the differentials in the $\hom_{\mathcal{C}}$-chain complexes.

More interestingly, given a dg category $\mathcal{C}$ one defines its associated *homotopy category* $\mathrm{ho}(\mathcal{C})$ by taking the homology of the $\hom_{\mathcal{C}}$-chain complexes:

$$\mathrm{obj}(\mathrm{ho}(\mathcal{C})) = \mathrm{obj}(\mathcal{C}) \quad \text{and} \quad \hom_{\mathrm{ho}\,\mathcal{C}}(X,Y) = H_*(\hom_{\mathcal{C}}(X,Y)) \ .$$

This is another graded category.

Two objects X and Y of a dg category $\mathcal{C}$ are called *dg isomorphic* if there exists a closed isomorphism of degree zero in $\hom_{\mathcal{C}}(X,Y)$, i.e., $f \in \hom_{\mathcal{C}}(X,Y)_0$ with $\partial f = 0$; they are *homotopy equivalent* if there exists an isomorphism of degree zero in $\hom_{\mathrm{ho}\,\mathcal{C}}(X,Y)$. Of course, dg isomorphic objects are in particular homotopy equivalent.

Let $f\colon \mathcal{B} \to \mathcal{C}$ be a dg functor. We say that f is a *dg equivalence* if f is fully faithful and every object of $\mathcal{C}$ is dg isomorphic to an object in the image of f. We say that f is a *quasi-equivalence* if $\mathrm{ho}(f)$ is fully faithful and every object of $\mathcal{C}$ is homotopy equivalent to an object in the image of f. Of course, dg equivalences are in particular quasi-equivalences.

Example 2.4.2. Given a ($\mathfrak{K}$-linear) category $\mathcal{E}$ the categories $\mathsf{ch}\,\mathcal{E}$ and $\mathsf{ch}^{\mathrm{b}}\mathcal{E}$ defined in section 2.3 are (the motivating) examples of dg categories, with respect to the $\hom_{\mathsf{ch}\,\mathcal{E}}$-chain complexes defined in (2.3.2). The units and compositions are given by chain homomorphisms $S^0\mathfrak{K} \to \hom_{\mathsf{ch}\,\mathcal{E}}(X,X)$ and

$$\hom_{\mathcal{C}}(Y,Z) \otimes_{\mathfrak{K}} \hom_{\mathcal{C}}(X,Y) \to \hom_{\mathcal{C}}(X,Z)$$

for all objects X, Y, and Z of $\mathcal{C}$: Indeed, one readily verifies that for every $f \in \hom_{\mathcal{C}}(Y,Z)_s$ and $g \in \hom_{\mathcal{C}}(X,Y)_t$ the following "Leibniz rule" holds:

$$\partial(fg) = (\partial f)g + (-1)^s f(\partial g) \ .$$

The associated homotopy category $\mathrm{ho}(\mathsf{ch}\,\mathcal{E})$ is the chain homotopy category: For every $n \in \mathbb{Z}$, one has that $\hom_{\mathrm{ho}(\mathsf{ch}\,\mathcal{E})}(X,Y)_n = H_n(\hom_{\mathsf{ch}\,\mathcal{E}}(X,Y))$ is the module of chain homotopy classes of chain homomorphisms $\Sigma^n X \to Y$.

Example 2.4.3. A dg category with one object is nothing but a so-called dga, or differential graded ($\mathfrak{K}$-)algebra. Thus a dg category can be thought of as dga with many objects. Given a dga or more generally a dg category $\mathcal{C}$, a left *dg module over* $\mathcal{C}$ is a dg functor $M\colon \mathcal{C} \to \mathsf{ch}\,\mathsf{mod}/\mathfrak{K}$. We denote by $\mathsf{dgmod}/\mathcal{C}$ the dg category they form.

For every dg category $\mathcal{C}$ there is a natural dg functor $h\colon \mathcal{C} \to \mathtt{dgmod}/\mathcal{C}^{\mathrm{op}}$, the *Yoneda embedding*, defined by sending an element X of $\mathcal{C}$ to the dg functor $\hom_{\mathcal{C}}(?, X)$. As in the classical, non enriched case one verifies that the Yoneda embedding is fully faithful

$$\hom_{\mathcal{C}}(X, Y) \cong \hom_{\mathtt{dgmod}/\mathcal{C}^{\mathrm{op}}}(h(X), h(Y)) \ ,$$

and $\hom_{\mathtt{dgmod}/\mathcal{C}^{\mathrm{op}}}(h(X), M) = M(X)$.

Definition 2.4.4. A *dg category with duality* is a category with duality $(\mathcal{C}, {}^{\vee}, \mu)$ in the sense of definition 2.1.1 such that $\mathcal{C}$ is a dg category, the functor ${}^{\vee}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is a dg functor, and μ is a natural transformation of dg functors.

A *morphism* of dg categories with duality is a morphism (F, η) of categories with duality as in definition 2.1.3 but in the enriched sense, i.e., the functor F is a dg functor and η is a natural transformation of dg functors.

In particular, for every object X in a dg category with duality $\mathcal{C}$, the chain complex $\hom_{\mathcal{C}}(X, X^{\vee})$ has an action of C_2. The homotopy category of a dg category with duality is a graded category with duality (in the obvious sense).

Example 2.4.5. Given an exact category with duality $\mathcal{E}$ in the sense of definition 2.2.5 then $\mathsf{ch}\,\mathcal{E}$ and $\mathsf{ch}^{\mathsf{b}}\mathcal{E}$ are dg categories with duality, see the end of the previous section 2.3.

2.5. **Pre-triangulated dg categories with duality.** The chief examples of dg categories are the categories of chain complexes in additive categories (see example 2.4.2) or of dg modules over some given dg category (see example 2.4.3). As is well known, both these examples have more structure, given by suspensions and cones. In this section we define what suspensions and cones are in arbitrary dg categories (where they in general need not exist), dealing first with suspensions and then, in a completely parallel manner, with cones. This leads to the definition of pre-triangulated dg categories, which in a nutshell are dg categories where suspensions and cones exist up to homotopy, and whose homotopy categories are therefore naturally triangulated.

Pre-triangulated dg categories were introduced, under the name of framed or enhanced triangulated categories, by Bondal and Kapranov [BK90]—and subsequently they were also studied by Keller [Kel99], who called them exact dg categories. Our presentation here also owes much to the more recent works of Bondal, Larsen, and Lunts [BLL04] and Drinfeld [Dri04].

Definition 2.5.1 (Compare (2.3.3))**.** Let $\mathcal{C}$ be a dg category. Let X be an object of $\mathcal{C}$ and let $n \in \mathbb{Z}$ be an integer. An object X' of $\mathcal{C}$ is called an *n-suspension of X* if there exists a closed isomorphism of degree n from X to X', i.e., if there exist homomorphisms $f \in \hom_{\mathcal{C}}(X, X')_n$ and $f' \in \hom_{\mathcal{C}}(X', X)_{-n}$ with

$$\partial f = 0 \ , \quad \partial f' = 0 \ , \quad \text{and} \quad f'f = \mathrm{id}_X \ , \quad ff' = \mathrm{id}_{X'} \ .$$

Notice that if X' is an n-suspension of X and X'' is an m-suspension of X', then X'' is an $(n+m)$-suspension of X. Moreover, if X' is an n-suspension of X, then X is a $(-n)$-suspension of X'.

Lemma 2.5.2. *Let $\mathcal{C}$ be a dg category, and let X be an object of $\mathcal{C}$ and $n \in \mathbb{Z}$. Suppose that there exists an n-suspension X' of X. Then:*

1. *X' is unique up to dg isomorphism.*
2. *In the opposite dg category $\mathcal{C}^{\mathrm{op}}$, X'^{op} is a $(-n)$-suspension of X^{op}.*
3. *If $\mathcal{D}$ is another dg category and $F\colon \mathcal{C} \to \mathcal{D}$ a dg functor, then $F(X')$ is an n-suspension of $F(X)$.*

Proof. Follows immediately from the definition. □

Definition 2.5.3. We say that a dg category $\mathcal{C}$ is *stable under suspensions* if for all objects X in $\mathcal{C}$ and for all $n \in \mathbb{Z}$ there exists an n-suspension of X in $\mathcal{C}$.

Example 2.5.4. Given a category $\mathcal{E}$, the categories $\mathsf{ch}\,\mathcal{E}$, $\mathsf{ch}^{\mathrm{b}}\mathcal{E}$ of section 2.3 are (the motivating) examples of dg categories stable under suspensions, see (2.3.3).

This applies in particular to the dg category $\mathsf{ch}\ \mathsf{mod}/\mathfrak{K}$, and therefore, given any dg category $\mathcal{C}$, also the functor dg category $\mathsf{dgmod}/\mathcal{C}$ is stable under suspensions.

Notice that this definition is self-dual, in the sense that a dg category $\mathcal{C}$ is closed under suspensions if and only if its opposite dg category $\mathcal{C}^{\mathrm{op}}$ is so, see lemma 2.5.2(2).

Definition 2.5.5. Given a dg category $\mathcal{C}$ we define a new dg category $\mathcal{C}^{\Sigma}$ whose objects are pairs $(X, m) \in \mathrm{obj}(\mathcal{C}) \times \mathbb{Z}$ and with

$$\hom_{\mathcal{C}^{\Sigma}}((X,m),(Y,n)) = \Sigma^{n-m} \hom_{\mathcal{C}}(X,Y)\ .$$

Lemma 2.5.6. *Let $\mathcal{C}$ be a dg category.*

1. *The dg category $\mathcal{C}^{\Sigma}$ contains $\mathcal{C}$ as a full dg subcategory and is stable under suspensions.*
2. *Every object of $\mathcal{C}^{\Sigma}$ can be obtained from objects of $\mathcal{C}$ by taking suspensions. Hence $\mathcal{C}$ is stable under suspensions if and only if the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\Sigma}$ is a dg equivalence.*
3. *The Yoneda embedding $h\colon \mathcal{C} \to \mathsf{dgmod}/\mathcal{C}^{\mathrm{op}}$ factorizes over the fully faithful dg functor*
$$h^{\Sigma}\colon \mathcal{C}^{\Sigma} \to \mathsf{dgmod}/\mathcal{C}^{\mathrm{op}} \quad , \quad (X,m) \mapsto \Sigma^m h(X)\ .$$

Now let $\mathcal{D}$ be another dg category and $F\colon \mathcal{C} \to \mathcal{D}$ a dg functor.

4. *If $\mathcal{D}$ is stable under suspensions then there exist a unique (up to dg isomorphism) dg functor $F'\colon \mathcal{C}^{\Sigma} \to \mathcal{D}$ that extends F.*
5. *If F is a quasi equivalence or a dg equivalence then the same is true for the induced dg functor $F^{\Sigma}\colon \mathcal{C}^{\Sigma} \to \mathcal{D}^{\Sigma}$.*

Proof. Follows immediately from the definition. □

Remark 2.5.7. If $\mathcal{C}$ is a dg category that is stable under suspensions, then there is a dg functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$, which is an equivalence of categories, i.e., a self-equivalence. Conversely, if $\mathcal{C}$ is a category together with a self-equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$, then $\mathcal{C}$ can be given the structure of a graded category by putting $\hom_{\mathcal{C}}(X,Y)_n = \hom_{\mathcal{C}}(\Sigma^n X, Y)$. With respect to this structure of a graded category, $\mathcal{C}$ is stable under suspensions and the functor Σ is a graded functor.

Definition 2.5.8 (Compare (2.3.4))**.** Let $\mathcal{C}$ be a dg category. Let $f\colon X \to Y$ be a closed homomorphism of degree zero in $\mathcal{C}$, i.e., $f \in \hom_{\mathcal{C}}(X,Y)_0$ with $\partial f = 0$. An object Z in $\mathcal{C}$ is called a *cone* of f, if there exist a 1-suspension X' of X and homomorphisms of degree zero

$$X' \xrightarrow{i} Z \xrightarrow{p} X' \ , \quad Y \xrightarrow{j} Z \xrightarrow{s} Y \ ,$$

satisfying

$$pi = \mathrm{id}_{X'} \ , \quad sj = \mathrm{id}_Y \ , \quad si = 0 \ , \quad pj = 0 \ , \quad ip \oplus js = \mathrm{id}_{Y \oplus X'} \ ,$$

and

$$\partial j = 0 \ , \quad \partial p = 0 \ , \quad \partial i = jf \ , \quad \partial s = -fp \ .$$

Lemma 2.5.9. *Let $\mathcal{C}$ be a dg category and let $f\colon X \to Y$ be a closed homomorphism of degree zero in $\mathcal{C}$. Suppose that there exists a cone Z of f. Then:*

(1) Z is unique up to dg isomorphism.
(2) If a (-1)-suspension Z' of Z exists, then in the opposite dg category $\mathcal{C}^{\mathrm{op}}$ a cone of f^{op} is given by Z'^{op} (which is a 1-suspension of Z^{op} in $\mathcal{C}^{\mathrm{op}}$).
(3) If $\mathcal{D}$ is another dg category and $F\colon \mathcal{C} \to \mathcal{D}$ a dg functor, then $F(Z)$ is a cone of $F(f)$.

Proof. (1) The first set of conditions in the definition of a cone means that Z is the direct sum of X' and Y in the graded category $\mathcal{C}^{\mathrm{gr}}$ underlying $\mathcal{C}$. Thus for any object W in $\mathcal{C}$ there are isomorphisms of graded modules

$$\hom_{\mathcal{C}}(W,Z) \cong \hom_{\mathcal{C}}(W,X') \oplus \hom_{\mathcal{C}}(W,Y) \ ,$$
$$\hom_{\mathcal{C}}(Z,W) \cong \hom_{\mathcal{C}}(X',W) \oplus \hom_{\mathcal{C}}(Y,W) \ ,$$

given by composing with i and j (or with p and s). Then the second set of conditions determines the differentials of $\hom_{\mathcal{C}}(W,Z)$ and $\hom_{\mathcal{C}}(Z,W)$.

(2) and (3) follow immediately from the definition. □

Definition 2.5.10. A dg category $\mathcal{C}$ is *strongly pre-triangulated* if it is stable under suspensions and for every closed homomorphism of degree zero f in $\mathcal{C}$ there exists a cone of f.

The property of being strongly pre-triangulated is clearly preserved under dg equivalences of dg categories. The corresponding definition that is invariant under quasi equivalences is given in definition 2.5.14 below.

Example 2.5.11. Given an additive category $\mathcal{E}$, the categories $\mathsf{ch}\,\mathcal{E}$, $\mathsf{ch}^{\mathrm{b}}\mathcal{E}$ of section 2.3 are (the motivating) examples of strongly pre-triangulated dg categories, see (2.3.3) and (2.3.4).

Similarly, given any dg category $\mathcal{C}$, the dg category $\mathsf{dgmod}/\mathcal{C}$ is strongly pre-triangulated.

Notice that this definition is self-dual, in the sense that a dg category $\mathcal{C}$ is strongly pre-triangulated if and only if its opposite dg category $\mathcal{C}^{\mathrm{op}}$ is so, see lemma 2.5.9(2).

The following definition is probably better understood if read together with lemma 2.5.13 below.

Definition 2.5.12. Given a dg category $\mathcal{C}$ we define a new dg category $\mathcal{C}^{\text{p-tr}}$, whose objects are the so-called *one-sided twisted complexes in* $\mathcal{C}$. A one-sided twisted complex is a finite sequence $(X_1, m_1), \ldots, (X_r, m_r)$ of objects in $\mathcal{C}^{\Sigma}$, with $r \geq 0$, together with homomorphisms $a_{i,j}$ of degree -1 from (X_j, n_j) to (X_i, n_i) for every $1 \leq i < j \leq r$:

$$\begin{aligned} a_{i,j} \in & \hom_{\mathcal{C}^{\Sigma}}((X_j, n_j), (X_i, n_i))_{-1} = (\Sigma^{n_i - n_j} \hom_{\mathcal{C}}(X_j, X_i))_{-1} \\ &= \hom_{\mathcal{C}}(X_j, X_i)_{-n_i - n_j - 1} \ , \end{aligned}$$

satisfying the following conditions: Write the homomorphisms $a_{i,j}$'s as a strictly upper triangular matrix $A = (a_{i,j})_{1 \leq i,j \leq r}$; then the following equality must hold:

$$\partial_{\text{ew}} A + A^2 = 0 \ ,$$

where $\partial_{\text{ew}} A = (\partial a_{i,j})$, i.e., the differential ∂ is applied entry-wise to the matrix A, and $A^2 = (\sum_{k=1}^{r} a_{k,j} a_{i,k})$ is matrix multiplication. The empty sequence for $r = 0$ represents the zero object.

Given two one-sided twisted complexes $X = ((X_1, m_1), \ldots, (X_r, m_r), A)$ and $Y = ((Y_1, n_1), \ldots, (Y_r, n_s), B)$ as above define

$$\begin{aligned} \hom_{\mathcal{C}^{\text{p-tr}}}(X, Y)_l &= \bigoplus_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}} \hom_{\mathcal{C}^{\Sigma}}((X_i, m_i), (Y_j, n_j))_l \\ &= \bigoplus_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}} \hom_{\mathcal{C}}(X_i, Y_j)_{l - n_j + m_i} \ . \end{aligned}$$

An element in $\hom_{\mathcal{C}^{\text{p-tr}}}(X, Y)_l$ is thought of as matrix $C = (c_{i,j})$ whose (i,j)-entry is a homomorphisms of degree l from (X_i, m_j) to (Y_j, n_j). Then the differential in $\hom_{\mathcal{C}^{\text{p-tr}}}(X, Y)$ is defined by the following formula:

$$\partial C = \partial_{\text{ew}} C + BC - (-1)^s CA \ .$$

Lemma 2.5.13. *Let $\mathcal{C}$ be a dg category.*

(1) *The dg category $\mathcal{C}^{\text{p-tr}}$ contains $\mathcal{C}^{\Sigma}$ (and hence also $\mathcal{C}$) as a full dg subcategory and is strongly pre-triangulated.*

(2) *Every object of $\mathcal{C}^{\text{p-tr}}$ can be obtained from objects of $\mathcal{C}^{\Sigma}$ by taking iterated cones of closed homomorphisms of degree zero. Hence $\mathcal{C}$ is strongly pre-triangulated if and only if the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{p-tr}}$ is a dg equivalence.*

(3) *The embedding $h^{\Sigma} \colon \mathcal{C}^{\Sigma} \to \mathtt{dgmod}/\mathcal{C}^{\text{op}}$ (and hence also the Yoneda embedding $h \colon \mathcal{C} \to \mathtt{dgmod}/\mathcal{C}^{\text{op}}$) factorizes over the fully faithful dg functor*

$$h^{\text{p-tr}} \colon \mathcal{C}^{\text{p-tr}} \to \mathtt{dgmod}/\mathcal{C}^{\text{op}} \ ,$$
$$((X_1, m_1), \ldots, (X_r, m_r), A) \mapsto \Sigma^{m_1} h(X_1) \oplus \ldots \oplus \Sigma^{m_r} h(X_r) \ ,$$

where the differential on the dg module $\Sigma^{m_1} h(X_1) \oplus \ldots \oplus \Sigma^{m_r} h(X_r)$ is $\partial + A$.

Now let $\mathcal{D}$ be another dg category and $F \colon \mathcal{C} \to \mathcal{D}$ a dg functor.

(4) *If $\mathcal{D}$ is a strongly pre-triangulated then there exist a unique (up to dg isomorphism) dg functor $F' \colon \mathcal{C}^{\text{p-tr}} \to \mathcal{D}$ that extends F.*

(5) *If F is a quasi equivalence or a dg equivalence then the same is true for the induced dg functor $F^{\text{p-tr}}\colon \mathcal{B}^{\text{p-tr}} \to \mathcal{C}^{\text{p-tr}}$.*

Proof. (1) We need to prove that $\mathcal{C}^{\text{p-tr}}$ is strongly pre-triangulated, the other assertion being obvious. First of all notice that $\mathcal{C}^{\text{p-tr}}$ is stable under suspensions: An n-suspension of $X = ((X_1, m_1), \dots, (X_r, m_r), A)$ is $X = ((X_1, m_1 + n), \dots, (X_r, m_r + n), (-1)^n A)$

A cone of a closed homomorphism of degree zero

$$C\colon ((X_1, m_1), \dots, (X_r, m_r), A) \to ((Y_1, n_1), \dots, (Y_r, n_s), B)$$

is $((Y_1, n_1), \dots, (Y_r, n_s), (X_1, m_1{+}1), \dots, (X_r, m_r{+}1), \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right))$. For example, if $f\colon X \to Y$ is a closed homomorphism of degree 0 in $\mathcal{C}$, then a cone of f is $((Y, 0), (X, 1), \left(\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right))$.

(2) Given any one-sided twisted complex $X = ((X_1, m_1), \dots, (X_r, m_r), A)$, consider $X' = ((X_1, m_1), \dots, (X_{r-1}, m_{r-1}), A')$, where $A' = (a_{i,j})_{1 \le i,j \le r-1}$ is the matrix obtained from A by deleting the last column and the last row. Then X is a cone of the closed homomorphism of degree zero

$$(a_{i,r})_{1 \le i \le r-1}\colon ((X_r, m_r - 1), 0) \to X' .$$

The remaining claims follow immediately from the definition. □

In particular we have that a dg category $\mathcal{C}$ is strongly pre-triangulated if the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{p-tr}}$ is a dg equivalence, see lemma 2.5.13(2) above. This motivates the following definition.

Definition 2.5.14. A dg category $\mathcal{C}$ is *pre-triangulated* if the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{p-tr}}$ is a quasi-equivalence.

Notice that the property of being pre-triangulated is invariant under quasi-equivalences of dg categories.

Lemma 2.5.15. *Let $\mathcal{C}$ be a pre-triangulated dg category. Then* $\operatorname{ho}\mathcal{C}$ *is a triangulated category.*

Proof. (The definition of a triangulated category is recalled in the next section 2.6.)

It is well known that the homotopy category $\operatorname{ho}(\mathsf{dgmod}/\mathcal{C}^{\text{op}})$ of (right) dg modules over a dg category $\mathcal{C}$ is a triangulated category—see for example [Kel94]. Lemma 2.5.13(2) above implies that there exists a fully faithful embedding $\operatorname{ho}(h^{\text{p-tr}})$ of $\operatorname{ho}(\mathcal{C}^{\text{p-tr}})$ in $\operatorname{ho}(\mathsf{dgmod}/\mathcal{C}^{\text{op}})$ as a triangulated subcategory. And finally, by assumption, the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^{\text{p-tr}}$ yields an equivalence of categories $\operatorname{ho}(\mathcal{C}) \simeq \operatorname{ho}(\mathcal{C}^{\text{p-tr}})$. □

Finally we define pre-triangulated categories with duality.

Definition 2.5.16. A *pre-triangulated dg category with duality* is simply a dg category with duality $(\mathcal{C}, {}^\vee, \mu)$ in the sense of definition 2.4.4 such that the dg category $\mathcal{C}$ is pre-triangulated.

Notice that there is no need to impose further conditions on the dg functor ${}^\vee$, because of lemma 2.5.2(3) and lemma 2.5.9(3).

Example 2.5.17. Given an exact category $\mathcal{E}$ with duality, the categories $\mathsf{ch}\mathcal{E}$, $\mathsf{ch}^{\mathrm{b}}\mathcal{E}$ of section 2.3 are (the motivating) examples of (strongly) pre-triangulated dg categories with duality.

Addendum 2.5.18 (to lemma 2.5.15)**.** *Let $\mathcal{C}$ be a pre-triangulated dg category with duality. Then* $\operatorname{ho}\mathcal{C}$ *is a triangulated category with duality.*

Proof. Follows directly from the definitions (the definition of a triangulated category with duality is given in the next section 2.6). □

The following example shows that not all triangulated categories (with duality) arise as homotopy categories of pre-triangulated dg categories.

Example 2.5.19. The stable homotopy category of finite spectra $\mathfrak{S}$, that is, the most important triangulated category in the world (of topology), is not a triangulated subcategory of the homotopy category of a pre-triangulated dg category.

The reason for this is the well-known fact that multiplication by 2 on the mod 2 Moore spectrum $\mathbb{M}(\mathbb{Z}_2) = \operatorname{cone}(2\operatorname{id}_{\mathbb{S}}\colon \mathbb{S} \to \mathbb{S})$ in $\mathfrak{S}$ is non-trivial: $2\operatorname{id}_{\mathbb{M}(2)} \neq 0$. This implies, in particular, that $\mathbb{M}(2)$ cannot be a ringspectrum, i.e., a monoid in the closed symmetric monoidal category $\mathfrak{S}$. It is known, moreover, that the additive order of the identity map of $\mathbb{M}(2)$ is equal to 4.

On the other hand, given an object in a pre-triangulated dg category, any closed endomorphism of degree zero induces a null-homotopic endomorphism on its own cone.

Moreover, the stable homotopy category of finite spectra is a triangulated category with duality in the sense of definition 2.6.6 in the next section, the dual of a finite spectrum $\mathbb{E}$ being $\hom_{\mathfrak{S}}(\mathbb{E}, \mathbb{S})$, its so-called Spanier-Whitehead dual.

2.6. **Triangulated categories with duality.** Here we first recall the definition of triangulated categories, as well as some basic facts about them, and then introduce triangulated categories with duality, following Paul Balmer's definition.

Definition 2.6.1 (triangulated category)**.** Let $\mathcal{T}$ be an additive category together with an additive self-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$, called *suspension* (also known as translation or shift). A *triangle* in $\mathcal{T}$ is a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

of composable homomorphisms. Homomorphisms of triangles are defined in the obvious way as commutative diagrams.

A *triangulated category* is an additive category $\mathcal{T}$ together with an additive self-equivalence $\Sigma\colon \mathcal{T} \to \mathcal{T}$ and a choice of a class of triangles, called *exact triangles* (an equivalent term is distinguished triangles), closed under isomorphism and satisfying the following axioms.

(1) For every object X in $\mathcal{T}$ the triangle

$$0 \to X \xrightarrow{\operatorname{id}_X} X \to 0$$

is exact.

(2) Every homomorphism $X \xrightarrow{f} Y$ fits into an exact triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

(the object Z is then called a *cone of f*).

(3) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'$ are exact triangles and $X \xrightarrow{x} X'$ and $Y \xrightarrow{y} Y'$ are homomorphisms such that $yf = f'x$, then there exists a homomorphism $Z \xrightarrow{z} Z'$ (not necessarily unique) such that $zg = g'y$ and $(\Sigma x)h = h'z$. In other words, every commutative diagram whose rows are exact triangles

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \exists z} & & \downarrow{\scriptstyle \Sigma x} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X' \end{array}$$

can be completed to a homomorphism of triangles.

(4) A triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is exact if and only if

$$Y \xrightarrow{-g} Z \xrightarrow{-h} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y$$

is exact.

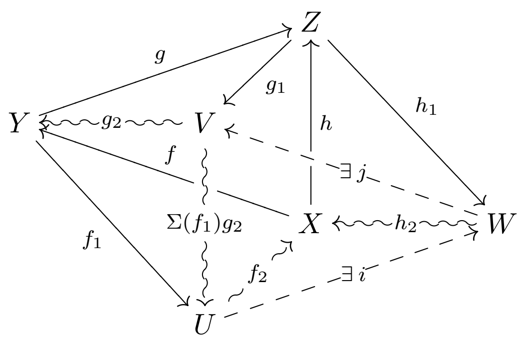
(5) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be any pair of composable homomorphisms of $\mathcal{T}$ and let $h = gf$ denote their composition. Then given any exact triangles over (i.e., any choice of cones of) f, g, and $h = gf$

$$X \xrightarrow{f} Y \xrightarrow{f_1} U \xrightarrow{f_2} \Sigma X \ ,$$

$$Y \xrightarrow{g} Z \xrightarrow{g_1} V \xrightarrow{g_2} \Sigma Y \ ,$$

$$X \xrightarrow{h} Z \xrightarrow{h_1} W \xrightarrow{h_2} \Sigma X \ ,$$

there are homomorphisms $i\colon U \to V$ and $j\colon W \to V$ such that the diagram



is an (enhanced) octahedron, where the homomorphisms of degree one are displayed as $\rightsquigarrow$. Explicitly this means that:

(a) the triangle

$$U \xrightarrow{i} W \xrightarrow{j} V \xrightarrow{\Sigma(f_1)g_2} \Sigma U$$

is exact;

(b) $jh_1 = g_1$ and $h_2 i = f_2$;

(c) $if_1 = h_1 g$ and $g_2 j = \Sigma(f)h_2$;

(d) the triangles

$$Y \xrightarrow{\quad k \quad} W \xrightarrow{\begin{pmatrix} -h_2 \\ j \end{pmatrix}} \Sigma(X) \oplus V \xrightarrow{(\Sigma(f)\ g_2)} \Sigma Y \ ,$$

$$Y \xrightarrow{\begin{pmatrix} g \\ f_1 \end{pmatrix}} Z \oplus U \xrightarrow{(h_1\ -i)} W \xrightarrow{\quad l \quad} \Sigma Y$$

are exact, where $k = if_1 = h_1 g$ and $l = g_2 j = \Sigma(f)h_2$ (see the preceding axiom).

A *triangulated functor* between triangulated categories $\mathcal{S}$ and $\mathcal{T}$ is an additive functor $F\colon \mathcal{S} \to \mathcal{T}$ together with a natural isomorphism $\zeta\colon F\Sigma \xrightarrow{\cong} \Sigma F$ such that for any exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{S}$ the triangle $F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\zeta_X F(h)} \Sigma F(X)$ is exact in $\mathcal{T}$.

A full additive subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *triangulated subcategory* if $\Sigma(\mathcal{S}) = \mathcal{S}$ and for every exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{T}$ if any two of the objects X, Y, and Z are in $\mathcal{S}$ then so is the third.

In the definition 2.6.1 of a triangulated category, the last axiom (5) is known as the octahedron (or composition) axiom. We remark that we assume it in its so-called enriched version, i.e., we always additionally assume that (5d) holds. This enriched octahedron axiom, which is satisfied in all the known examples of triangulated categories, was first introduced by Beĭlinson, Bernstein, and Deligne [BBD82].

Example 2.6.2. The main examples for us of triangulated categories are given by homotopy categories of pre-triangulated dg categories, see lemma 2.5.15, like for example $\mathrm{ho}(\mathsf{ch}\,\mathcal{E})$ (or $\mathrm{ho}(\mathsf{ch}^{\mathrm{b}}\mathcal{E})$), the chain homotopy category of (bounded) chain complexes in an additive category $\mathcal{E}$.

Another fundamental example is given by the stable homotopy category of spectra, see remark 2.5.19 above, but this cannot arise as homotopy category of a pre-triangulated dg category.

Notice that the definition of triangulated category is self-dual, in the sense that an additive category $\mathcal{T}$ is triangulated if and only if its opposite category $\mathcal{T}^{\mathrm{op}}$ is triangulated, where $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ is an exact triangle in $\mathcal{T}$ if and only if $(\Sigma^{-1}Z)^{\mathrm{op}} \xleftarrow{\Sigma^{-1}(h^{\mathrm{op}})} X^{\mathrm{op}} \xleftarrow{f^{\mathrm{op}}} Y^{\mathrm{op}} \xleftarrow{g^{\mathrm{op}}} Z^{\mathrm{op}}$ is an exact triangle in $\mathcal{T}^{\mathrm{op}}$. Notice that if Σ is the suspension in $\mathcal{T}$ then the suspension in $\mathcal{T}^{\mathrm{op}}$ is Σ^{-1}.

Now we recall Verdier's fundamental definition of triangulated quotients.

Definition 2.6.3. Given a triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$, the *quotient* $\mathcal{T}/\mathcal{S}$ is defined as the localization of $\mathcal{T}$ with respect to the multiplicative "set" of morphisms whose cone is isomorphic to an object of $\mathcal{S}$. (The usual set-theoretic problems are to be addressed at this point, as is customary when dealing with localization of categories. We postpone this issue until the next section 2.7.) The quotient $\mathcal{T}/\mathcal{S}$ is a triangulated category in a canonical way: Its exact triangles are those isomorphic to images of exact triangles in $\mathcal{T}$.

Notice that we do not require $\mathcal{S}$ to be *thick*, i.e., closed under direct summands. An object of $\mathcal{T}$ has zero image in $\mathcal{T}/\mathcal{S}$ if and only if it belongs to $\overline{\mathcal{S}}$, the *thick closure* of $\mathcal{S}$ in $\mathcal{T}$, i.e., the smallest thick triangulated subcategory of $\mathcal{T}$ containing $\mathcal{S}$. Notice also that $\mathcal{T}/\mathcal{S} = \mathcal{T}/\overline{\mathcal{S}}$.

We record, without proof, the following proposition due to Verdier [Ver96].

Proposition 2.6.4. *Let $\mathcal{S}$ be a triangulated subcategory of a triangulated category $\mathcal{T}$. For any object Y of $\mathcal{T}$ define a category $\mathcal{Q}_Y$ as the full subcategory of $Y\downarrow\mathcal{T}$ of homomorphisms $Y \to Z$ in $\mathcal{T}$ whose cone is isomorphic to an object of $\mathcal{S}$. Then the category $\mathcal{Q}_Y$ is right-filtering, and for all $X, Y \in \operatorname{obj}\mathcal{T}/\mathcal{S} = \operatorname{obj}\mathcal{T}$ and for all $n \in \mathbb{Z}$ there exists a natural isomorphism*

$$\operatorname*{colim}_{(Y\to Z)\in\mathcal{Q}_Y} \hom_{\mathcal{T}}(X,Z)_n \xrightarrow{\cong} \hom_{\mathcal{T}/\mathcal{S}}(X,Y)_n \ .$$

Proof. See [Ver96]. □

Example 2.6.5 (and definition of derived categories). Let $\mathcal{E}$ be an exact category. Then, as recalled in section 2.3 above (see also [Nee90, lemma 1.1 on page 389]), the categories $\operatorname{ho}(\mathsf{ach}\,\mathcal{E})$ and $\operatorname{ho}(\mathsf{ach}^{\mathsf{b}}\mathcal{E})$ of acyclic complexes are triangulated subcategories of $\operatorname{ho}(\mathsf{ch}\mathcal{E})$ and $\operatorname{ho}(\mathsf{ch}^{\mathsf{b}}\mathcal{E})$, respectively. Hence the quotient triangulated categories

$$\mathcal{D}(\mathcal{E}) = \operatorname{ho}(\mathsf{ch}\,\mathcal{E})/\operatorname{ho}(\mathsf{ach}\,\mathcal{E}) \quad \text{and} \quad \mathcal{D}^b(\mathcal{E}) = \operatorname{ho}(\mathsf{ch}^{\mathsf{b}}\mathcal{E})/\operatorname{ho}(\mathsf{ach}^{\mathsf{b}}\mathcal{E})$$

are defined, and are called the *derived category* and the *bounded derived category* of the exact category $\mathcal{E}$.

As Thomason [TT90] and Neeman [Nee90] noticed, without additional assumption on the exact category $\mathcal{E}$ the triangulated subcategories of acyclic complexes are in general not thick (but nonetheless the derived categories are defined as above). More precisely, Neeman proved that the triangulated subcategory $\operatorname{ho}(\mathsf{ach}\,\mathcal{E})$ is thick in $\operatorname{ho}(\mathsf{ch}\,\mathcal{E})$ if and only if idempotents split in $\mathcal{E}$, i.e., for every homomorphism $e\colon M \to M$ with $e^2 = e$ there exist homomorphisms $M \xrightarrow{f} N \xrightarrow{g} M$ with $gf = e$ and $fg = \mathrm{id}_N$ (see [Nee90, lemma 1.2 on page 390 and remark 1.8 on page 392]). And Thomason essentially proved that the triangulated subcategory $\operatorname{ho}(\mathsf{ach}^{\mathsf{b}}\mathcal{E})$ is thick in $\operatorname{ho}(\mathsf{ch}^{\mathsf{b}}\mathcal{E})$ if and only if split epimorphisms are admissible in $\mathcal{E}$, i.e., every homomorphism $f\colon M \to N$ such that there exists a homomorphisms $s\colon N \to M$ with $fs = \mathrm{id}_N$ is an admissible epimorphism (see [Nee90, remarks 1.9 and 1.10 on page 393-394]). Notice that if in an exact category idempotents split then automatically split epimorphisms are admissible (see [TT90, lemma A.6.2 on page 399]).

Notice that the derived categories are the localizations of the categories of (bounded) chain complexes in $\mathcal{E}$ with respect to the so-called quasi-isomorphisms, i.e., chain homomorphisms whose cone is acyclic.

Now we define categories with duality in the triangulated setting, following Balmer [Bal00].

Definition 2.6.6. A *triangulated category with duality* is a category with duality $(\mathcal{T}, {}^\vee, \mu)$ in the sense of definition 2.1.1, such that $\mathcal{T}$ is a triangulated category, the functor ${}^\vee\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{T}$ is a triangulated functor, i.e., there exists a natural isomorphism $\zeta\colon {}^\vee \circ \Sigma \xrightarrow{\cong} \Sigma^{-1} \circ {}^\vee$ such that for any exact triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$ in $\mathcal{T}$ the triangle $Z^\vee \xrightarrow{g^\vee} Y^\vee \xrightarrow{f^\vee} X^\vee \xrightarrow{\Sigma(h^\vee)\zeta_X} \Sigma(Z^\vee)$ is exact in $\mathcal{T}^{\mathrm{op}}$, and furthermore the natural isomorphism μ between the identity and the double dual is compatible with the triangulation, i.e., for every object X of $\mathcal{T}$ one has $\mu_{\Sigma X} = \Sigma(\mu_X)$. (Observe that for every pair of objects X and Y in $\mathcal{T}$ one automatically has that $\mu_{X\oplus Y} = \mu_X \oplus \mu_Y$, as explained after definition 2.2.5 above.)

A *morphism* of triangulated categories with duality is a morphism (F, η) of the underlying categories with duality in the sense of definition 2.1.3 such that the functor F is a triangulated functor.

Example 2.6.7. Given an exact category with duality $\mathcal{E}$ the homotopy category $\mathrm{ho}(\mathsf{ch}^{\mathsf{b}}\mathcal{E})$ is a triangulated category with duality, see addendum 2.5.18.

Let $\mathcal{S}$ be a triangulated subcategory of a category with duality $(\mathcal{T}, {}^\vee, \mu)$. We say that $\mathcal{S}$ is *stable under the duality* if $\mathcal{S}^\vee = \mathcal{S}$. Then it follows from the definition of the quotient that $\mathcal{T}/\mathcal{S}$ inherits the structure of a triangulated category with duality. In this case we always use the symbols ${}^\vee$ and μ also for the duality in the quotient.

Example 2.6.8. Given an exact category with duality $\mathcal{E}$, the triangulated subcategories of acyclic complexes are stable under the duality, and therefore the derived categories

$$\mathcal{D}(\mathcal{E}) = \mathrm{ho}(\mathsf{ch}\,\mathcal{E})/\,\mathrm{ho}(\mathsf{ach}\,\mathcal{E}) \quad \text{and} \quad \mathcal{D}^b(\mathcal{E}) = \mathrm{ho}(\mathsf{ch}^{\mathsf{b}}\mathcal{E})/\,\mathrm{ho}(\mathsf{ach}^{\mathsf{b}}\mathcal{E})$$

are triangulated categories with duality.

2.7. **Differential graded quotients, and dualities.** In the final section of this chapter we present Drinfeld's definition [Dri04] of quotients of dg categories, adapting his arguments to the presence of dualities.

Definition 2.7.1. Let $\mathcal{C}$ be a dg category, and $\mathcal{B}$ a full dg subcategory of $\mathcal{C}$. We define the *quotient* dg category $\mathcal{C}/\mathcal{B}$ as the dg category obtained from $\mathcal{C}$ by adding for every object B in $\mathcal{B}$ a new homomorphism of degree one

$$\gamma_B \in \hom_{\mathcal{C}/\mathcal{B}}(B,B)_1 \quad \text{with} \quad \partial\gamma_B = \mathrm{id}_B$$

and no further objects nor further relations between the homomorphisms. So for all $X, Y \in \operatorname{obj}\mathcal{C}/\mathcal{B} = \operatorname{obj}\mathcal{C}$ there is an isomorphism of graded modules between

$$\bigoplus_{\substack{N\in\mathbb{N}\\ B_1,\dots,B_N\\ \in \operatorname{obj}\mathcal{B}}} \hom_{\mathcal{C}}(B_N, Y)\otimes S^1\mathfrak{K}\otimes \hom_{\mathcal{C}}(B_{N-1}, B_N)\otimes_{\mathfrak{K}}\cdots\otimes S^1\mathfrak{K}\otimes\hom_{\mathcal{C}}(X, B_1)$$

and $\hom_{\mathcal{C}/\mathcal{B}}(X,Y)$, under which $f_n \otimes 1 \otimes f_{n-1} \otimes \cdots \otimes 1 \otimes f_0$ (where 1 is the generator of $S^1\mathfrak{K}$ in degree one) corresponds to $f_n\gamma_{B_N} f_{n-1}\cdots\gamma_{B_1} f_0$.

Notice that the quotient depends on the ground ring $\mathfrak{K}$, even if it is hidden from the notation.

Example 2.7.2. Let R be a dga and $\mathcal{B}$ be the corresponding dg category with just one object B with $\hom_{\mathcal{B}}(B,B) = R$. Then the quotient dg category $\mathcal{B}/\mathcal{B}$ has again only one object B and the dga $\hom_{\mathcal{B}/\mathcal{B}}(B,B)$ is obtained from R

by adding a new generator γ of degree one with $\partial\gamma = 1$. As a dg module over R one has that $\hom_{\mathcal{B}/\mathcal{B}}(B, B) = \operatorname{cone}(R \to \bar{R})$, where $\bar{R}$ is the bar resolution of R as a dg bimodule over R. Certainly, $\hom_{\mathcal{B}/\mathcal{B}}(B, B)$ has trivial homology. (This example is due to Drinfeld, [Dri04, example 3.2 on page 653]).

Now suppose that $\mathcal{C}$ is a pre-triangulated dg category, and that $\mathcal{B}$ is a full pre-triangulated dg subcategory of $\mathcal{C}$. From the definition it follows that the quotient $\mathcal{C}/\mathcal{B}$ is again a pre-triangulated dg category. The triangulated functor $\operatorname{ho}(\mathcal{C}) \to \operatorname{ho}(\mathcal{C}/\mathcal{B})$ by definition sends $\operatorname{ho}(\mathcal{B})$ to zero, and hence it induces a triangulated functor

$$\Psi\colon \operatorname{ho}(\mathcal{C})/\operatorname{ho}(\mathcal{B}) \to \operatorname{ho}(\mathcal{C}/\mathcal{B}) \ . \tag{2.7.3}$$

Drinfeld [Dri04] proved that under suitable flatness conditions (that are for example automatically satisfied when working over a field $\mathfrak{K}$, see definition 2.7.4) the functor Ψ is an equivalence of triangulated categories, see theorem 2.7.5 below. On the other hand, if these flatness conditions are not satisfied, one can always replace $\mathcal{C}$ and $\mathcal{B}$ by quasi-equivalent dg categories that do satisfy the conditions, see theorem 2.7.9. By carefully going through Drinfeld's proofs, one can show that everything works as well in the presence of dualities, see the addenda 2.7.6 and 2.7.10 to Drinfeld's theorems.

Definition 2.7.4. A chain complex C of modules over $\mathfrak{K}$ is called *homotopically flat (over $\mathfrak{K}$)* if for every acyclic chain complex A of modules over $\mathfrak{K}$, their tensor product $A \otimes_{\mathfrak{K}} C$ is again acyclic.

A dg category $\mathcal{C}$ is called homotopically flat (over $\mathfrak{K}$) if for every pair of objects X and Y in $\mathcal{C}$ the chain complex $\hom_{\mathcal{C}}(X, Y)$ of modules over $\mathfrak{K}$ is homotopically flat (over $\mathfrak{K}$).

(Notice that this condition is automatically satisfied if $\mathfrak{K}$ is a field.)

Theorem 2.7.5 (Drinfeld)**.** *Let $\mathcal{C}$ be a pre-triangulated dg category and $\mathcal{B}$ a full pre-triangulated dg subcategory of $\mathcal{C}$. Assume that $\mathcal{C}$ is homotopically flat. Then the functor $\Psi\colon \operatorname{ho}(\mathcal{C})/\operatorname{ho}(\mathcal{B}) \to \operatorname{ho}(\mathcal{C}/\mathcal{B})$ is an equivalence of triangulated categories.*

Addendum 2.7.6. *If in the situation of theorem 2.7.5 the category $\mathcal{C}$ is a pre-triangulated dg category with duality, and if the dg subcategory $\mathcal{B}$ is stable under the duality, then the quotient category $\mathcal{C}/\mathcal{B}$ inherits the structure of a pre-triangulated category with duality and the functor Ψ induces an equivalence of triangulated categories with duality.*

Proof. Actually, the theorem holds true if either one of the following two conditions (that are weaker than and implied by $\mathcal{C}$ being homotopically flat) is satisfied:

(1) for all objects X in $\mathcal{C}$ and B in $\mathcal{B}$ the chain complex $\hom_{\mathcal{C}}(X, B)$ is homotopically flat;
(2) for all objects X in $\mathcal{C}$ and B in $\mathcal{B}$ the chain complex $\hom_{\mathcal{C}}(B, X)$ is homotopically flat.

We now prove the theorem and the addendum if the first condition above holds (the other case being dual).

First of all notice that, as far as the addendum is concerned, putting

$$(\gamma_B)^\vee = \gamma_{B^\vee}$$

gives $\mathcal{C}/\mathcal{B}$ the structure of a pre-triangulated dg category with duality.

Notice that the functor Ψ is (essentially) surjective, therefore all we are left to show is that it is fully faithful, i.e., that for all $n \in \mathbb{Z}$ the homomorphism

$$\Psi\colon\ \hom_{\mathrm{ho}(\mathcal{C})/\,\mathrm{ho}(\mathcal{B})}(X,Y)_n \to \hom_{\mathrm{ho}(\mathcal{C}/\mathcal{B})}(X,Y)_n \tag{2.7.7}$$

is bijective. It is indeed clear that it is compatible with the duality.

Applying proposition 2.6.4 and the definition of $\hom_{\mathrm{ho}(\mathcal{C})}$, one sees that the left-hand side can be computed as

$$\operatorname*{colim}_{(Y\to Z)\in \mathcal{Q}_Y} H_n(\hom_{\mathcal{C}}(X,Z)) \xrightarrow{\cong} \hom_{\mathrm{ho}(\mathcal{C})/\,\mathrm{ho}(\mathcal{B})}(X,Y)_n$$

where $\mathcal{Q}_Y$ is the right filtering subcategory of $Y{\downarrow}\,\mathrm{ho}(\mathcal{C})$ of degree zero homomorphisms $Y \to Z$ in $\mathrm{ho}(\mathcal{C})$ whose cone is isomorphic to an object of $\mathrm{ho}(\mathcal{B})$. On the other hand, the left-hand side can be computed as

$$\operatorname*{colim}_{(Y\to Z)\in \mathcal{Q}_Y} H_n(\hom_{\mathcal{C}/\mathcal{B}}(X,Z)) \xrightarrow{\cong} \hom_{\mathrm{ho}(\mathcal{C}/\mathcal{B})}(X,Y)_n\ ,$$

because $H_n(\hom_{\mathcal{C}/\mathcal{B}}(X,Y)) \cong \hom_{\mathrm{ho}(\mathcal{C}/\mathcal{B})}(X,Y)_n$ and every morphism $Y \to Z$ in $\mathcal{Q}_Y$ induces an isomorphism $H_n(\hom_{\mathcal{C}/\mathcal{B}}(X,Y)) \xrightarrow{\cong} H_n(\hom_{\mathcal{C}/\mathcal{B}}(X,Z))$ since $\hom_{\mathcal{C}/\mathcal{B}}(X,B)$ is acyclic for every object B of $\mathcal{B}$ (acyclicity is clear since B is homotopy equivalent to 0 as an object of $\mathcal{C}/\mathcal{B}$.)

We see then that the homomorphism Φ of (2.7.3) is induced by the natural chain homomorphisms $\hom_{\mathcal{C}}(X,Z) \to \hom_{\mathcal{C}/\mathcal{B}}(X,Z)$. This is obviously injective, and its cokernel is the union of an increasing sequence of chain complexes whose N-th successive quotient is isomorphic to

$$\bigoplus_{\substack{B_1,\dots,B_N\\ \in \operatorname{obj}\mathcal{B}}} \hom_{\mathcal{C}}(B_N,Z)\otimes S^1\mathfrak{K}\otimes\hom_{\mathcal{C}}(B_{N-1},B_N)\otimes_{\mathfrak{K}}\cdots\otimes S^1\mathfrak{K}\otimes\hom_{\mathcal{C}}(X,B_1)\ .$$

Write $F_N(X,Z)$ for the chain complex above. We need then only to prove that $\operatorname{colim}_{(Y\to Z)\in\mathcal{Q}_Y} H_n(F_N(X,Z)) = 0$ for all $N \geq 1$ and for all $n \in \mathbb{Z}$.

Notice that the dg functor that sends Z to $F_N(X,Z)$ is a direct sum of functors of the form $\hom_{\mathcal{C}}(B,Z)\otimes_{\mathfrak{K}} G_N(X,B)$ with B in $\mathcal{B}$, where

- $\operatorname{colim}_{(Y\to Z)\in\mathcal{Q}_Y} H_n(\hom_{\mathcal{C}}(B,Z)) = 0$ for all $n \in \mathbb{Z}$;
- $G_N(X,B)$ is homotopically flat by assumption.

Therefore the theorem follows from the following lemma due to Spaltenstein, see [Spa88; Dri04, lemma 8.4 on page 667]

Lemma 2.7.8. *Let $\mathcal{Q}$ be a right-filtering category and let*

$$F\colon\ \mathcal{Q} \to \mathrm{ho}(\mathsf{ch\,mod}/\mathfrak{K})$$

be a functor such that

$$\operatorname*{colim}_{q\in\mathcal{Q}} H_n(F(q)) = 0$$

for all $n \in \mathbb{Z}$. Then for every homotopically flat chain complex of modules G

$$\operatorname*{colim}_{q\in\mathcal{Q}} H_n(F(q)\otimes_{\mathfrak{K}} G) = 0$$

for every $n \in \mathbb{Z}$.

(Notice that the lemma would be obvious if the functor F lifted from the homotopy category to the category of chain complexes itself.) □

Theorem 2.7.9. *Let* $\mathcal{C}$ *be any pre-triangulated dg category. Then there exists a homotopically flat pre-triangulated dg category* $\mathcal{C}'$ *and a quasi-equivalence* $\mathcal{C}' \to \mathcal{C}$.

Addendum 2.7.10. *If in the situation of theorem 2.7.9 above* $\mathcal{C}$ *is a pre-triangulated dg category with duality, then* $\mathcal{C}'$ *can be chosen as a homotopically flat pre-triangulated dg category with duality, and the quasi-equivalence* $\mathcal{C}' \to \mathcal{C}$ *induces a quasi-equivalence of dg-categories with duality.*

3. Witt groups and L-theory

3.1. **Witt groups of exact categories.** In this section we recall the definition of the Witt group of an exact category, following Balmer's generalization of Knebusch's original definition [Kne77], which dealt only with the case of vector bundles over a scheme.

Let $\mathcal{E} = (\mathcal{E}, {}^\vee, \mu)$ be an exact category with duality, see definition 2.2.5. Recall that then for every object M of $\mathcal{E}$ the duality defines a linear action of the cyclic group of order two $\mathrm{C}_2 = \langle\, \tau \,|\, \tau^2 = 1 \,\rangle$ on the module $\hom_{\mathcal{E}}(M, M^\vee)$ by putting $\tau(f) = \tau_{M,M}(f) = f^\vee \mu_M$; henceforth we view $\hom_{\mathcal{E}}(M, M^\vee)$ as a $\mathfrak{K}\mathrm{C}_2$-module.

Definition 3.1.1. The *symmetric Q-module* of an object M of $\mathcal{E}$ is defined as the fixed-point module

$$\begin{aligned} Q^0_{\mathcal{E}}(M) &= \hom_{\mathcal{E}}(M, M^\vee)^{\mathrm{C}_2} = H^0(\mathrm{C}_2; \hom_{\mathcal{E}}(M, M^\vee)) \\ &= \hom_{\mathfrak{K}\mathrm{C}_2}(\mathfrak{K}, \hom_{\mathcal{E}}(M, M^\vee)) \ . \end{aligned}$$

Its elements are called *symmetric forms* on M. Explicitly, a symmetric form on M is a homomorphism $\phi\colon M \to M^\vee$ that is symmetric with respect to the duality, i.e., the square

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M^\vee \\ {\scriptstyle \mu_M}\downarrow & & \big\| {\scriptstyle \mathrm{id}_{M^\vee}} \\ M^{\vee\vee} & \xrightarrow[\phi^\vee]{} & M^\vee \end{array}$$

commutes. A symmetric form is called *non-degenerate* if it is an isomorphism. A *(non-degenerate) symmetric space* in $\mathcal{E}$ is an object M of $\mathcal{E}$ together with a (non-degenerate) symmetric form.

An *isometric homomorphism* of symmetric spaces (M_1, ϕ_1) and (M_2, ϕ_2) is a homomorphism $f\colon M_1 \to M_2$ respecting the symmetric forms, i.e., such that the square

$$\begin{array}{ccc} M_1 & \xrightarrow{\phi_1} & M_1^\vee \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle f^\vee} \\ M_2 & \xrightarrow{\phi_2} & M_2^\vee \end{array}$$

commutes: $f^\vee \phi_2 f = \phi_1$. An *isometry* is an isometric isomorphism.

The *orthogonal sum* of two symmetric spaces (M_1, ϕ_1) and (M_2, ϕ_2) is defined as

$$(M_1, \phi_1) \perp (M_2, \phi_2) = \left(M_1 \oplus M_2, \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} \right) \ .$$

Analogously one defines the *quadratic Q-module* as the coinvariant module

$$\begin{aligned} Q_0^{\mathcal{E}}(M) &= \hom_{\mathcal{E}}(M, M^\vee)_{\mathrm{C}_2} = H_0(\mathrm{C}_2; \hom_{\mathcal{E}}(M, M^\vee)) \\ &= \mathfrak{K} \otimes_{\mathfrak{K}\mathrm{C}_2} \hom_{\mathcal{E}}(M, M^\vee) \ , \end{aligned}$$

and *quadratic forms* and *spaces* and their homomorphisms correspondingly.

The norm homomorphism

$$1 + \tau\colon Q_0^{\mathcal{E}}(M) \to Q^0_{\mathcal{E}}(M)$$

is called *symmetrization* and a quadratic form $\psi \in Q_0^{\mathcal{E}}(M)$ is called non-degenerate if its symmetrization $(1+\tau)\psi = \psi + \psi^\vee \mu_M \in Q^0_{\mathcal{E}}(M)$ is non-degenerate.

We remark that some authors, including Balmer, reserve the name symmetric space for what is for us a non-degenerate symmetric space.

The property of being non-degenerate is obviously invariant under isometries, and the orthogonal sum of non-degenerate spaces is non-degenerate again.

These definitions depend of course on the identification μ between objects of $\mathcal{E}$ and their double duals. In particular, given an exact category with duality $(\mathcal{E}, {}^\vee, \mu)$, by changing the sign of the natural isomorphism μ we get a new exact category with duality $(\mathcal{E}, {}^\vee, -\mu)$, and symmetric forms in the latter category $(\mathcal{E}, {}^\vee, -\mu)$ are classically known as skew-symmetric forms in the former category $(\mathcal{E}, {}^\vee, \mu)$.

Notice that the Q-modules define contravariant functors

$$Q^0_{\mathcal{E}}, Q_0^{\mathcal{E}} \colon \mathcal{E}^{\mathrm{op}} \to \mathtt{mod}/\mathfrak{K} \,,$$

and the symmetrization a natural transformation between them.

Definition 3.1.2. Let $\mathcal{E}$ be any ($\mathfrak{K}$-linear) category. We say that 2 *is invertible in* $\mathcal{E}$, or equivalently that $\mathcal{E}$ contains $\frac{1}{2}$, and write in symbols $\frac{1}{2} \in \mathcal{E}$, if every homomorphism in $\mathcal{E}$ is uniquely 2-divisible, i.e., in other words, if $\mathcal{E}$ is a $\mathfrak{K}[\frac{1}{2}]$-category.

Notice that if $\frac{1}{2} \in \mathcal{E}$ then the symmetrization defines an isomorphism $Q_0^{\mathcal{E}}(M) \cong Q^0_{\mathcal{E}}(M)$.

Example 3.1.3. Given any object L of $\mathcal{E}$ one can define a non-degenerate symmetric space $H(L) = \left(L \oplus L^\vee, \left(\begin{smallmatrix} 0 & \mathrm{id}_{L^\vee} \\ \mu_L & 0 \end{smallmatrix}\right)\right)$, called the *hyperbolic space over* L. Notice that here one uses assumption (2.1.2) that $(\mu_L)^\vee = (\mu_{L^\vee})^{-1}$.

Lemma 3.1.4. *For all pair of objects M_1 and M_2 of $\mathcal{E}$ one has*

$$Q^0_{\mathcal{E}}(M_1 \oplus M_2) \cong Q^0_{\mathcal{E}}(M_1) \oplus Q^0_{\mathcal{E}}(M_2) \oplus \hom_{\mathcal{E}}(M_1, M_2^\vee),$$
$$Q_0^{\mathcal{E}}(M_1 \oplus M_2) \cong Q_0^{\mathcal{E}}(M_1) \oplus Q_0^{\mathcal{E}}(M_2) \oplus \hom_{\mathcal{E}}(M_1, M_2^\vee).$$

Proof. One has $(M_1 \oplus M_2)^\vee \cong M_1^\vee \oplus M_2^\vee$ and

$$\begin{aligned} &\hom_{\mathcal{E}}(M_1 \oplus M_2, M_1^\vee \oplus M_2^\vee) \\ \cong\ &\hom_{\mathcal{E}}(M_1, M_1^\vee) \oplus \hom_{\mathcal{E}}(M_2, M_2^\vee) \oplus \hom_{\mathcal{E}}(M_1, M_2^\vee) \oplus \hom_{\mathcal{E}}(M_2, M_1^\vee). \end{aligned}$$

The group C_2 acts on both sides of this equation, via the canonical action recalled above on the left-hand side and on the first two direct summands of the right-hand side, and on the last two direct summands by interchanging them. The canonical isomorphism is C_2-equivariant with respect to these actions, and the claims follow immediately from this. □

Definition 3.1.5. Assume that the category $\mathcal{E}$ is essentially small, i.e., the isomorphism classes of objects of $\mathcal{E}$ form a set.

The *symmetric Witt monoid* of $\mathcal{E}$ is defined as the (obviously abelian) monoid of isometry classes of non-degenerate symmetric spaces in $\mathcal{E}$ with respect to the orthogonal sum, and it is denoted $MW^0(\mathcal{E}) = MW^0(\mathcal{E}, {}^\vee, \mu)$. The *quadratic Witt monoid* $MW_0(\mathcal{E})$ is defined completely analogously.

Notice that so far one only needs the additive structure. The exact structure is going to be used now in the definition of metabolic spaces and hence of the Witt groups.

Definition 3.1.6. A symmetric space (M, ϕ) is called *metabolic* if there exists a short exact sequence

$$L \rightarrowtail^{\iota} M \overset{\iota^\vee \phi}{\twoheadrightarrow} L^\vee ,$$

where L is then called a *lagrangian* of (M, ϕ). If the above sequence splits then (M, ϕ) is called *split metabolic*. A quadratic space is called (split) metabolic if its symmetrization is.

Notice that a non-degenerate symmetric space (M, ϕ) is split metabolic if and only if it is isometric to a non-degenerate symmetric space of the form $\left(L \oplus L^\vee, \begin{pmatrix} 0 & \mathrm{id}_{L^\vee} \\ \mu_L & \psi \end{pmatrix}\right)$ for some $\psi \in Q^0_{\mathcal{E}}(L)$. In particular, hyperbolic spaces (see example 3.1.3) are split metabolic, with $\psi = 0$.

If $(M, \phi) \cong \left(L \oplus L^\vee, \begin{pmatrix} 0 & \mathrm{id}_{L^\vee} \\ \mu_L & \psi \end{pmatrix}\right)$ is a non-degenerate split-metabolic space and $\psi = 2\xi$, then $\begin{pmatrix} 1 & -\xi \\ 0 & 1 \end{pmatrix}$ defines an isometry between (M, ϕ) and $H(L)$. In particular, if $\frac{1}{2} \in \mathcal{E}$ then all non-degenerate split metabolic spaces are hyperbolic; and in a split exact category with duality $\mathcal{E}$ where $\frac{1}{2} \in \mathcal{E}$ metabolic and hyperbolic spaces coincide.

Conversely, if $\frac{1}{2} \notin \mathcal{E}$ then there might exist split metabolic forms that are not hyperbolic. For example, let $\mathcal{E} = \mathtt{fgp}/\mathbb{Z}$ (see example 2.2.6). Then the split metabolic non-degenerate space $(\mathbb{Z} \oplus \mathbb{Z}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix})$ is not isometric to $H(\mathbb{Z})$.

Finally, one can show that there are in general metabolic spaces that are not split metabolic, see [KO91, last remark on page 444]].

Notice that for any non-degenerate symmetric space (M, ϕ) the orthogonal sum $(M, \phi) \perp (M, -\phi)$ is split metabolic: Indeed, one always has the following split exact sequence

$$M \overset{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}{\rightarrowtail} M \oplus M \overset{(\phi\ -\phi)}{\twoheadrightarrow} M^\vee ,$$

and of course $(\phi\ -\phi) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^\vee \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}$.

The orthogonal sum of metabolic symmetric spaces is again metabolic, furthermore being metabolic is invariant under isometries, and the unique symmetric space structure on the zero object is obviously non-degenerate and metabolic. Therefore the set of isometry classes of non-degenerate metabolic symmetric spaces forms a submonoid $NW^0(\mathcal{E})$ of $MW^0(\mathcal{E})$, so we can consider the quotient monoid $MW^0(\mathcal{E})/NW^0(\mathcal{E})$. Since, as we have just seen, for every $(M, \phi) \in MW^0(\mathcal{E})$ there is another element $(M, -\phi) \in MW^0(\mathcal{E})$ such that $(M, \phi) \perp (M, -\phi) \in NW^0(\mathcal{E})$, the quotient monoid is in fact a group.

Definition 3.1.7. The *symmetric Witt group* of $\mathcal{E}$ is defined as the (obviously abelian) quotient group of $MW^0(\mathcal{E})$ by the submonoid $NW^0(\mathcal{E})$ of isometry classes of non-degenerate metabolic symmetric spaces in $\mathcal{E}$, and it is denoted $W^0(\mathcal{E}) = W^0(\mathcal{E}, ^\vee, \mu) = MW^0(\mathcal{E})/NW^0(\mathcal{E})$.

The *quadratic Witt group* $W_0(\mathcal{E})$ is defined completely analogously.

Notice that the symmetrization defines a homomorphism $W_0(\mathcal{E}) \to W^0(\mathcal{E})$ and that this is an isomorphism if $\frac{1}{2} \in \mathcal{E}$.

From the definition it follows that the following conditions are equivalent for a non-degenerate symmetric form (M, ϕ):

(i) (M, ϕ) represents zero in the Witt group $W^0(\mathcal{E})$;
(ii) there exists a non-degenerate metabolic space (H, ξ) such that the orthogonal sum $(M, \phi) \perp (H, \xi)$ is metabolic;
(iii) there exists a non-degenerate split metabolic space (H, ξ) such that the orthogonal sum $(M, \phi) \perp (H, \xi)$ is metabolic;
(iv) there exists a non-degenerate metabolic space (H, ξ) such that the orthogonal sum $(M, \phi) \perp (H, \xi)$ is split metabolic.

However, these conditions do not imply in general that (M, ϕ) itself is metabolic, not even in split exact categories with duality where 2 is invertible, as Ojanguren's example discussed in [Bal05, example 2.6] shows.

Example 3.1.8. If R is a ring with involution one simply writes $W^0(R)$ for $W^0(\mathtt{fgp}/R)$, see example 2.2.6. Here we recall some well-known computations. First of all, one has the following isomorphisms:

$$W^0(\mathbb{Z}) \cong W^0(\mathbb{R}) \cong \mathbb{Z} \ ,$$

the first induced by the change of ring map, the second by the signature. Furthermore

$$W^0(\mathbb{Q}) \cong W^0(\mathbb{Z}) \oplus \bigoplus_{p \text{ prime}} W^0(\mathbb{F}_p) \ ,$$

and for any finite field $\mathbb{F}_{p^n}$ one has

$$W^0(\mathbb{F}_{p^n}) \cong \begin{cases} \mathbb{Z}/2 & \text{if } p^n \text{ is even,} \\ (\mathbb{Z}/2)[\epsilon]/\epsilon^2 & \text{if } p^n \equiv 1 \pmod 4 \ , \\ \mathbb{Z}/4 & \text{if } p^n \equiv 3 \pmod 4 \ . \end{cases}$$

The proofs of these facts can be found in [MH73].

Example 3.1.9. If X is a scheme and $\mathcal{L}$ a line bundle over X one simply writes $W^0(X; \mathcal{L})$ for $W^0(\mathtt{vect}/X)$ with respect to the $\mathcal{L}$-twisted duality, and simply $W^0(X)$ when $\mathcal{L} = \mathcal{O}_X$, see example 2.2.7. This corresponds to Knebusch's original definition [Kne77]. The first non-trivial computation in the non-affine case is the following: If K is a field of characteristic different from 2 and n is a positive integer, then

$$W^0(\mathbb{P}^n_K) \cong W^0(K) \ ,$$

see [Ara80, Gil01].

3.2. **Witt groups of triangulated categories.** In this section we present the definition and main properties of triangular Witt groups, that were introduced and studied by Paul Balmer in a series of papers around the turn of the century, see in particular [Bal99, Bal00, Bal01a] and his surveys [Bal04, Bal05].

Let $\mathcal{T} = (\mathcal{T}, ^\vee, \mu)$ be a triangulated category with duality, see definition 2.6.6. As explained in remark 2.5.7 we can view the triangulated category $\mathcal{T}$ as a graded category. Recall that for every object X of $\mathcal{T}$ the duality defines an action of the cyclic group of order two C_2 on the graded module $\hom_{\mathcal{T}}(X, X^\vee)$: For every homomorphism $f \in \hom_{\mathcal{T}}(X, X^\vee)_n$ of degree n, i.e., $f\colon \Sigma^n X \to X^\vee$, one puts $\tau(f) = \tau_{X,X}(f) = \Sigma^n(f^\vee \mu_X)$; henceforth we view $\hom_{\mathcal{T}}(X, X^\vee)$ as a graded $\mathfrak{K}\mathrm{C}_2$-module.

Symmetric forms and spaces are defined in complete analogy with the case of exact categories.

Definition 3.2.1. Let $n \in \mathbb{Z}$ be an integer. The *symmetric graded Q-module* of an object X of $\mathcal{T}$ is defined as the fixed-point graded module

$$\begin{aligned} Q^n_{\mathcal{T}}(X) &= \left(\hom_{\mathcal{T}}(X, X^\vee)^{\mathrm{C}_2}\right)_n = H^n(\mathrm{C}_2; \hom_{\mathcal{T}}(X, X^\vee)) \\ &= \hom_{\mathfrak{K}\mathrm{C}_2}(\mathfrak{K}, \hom_{\mathcal{T}}(X, X^\vee))_n \ . \end{aligned}$$

Its elements are called *n-dimensional symmetric forms* on X. Explicitly, an n-dimensional symmetric form is a homomorphism $\phi\colon \Sigma^n X \to X^\vee$ which is symmetric with respect to the duality, i.e., the following square

$$\begin{array}{ccc} \Sigma^n X & \xrightarrow{\phi} & X^\vee \\ {\scriptstyle \Sigma^n \mu_X}\downarrow & & \Vert {\scriptstyle \mathrm{id}_{X^\vee}} \\ \Sigma^n X^{\vee\vee} & \xrightarrow[\Sigma^n \phi^\vee]{} & X^\vee \end{array}$$

commutes. An n-dimensional symmetric form is called *non-degenerate* if it is an isomorphism. A *n-dimensional (non-degenerate) symmetric space* is an object X together with a (non-degenerate) n-dimensional symmetric form.

An *isometric homomorphism* of n-dimensional symmetric spaces (X_1, ϕ_1) and (X_2, ϕ_2) is a homomorphism $f\colon X_1 \to X_2$ of degree zero respecting the symmetric forms, i.e., such that the square

$$\begin{array}{ccc} \Sigma^n X_1 & \xrightarrow{\phi_1} & X_1^\vee \\ {\scriptstyle \Sigma^n f}\downarrow & & \uparrow {\scriptstyle f^\vee} \\ \Sigma^n X_2 & \xrightarrow{\phi_2} & X_2^\vee \end{array}$$

commutes: $f^\vee \phi_2 \Sigma^n(f) = \phi_1$. An *isometry* is an isometric isomorphism.

The *orthogonal sum* of two n-dimensional symmetric spaces (X_1, ϕ_1) and (X_2, ϕ_2) is defined as

$$(X_1, \phi_1) \perp (X_2, \phi_2) = \left(X_1 \oplus X_2, \left(\begin{smallmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{smallmatrix}\right)\right) \ .$$

Analogously one defines the *quadratic graded Q-module* as the coinvariant module

$$\begin{aligned} Q_n^{\mathcal{T}}(M) &= \left(\hom_{\mathcal{T}}(X, X^\vee)_{\mathrm{C}_2}\right)_n = H_n(\mathrm{C}_2; \hom_{\mathcal{T}}(X, X^\vee)) \\ &= \left(\mathfrak{K} \otimes_{\mathfrak{K}\mathrm{C}_2} \hom_{\mathcal{T}}(X, X^\vee)\right)_n \ , \end{aligned}$$

and *quadratic forms* and *spaces* and their homomorphisms correspondingly.

The norm homomorphism

$$1+\tau\colon Q_n^{\mathcal{T}}(M)\to Q_{\mathcal{T}}^n(M)$$

is called *symmetrization* and a quadratic form $\psi\in Q_n^{\mathcal{T}}(M)$ is called non-degenerate if its symmetrization $(1+\tau)\psi=\psi+\psi^{\vee}\mu_M\in Q_{\mathcal{T}}^n(M)$ is non-degenerate.

We remark that our n-dimensional non-degenerate symmetric spaces are called by Balmer "symmetric spaces for the $(-n)$-shifted duality" (notice the minus sign). Moreover, Balmer does not consider the quadratic counterparts, since he primarily focuses on cases where 2 is invertible and therefore symmetric and quadratic spaces coincide.

Notice that the graded Q-modules define contravariant functors

$$Q_{\mathcal{T}}^n, Q_n^{\mathcal{T}}\colon \mathcal{T}^{\mathrm{op}}\to \mathsf{mod}/\mathfrak{K}\ ,$$

and the symmetrization a natural transformation between them, which is an isomorphism if $\frac{1}{2}\in\mathcal{T}$.

Lemma 3.2.2. *For all pairs of objects X_1 and X_2 of $\mathcal{T}$ and for every $n\in\mathbb{Z}$ one has*

$$Q_{\mathcal{T}}^n(X_1\oplus X_2)\cong Q_{\mathcal{T}}^n(X_1)\oplus Q_{\mathcal{T}}^n(X_2)\oplus \hom_{\mathcal{T}}(X_1,X_2^{\vee})_n,$$
$$Q_n^{\mathcal{T}}(X_1\oplus X_2)\cong Q_n^{\mathcal{T}}(X_1)\oplus Q_0^{\mathcal{T}}(X_2)\oplus \hom_{\mathcal{T}}(X_1,X_2^{\vee})_n.$$

Proof. The proof of lemma 3.1.4 goes through completely unmodified in the graded setting. □

Definition 3.2.3. Assume that the category $\mathcal{T}$ is essentially small, i.e., the isomorphism classes of objects of $\mathcal{T}$ form a set. Let $n\in\mathbb{Z}$ be an integer.

The *n-dimensional symmetric Witt monoid* of $\mathcal{T}$ is defined as the (obviously abelian) monoid of isometry classes of non-degenerate n-dimensional symmetric spaces in $\mathcal{T}$ with respect to the orthogonal sum, and it is denoted $MW^n(\mathcal{T})=MW^n(\mathcal{T},{}^{\vee},\mu)$. The *quadratic Witt monoid* $MW_n(\mathcal{T})$ is defined completely analogously.

Notice again that Balmer writes MW^{-n} for what we decided to denote MW^n.

Lemma 3.2.4. *Let $(\mathcal{T},{}^{\vee},\mu)$ be a triangulated category with duality. Then for all $n\in\mathbb{Z}$ one has that $MW^{n+2}(\mathcal{T},{}^{\vee},\mu)=MW^n(\mathcal{T},{}^{\vee},-\mu)$. In particular, $MW^{n+4}(\mathcal{T},{}^{\vee},\mu)=MW^n(\mathcal{T},{}^{\vee},\mu)$.*

Proof. Indeed, the triangulated functor Σ^2 induces an equivalence of triangulated categories with duality between $(\mathcal{T},{}^{\vee},\mu)$ and $(\mathcal{T},{}^{\vee},-\mu)$. □

In this section, up to the definition above, only the graded structure of a triangulated category has been used. The triangulated structure is going to be used now in the definition of metabolic spaces and hence of the Witt groups.

Definition 3.2.5. An n-dimensional symmetric space (X,ϕ) is called *metabolic* (also called neutral by Balmer) if there exist an object L (called a *lagrangian*) and homomorphisms $\iota\colon L \to \Sigma^n X$ and $\delta\colon \Sigma^{-1}(L^\vee) \to L$ such that the following triangle is exact

$$\Sigma^{-1}(L^\vee) \xrightarrow{\delta} L \xrightarrow{\iota} \Sigma^n X \xrightarrow{\iota^\vee \phi} L^\vee$$

and $\Sigma^{-1}(\delta^\vee) = \mu_L \delta$.

As in the exact case, one sees immediately that the property of being metabolic is invariant under isometries, and that the orthogonal sum of two metabolic spaces is again metabolic. Moreover, if ϕ is an n-dimensional symmetric form on X, then $(X, \phi) \perp (X, -\phi)$ is metabolic. Hence we can consider the submonoid $NW^n(\mathcal{T}) = NW^n(\mathcal{T}, {}^\vee, \mu)$ of $MW^n(\mathcal{T})$ consisting of isometry classes of n-dimensional non-degenerate metabolic spaces, and as in the exact case the quotient monoid $MW^n(\mathcal{T})/NW^n(\mathcal{T})$ is a group.

Definition 3.2.6. The n-dimensional symmetric Witt group $W^n(\mathcal{T}) = W^n(\mathcal{T}, {}^\vee, \mu)$ of a triangulated category with duality $\mathcal{T}$ is defined as the quotient $MW^n(\mathcal{T})/NW^n(\mathcal{T})$.

Again, we warn the reader that Balmer writes W^{-n} for what we decided to denote W^n.

From lemma 3.2.4 above one obtains 4-periodicity for the Witt groups.

Proposition 3.2.7. *Let $(\mathcal{T}, {}^\vee, \mu)$ be a triangulated category with duality. Then for all $n \in \mathbb{Z}$ one has that $W^{n+2}(\mathcal{T}, {}^\vee, \mu) = W^n(\mathcal{T}, {}^\vee, -\mu)$. In particular, $W^{n+4}(\mathcal{T}, {}^\vee, \mu) = W^n(\mathcal{T}, {}^\vee, \mu)$.*

As in the case of exact categories with duality, it follows from the definition that the following conditions are equivalent for an n-dimensional non-degenerate symmetric space (X, ϕ):

(i) (X, ϕ) represents zero in the n-dimensional Witt group $W^n(\mathcal{T})$;
(ii) there exists an n-dimensional non-degenerate metabolic space (H, ξ) such that the orthogonal sum $(M, \phi) \perp (H, \xi)$ is metabolic.

Unlike in the exact case, in triangulated categories with duality in which 2 is invertible every non-degenerate symmetric space that represents zero in the Witt group is itself metabolic.

Theorem 3.2.8 (Balmer)**.** *Let $\mathcal{T}$ be a triangulated category with duality such that $\frac{1}{2} \in \mathcal{T}$, and let (X, ϕ) be an n-dimensional non-degenerate symmetric space. Then (X, ϕ) represents zero in the n-dimensional Witt group $W^n(\mathcal{T})$ if and only if (X, ϕ) is metabolic.*

Proof. Since we are not going to use this result we simply refer to [Bal00, theorem 3.5 on page 326, proof 3.6 on pages 326–330] for the involved and lengthy proof. □

The following fundamental lemma is instrumental for the study of Witt groups of triangulated categories for which 2 is invertible.

Lemma 3.2.9 (Balmer)**.** *Let $\mathcal{T}$ be a triangulated category with duality. Assume that 2 is invertible in $\mathcal{T}$. Given an $(n+1)$-dimensional symmetric space $(X, \phi \in Q^{n+1}_{\mathcal{T}}(X))$, consider a cone C of $\Sigma^{-n-1}\phi\colon X \to \Sigma^{-n-1}(X^\vee)$.*

Then there is one and up to isometry only one n-dimensional non-degenerate symmetric space $(C, \partial\phi \in Q^n_{\mathcal{T}}(C))$ such that the diagram

$$\begin{array}{ccccccc}
\Sigma^{n+1}X & \xrightarrow{\phi} & X^\vee & \xrightarrow{f} & \Sigma(\Sigma^n C) & \xrightarrow{g} & \Sigma^{n+2}X \\
\Sigma^{n+1}\mu_X \big\downarrow & & \big\| & & \cong \big\downarrow \Sigma(\partial\phi) & & \big\downarrow \Sigma^{n+2}\mu_X \\
\Sigma^{n+1}(X^{\vee\vee}) & \xrightarrow[\Sigma^{n+1}\phi^\vee]{} & X^\vee & \xrightarrow[(-1)^n g^\vee]{} & \Sigma(C^\vee) & \xrightarrow[f^\vee]{} & \Sigma^{n+2}(X^{\vee\vee})
\end{array}$$

commutes.

Definition 3.2.10. The n-dimensional non-degenerate symmetric space $(C, \partial\phi \in Q^n_{\mathcal{T}}(C))$ of lemma 3.2.9 above is called the *boundary* of the $(n+1)$-dimensional symmetric space $(X, \phi \in Q^{n+1}_{\mathcal{T}}(X))$, and one usually writes ∂X for C. (Notice that ∂X depends on the symmetric form on X, even though it is hidden in the notation.)

Proof. ([Bal00, lemma 2.6 on page 320].) The existence of an isomorphism $\partial\phi\colon \Sigma^n C \to C^\vee$ completing the diagram above is just an axiom in any triangulated category. Now use that 2 is invertible in $\mathcal{T}$ and replace $\partial\phi$ with the symmetric isomorphism $\frac{1}{2}((\partial\phi) + (\partial\phi)^\vee)$. □

The assumption that 2 is invertible is crucial in the proof of the above lemma. When the assumption is not satisfied it seems impossible to prove the lemma using only the triangulated structure. But if the triangulated category with duality at hand arises as the homotopy category of a pre-triangulated dg category with duality, then the lemma is true, compare lemma 3.3.13 below.

Corollary 3.2.11. *Let $\mathcal{T}$ be a triangulated category with duality. Assume that 2 is invertible in $\mathcal{T}$. Let $n \in \mathbb{Z}$ be any integer. Then an n-dimensional non-degenerate symmetric space $(X, \phi \in Q^n_{\mathcal{T}}(X))$ is metabolic if and only if it is isomorphic to the cone $(\partial W, \partial\psi \in Q^n_{\mathcal{T}}(W)$ of an $(n+1)$-dimensional symmetric space $(W, \psi \in Q^{n+1}_{\mathcal{T}}(W))$.*

Proof. Follows at once from lemma 3.2.9 above. □

We close this section by recalling some deep theorems of Balmer's about triangular Witt groups in the cases where 2 is invertible.

The first one proves that the classical Witt group of an exact category with duality where 2 is invertible can be recovered from the triangular Witt group of its bounded derived category.

Theorem 3.2.12 (Balmer)**.** *If $\mathcal{E}$ is an exact category with duality such that 2 is invertible in $\mathcal{E}$, then there is an isomorphism*

$$W(\mathcal{E}) \cong W^0(\mathcal{D}^b(\mathcal{E}))\ .$$

Proof. This is proved in [Bal01a] under the mild additional assumption that all split epimorpshims in $\mathcal{E}$ are admissible (see example 2.6.5 for a discussion of this condition), and the general case is deduced from this in [BW02, remarks following theorem 1.4]. □

The next localization theorem provides the key computational tool for triangular Witt groups.

Theorem 3.2.13. *Let $\mathcal{T}$ be a triangulated category with duality, and assume that 2 is invertible in $\mathcal{T}$. Let $\mathcal{S}$ be a thick triangulated subcategory of $\mathcal{T}$ stable under the duality. Then there is a 12-term periodic long exact sequence of triangular Witt groups*

$$\ldots \to W^{n+1}(\mathcal{T}/\mathcal{S}) \to W^n(\mathcal{S}) \to W^n(\mathcal{T}) \to W^n(\mathcal{T}/\mathcal{S}) \to \ldots \ .$$

Proof. This is proved in [Bal00, theorem 6.2] under the mild additional assumption that $\mathcal{T}$ is "weakly cancellative", which is then removed in [BW02, theorem 2.1]. □

This theorem is typically applied to the derived category of vector bundles over a "well-behaved" scheme where 2 is invertible and the localization to an open subscheme. This is the key ingredient in showing that triangular Witt groups yield a generalized cohomology theory, see [Bal01b], which is beyond the scope of this article.

3.3. **Algebraic L-theory of pre-triangulated dg categories.** We now define L-theory groups for pre-triangulated dg categories. Ranicki first developed an algebraic theory of L-groups using chain complexes of modules over a ring with involution, thereby expressing Wall's L-group in a unified and more conceptual way. He then extended his theory as to encompass chain complexes in additive categories with so-called chain duality. We now take his ideas one little step further, in order to deal with any pre-triangulated dg category with duality. The guiding principle is that algebraic L-theory á la Ranicki relies not primarily on the fact that one is working with categories whose objects are chain complexes, but rather it is instrumental that the "morphisms" in the categories are organized in chain complexes, i.e., one is working with dg categories (with duality). In this setting one can define symmetric and quadratic (and hyperquadratic) spaces, that Ranicki calls—in analogy with the algebraic topological motivating examples—symmetric or quadratic (or hyperquadratic) algebraic Poincaré complexes. In order to define the neutral elements in the yet-to-be-defined L-groups, one then needs an extra structure: suspensions and cones, i.e., one needs to work with pre-triangulated dg categories (with duality). The resulting L-theory turns out to depend essentially only on the corresponding triangulated categories (with duality), but one needs to have pre-triangulated dg models even in order to define the L-groups. The relation to Balmer's Witt groups is the following: when 2 is invertible then the new L-groups of a pre-triangulated dg category with duality coincide with Balmer's Witt groups of the associated triangulated category with duality.

First we define what symmetric and quadratic forms are in pre-triangulated dg categories with duality.

As always, let $\mathrm{C}_2 = \langle\, \tau \mid \tau^2 = 1 \,\rangle$ denote the cyclic group of order two. Let W be the standard free $\mathfrak{K}\mathrm{C}_2$-resolution of the trivial $\mathfrak{K}\mathrm{C}_2$-module $\mathfrak{K}$:

$$W \ : \qquad \ldots \longrightarrow \underset{j=3}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1-\tau} \underset{j=2}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1+\tau} \underset{j=1}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1-\tau} \underset{j=0}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1+\tau} \mathfrak{K} \ .$$

Notice that its dual in the category of chain complexes of (not necessarily bounded) finitely generated projective modules over $\mathfrak{K}\mathrm{C}_2$, with respect to the standard duality defined by sending X to $X^\vee = \hom_{\mathsf{ch\,mod}/\mathfrak{K}\mathrm{C}_2}(X, S^0(\mathfrak{K}\mathrm{C}_2))$, is

$$W^\vee \ : \qquad \underset{j=0}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{\tau-1} \underset{j=-1}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{\tau+1} \underset{j=-2}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{\tau-1} \underset{j=-3}{\mathfrak{K}\mathrm{C}_2} \longrightarrow \dots \ .$$

Furthermore, let $\widehat{W}$ denote the free Tate $\mathfrak{K}\mathrm{C}_2$-resolution of the trivial $\mathfrak{K}\mathrm{C}_2$-module $\mathfrak{K}$:

$$\widehat{W} \ : \quad \dots \longrightarrow \underset{j=2}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1+\tau} \underset{j=1}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1-\tau} \underset{j=0}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1+\tau} \underset{j=-1}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1-\tau} \underset{j=-2}{\mathfrak{K}\mathrm{C}_2} \xrightarrow{1+\tau} \dots \ .$$

Notice that the cone of the inclusion chain homomorphism $W \to \widehat{W}$ is isomorphic to $\Sigma^{-1}(W^\vee)$.

Let $\mathcal{C}$ be a pre-triangulated dg category with duality, see definition 2.4.4. Recall that then for every object X of $\mathcal{C}$ the duality defines an action of C_2 on the chain complex (of modules over $\mathfrak{K}$) $\hom_{\mathcal{C}}(X, X^\vee)$, that we can thus consider as a chain complex of modules over $\mathfrak{K}\mathrm{C}_2$. Let n be an integer.

Definition 3.3.1. The *n-dimensional symmetric Q-module* of an object X of $\mathcal{C}$ is defined as the n-th group hypercohomology of the group C_2 with coefficients in the chain complex $\hom_{\mathcal{C}}(X, X^\vee)$

$$\begin{aligned} Q^n_{\mathcal{C}}(X) &= H^n(\mathrm{C}_2; \hom_{\mathcal{C}}(X, X^\vee)) \\ &= H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(W, \hom_{\mathcal{C}}(X, X^\vee))) \ . \end{aligned}$$

Its elements should be thought of as n-dimensional symmetric forms over X, see below. An *n-dimensional symmetric space in $\mathcal{C}$* is an object X of $\mathcal{C}$ together with an element $\varphi \in Q^n_{\mathcal{C}}(X)$.

Analogously, one defines the *n-dimensional quadratic Q-module* of an object X of $\mathcal{C}$ as the n-th group homology of the group C_2 with coefficients in the chain complex $\hom_{\mathcal{C}}(X, X^\vee)$

$$\begin{aligned} Q^{\mathcal{C}}_n(X) &= H_n(\mathrm{C}_2; \hom_{\mathcal{C}}(X, X^\vee)) \\ &= H_n(W \otimes_{\mathfrak{K}\mathrm{C}_2} \hom_{\mathcal{C}}(X, X^\vee)) = H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(W^\vee, \hom_{\mathcal{C}}(X, X^\vee))) \ , \end{aligned}$$

and the *n-dimensional hyperquadratic Q-module* of an object X of $\mathcal{C}$ as the n-th Tate cohomology of the group C_2 with coefficients in the chain complex $\hom_{\mathcal{C}}(X, X^\vee)$

$$\begin{aligned} \widehat{Q}^n_{\mathcal{C}}(X) &= \widehat{H}^n(\mathrm{C}_2; \hom_{\mathcal{C}}(X, X^\vee)) \\ &= H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(\widehat{W}, \hom_{\mathcal{C}}(X, X^\vee))) \ . \end{aligned}$$

Correspondingly one speaks of n-dimensional quadratic and hyperquadratic spaces in $\mathcal{C}$.

Notice that we deviate from Ranicki's terminology, in order to maintain the analogy with the exact and triangulated cases—Ranicki's terminology in turn is motivated from the topological examples. Namely, Ranicki calls "symmetric (or quadratic or hyperquadratic) algebraic complex" what we baptized a symmetric (or quadratic or hyperquadratic) space. See also the remark following definition 3.3.4 below.

The fundamental property of the Q-groups defined above is their homotopy invariance: If $f\colon X \to X'$ is a homotopy equivalence, then $Q^n(f)$ and $Q_n(f)$ are isomorphisms. See lemma 3.3.13 below.

Example 3.3.2. More explicitly, if $\mathcal{C} = \mathtt{ch}^{\mathrm{b}}\mathcal{E}$ for some exact category with duality $\mathcal{E}$, then any element $\varphi \in Q^n_{\mathtt{ch}\,\mathcal{E}}(X)$ is represented by an n-chain in $\hom_{\mathfrak{K}\mathrm{C}_2}(W, \hom_{\mathcal{C}}(X, X^\vee))$ (unique modulo boundaries), and such an n-chain is given by a sequence of chains

$$\left(\varphi_s \in \hom_{\mathcal{C}}(X, X^\vee)_s = \prod_{r\in\mathbb{Z}} \hom_{\mathcal{E}}(X_{n+s-r}, X_r^\vee)\right)_{s\geq 0}$$

such that for all $s \geq 0$ and all $r \in \mathbb{Z}$

$$\partial_X \varphi_s - (-1)^{n+s}\varphi_s \partial_{X^\vee} = (-1)^n(\varphi_{s-1} + (-1)^s \tau\varphi_{s-1})\colon X^{n+s-r-1} \to X_r$$

(where $\varphi_{-1} = 0$). In particular this means that φ_0 is a chain homomorphism, φ_1 is a chain homotopy between φ_0 and $\varphi_0^\vee$, and so on.... An analogous formula for $Q_n^{\mathtt{ch}\,\mathcal{E}}(X)$ and $\widehat{Q}^n_{\mathtt{ch}\,\mathcal{E}}(X)$ is available, too.

Notice that the Q-modules define contravariant functors

$$Q_n^{\mathcal{C}}, Q^n_{\mathcal{C}}, \widehat{Q}^n_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \to \mathtt{mod}/\mathfrak{K}\ ,$$

and for every object X in $\mathcal{C}$ there is a natural long exact sequence

$$(3.3.3) \qquad \dots \to Q_n^{\mathcal{C}}(X) \xrightarrow{1+\tau} Q^n_{\mathcal{C}}(X) \xrightarrow{J} \widehat{Q}^n_{\mathcal{C}}(X) \xrightarrow{H} Q_{n-1}^{\mathcal{C}}(X) \to \dots$$

induced by the exact triangle $W^\vee \to W \to \widehat{W} \to \Sigma W^\vee$. If 2 is invertible in $\mathcal{C}$ then $\widehat{Q}^n_{\mathcal{C}}(X) = 0$ for every $n \in \mathbb{Z}$ and every object X of $\mathcal{C}$, therefore the so-called symmetrization homomorphism $1 + \tau$ induces an isomorphism $Q_n^{\mathcal{C}}(X) \cong Q^n_{\mathcal{C}}(X)$.

There are homomorphisms $\mathrm{ev}\colon Q^n_{\mathcal{C}}(X) \to Q^n_{\mathrm{ho}\,\mathcal{C}}(X)$ and $\mathrm{ev}\colon Q_n^{\mathcal{C}}(X) \to Q_n^{\mathrm{ho}\,\mathcal{C}}(X)$, induced by the obvious graded functor $\mathcal{C} \to \mathrm{ho}\,\mathcal{C}$ which is the identity on objects, such that the following diagram

$$\begin{array}{ccccc} Q_n^{\mathcal{C}}(X) & \xrightarrow{\mathrm{ev}} & Q_n^{\mathrm{ho}\,\mathcal{C}}(X) & & \\ {\scriptstyle 1+\tau}\downarrow & & \downarrow{\scriptstyle 1+\tau} & & \\ Q^n_{\mathcal{C}}(X) & \xrightarrow[\mathrm{ev}]{} & Q^n_{\mathrm{ho}\,\mathcal{C}}(X) & \subset & \hom_{\mathrm{ho}_{\mathcal{C}}}(X, X^\vee)_n = \hom_{\mathrm{ho}\,\mathcal{C}}(\Sigma^n X, X^\vee) \end{array}$$

commutes. In particular, any n-dimensional symmetric or quadratic space in $\mathcal{C}$ defines an n-dimensional symmetric or quadratic space in the associated homotopy category $\mathrm{ho}(\mathcal{C})$.

Definition 3.3.4. An n-dimensional symmetric space $(X, \varphi \in Q^n_{\mathcal{C}}(X))$ in $\mathcal{C}$ is *non-degenerate* if the corresponding n-dimensional symmetric space $(X, \mathrm{ev}(\varphi) \in Q^n_{\mathrm{ho}\,\mathcal{C}}(X))$ is non-degenerate, i.e., if $\mathrm{ev}(\varphi)\colon \Sigma^n X \to X^\vee$ is an isomorphism in the associated triangulated homotopy category $\mathrm{ho}(\mathcal{C})$. Completely analogously one defines n-dimensional non-degenerate quadratic spaces.

Again, we deviate from Ranicki's terminology in that we call non-degenerate what he refers to as “Poincaré”: e.g., a “symmetric algebraic Poincaré complex” in the sense of Ranicki is called here a non-degenerate symmetric space.

Example 3.3.5. The fundamental examples of non-degenerate symmetric spaces arise from algebraic topology as follows. Let X be a connected finite n-dimensional Poincaré complex (we refer to [Wal99] for the definition of this term). Denote by $C_*(\widetilde{X})$ the cellular chain complex of the universal cover of X: It is a bounded chain complex of finitely generated free modules over the ring with involution $\mathbb{Z}^\omega[\pi_1(X)]$, the integral group ring of the fundamental group of X with the involution twisted by the orientation homomorphism ω of X (see example 2.2.6). Then $C_*(\widetilde{X})^\vee$ is a non-degenerate n-dimensional symmetric space in the category of bounded chain complex of finitely generated free modules over $\mathbb{Z}^\omega[\pi_1(X)]$. This is Ranicki's so-called “symmetric construction”, see [Ran80b, proposition 2.1]. Notice that Ranicki works with $C_*(\widetilde{X})$ and not with its dual as above, because his $Q^n(C_*(\widetilde{X}))$ actually coincides with our $Q^n(C_*(\widetilde{X})^\vee)$. This example also explains Ranicki's standard choice of terminology.

Lemma 3.3.6. *For all $n \in \mathbb{Z}$ and for all pairs X_1 and X_2 of objects of $\mathcal{C}$ there are natural direct sum decompositions of modules*

$$Q^n_{\mathcal{C}}(X_1 \oplus X_2) \cong Q^n_{\mathcal{C}}(X_1) \oplus Q^n_{\mathcal{C}}(X_2) \oplus \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n,$$
$$Q_n^{\mathcal{C}}(X_1 \oplus X_2) \cong Q_n^{\mathcal{C}}(X_1) \oplus Q_n^{\mathcal{C}}(X_2) \oplus \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n,$$

which are compatible with the natural symmetrization homomorphism $1+\tau$. The homomorphism $1+\tau\colon \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n \to \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n$ is an isomorphism. Moreover there are natural isomorphisms

$$\widehat{Q}^n_{\mathcal{C}}(X_1 \oplus X_2) \cong \widehat{Q}^n_{\mathcal{C}}(X_1) \oplus \widehat{Q}^n_{\mathcal{C}}(X_2)\ .$$

Under this isomorphisms the long exact sequence (3.3.3) for $X = X_1 \oplus X_2$ corresponds to the direct sum of the corresponding long exact sequences for X_1 and X_2 and of the long exact sequence

$$\ldots \to 0 \to \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n \xrightarrow{1+\tau} \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n \to 0 \to$$
$$\to \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_{n-1} \xrightarrow{1+\tau} \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_{n-1} \to 0 \to \ldots\ .$$

Proof. The proof of lemma 3.1.4 goes through completely unmodified in the dg setting. □

Using the sum formulas from the above lemma one defines the *orthogonal sum* $(X_1, \varphi_1) \perp (X_2, \varphi_2)$ of two n-dimensional symmetric spaces $(X_1, \varphi_1 \in Q^n_{\mathcal{C}}(X))$, $(X_2, \varphi_2 \in Q^n_{\mathcal{C}}(X))$ as the n-dimensional symmetric space

$$\big(X_1 \oplus X_2, (\varphi_1, \varphi_2) \in Q^n_{\mathcal{C}}(X_1) \oplus Q^n_{\mathcal{C}}(X_2) \subseteq Q^n_{\mathcal{C}}(X_1 \oplus X_2)\big).$$

Analogously for the quadratic and hyperquadratic case.

Definition 3.3.7. Assume that the category $\mathcal{C}$ is essentially small, i.e., the quasi isomorphism classes of objects in $\mathcal{C}$ form a set.

The *n-dimensional symmetric L-monoid* of $\mathcal{C}$ is defined as the (obviously abelian) monoid of isometry classes of non-degenerate n-dimensional symmetric spaces in $\mathcal{C}$ with respect to the orthogonal sum, and it is denoted $ML^n(\mathcal{C}) = ML^n(\mathcal{C}, ^\vee, \mu)$. The *$n$-dimensional quadratic L-monoid* is defined completely analogously.

Lemma 3.3.8. *Let $(\mathcal{C}, ^\vee, \mu)$ be a pre-triangulated dg category with duality. Then for all $n \in \mathbb{Z}$ one has that $ML^{n+2}(\mathcal{T}, ^\vee, \mu) = ML^n(\mathcal{T}, ^\vee, -\mu)$. In particular, $ML^{n+4}(\mathcal{T}, ^\vee, \mu) = ML^n(\mathcal{T}, ^\vee, \mu)$. Exactly the same holds in the quadratic case.*

Proof. Indeed, the dg functor Σ^2 induces a quasi-equivalence of pre-triangulated dg cateogires with duality between $(\mathcal{C}, ^\vee, \mu)$ and $(\mathcal{C}, ^\vee, -\mu)$. □

Now we want to define what elements should represent zero in the yet-to-be-defined L-groups, i.e., what are the analogs of metabolic spaces in the L-theoretic setting.

Let $f\colon X \to Y$ be a closed homomorphism of degree zero in $\mathcal{C}$. Consider the induced chain homomorphism of chain complexes of modules over $\mathfrak{K}\mathrm{C}_2$

$$\hom_{\mathcal{C}}(f, f^\vee)\colon \hom_{\mathcal{C}}(Y, Y^\vee) \to \hom_{\mathcal{C}}(X, X^\vee)$$

and take its mapping cone:

$$\mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)) \ .$$

Definition 3.3.9. The *n-dimensional relative symmetric, quadratic, and hyperquadratic Q-modules* of a closed homomorphism of degree zero are defined as

$$Q^n_{\mathcal{C}}(f) = H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(W, \mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)))) \ ,$$
$$Q_n^{\mathcal{C}}(f) = H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(W^\vee, \mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)))) \ ,$$
$$\widehat{Q}^n_{\mathcal{C}}(f) = H_n(\hom_{\mathfrak{K}\mathrm{C}_2}(\widehat{W}, \mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)))) \ .$$

One then gets a ladder of long exact sequences

$$\begin{array}{ccccccccc} \cdots \longrightarrow & Q^{n+1}_{\mathcal{C}}(f) & \xrightarrow{\partial} & Q^n_{\mathcal{C}}(Y) & \longrightarrow & Q^n_{\mathcal{C}}(X) & \longrightarrow & Q^n_{\mathcal{C}}(f) & \longrightarrow \cdots \\ & \uparrow{\scriptstyle 1+\tau} & & \uparrow{\scriptstyle 1+\tau} & & \uparrow{\scriptstyle 1+\tau} & & \uparrow{\scriptstyle 1+\tau} & \\ \cdots \longrightarrow & Q_{n+1}^{\mathcal{C}}(f) & \xrightarrow{\partial} & Q_n^{\mathcal{C}}(Y) & \longrightarrow & Q_n^{\mathcal{C}}(X) & \longrightarrow & Q_n^{\mathcal{C}}(f) & \longrightarrow \cdots \ . \end{array}$$

Notice that there is a natural group homomorphism

$$\mathrm{ev}\colon Q^{n+1}_{\mathcal{C}}(f) \to \hom_{\mathrm{ho}\,\mathcal{C}}(X, \mathrm{cone}(f^\vee))_{n+1}$$

induced by the commutative diagram

$$\begin{array}{ccccc} \hom_{\mathcal{C}}(Y, Y^\vee) & \longrightarrow & \hom_{\mathcal{C}}(X, X^\vee) & \longrightarrow & \mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)) \\ \downarrow & & \| & & \downarrow \\ \hom_{\mathcal{C}}(X, Y^\vee) & \longrightarrow & \hom_{\mathcal{C}}(X, X^\vee) & \longrightarrow & \hom_{\mathcal{C}}(X, \mathrm{cone}(f^\vee)) \ . \end{array}$$

Definition 3.3.10. An *$(n+1)$-dimensional symmetric (quadratic) pair in $\mathcal{C}$* is a closed homomorphism of degree zero $f\colon X \to Y$ together with an element $\psi \in Q^{n+1}_{\mathcal{C}}(f)$ (respectively, $\psi \in Q_{n+1}^{\mathcal{C}}(f)$) such that $\mathrm{ev}(\psi) \in \hom_{\mathrm{ho}\,\mathcal{C}}(X, \mathrm{cone}(f^\vee))_{n+1}$ (respectively, $\mathrm{ev}((1+\tau)\psi)$) is an isomorphism.

Notice that in this case Y together with $\partial(\psi)$ is an n-dimensional non-degenerate space, which is called the *boundary* of the pair.

Definition 3.3.11. Two n-dimensional non-degenerate symmetric spaces $(X_1, \varphi_1 \in Q^n_{\mathcal{C}}(X_1))$, $(X_2, \varphi_2 \in Q^n_{\mathcal{C}}(X_2))$ are *bordant* if there exists an $(n+1)$-dimensional symmetric pair $(\begin{pmatrix} f_1 & f_2 \end{pmatrix} : X_1 \oplus X_2 \to Y, \psi \in Q^{n+1}_{\mathcal{C}}(\begin{pmatrix} f_1 & f_2 \end{pmatrix}))$ called *bordism* such that

$$\partial(\psi) = (\varphi_1, -\varphi_2, 0) \in Q^n_{\mathcal{C}}(X_1) \oplus Q^n_{\mathcal{C}}(X_2) \oplus \hom_{\mathrm{ho}\,\mathcal{C}}(X_1, X_2^\vee)_n \cong Q^n_{\mathcal{C}}(X_1 \oplus X_2).$$

We also call null-bordant n-dimensional non-degenerate symmetric spaces *metabolic*, in analogy with the standard terminology in the exact and triangulated cases.

Bordism of non-degenerate quadratic spaces is defined completely analogously.

Lemma 3.3.12. *Let $f \colon X \to Y$ be a closed morphism of degree zero, and denote its cone by $Z = \mathrm{cone}(f)$. Then the relative Q-groups of f and the absolute Q-groups of Z in the symmetric and quadratic case are related by the following long exact sequences*

$$\ldots \to \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_n \to Q^n_{\mathcal{C}}(f) \to Q^n_{\mathcal{C}}(Z) \to \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_{n-1} \to \ldots$$

For the hyperquadratic Q-groups there are isomorphisms

$$\widehat{Q}^n_{\mathcal{C}}(f) \cong \widehat{Q}^n_{\mathcal{C}}(Z) \ .$$

One has the following commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & \widehat{Q}^{n-1}_{\mathcal{C}}(f) & \to & \widehat{Q}^{n-1}_{\mathcal{C}}(Z) & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \to & \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_n & \longrightarrow & Q^{\mathcal{C}}_n(f) & \longrightarrow & Q^{\mathcal{C}}_n(Z) & \longrightarrow & \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_{n-1} & \to & \ldots \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \downarrow\cong & & \\
\ldots & \to & \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_n & \longrightarrow & Q^n_{\mathcal{C}}(f) & \longrightarrow & Q^n_{\mathcal{C}}(Z) & \longrightarrow & \hom_{\mathrm{ho}\,\mathcal{C}}(Z, X^\vee)_{n-1} & \to & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & 0 & \longrightarrow & \widehat{Q}^n_{\mathcal{C}}(f) & \longrightarrow & \widehat{Q}^n_{\mathcal{C}}(Z) & \longrightarrow & 0 & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

whose rows and columns are exact.

Proof. This lemma is the natural generalization of the sum formulas of lemma 3.1.4 above, and the proof is also completely analogous.

One verifies that the cone of the natural chain homomorphism of chain complexes of modules over $\mathfrak{K}\mathrm{C}_2$

$$\mathrm{cone}(\hom_{\mathcal{C}}(f, f^\vee)) \to \hom_{\mathcal{C}}(Z, Z^\vee)$$

is naturally isomorphic to

$$\hom_{\mathcal{C}}(Z, X^\vee) \oplus \hom_{\mathcal{C}}(X, Z^\vee)$$

where the C_2-action swaps the two direct summands via the duality. The claims then follow exactly as in lemma 3.1.4. □

This lemma measures the failure of the quadratic and symmetric Q-groups of being cohomological functors. On the other hand, the lemma shows that hyperquadratic Q-groups are cohomological. Its main application is the following lemma, which provides the analog of lemma 3.2.9 above in the L-theory setting.

Lemma 3.3.13. *Let $\mathcal{C}$ be a pre-triangulated dg category with duality. Given an $(n+1)$-dimensional symmetric space $(X, \varphi \in Q^{n+1}_{\mathcal{C}}(X))$, consider a cone C of $\Sigma^{-n-1}\operatorname{ev}(\varphi)\colon X \to \Sigma^{-n-1}(X^\vee)$. Then there is one and up to isometry only one n-dimensional non-degenerate symmetric space $(C, \partial\varphi \in Q^n_{\mathcal{C}}(C))$ such that the diagram*

$$\begin{array}{ccccccc}
\Sigma^{n+1}X & \xrightarrow{\operatorname{ev}\varphi} & X^\vee & \xrightarrow{f} & \Sigma(\Sigma^n C) & \xrightarrow{g} & \Sigma^{n+2}X \\
\downarrow{\scriptstyle \Sigma^{n+1}\mu_X} & & \| & & \cong \downarrow{\scriptstyle \Sigma(\operatorname{ev}\partial\varphi)} & & \downarrow{\scriptstyle \Sigma^{n+2}\mu_X} \\
\Sigma^{n+1}(X^{\vee\vee}) & \xrightarrow[\Sigma^{n+1}\operatorname{ev}\varphi^\vee]{} & X^\vee & \xrightarrow[(-1)^n g^\vee]{} & \Sigma(C^\vee) & \xrightarrow[f^\vee]{} & \Sigma^{n+2}(X^{\vee\vee})
\end{array}$$

commutes. Ditto for the quadratic case.

Definition 3.3.14. The n-dimensional non-degenerate symmetric space $(C, \partial\phi \in Q^n_{\mathcal{C}}(C))$ of lemma 3.3.13 above is called the *boundary* of the $(n+1)$-dimensional symmetric space $(X, \phi \in Q^{n+1}_{\mathcal{C}}(X))$, and one usually writes ∂X for C. (Notice that ∂X depends on the symmetric form on X, even though it is hidden in the notation.) Ditto for the quadratic case.

Proof. We prove the lemma in the symmetric case, the quadratic case being completely analogous.

Let $(X, \varphi \in Q^{n+1}_{\mathcal{C}}(X))$ be an $(n+1)$-dimensional symmetric space, and let C be a cone of $\operatorname{ev}\varphi\colon \Sigma^{n+1}X \to X^\vee$. Let $f\colon \operatorname{cone}(f) \to \Sigma X$ be the natural homomorphism The lemma 3.3.13 □

The following corollary is the analog of corollary 3.2.11 in the L-theory setting.

Corollary 3.3.15. *Let $\mathcal{C}$ be a pre-triangulated category with duality, and consider a pair of non-degenerate n-dimensional symmetric spaces (X_1, φ_1) and (X_2, φ_2). Then they are bordant if and only if there exists an $(n+1)$-dimensional symmetric space (Y, ψ) whose boundary $(\partial Y, \partial\psi)$ is isometric to the orthogonal sum $(X_1, \varphi_1) \perp (X_2, -\varphi_2)$.*

Proof. Follows at once from lemma 3.3.13 above. □

As in the exact and triangulated case, one sees immediately that the property of being metabolic (i.e., null-bordant) is invariant under isometries, and that the orthogonal sum of two metabolic spaces is again metabolic. Moreover, if φ is an n-dimensional symmetric form on X, then $(X, \varphi) \perp (X, -\varphi)$ is metabolic. Hence we can consider the submonoid $NL^n(\mathcal{C}) = NL^n(\mathcal{C}, {}^\vee, \mu)$ of $ML^n(\mathcal{C})$ consisting of isometry classes of n-dimensional non-degenerate metabolic spaces, and as in the exact and triangulated case the quotient monoid $ML^n(\mathcal{C})/NL^n(\mathcal{C})$ is a group. Everything applies to the quadratic case as well.

Definition 3.3.16. Let $\mathcal{C}$ be an essentially small pre-triangulated dg category with duality.

The *n-dimensional symmetric L-theory group* of $\mathcal{C}$ is defined as the quotient $L^n(\mathcal{C}) = L^n(\mathcal{C}, ^\vee, \mu) = ML^n(\mathcal{C})/NL^n(\mathcal{C})$.

The *n-th quadratic L-theory group* $L_n(\mathcal{C})$ of $\mathcal{C}$ is defined analogously using n-dimensional non-degenerate quadratic spaces.

Notice that the natural transformation $1 + \tau\colon Q_n^{\mathcal{C}} \to Q^n_{\mathcal{C}}$ defines a group homomorphism $s\colon L_n(\mathcal{C}) \to L^n(\mathcal{C})$, called *symmetrization*.

From lemma 3.3.8 above one obtains 4-periodicity for the Witt groups.

Proposition 3.3.17. *Let $(\mathcal{C}, ^\vee, \mu)$ be a pre-triangulated dg category with duality. Then for all $n \in \mathbb{Z}$ one has that $L^{n+2}(\mathcal{C}, ^\vee, \mu) = L^n(\mathcal{C}, ^\vee, -\mu)$. In particular, $L^{n+4}(\mathcal{C}, ^\vee, \mu) = L^n(\mathcal{C}, ^\vee, \mu)$. Similarly in the quadratic case.*

The following proposition shows that when both are defined, that is, for triangulated categories with duality where 2 is invertible and which arise as homotopy categories of pre-triangulated dg categories with duality, Balmer's Witt groups and our L-groups are isomorphic.

Proposition 3.3.18. *Let $\mathcal{C}$ be a pre-triangulated dg category with duality. Assume that 2 is invertible in $\mathcal{C}$. Then for all $n \in \mathbb{Z}$*

$$L_n(\mathcal{C}) \cong L^n(\mathcal{C}) \cong W^n(\operatorname{ho}\mathcal{C}) \ .$$

Proof. Since 2 is invertible in $\mathcal{C}$ the symmetrization $1 + \tau\colon Q_n^{\mathcal{C}} \to Q^n_{\mathcal{C}}$ is actually a natural isomorphism, whence the first isomorphism above. The second isomorphism follows from lemmas 3.2.9 and 3.3.13 above together with their respective corollaries. ☐

Example 3.3.19. Let R be a ring with involution, see example 2.2.6, and assume that $\frac{1}{2} \in R$. Then for the L-theory and Witt groups of R one has

$$L_n(\mathsf{ch}^{\mathsf{b}}\mathsf{fgp}/R) \cong L^n(\mathsf{ch}^{\mathsf{b}}\mathsf{fgp}/R) \cong W^n(\operatorname{ho}(\mathsf{ch}^{\mathsf{b}}\mathsf{fgp}/R)) \ .$$

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