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MIXED HODGE STRUCTURE ON THE VANISHING COHOMOLOGY

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Abstract

We construct a mixed Hodge structure on the cohomology of the Milnor fiber associated to an isolated singularity of a complex hypersurface. We determine the relations it has with monodromy, intersection form and local cohomology.

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Introduction

Let $P \in \mathbb{C}[z_0, \cdots, z_n]$ with $P(0)=0$. Assume that $0 \in \mathbb{C}^{n+1}$ is a critical point of P. Denote B the open ball in $\mathbb{C}^{n+1}$ with center 0 and radius $\varepsilon > 0$. There exists $\eta > 0$ such that $0 < |t| < \eta$ implies that $B_t = P^{-1}(t) \cap B$ is a complex manifold. In this paper we follow a suggestion of Deligne and construct a mixed Hodge structure on the cohomology of B_t (the vanishing cohomology) in the case that P has an isolated critical point at 0.

Let S be the disk with center 0 and radius η. Denote $X' = P^{-1}(S) \cap B$. Let $\rho : X \to X'$ be a resolution of P, i.e. a proper map which is an isomorphism outside $\rho^{-1}(0)$ such that $(P\rho)^{-1}(0)$ is a union of smooth divisors on X with normal crossings. Let e be the least common multiple of the multiplicities occurring in the fiber $(P\rho)^{-1}(0)$. Let $\tilde{S}$ be the disk with radius $\eta^{1/e}$ and define $\sigma : \tilde{S} \to S$ by $\sigma(t) = t^e$. Let $\tilde{X}$ be the normalization of the fiber product $X \times_S \tilde{S}$ and let $\pi : \tilde{X} \to X$ be the natural map. Let $D = (P\rho\pi)^{-1}(0)$ and denote $D_0, \cdots, D_m$ its irreducible components. The cohomology groups of the D_i

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and their multiple intersections are the constituents of the mixed Hodge structure on $H^*(B_t)$.

The spaces $\tilde{X}$ and D_i have singularities, but still their cohomology carries a pure Hodge structure because they only have quotient singularities.

In chapter 1 we develop a De Rham cohomology for projective varieties with quotient singularities. In chapter 2 we use this to get hold of the limit Hodge structure of a one parameter family of projective varieties over the punctured disk, whose monodromy is not unipotent. We use the resulting construction to put a mixed Hodge structure on $H^*(B_t)$ in chapter 3, after we have computed the mixed Hodge structure for a projective variety with only one singular point. Chapter 4 contains the study of the relations with problems concerning finiteness of monodromy, intersection form and local cohomology. Finally in chapter 5 we list some open problems.

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1. Projective *V*-manifolds

(1.1) Definition. A V-manifold of dimension n is a complex analytic space which admits an open covering $\{U_i\}$ such that each U_i is analytically isomorphic to Z_i/G_i where $Z_i \subset \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $GL(n, \mathbb{C})$.

V-manifolds have been classified locally by D. Prill [15].

To state the result we need the following.

(1.2) Definition. A finite subgroup G of $GL(n,\mathbb{C})$ is called *small* if no element of G has 1 as an eigenvalue of multiplicity precisely $n-1$. In other words: G contains no rotations around hyperplanes other than the identity.

For every finite subgroup G of $GL(n,\mathbb{C})$ denote G_{big} the normal subgroup of G which is generated by all rotations around hyperplanes. Then the G_{big}-invariant polynomials form a polynomial algebra and hence $\mathbb{C}^n/G_{\text{big}}$ is isomorphic to $\mathbb{C}^n$.

The group G/G_{big} maps isomorphically to a small subgroup of $GL(n,\mathbb{C})$, once a basis of invariant polynomials has been chosen. Hence local classification of V-manifolds reduces to the classification of actions of small subgroups of $GL(n,\mathbb{C})$.

(1.3) Theorem. *Let G_1 and G_2 be small subgroups of $GL(n,\mathbb{C})$. Then $\mathbb{C}^n/G_1 \cong \mathbb{C}^n/G_2$ if and only if G_1 and G_2 are conjugate subgroups.* Cf. [15].

We are interested in the Hodge theory of projective V-manifolds. The following proposition shows that we can expect an analogous situation as in the smooth projective case:

(1.4) Proposition. *Every V-manifold is a rational homology manifold.*

Proof. In view of the local classification we need only compute the local cohomology groups of $0 \in \mathbb{C}^n/G$ where $G \subset GL(n,\mathbb{C})$ is a small subgroup. Choose a G-invariant metric on $\mathbb{C}^n$. Let D be the ball with radius 1 and let S be its boundary. Then $H^k_{\{0\}}(\mathbb{C}^n/G, \mathbb{Q}) \cong H^k(\mathbb{C}^n/G, \mathbb{C}^n/G\text{-}\{0\}, \mathbb{Q}) \cong H^k(D/G, D/G\text{-}\{0\}, \mathbb{Q}) \cong \tilde{H}^{k-1}(D/G\text{-}\{0\}, \mathbb{Q}) \cong \tilde{H}^{k-1}(D\text{-}\{0\}, \mathbb{Q})^G \cong \tilde{H}^{k-1}(S, \mathbb{Q})^G = 0$ for $k \neq 2n$ and $=\mathbb{Q}$ for $k = 2n$. This follows from the fact that S is an oriented S^{2n-1} and that all elements of G preserve its orientation.

(1.5) Corollary. *If X is a complete algebraic V-manifold, then the canonical Hodge structure on $H^k(X)$ is purely of weight k for all $k \geqq 0$. If $p: \tilde{X} \to X$ is a resolution of singularities for X, then the map $p^*: H^k(X) \to H^k(\tilde{X})$ is injective for all $k \geqq 0$.* Cf. [5], Th. (8.2.4).

(1.6) In the following sections we will construct a complex $\tilde{\Omega}^\cdot_X$ on the projective V-manifold X with the following properties:

(i) $\tilde{\Omega}^p_X$ is a coherent analytic sheaf on X for every integer p; $\tilde{\Omega}^p_X \neq 0$ if and only if $0 \leqq p \leqq \dim X$; the maps $d: \tilde{\Omega}^p_X \to \tilde{\Omega}^{p+1}_X$ are $\mathbb{C}$-linear;

(ii) $\tilde{\Omega}^\cdot_X$ is a resolution of the constant sheaf $\mathbb{C}$ on X;

(iii) the spectral sequence of hypercohomology

$$E_1^{pq} = H^q(X, \tilde{\Omega}^p_X) \Rightarrow \boldsymbol{H}^{p+q}(X, \tilde{\Omega}^\cdot_X) = H^{p+q}(X, \mathbb{C})$$

degenerates at $E_1 (E_1 = E_\infty)$ and the induced filtration on $H^{p+q}(X,\mathbb{C})$ coincides with the canonical Hodge filtration.

(1.7) Definition. Let X be a V-manifold and denote Σ its singular locus. Denote $j: X - \Sigma \to X$ the inclusion map.

Then we define $\tilde{\Omega}^\cdot_X = j_* \tilde{\Omega}^\cdot_{X-\Sigma}$.

(1.8) Lemma. *Let D be an open ball with center 0 in $\mathbb{C}^n$. Let G be a small subgroup of $GL(n,\mathbb{C})$ which leaves D invariant and let $U = D/G$. Denote $\rho: D \to U$ the quotient map. Then $\Gamma(U, \tilde{\Omega}^p_U) = \Gamma(D, \Omega^p_D)^G$ for all $p \geqq 0$, i.e. $\tilde{\Omega}^\cdot_U \cong (\rho_* \Omega^\cdot_D)^G$.*

Proof. Denote $\Sigma = \operatorname{Sing}(U)$ and $N = \rho^{-1}(\Sigma)$. Then $\Gamma(U, \tilde{\Omega}^p_U) = \Gamma(U - \Sigma, \Omega^p_{U-\Sigma}) = \Gamma(D - N, \Omega^p_D)^G$ because $\rho: D - N \to U - \Sigma$ is smooth (use that G is small). Moreover $\Gamma(D - N, \Omega^p_D) = \Gamma(D, \Omega^p_D)$ because D is smooth, Ω^p_D is locally free on D and N has codimension at least two in D.

(1.9) Corollary. *$\tilde{\Omega}^\cdot_X$ is a resolution of the constant sheaf $\mathbb{C}_X$ for every V-manifold X.*

PROOF. Take an open subset $U = D/G$ as in (1.8). The sequence

$$0 \to \mathbb{C} \to \Gamma(D, \mathcal{O}_D) \xrightarrow{d} \Gamma(D, \Omega_D^1) \to \cdots$$

is a G-equivariant exact sequence of $\mathbb{C}$-vector spaces, hence the sequence of G-invariants is also exact.

(1.10) COROLLARY. *$\tilde{\Omega}_X^p$ is coherent for all $p \geqq 0$.*

PROOF. The local quotient maps $\rho: D \to U$ are proper.

(1.11) LEMMA. *Let $\pi: \tilde{X} \to X$ be a resolution of singularities for the V-manifold X. Then $\tilde{\Omega}_X^{\cdot} \cong \pi_* \Omega_{\tilde{X}}^{\cdot}$.*

PROOF. Denote $\Sigma = \mathrm{Sing}(X)$, $D = \pi^{-1}(\Sigma)$. Then D is a divisor with normal crossings on $\tilde{X}$. Denote $j: X - \Sigma \to X$ and $i: \tilde{X} - D \to \tilde{X}$ the inclusion maps. Then the map $\Omega_{\tilde{X}}^{\cdot} \to i_* \Omega_{\tilde{X}-D}^{\cdot}$ induces an injective map $\pi_* \Omega_{\tilde{X}}^{\cdot} \to \pi_* i_* \Omega_{\tilde{X}-D}^{\cdot} = j_* \pi_* \Omega_{\tilde{X}-D}^{\cdot} = \tilde{\Omega}_X^{\cdot}$. We have to show that this map is surjective. Let p be a non-negative integer. Assume $X = Z/G$ with Z an open ball in $\mathbb{C}^n$ and $G \subset GL(n, \mathbb{C})$ a small subgroup. The quotient sheaf $\tilde{\Omega}_X^p / \pi_* \Omega_{\tilde{X}}^p$ has support on Σ, so for every holomorphic function f on X which vanishes on Σ there exists an integer k such that $f^k \tilde{\Omega}_X^p \subset \pi_* \Omega_{\tilde{X}}^p$. Let ω be a holomorphic p-form on $\tilde{X} - D$ and let x be a smooth point of D. We show that ω can be extended to a holomorphic p-form on a neighborhood of x in $\tilde{X}$. First observe that ω is meromorphic along D: if f is a holomorphic function on X such that $f\tilde{\Omega}_X^p \subset \pi_* \Omega_{\tilde{X}}^p$ then $\pi^* f \cdot \omega$ extends to a holomorphic p-form on $\tilde{X}$.

Let $z_1, \cdots, z_n$ be holomorphic coordinates on a neighborhood W of x in $\tilde{X}$, centered at x, such that D is given locally by the equation $z_1 = 0$. Write

$$\omega = \sum_{1 \leqq i_1 < \cdots < i_p \leqq n} a_{i_1 \cdots i_p} \, dz_{i_1} \wedge \cdots \wedge dz_{i_p}$$

with $a_{i_1 \cdots i_p}$ meromorphic along D. We show that $a_{i_1 \cdots i_p}$ is in fact holomorphic on W.

We may assume that x is not contained in the support of the divisor of zeroes of $a_{i_1 \cdots i_p}$, because this divisor intersects D in a set of codimension at least 2 in $\tilde{X}$ and it is sufficient to extend ω to the complement of a set of codimension two in $\tilde{X}$. So we may even suppose that $a_{i_1 \cdots i_p}$ does not vanish in any point of W. Then one may write $a_{i_1 \cdots i_p} = z_1^m \cdot b_{i_1 \cdots i_p}$ for some integer m, depending on $i_1, \cdots, i_p$ and some holomorphic function $b_{i_1 \cdots i_p}$ which is invertible on W.

If $i_1 = 1$, consider the $2p$-chain Γ on W given by the equations $z_j = 0 (j \neq i_1, \cdots, i_p)$. Let $C = \rho^{-1}\pi(\Gamma)$ and let η be the holomorphic G-invariant p-form on Z corresponding to ω. Then if $g = |G|$:

$$\int |z_1|^{2m} |b_{i_1 \cdots i_p}|^2 \, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}$$
$$= \int_\Gamma \omega \wedge \bar{\omega} = \frac{1}{g} \int_C \eta \wedge \bar{\eta} < \infty$$

hence $m \geqq 0$, so $a_{i_1 \cdots i_p}$ is holomorphic on W. If $i_1 \neq 1$, consider for all $t \in \mathbb{C}$ the $2p$-chain $\Gamma(t)$ on W given by the equations $z_1 = t$, $z_j = 0$ $(j \neq 1, i_1, \cdots, i_p)$. Let $C(t) = \rho^{-1}\pi(\Gamma(t))$ and for $t \neq 0$ define

$$\alpha(t) = \int_{\Gamma(t)} \omega \wedge \bar{\omega} = \frac{1}{g} \int_{C(t)} \eta \wedge \bar{\eta}.$$

Then $\lim_{t \to 0} \alpha(t) = 1/g \int_{C(0)} \eta \wedge \bar{\eta}$ exists. On the other hand $\alpha(t) \sim c|t|^{2m}$ for some $c > 0$. This implies that $m \geqq 0$.

(1.12) THEOREM. *Let X be a projective V-manifold. Then the spectral sequence of hypercohomology*

$$E_1^{pq} = H^q(X, \tilde{\Omega}_X^p) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1 and the induced filtration on $H^{p+q}(X, \mathbb{C})$ coincides with the Hodge filtration.

PROOF. It follows from [18], proposition IV.12 or from [12], remark 2.3 that every V-manifold is Cohen-Macaulay. One concludes from [9], section 3.2 that $\tilde{\Omega}_X^n$ $(n = \dim(X))$ is the canonical dualizing sheaf on X. The cup product $\tilde{\Omega}_X^p \otimes \tilde{\Omega}_X^{n-p} \to \tilde{\Omega}_X^n$ defines an injective sheaf homomorphism $\tau: \tilde{\Omega}_X^p \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\tilde{\Omega}_X^{n-p}, \tilde{\Omega}_X^n)$ for every $p \geqq 0$, given by $(\tau(\omega_1))(\omega_2) = \omega_1 \wedge \omega_2$. We first show that τ is surjective for all p. Let $x \in X$ and let $U = Z/G$ be a neighborhood of x in X as in the proof of lemma (1.11). Denote $A = \mathcal{O}_{X,x}$, $B = \mathcal{O}_{Z,0}$, $N = \Omega_{Z,0}^{n-p}$, $M = \Omega_{Z,0}^n$. We have to show that $\mathrm{Hom}_B(N, M)^G = \mathrm{Hom}_A(N^G, M^G)$.

One has $\mathrm{Hom}_A(N^G, M^G) = \mathrm{Hom}_A(N^G, M)^G = \mathrm{Hom}_B(BN^G, M)^G$. The natural map $BN^G \to N$ induces a map $\mathrm{Hom}_B(N, M) \to \mathrm{Hom}_B(BN^G, M)$. It is sufficient to show that this map is an isomorphism. Let $R = N/BN^G$. Because $\rho: Z \to U$ is étale outside $S = \rho^{-1}(\Sigma)$ where $\Sigma = \mathrm{Sing}(X)$, R has support contained in S. This implies that $\dim(R) \leqq n - 2$. Because $M = A$ is Cohen-Macaulay, depth $(M_p) \geqq 2$ for every prime ideal $p \in \mathrm{Spec}(A)$ with $I(S) \subset p$. Hence $\mathrm{Ext}_A^i(R, M) = 0$ for $i < 2$ (cf. [10], Proposition 3.7). Therefore $\mathrm{Hom}_B(N, M)$ and $\mathrm{Hom}_B(BN^G, M)$ are isomorphic.

By Grothendieck duality the pairing

$$H^q(X, \tilde{\Omega}_X^p) \otimes H^{n-q}(X, \tilde{\Omega}_X^{n-p}) \to H^n(X, \tilde{\Omega}_X^n) = \mathbb{C}$$

is non-singular for every $p, q \geqq 0$.

Let $\pi:\tilde{X}\to X$ be a resolution of singularities for X. Then there is a morphism of spectral sequences

$$\begin{array}{ccc} E_1^{pq}=H^q(X,\tilde{\Omega}_X^p) & \Rightarrow & H^{p+q}(X,\mathbb{C}) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ E_1^{pq}=H^q(\tilde{X},\Omega_{\tilde{X}}^p) & \Rightarrow & H^{p+q}(\tilde{X},\mathbb{C}) \end{array}$$

If $\omega\in H^q(X,\tilde{\Omega}_X^p)$, there exists $\eta\in H^{n-q}(X,\tilde{\Omega}_X^{n-p})$ with $\omega\wedge\eta\neq 0$. Then $\pi^*(\omega)\wedge\pi^*(\eta)=\pi^*(\omega\wedge\eta)\neq 0$ so $\pi^*(\omega)\neq 0$. Hence π^* is injective on E_1. Because the spectral sequence for $\tilde{X}$ degenerates at E_1, so does the spectral sequence for X. Finally the induced filtration on $H^{p+q}(X,\mathbb{C})$ has to be the same as the Hodge filtration, because we get it by intersection of the Hodge filtration on $H^{p+q}(\tilde{X},\mathbb{C})$ with $H^{p+q}(X,\mathbb{C})$ and the Hodge filtration is functorial.

We conclude this chapter with a generalization of the structure theorems for the cohomology of smooth projective varieties to the case of projective V-manifolds.

(1.13) THEOREM. *Let X be a projective V-manifold of dimension n. Let $L\in H^2(X,\mathbb{Z})$ be the cohomology class of an ample divisor on X. Then for all $q\in\mathbb{N}$ the map $\omega\mapsto L^q\wedge\omega$ induces an isomorphism between $H^{n-q}(X,\mathbb{C})$ and $H^{n+q}(X,\mathbb{C})$.*

PROOF. Let $\pi:\tilde{X}\to X$ be a resolution of singularities for X. Then π is a projective map and there exists a divisor D on $\tilde{X}$ which is very ample for π and such that, if $[D]$ is the cohomology class of D, then $[D]\wedge\pi^*(\eta)=0$ for all $\eta\in H^i(X,\mathbb{C})$ and all i, i.e. $\pi_*[D]=0$. It follows from [11], prop. (4.4.10)(ii) that for k sufficiently large $C=[D]+k\pi^*L$ is the cohomology class of a very ample divisor on $\tilde{X}$. Hence for all $q\in\mathbb{N}$ the map $\omega\mapsto C^q\wedge\omega$ induces an isomorphism between $H^{n-q}(\tilde{X},\mathbb{C})$ and $H^{n+q}(\tilde{X},\mathbb{C})$. For $\eta\in H^{n-q}(X,\mathbb{C})$ one has $C\wedge\pi^*(\eta)=k\pi^*L\wedge\pi^*(\eta)=k\pi^*(L\wedge\eta)$ so $C^q\wedge\pi^*\eta=k^q\pi^*(L^q\wedge\eta)$. Because π^* is injective, the map $\omega\mapsto L^q\wedge\omega$ is injective on $H^{n-q}(X,\mathbb{C})$. By Poincaré duality $H^{n-q}(X,\mathbb{C})$ and $H^{n+q}(X,\mathbb{C})$ have equal dimensions, hence L^q is an isomorphism.

(1.14) COROLLARY. *Define the primitive cohomology groups $P^k(X,\mathbb{C})(k\in\mathbb{Z})$ by $P^k(X,\mathbb{C})=\operatorname{Ker}(L^{n-k+1}:H^k(X,\mathbb{C})\to H^{2n-k+2}(X,\mathbb{C}))$. Then for all $q\geqq 0$ one has the primitive decomposition*

$$H^q(X,\mathbb{C})=\bigoplus_{r\geqq 0}L^rP^{q-2r}(X,\mathbb{C}).$$

(1.15) REMARK. Obviously the primitive decomposition depends on the choice of the ample divisor L on X. However if one chooses ample divisors L and C on X resp. $\tilde{X}$ as in the proof of (1.13), the map π^* will preserve the primitive decompositions. In particular one concludes that the hermitian form Q on $H^{n-q}(X,\mathbb{C})$ defined by

$$Q(x,y)=(-1)^{(n-q)(n-q-1)/2}Cx\wedge L^q\bar{y}[X]$$

where C is Weil's operator and the bar denotes complex conjugation with respect to $H^{n-q}(X,\mathbb{R})$, induces a positive definite hermitian form on $P^{n-q}(X,\mathbb{C})$.

The following is a generalization of [4] to the case of algebraic V-manifolds.

(1.16) DEFINITION. Let X be a V-manifold. A divisor Y on X is called a divisor with V-normal crossings if locally on X one has $(X,Y)=(Z,D)/G$ with $Z\subset\mathbb{C}^n$ an open domain, $G\subset GL(n,\mathbb{C})$ a small subgroup acting on Z and $D\subset Z$ a G-invariant divisor with normal crossings.

(1.17) DEFINITION. Let X be a V-manifold and Y a divisor with V-normal crossings on X. Define the complex $\tilde{\Omega}_X^{\cdot}(\log Y)$ on X by

$$\tilde{\Omega}_X^{\cdot}(\log Y)=j_*\tilde{\Omega}_{X-\Sigma}^{\cdot}(\log(Y-\Sigma))$$

where $\Sigma=\operatorname{Sing}(X)$ and $j:X-\Sigma\to X$ is the inclusion map. One checks like before that if $(X,Y)=(Z,D)/G$ as in (1.16) and if $\rho:Z\to X$ is the quotient map, then

$$\tilde{\Omega}_X^{\cdot}(\log Y)=(\rho_*\Omega_Z^{\cdot}(\log D))^G.$$

If $\pi:\tilde{X}\to X$ is a resolution of singularities for X such that the total transform $\tilde{Y}$ of Y is a divisor with normal crossings on $\tilde{X}$, then

$$\tilde{\Omega}_X^{\cdot}(\log Y)=\pi_*\Omega_{\tilde{X}}^{\cdot}(\log\tilde{Y}).$$

(1.18) DEFINITION. The weight filtration W on $\tilde{\Omega}_X^{\cdot}$ $(\log Y)$ is defined by

$$W_k\tilde{\Omega}_X^p(\log Y)=\tilde{\Omega}_X^k(\log Y)\wedge\tilde{\Omega}_X^{p-k}\qquad(k\in\mathbb{Z})$$

and the Hodge filtration F is given by

$$\begin{aligned} F^k\tilde{\Omega}_X^p(\log Y)&=\tilde{\Omega}_X^p(\log Y) && \text{if } p\geqq k;\\ F^k\tilde{\Omega}_X^p(\log Y)&=0 && \text{if } p<k. \end{aligned}$$

Thus W is increasing and F is decreasing.

Assume that Y is a union of irreducible components $Y_1,\cdots,Y_m$ without self-intersection. Denote $\tilde{Y}^{(p)}$ the disjoint union of all p-fold intersections $Y_{i_1}\cap\cdots\cap Y_{i_p}$ for $1\leqq i_1<\cdots<i_p\leqq m$.

Denote $a_p : \tilde{Y}^{(p)} \to X$ the natural map. Analogous to the smooth case one has a residue map

$$R : W_k \tilde{\Omega}^p_X(\log Y) \to (a_k)_* \tilde{\Omega}^{p-k}_{\tilde{Y}^{(k)}} \qquad (p, k \geqq 0).$$

(Remark that $\tilde{Y}^{(k)}$ is a V-manifold for every $k \geqq 0$).

(1.19) LEMMA. *R induces for every $k \geqq 0$ an isomorphism of complexes*

$$Gr^W_k \tilde{\Omega}^{\cdot}_X(\log Y) \to (a_{k*}) \tilde{\Omega}^{\cdot}_{\tilde{Y}^{(k)}}[-k].$$

PROOF. Analogous to corollary (1.9).

This lemma together with theorem (1.12) show that one may use the bifiltered complex $(\tilde{\Omega}^{\cdot}_X(\log Y), F, W)$ to compute the canonical mixed Hodge structure on $H^k(X-Y)(k \geqq 0)$. In particular one has the spectral sequence

$$E_1^{-n,k+n} = H^{k-n}(\tilde{Y}^{(n)}, \mathbb{Q})(-n) \Rightarrow H^k(X-Y, \mathbb{Q}).$$

With $E_2^{-n,k+n} = E_\infty^{-n,k+n} = Gr^W_{n+k} H^k(X-Y, \mathbb{Q})$. Cf. [5], where the notion of cohomological mixed Hodge complex is used.

(1.20) EXAMPLE. Let $f(z_0, \cdots, z_n)$ be a quasi-homogeneous polynomial. This means that there exist positive rational numbers $w_0, \cdots, w_n$ such that for all $t \in \mathbb{C}$ one has

$$f(t^{w_0} z_0, \cdots, t^{w_n} z_n) = t f(z_0, \cdots, z_n).$$

One may compute the mixed Hodge structure on the affine variety $X \subset \mathbb{C}^{n+1}$ with equation $f(z) = 1$ in the following way, provided f has an isolated critical point at 0.

Write $w_i = u_i/v_i$ with $(u_i, v_i) = 1$. Let $d = lcm(v_0, \cdots, v_n)$ and define $b_i = dw_i$, $i = 0, \cdots, n$.

Define $(n+1) \times (n+1)$-matrices $g^{(0)}, \cdots, g^{(n)}$ by $g^{(i)}_{jk} = 0$ if $j \neq k$, $g^{(i)}_{kk} = 1$ if $i \neq k$ and $g^{(k)}_{kk} = \exp(2\pi i/b_k)$.

Let G be the subgroup of $PGL(n+2, \mathbb{C})$ generated by the elements

$$\begin{bmatrix} g^{(k)} & 0 \\ 0 & 1 \end{bmatrix} \qquad (k = 0, \cdots, n).$$

Then G acts on $\mathbb{P}^{n+1}(\mathbb{C})$. Let M be the quotient $\mathbb{P}^{n+1}/G$.

Denote $h(y_0, \cdots, y_n) = f(y_0^{b_0}, \cdots, y_n^{b_n})$. Let $Z \subset \mathbb{C}^{n+1}$ be given by the equation $h(y) = 1$. Denote $\bar{Z} \subset \mathbb{P}^{n+1}$ its projective closure. Then G leaves the pair $(Z, \bar{Z})$ invariant. Moreover $X = Z/G$. If f has an isolated singularity at 0 then $\bar{X} = \bar{Z}/G$ and $\bar{X} - X$ are V-manifolds (though $\bar{Z}$ need not be smooth).

The exact sequence

$$0 \to P^n(\bar{X}) \to H^n(X) \to P^{n-1}(\bar{X} - X) \to 0$$

shows that $Gr^W_k H^n(X) = 0$ for $k \neq n$, $n+1$.

See [20] for a more detailed description.

2. Limits of Hodge structures

The purpose of this chapter is: to give an explicit description for the limit Hodge structure associated to a projective one parameter family over the punctured disk. This has been done in [19] for families with unipotent monodromy.

(2.1) NOTATIONS. S is the unit disk in the complex plane, $S^* = S - \{0\}$ and $H = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ is the universal covering of S^* by the map $z \to \exp(2\pi i z)$.

X is a smooth, closed, connected subvariety of dimension $n+1$ of $\mathbb{P}^r \times S$ for some $r > 0$. We denote $f : X \to S$ the projection on S. We assume that f is surjective, smooth in every point $x \in X$ with $f(x) \neq 0$ and that $f^{-1}(0)$ is a union $E_0 \cup \cdots \cup E_m$ of smooth divisors on X which cross normally. Denote e_i the multiplicity of E_i and let $e = lcm(e_0, \cdots, e_m)$.

Let $\tilde{S}$ be another copy of the unit disk and define $\sigma : \tilde{S} \to S$ by $\sigma(t) = t^e$. Denote $\tilde{X}$ the normalization of $X \times_S \tilde{S}$ and let $\pi : \tilde{X} \to X$ and $\tilde{f} : \tilde{X} \to \tilde{S}$ be the natural maps.

Denote $D_i = \pi^{-1}(E_i)$, $i = 0, \cdots, m$. Let $D = \bigcup_{i=0}^m D_i$. Let $X_\infty = X \times_S H$

$$\begin{array}{ccccc} D_i & \longrightarrow & \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{S} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \sigma} \\ E_i & \longrightarrow & X & \xrightarrow{f} & S \end{array}$$

(2.2) LEMMA. *$\tilde{X}$ is a V-manifold and $\tilde{f}^{-1}(0) = D$ is a reduced divisor with V-normal crossings on $\tilde{X}$.*

PROOF. Cover X with coordinate neighborhoods A with coordinates $z_0, \cdots, z_n$ such that there exist integers $\nu \geqq 0$, $d_0, \cdots, d_\nu \geqq 1$ with $f(z_0, \cdots, z_n) = z_0 \cdots z_\nu$. Fix one such A.

Let $\tilde{A}$ be the normalization of $A \times_S \tilde{S}$. Then $\tilde{A}$ is isomorphic to an open analytic subset of the normalization of

$$\{(w, z_0, \cdots, z_n) \in \mathbb{C}^{n+2} \mid z_0^{d_0} \cdots z_\nu^{d_\nu} = w^e\}.$$

Let $d = gcd(d_0, \cdots, d_\nu)$, $e' = e/d$, $d'_i = d_i/d$ $(i = 0, \cdots, \nu)$ and $\zeta = \exp(2\pi i/d)$.

Then the normalization $\tilde{R}$ of the ring

$$R=\mathbb{C}[z_0,\cdots,z_n,w]/z_0^{d_0}\cdots z_\nu^{d_\nu}-w^e)$$

is isomorphic to the direct product of d isomorphic copies of the normalization of the ring

$$\mathbb{C}[z_0,\cdots,z_n,w]/(z_0^{d_0'}\cdots z_\nu^{d_\nu'}-w^{e'})$$

so to compute it we only need to consider the case $d=1$.

Denote $c_i=e/d_i$ for $i=0,\cdots,\nu$. Denote $R_1=\mathbb{C}[y_0,\cdots,y_n]$; consider R as a subring of R_1 by putting $z_i=y_i^{c_i}$ if $0\leqq i<\nu$, $z_i=y_i$ for $i>\nu$ and $w=y_0\cdots y_\nu$.

The group $G=\mathbb{Z}/(c_0)\times\cdots\times\mathbb{Z}/(c_\nu)$ acts on R_1 by

$$(a_0,\cdots,a_\nu)\cdot y_j=\exp(2\pi i a_j/c_j)\cdot y_j \quad \text{if} \quad j\leqq\nu;$$
$$(a_0,\cdots,a_\nu)\cdot y_j=y_j \quad \text{if} \quad j>\nu.$$

The ring of G-invariants R_1^G is just $\mathbb{C}[z_0,\cdots,z_n]$. Let $G'=\{g\in G \mid gw=w\}$. Then R is isomorphic to $R_1^{G'}$, because R_1 integrally closed and $G'=\mathrm{Gal}(K_1/K)$ where K_1 and K are the fields of fractions of R_1 and R respectively. This shows that $\mathrm{Spec}(\tilde{R})$ and hence $\tilde{A}$ and $\tilde{X}$ are V-manifolds. The action of G' on R_1 identifies G' with a small subgroup of $GL(n+1,\mathbb{C})$, because no element of G of the form $(0,\cdots,0,a_i,0,\cdots,0)$ with $a_i\neq 0$ leaves w invariant.

Locally D is the quotient under G' of the reduced divisor $y_0\cdots y_\nu=0$ and every generic point of this divisor has a trivial isotropy group. Hence D is reduced.

(2.3) Remark. With the same notations a basis for $\tilde{R}$ as a free module over $\mathbb{C}[z_0,\cdots,z_n]$ is obtained as follows. Write $k=q_{ik}c_i+r_{ik}$ $(k=0,\cdots,e-1;\ i=0,\cdots,\nu)$ with $q_{ik}\in\mathbb{Z}$ and $0\leqq r_{ik}<c_i$. Define $x_k=\prod_{i=0}^{\nu} y_i^{r_{ik}}=w^k\prod_{i=0}^{\nu} z_i^{-q_{ik}}$. Then $x_0,\cdots,x_{e-1}$ form a basis for $\tilde{R}$ as a $\mathbb{C}[z_0,\cdots,z_n]$-module. The group $G/G'=\mathbb{Z}(e)$ acts on $\tilde{X}$. This action coincides with the action of $\mathbb{Z}/(e)$ on $\tilde{X}$ which is induced by multiplication with eth roots of unity on $\tilde{S}$. One gets X back as the quotient of $\tilde{X}$ under this action.

(2.4) Example. Define $Y\subset P^2(\mathbb{C})\times S$ by the equation $x^2z-y^3=tz^3$ where (x,y,z) are homogeneous coordinates on $\mathbb{P}^2(\mathbb{C})$. If one blows up a point three times one obtains a manifold X with a projection $f:X\to S$ such that $f^{-1}(0)=E_0+2E_1+3E_2+6E_3$, the E_i intersecting like in fig. 1 and all E_i non-singular rational curves.

In this case X is smooth and D_0, D_1, D_2 are the disjoint union of 1,2 resp. 3 non-singular rational curves, while D_3 is an elliptic curve. See fig. 2.

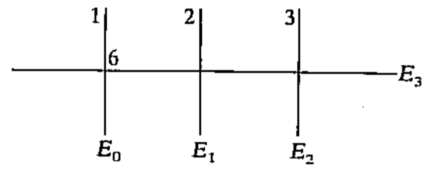

Figure 1.

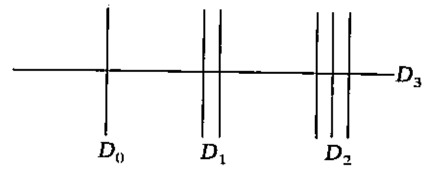

Figure 2.

(2.5) Remark. From (2.2) one can conclude that for every $p\geqq 0$ the sheaf $\tilde{\Omega}_{\tilde{X}}^p(\log D)$ is locally free of rank $\binom{n+1}{p}$ on $\tilde{X}$. In fact, on an open subset $\tilde{A}$ as above, $\tilde{\Omega}_{\tilde{X}}^1(\log D)$ is the free $\mathcal{O}_{\tilde{X}}$-module on the generators $dz_i/z_i(i=0,\cdots,\nu)$ and $dz_j(j>\nu)$ and $\tilde{\Omega}_{\tilde{X}}^p(\log D)=\Lambda_{\mathcal{O}_{\tilde{X}}}^p\tilde{\Omega}_{\tilde{X}}^1(\log D)$.

If w is a coordinate on $\tilde{S}$ with $w^e=z_0^{d_0}\cdots z_\nu^{d_\nu}$ on $\tilde{A}$, then $dw/w=\sum_{i=0}^{\nu}(d_i/e)\,dz_i/z_i=\sum_{i=0}^{\nu} dy_i/y_i$. This shows that $\tilde{f}^*\Omega_{\tilde{S}}^1(\log 0)$ is locally a direct factor of $\tilde{\Omega}_{\tilde{X}}^1(\log D)$.

(2.6) Definition.

$$\tilde{\Omega}_{\tilde{X}/\tilde{S}}^p(\log D)=\tilde{\Omega}_{\tilde{X}}^p(\log D)/\tilde{f}^*\Omega_{\tilde{S}}^1(\log 0)\wedge\tilde{\Omega}_{\tilde{X}}^{p-1}(\log D).$$

Then one has a complex $\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)$ of locally free sheaves of finite rank on $\tilde{X}$ and the differentials in this complex are $f^{\cdot}\mathcal{O}_{\tilde{S}}$-linear. Moreover

$$0\longrightarrow\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)[-1]\xrightarrow{\wedge dw/w}\tilde{\Omega}_{\tilde{X}}^{\cdot}(\log D)\longrightarrow\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)\longrightarrow 0$$

is an exact sequence of complexes on $\tilde{X}$.

Now we have all ingredients to describe the limit Hodge structure of the family $\tilde{f}:\tilde{X}\to\tilde{S}$ (or equivalently: of the family $f:X\to S$). We state the results without proofs, because these are quite the same as for the unipotent case. We refer to [5], for definitions of concepts in Hodge theory.

(2.7) Theorem. *For every $p\geqq 0$ the sheaf $\boldsymbol{R}^p\tilde{f}_*\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)$ is locally free of finite rank on $\tilde{S}$.*

The choice of a parameter w on $\tilde{S}$ determines an isomorphism

$$\psi_w:\boldsymbol{H}^p(D,\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)\otimes_{\mathcal{O}_X}\mathcal{O}_D)\xrightarrow{\sim}H^p(X_\infty,\mathbb{C}).$$

If w' is another parameter on S with $a=(w'w^{-1})(0)$ then $\psi_{w'}^{-1}\psi_w=\exp(-2\pi i\log(a)\,\mathrm{Res}_0(\bar{\nabla}))$ where

$$\bar{\nabla}:\boldsymbol{R}^p\tilde{f}_*\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}(\log D)\to\Omega_{\tilde{S}}^1(\log 0)\otimes_{\mathcal{O}_S}\boldsymbol{R}^p\tilde{f}_*\tilde{\Omega}_{\tilde{X}/\tilde{S}}^{\cdot}\log D)$$

is the Gauss-Manin connection; its residue $\mathrm{Res}_0(\bar{\nabla})$ is nilpotent.

(2.8) One puts a mixed Hodge structure on $H^p(X_\infty, \mathbb{C})$ as follows. First observe that $H^p(X_\infty, \mathbb{C}) = \boldsymbol{H}^p(D, \tilde{\Omega}^{\cdot}_{\tilde{X}/\tilde{S}}(\log D) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_D)$. One constructs a cohomological mixed Hodge complex

$$((A^{\cdot}_{\mathbb{Q}}, W), (A^{\cdot}_{\mathbb{C}}, F, W))$$

on D as in [19], but using as building blocks for $A^{\cdot}_{\mathbb{C}}$ the sheaves

$$A^{pq} = \tilde{\Omega}^{p+q+1}_{\tilde{X}}(\log D)/W_q\tilde{\Omega}^{p+q+1}_{\tilde{X}}(\log D).$$

This becomes a cohomological mixed Hodge complex by theorem (1.12). It satisfies

$$Gr_r^W A^{\cdot}_{\mathbb{Q}} = \bigoplus_{k \geqq 0, -r} (a_{r+2k+1})_* \mathbb{Q}_{\tilde{D}^{(r+2k+1)}}(-r-k)[-r-2k];$$

$$Gr_r^W A^{\cdot}_{\mathbb{C}} = \bigoplus_{k \geqq 0, -r} (a_{r+2k+1})_* \tilde{\Omega}^{\cdot}_{\tilde{D}^{(r+2k+1)}}[-r-2k].$$

(2.9) Corollary. *The spectral sequence of hypercohomology for the filtered complex* $(A^{\cdot}_{\mathbb{Q}}, W)$*:*

$$E_1^{-r,q+r} = \bigoplus_{k \geqq 0, -r} H^{q-r-2k}(\tilde{D}^{(2k+r+1)}, \mathbb{Q})(-r-k) \Rightarrow H^q(X_\infty, \mathbb{Q})$$

degenerates at E_2*; the spectral sequence of hypercohomology for the filtered complex* $(A^{\cdot}_{\mathbb{C}}, F)$*:*

$$E_1^{pq} = H^q(D, \tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_D) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1 *i.e.* $E_1^{pq} = Gr_F^p H^{p+q}(X_\infty, \mathbb{C})$.

(2.10) Corollary. *For every* $p, q \geqq 0$ *the sheaf* $R^q\tilde{f}_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)$ *is locally free on* $\tilde{S}$.

Proof. For $w \in \tilde{S}$ with $w \neq 0$ one has $R^q\tilde{f}_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)(w) \cong H^q(\tilde{X}_w, \Omega^p_{\tilde{X}_w})$. Because f is smooth and proper over the punctured disk, the sheaf $R^q\tilde{f}_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)$ is locally free on $\tilde{S}-\{0\}$. By semi-continuity one has

$$\dim_{\mathbb{C}} H^q(D, \tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_D) \geqq \dim_{\mathbb{C}} H^q(\tilde{X}_w, \Omega^p_{\tilde{X}_w}).$$

Because of (2.7) and (2.9) one has

$$\sum_{p+q=r} \dim_{\mathbb{C}} H^q(D, \tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_D) = \sum_{p+q=r} \dim_{\mathbb{C}} H^q(\tilde{X}_w, \Omega^p_{\tilde{X}_w})$$

for all $r \geqq 0$, both sides being equal to the rank of the locally free sheaf $\boldsymbol{R}^r\tilde{f}_*\tilde{\Omega}^{\cdot}_{\tilde{X}/\tilde{S}}(\log D)$. Thus we must have continuity of the dimensions. Because $\tilde{f}$ is flat we may conclude by [14], Corollary 2 of p. 50.

(2.11) Theorem. *For all* $p, q \geqq 0$ *the sheaves* $R^qf_*\Omega^p_{X/S}(\log E)$ *are locally free of finite rank on* S.

Proof. With the local computations of (2.2) one shows that $\Omega^p_{X/S}(\log E)$ is equal to the subsheaf of $\mathbb{Z}/(e)$-invariants of $\pi_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)$ and hence is a direct factor of it. Therefore it is sufficient to show that $R^qf_*(\pi_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D))$ is locally free of finite rank on S. One has

$$R^qf_*(\pi_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)) = R^q(f\pi)_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D) = R^q(\sigma\tilde{f})_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D) = \sigma_*R^q\tilde{f}_*\tilde{\Omega}^p_{\tilde{X}/\tilde{S}}(\log D)$$

because π and σ are finite maps. The last term is locally free on S by corollary (2.10).

(2.12) The maps $\tilde{f}: \tilde{X} - D \to \tilde{S} - \{0\}$ and $f: X - E \to S - \{0\}$ are C^∞-fibrations. For $w \in \tilde{S}$, $w \neq 0$ the fundamental groups $\pi_1(\tilde{S}-\{0\}, w)$ and $\pi_1(S-\{0\}, w^e)$ act on the cohomology of the fiber $\tilde{X}_w = X_{w^e}$. The actions of the positive generators of these groups extend to automorphisms $\tilde{T}$ resp. T of the sheaves $\boldsymbol{R}^q\tilde{f}_*\tilde{\Omega}^{\cdot}_{\tilde{X}/\tilde{S}}(\log D)$ resp. $\boldsymbol{R}^qf_*\Omega^{\cdot}_{X/S}(\log E)$. Denote $\tilde{T}_0$ resp. T_0 their fibers over $0 \in \tilde{S}$ resp. $0 \in S$. Denoting $\tilde{\nabla}$ and ∇ the corresponding Gauss-Manin connections, one has:

$$T_0 = \exp(-2\pi i \operatorname{Res}_0(\nabla)); \qquad \tilde{T}_0 = \exp(-2\pi i \operatorname{Res}_0(\tilde{\nabla})).$$

We fix parameters t, τ on $S, \tilde{S}$ with $t = \tau^e$. These choices determine isomorphisms

$$\boldsymbol{H}^q(D, \tilde{\Omega}^{\cdot}_{\tilde{X}/\tilde{S}}(\log D) \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{O}_D) \to H^q(X_\infty, \mathbb{C}) \leftarrow \boldsymbol{H}^q(E, \Omega^{\cdot}_{X/S}(\log E) \otimes_{\mathcal{O}_X} \mathcal{O}_E).$$

We consider T_0, $\tilde{T}_0$, $\operatorname{Res}_0(\nabla)$ and $\operatorname{Res}_0(\tilde{\nabla})$ as endomorphisms of $H^q(X_\infty)$ by means of these isomorphisms. Then $T_0^e = \tilde{T}_0$ is unipotent. Denote $N = \log(\tilde{T}_0) = \sum_{k=0}^{n+1}(-1)^{k+1}(T_0 - I)^k/k$ and denote γ_s the semisimple part of T_0.

(2.13) Theorem. *Let* $q \geqq 0$. *Then*

(1) *N is a morphism of Hodge structures of type $(-1,-1)$ on $H^q(X_\infty)$, i.e. $N(W_kH^q(X_\infty)) \subset W_{k-2}H^q(X_\infty)$ and $N(F^pH^q(X_\infty)) \subset F^{p-1}H^q(X_\infty)$ for all p, $k \geqq 0$.*

(2) *For every $r \geqq 0$ the map*

$$N^r: Gr^W_{q+r}H^q(X_\infty) \to Gr^W_{q-r}H^q(X_\infty)(-r)$$

is an isomorphism of Hodge structures.

(3) *γ_s is an isomorphism of mixed Hodge structures.*

Proof. One proves (1) and (2) in the same way as [19], theorem (5.9). To prove (3), consider the map $\lambda: \tilde{X} \to \tilde{X}$ induced by the map $\tau \to \exp(2\pi i\tau/e)$ on $\tilde{S}$. It induces an automorphism of $D_{i_1} \cap \cdots \cap D_{i_r}$ for every r-tuple

$(i_1, \cdots, i_r)$ with $0 \leqq i_1 < \cdots < i_r \leqq m$. Therefore λ^* acts as an automorphism on each of the spectral sequences of (2.9). We will show that $\gamma_s = \lambda^*$. For this we need $\lambda^* T_0 = T_0 \lambda^*$ and $T_0(\lambda^*)^{-1}$ unipotent. Take $t_0 \in S - \{0\}$, $\tau_0 \in \tilde{S}$ with $\tau_0^e = t_0$. Let $\zeta = \exp(2\pi i/e)$. Because $X_\infty \to H$ is a C^∞ fiber bundle, there exists a continuous family of diffeomorphisms

$$h_\alpha : X_{t_0} \to X_{t_0 \exp(2\pi i \alpha)} \qquad (\alpha \in \mathbb{R})$$

with $h_0 = id$ and $h_{\alpha+\alpha'} = h_\alpha \circ h_{\alpha'}$ for $\alpha, \alpha' \in \mathbb{R}$. Each h_α is uniquely determined up to homotopy. Hence the diagram

$$\begin{array}{ccc} \tilde{X}_{\tau_0} & \xrightarrow{h_1} & \tilde{X}_{\tau_0\zeta} \\ {\scriptstyle \lambda^{-1}}\downarrow & & \downarrow{\scriptstyle \lambda^{-1}} \\ \tilde{X}_{\tau_0\bar{\zeta}} & \xrightarrow{h_1} & \tilde{X}_{\tau_0} \end{array}$$

is commutative up to homotopy (recall that $\tilde{X}_\tau = X_{\tau^e}$ for $\tau \in \tilde{S} - \{0\}$). This implies that on $H^q(\tilde{X}_{\tau_0})$ one has the relation $h_1^*(\lambda^{-1})^* = (\lambda^{-1})^* h_1^*$. Taking limits for $t \to 0$ one gets the relation $T_0(\lambda^*)^{-1} = (\lambda^*)^{-1} T_0$ on $H^q(X_\infty)$, hence also $T_0 \lambda^* = \lambda^* T_0$.

Both T_0 and λ^* act on the spectral sequence

$$E_2^{pq} = H^p(E, i^\cdot R^q k_* \mathbb{C}_{X_\infty}) \Rightarrow H^{p+q}(X_\infty, \mathbb{C})$$

(cf. [19], (2.4) and (2.5)), where $k : X_\infty \to X$ is the natural map and $i : E \to X$ is the inclusion. In fact, denoting $\tilde{i} : D \to \tilde{X}$ and $\tilde{k} : X_\infty \to \tilde{X}$ the analogous maps for $\tilde{X}$, one has

$$i^\cdot R^q k_* \mathbb{C}_{X_\infty} = \pi_* \tilde{i}^\cdot R^q \tilde{k}_* \mathbb{C}_{X_\infty}$$

because π is finite; hence λ^* acts on the sheaf $i^\cdot R^q k_* \mathbb{C}_{X_\infty}$. Let $Q \in E$. Let $z_0, \cdots, z_n$ be coordinates on a neighborhood V of Q in X, centered at Q, such that $f(z_0, \cdots, z_n) = z_0^{d_0} \cdots z_\nu^{d_\nu}$ for some integers $\nu \geqq 0$, $d_0, \cdots, d_\nu \geqq 1$. For $\varepsilon > 0$, $0 < \eta \ll \varepsilon$ denote $V_{\varepsilon,\eta}$ the set $\{(z_0, \cdots, z_n) \in V \mid \sum_{i=0}^n |z_i|^2 < \varepsilon$ and $|f(z)| < \eta\}$. For ε, η sufficiently small one has (cf. [19], (2.8))

$$(i^\cdot R^q k_* \mathbb{C}_{X_\infty})_Q = H^q(k^{-1} V_{\varepsilon,\eta}, \mathbb{C}),$$

and $k^{-1} V_{\varepsilon,\eta}$ consists of $d = \gcd(d_0, \cdots, d_\nu)$ components, each of which has the homotopy type of a ν-dimensional torus $S^1 \times \cdots \times S^1$. Hence its cohomology ring is an exterior algebra on $H^0(k^{-1} V_{\varepsilon,\eta}, \mathbb{C})$ which is isomorphic to $\mathbb{C}[\tau]/(\tau^d - 1)$. Both T_0 and λ^* act on this algebra by cyclic permutation of the components, i.e. by the substitution $\tau \to \exp(2\pi i/d)\tau$ leaving a set of generators for the exterior algebra fixed. This implies that $T_0(\lambda^*)^{-1}$ acts as the identity on $E_2^{pq} = H^p(E, i^\cdot R^q k_* \mathbb{C}_{X_\infty})$ for all $p, q \geqq 0$, so $T_0(\lambda^*)^{-1}$ is unipotent on $H^{p+q}(X_\infty, \mathbb{C})$.

(2.14) EXAMPLE. Consider again example (2.4). One can show that $H^1(X_\infty) = H^1(D_3)$. Because D_3 is an elliptic curve with an automorphism λ of order 6, the Hodge filtration on $H^1(X_\infty, \mathbb{C})$ is given by

$$F^1 H^1(X_\infty, \mathbb{C}) = \mathbb{C}.(e_1 + \rho e_2) \qquad (\rho = \exp(2\pi i/6))$$

if e_1, e_2 is a basis for $H^1(X_\infty, \mathbb{Z})$.

(2.15) EXAMPLE (ordinary double point). Suppose $g : Y \to S$ is a family of projective varieties with Y smooth, g smooth outside a point $y_0 \in Y_0$ and y_0 a non-degenerate critical point of g. By blowing up y_0 in Y one obtains a family $f : X \to S$ satisfying the hypotheses of (2.1) with $e = 2$, $E = E_0 \cup E_1$; E_0 is a desingularization of Y_0, $E_1 = \mathbb{P}^{n+1}$ has multiplicity 2 and $E_0 \cap E_1$ is a non-singular quadric in E_1. Here $\tilde{X}$ is non-singular, $D_0 \cong E_0$ and D_1 is isomorphic to a non-singular quadric of dimension $n+1$ which is a 2-fold covering of $\mathbb{P}^{n+1} = E_1$, ramified along $E_0 \cap E_1$.

One obtains for $q \geqq 0$:

$$Gr_{q-1}^W H^q(X_\infty) = \mathrm{Coker}\,(H^{q-1}(D_0) \oplus H^{q-1}(D_1) \to H^{q-1}(D_0 \cap D_1));$$

$$Gr_q^W H^q(X_\infty) = H(H^{q-2}(D_0 \cap D_1)(-1) \to H^q(D_0) \oplus H^q(D_1) \to H^q(D_0 \cap D_1));$$

$$Gr_{q+1}^W H^q(X_\infty) = \mathrm{Ker}\,(H^{q-1}(D_0 \cap D_1)(-1) \to H^{q+1}(D_0) \oplus H^{q+1}(D_1));$$

$$Gr_r^W H^q(X_\infty) = 0 \quad \text{if} \quad r \neq q-1, q, q+1.$$

If n is even, $H^q(X_\infty)$ is purely of weight q for all $q \geqq 0$, the monodromy is trivial on $H^q(X_\infty)$ for $q \neq n$ and has order 2 with -1 as an eigenvalue of multiplicity one if $q = n$.

If n is odd, $H^q(X_\infty)$ is purely of weight q with trivial monodromy if $q \neq n$ and for $H^n(X_\infty)$ one gets: $Gr_{n-1}^W H^n(X_\infty) = \mathbb{Q}(-(n-1)/2)$, $Gr_{n+1}^W H^n(X_\infty) = \mathbb{Q}(-(n+1)/2)$, $Gr_n^W H^n(X_\infty) = H^n(D_0)$. The monodromy is unipotent, $(T-I)^2 = 0$ and $T - I$: $Gr_{n+1}^W H^n(X_\infty) \xrightarrow{\sim} Gr_{n-1}^W H^n(X_\infty)$.

(2.16) REMARK. One can deduce from theorems (2.10) and (2.13) that the mixed Hodge structure on $H^q(X_\infty)$, constructed in (2.8), coincides with the one, constructed by W. Schmid [16]. See appendix.

(2.17) The relation between the cup product and the mixed Hodge structure on $H^*(X_\infty)$ has been investigated by Schmid [16], §6. For future use we recall the result.

Because $\tilde{T}_0$ satisfies $\tilde{T}_0(x \wedge y) = \tilde{T}_0(x) \wedge \tilde{T}_0(y)$ for all $x, y \in H^*(X_\infty)$, i.e. $\tilde{T}_0$ is an isometry, N is an infinitesimal isometry, i.e. $N(x) \wedge y + x \wedge N(y) = 0$ for all $x, y \in H^*(X_\infty)$.

Recall that all X_t are embedded simultaneously in $\mathbb{P}^r$ for some $r>0$. Let L be the cohomology class of a hyperplane section of $X_t (t\neq 0)$, considered as an element of $H^2(X_\infty)$ by means of the natural map $H^2(X_t)\to H^2(X_\infty)$ (unique after the choice of $\alpha\in H$ with $t=\exp(2\pi i\alpha)$). Then L does not depend on the choice of t or α.

Because the map $H^*(X_t)\xrightarrow{\psi_t} H^*(X_\infty)$ is a ring homomorphism for cup product, one has for all $k\geqq 0$:

$$L^k: H^{n-k}(X_\infty)\to H^{n+k}(X_\infty),$$

where $L^k(\omega)=L^k\wedge\omega$ for $\omega\in H^{n-k}(X_\infty)$. Denote $P^q(X_\infty)$ the kernel of the map $L^{n-q+1}: H^q(X_\infty)\to H^{2n-q+2}(X_\infty)$ if $0\leqq q\leqq n$ and $P^q(X_\infty)=0$ for $q>n$ or $q<0$. Then for all $q\geqq 0$ one has a decomposition

$$H^q(X_\infty)=\bigoplus_{k\geqq 0} L^k P^{q-2k}(X_\infty).$$

Elements of $P^q(X_\infty)$ are called primitive classes. Consider on $P^q(X_\infty)$ the bilinear form

$$Q(x,y)=\int_{X_t} (-1)^{q(q-1)/2} L^{n-q}\psi_t^{-1}(x\wedge y).$$

Then Q does not depend on the choice of t, because all elements $(\psi_t)_*[X_t]\in H_{2n}(X_\infty)$ are the same.

Because L is a T_0-invariant class, $P^q(X_\infty)\subset H^q(X_\infty)$ also carries a mixed Hodge structure, and for all $r\geqq 0$ one has

$$N^r: Gr^W_{q+r}P^q(X_\infty)\xrightarrow{\sim} Gr^W_{q-r}P^q(X_\infty).$$

Denote

$$P_{q,r}(X_\infty)=\operatorname{Ker}(N^{r+1}: Gr^W_{q+r}P^q(X_\infty)\to Gr^W_{q-r-2}P^q(X_\infty)).$$

Then $P_{q,r}(X_\infty)$ carries a Hodge structure of weight $q+r$. Let

$$P_{q,r}(X_\infty)=\bigoplus_{a+b=q+r} P^{a,b}_{q,r}(X_\infty)$$

be its Hodge decomposition. Denote Q_r the bilinear form on $P_{q,r}(X_\infty)$ defined by $Q_r(x,y)=Q(\tilde{x},N^r\tilde{y})$, where $x,y\in P_{q,r}(X_\infty)$ and $\tilde{x},\tilde{y}$ are elements of $W_{q+r}P^q(X_\infty)$ whose classes mod W_{q+r-1} are x resp. y. The fact that N is an infinitesimal isometry implies that $Q_r(x,y)$ is well-defined.

(2.18) THEOREM. *With notations as in* (2.17) *one has:*
(i) $Q_r(x,y)=0$ *if* $x\in P^{a,b}_{q,r}$, $y\in P^{c,d}_{q,r}$ *and* $(a,b)\neq(d,c)$;
(ii) $i^{a-b}Q_r(x,\bar{x})>0$ *if* $x\in P^{a,b}_{q,r}$, $x\neq 0$.

This theorem describes completely the connection between cup product and mixed Hodge structure, for for all q, $r>0$ one has a double decomposition

$$Gr^W_{q+r}H^q(X_\infty)=\bigoplus_{k,j\geqq 0} L^k N^j P_{q-2k,r+2j}$$

Appendix

We compare the constructions of [16] and (2.8). We keep the notations of the preceeding section. The weight filtration on $H^q(X_\infty)$ is the same in both cases because it is characterized by the property that the map

$$N^r: Gr^W_{q+r}H^q(X_\infty)\to Gr^W_{q-r}H^q(X_\infty)$$

is an isomorphism for all $r\geqq 0$.

Let us recall the construction by Schmid of the Hodge filtration. For $\alpha\in H$ denote $e(\alpha)=\exp(2\pi i\alpha)\in\bar{S}$ and denote $i_\alpha: X_{e(\alpha)}\to X_\infty$ the natural inclusion. The corresponding map

$$i^*_\alpha: H^q(X_\infty)\to H^q(X_{e(\alpha)})$$

is an isomorphism. Denote F_α the filtration on $H^q(X_\infty)$ obtained by pulling back the Hodge filtration on $H^q(X_{e(\alpha)})$ by means of i^*_α. Then $\exp(-\alpha N)F_\alpha=\exp(-(\alpha+1)N)F_{\alpha+1}$ so for $w\in\bar{S}$, $w\neq 0$ we may define a filtration $\tilde{F}_w$ on $H^q(X_\infty)$ by $\tilde{F}_w=\exp(-N\log(w))F_{\log(w)}$. The filtrations $\tilde{F}_w$, considered as points of a suitable product of Grassmann varieties, tend to a limit F_∞ as w goes to 0. We have to show that $F_\infty=$ our Hodge filtration F.

Put $A=H^0(\bar{S},\mathcal{O}_{\bar{S}})$, $M=H^0(\bar{S},\boldsymbol{R}^q\bar{f}_*\bar{\Omega}^{\cdot}_{\bar{X}/\bar{S}}(\log D))$, $M_w=\boldsymbol{R}^q\bar{f}_*\bar{\Omega}^{\cdot}_{\bar{X}/\bar{S}}(\log D)(w)$ for $w\in\bar{S}$ and $p_w: M\to M_w$ the canonical map (evaluation at w). Start with a basis $\boldsymbol{e}(0)$ of M_0. Claim: there exists a basis $\boldsymbol{e}$ of M over A, such that $p_0(\boldsymbol{e})=\boldsymbol{e}(0)$ and such that the Gauss-Manin connection $\bar{\nabla}$ satisfies $\bar{\nabla}\boldsymbol{e}=N\boldsymbol{e}\otimes d\tau/\tau$ with N a constant nilpotent matrix. The $\mathbb{C}$-vector space $V\subset M$ generated by the e_i is characterized by $V=\{m\in M\,|(\bar{\nabla}_{\tau d/d\tau})^k m=0$ for k sufficiently large$\}$. The proof is straightforward. One uses that $\bar{\nabla}$ has a nilpotent residue at 0. A multivalued horizontal section of $\boldsymbol{R}^q\bar{f}_*\bar{\Omega}^{\cdot}_{\bar{X}/\bar{S}}(\log D)$ over $\bar{S}$ is an element $s=\sum_{i=0}^n m_i(\log\tau)^i\in M\otimes_A A[\log\tau]$, which satisfies the relation

$$\sum_{i=0}^n \bar{\nabla}(m_i)(\log\tau)^i+\sum_{i=0}^n i m_i(\log\tau)^{i-1}\,d\tau/\tau=0.$$

The space of all multivalued horizontal sections $\tilde{V}$ is isomorphic with V by the map $y_0: m\to\exp(2\pi i N\log\tau)m (m\in\tilde{V})$.

The computations in [16], §2 show that $\tilde{V}$ and $H^q(X_\infty)$ are canonically isomorphic. The map $y_0^{-1}(p_0)^{-1}: M_0\to H^q(X_\infty)$ is the same as the map, denoted by ψ_t in [19], (4.24). This leads to the following abstract setting.

For G a filtration on M by direct summands one may construct filtrations G_∞ and G'_∞ on $\tilde{V}$ in the following way.

For $\alpha \in H$ denote $y_\alpha : \tilde{V} \to M_{e(\alpha)}$ the isomorphism given by $y_\alpha(s) = \exp(2\pi i\alpha N)p_{e(\alpha)}(s)$. Then for $s \in V$ one has the relation

$$y_\alpha(\exp(-2\pi i N \log \tau)s) = p_{e(\alpha)}(s).$$

Let G_α be the filtration $y_\alpha^{-1}p_{e(\alpha)}G$ of $\tilde{V}$. Then the filtrations $(G_\alpha)_{\alpha\in H}$ have a limit G_∞ uniformly on every vertical strip $R_{a,b} = \{z \in H \mid a < \text{Re}(z) < b\}$. (This is Schmid's construction). We define $G'_\infty = y_0^{-1}p_0G$. We have to show that $G'_\infty = G_\infty$.

Assume that G is a decreasing filtration with $G^0 = M$. Let $a_j = \text{rank of } G^j$. Choose an A-basis $e_1, \cdots, e_{a_0}$ for M such that $G^j = Ae_1 + \cdots + Ae_{a_j}$ for $j \geq 0$. Let $f_1, \cdots, f_{a_0}$ be the uniquely determined basis of V with $p_0(f_i) = p_0(e_i)$ for $i = 1, \cdots, a_0$. Write $e_i = Cf_i$ with $C \in \text{Aut}_A(M)$; then $C(0) = I$.

Let $g_i = \exp(-2\pi i N \log \tau)f_i \in \tilde{V}$. For $\alpha \in H$ let $h_i(\alpha) = y_\alpha^{-1}p_{e(\alpha)}(e_i)$. Choose norms on $M_w (w \in \tilde{S})$ and $\tilde{V}$ such that the $p_w(e_i)$ (resp. g_i) form an orthonormal basis. Then for $\alpha \in H$ with $w = e(\alpha)$ one has:

$$\begin{aligned} h_i(\alpha) &= y_\alpha^{-1}p_w(e_i) \\ &= y_\alpha^{-1}p_w(f_i + (C-I)f_i) \\ &= g_i + y_\alpha^{-1}p_w(C-I)f_i \\ &= g_i + y_\alpha^{-1}(C(w)-I)p_w(f_i). \end{aligned}$$

So

$$\|h_i(\alpha) - g_i\| \leq \|y_\alpha^{-1}\| \cdot \|C(w) - I\| \cdot \|p_w(f_i)\|.$$

On a vertical strip $R_{a,b}$, $\|y_\alpha^{-1}\|$ and $\|p_w(f_i)\|$ grow at most as a power of $\text{Im}(\alpha)$. Moreover $\|C(w) - I\|$ goes to zero at least as fast as $|w| = \exp(-2\pi \,\text{Im}(\alpha))$, so $\lim_{\text{Im}\,\alpha\to\infty} \|h_i(\alpha) - g_i\| = 0$ uniformly on $R_{a,b}$. This shows in particular that the filtration G'_∞, which is determined by the basis f_i, coincides with the filtration G_∞.

3. Isolated singularities of hypersurfaces

(3.1) Let $P \in \mathbb{C}[z_0, \cdots, z_n]$ be a polynomial with $P(0) = 0$, such that 0 is an isolated critical point of P. This means that 0 is an isolated point of the set $\{Q \in \mathbb{C}^{n+1} \mid \partial P/\partial z_i(Q) = 0 \text{ for all } i\}$. We are interested in the map germ $P : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. We will construct a mixed Hodge structure on the cohomology of $B_\eta = \{z \in \mathbb{C}^{n+1} \mid P(z) = \eta \text{ and } |z| < \varepsilon\}$ (ε, η small enough, $0 < \eta \ll \varepsilon$).

(3.2) One can embed the map germ P in a projective family as follows. Brieskorn ([3], §1.1) has shown that there exists a homogeneous polynomial $\tilde{P} \in \mathbb{C}[z_0, \cdots, z_{n+1}]$ such that the hypersurface $Y_0 \subset \mathbb{P}^{n+1}$ given by the equation $\tilde{P} = 0$ has a unique singular point $y_0 = (0, \cdots, 0, 1)$ which is analytically equivalent to the singularity of the zero set of P at 0. One constructs such a $\tilde{P}$ by adding to P a homogeneous polynomial $R(z_0, \cdots, z_n)$ of sufficient generality and high degree and then homogenizing. Let $N = \deg(\tilde{P})$. Then for all t sufficiently close to 0, $t \neq 0$, the zero set of $\tilde{P} - tz_{n+1}^N$ in $\mathbb{P}^{n+1}$ is smooth. Choose $\eta > 0$ such that this is the case for $0 < |t| < \eta$. Define $Y = \{(z, t) \in \mathbb{P}^{n+1} \times \mathbb{C} \mid \tilde{P}(z) - tz_{n+1}^N = 0 \text{ and } |t| < \eta\}$. Let $f_1 : Y \to S = \{t \in \mathbb{C} \mid |t| < \eta\}$ be the projection on the second factor. Denote $Y_t = f_1^{-1}(t)$ for $t \in S$. Then y_0 is the only critical point of f_1. Let $\rho : X \to Y$ be a resolution of singularities for the map f_1. This means that ρ is proper, induces an isomorphism between $X - \rho^{-1}(y_0)$ and $Y - \{y_0\}$ and $(f_1\rho)^{-1}(0)$ is a divisor with normal crossings on X. Put $f = f_1\rho$ and $f^{-1}(0) = E_0 \cup \cdots \cup E_m$ where the restriction of ρ to E_0 makes E_0 into a resolution of the singularity of Y_0.

(3.3) An exact sequence

Let B be a small ball in Y with center y_0 and radius $\varepsilon > 0$ so small that all spheres with center y_0 and radius less than ε intersect Y_0 transversally. Choose $\eta' > 0$ such that all fibers Y_t with $|t| < \eta'$ intersect B transversally. Define $B_t = Y_t \cap B$ for $|t| < \eta'$. Then B_t is diffeomorphic with the Milnor fiber of P (cf. [13], Theorem 5.11) if $t \neq 0$. One can construct a homeomorphism between Y_0 and Y_t/B_t ($t \neq 0$), hence $\tilde{H}^i(Y_0) \cong H^i(Y_t, B_t)$ for all i. The exact sequence of relative cohomology gives $0 \to \tilde{H}^n(Y_0) \to H^n(Y_t) \to H^n(B_t) \to H^{n+1}(Y_0) \to H^{n+1}(Y_t) \to 0$ because $H^i(B_t) = 0$ for $i \neq 0, n$ (cf. [13], Theorem 6.5).

Define B_∞ analogously to X_∞ (cf. (2.1)). Then passing to the limit as t tends to 0 one obtains an exact sequence $0 \to \tilde{H}^n(Y_0) \to H^n(X_\infty) \to H^n(B_\infty \to H^{n+1}(Y_0) \to H^{n+1}(X_\infty) \to 0$ which is T_0-equivariant because one may take the geometric monodromy $h : Y_t \to Y_t$ of the family Y to be the identity outside B_t.

We will put a mixed Hodge structure on $H^n(B_\infty)$ such that the above sequence becomes an exact sequence of mixed Hodge structures.

(3.4) Construction. We first compute the canonical mixed Hodge structure on the cohomology of Y_0 as defined by Deligne [5]. Remark that the map $f : X \to S$ satisfies the hypotheses of (2.2). We preserve the notations of (2.2). Because Y_0 is reduced, $e_0 = 1$ (if $n > 0$), so $\pi : D_0 \xrightarrow{\sim} E_0$. Define $C_i = D_i \cap D_0 = E_i \cap E_0$ ($i = 1, \cdots, m$). Then $C = C_1 \cup \cdots \cup C_m$ is a divisor with normal crossings on D_0. The cohomology of $D_0 - C \cong Y_0 - \{y_0\}$ can be computed in terms of the cohomology of D_0 and $\tilde{C}^{(i)}$, $i > 0$, as in [4]. Dualizing it one obtains the cohomology groups with compact support

$H^q_c(Y_0-\{y_0\})=\tilde{H}^q(Y_0)$. This shows that one may use the following cohomological mixed Hodge complex $A^{\cdot}(Y_0)$ on D_0 to compute the mixed Hodge structure on $H^*(Y_0)$.

Let $j: D_0-C\to D_0$ be the inclusion and denote $c_p:\tilde{C}^{(p)}\to D_0$ the projection $(p>0)$.

(i) Let $A^{\cdot}(Y_0)_{\mathbb{Z}}=j_!\mathbb{Z}_{D_0-C}$ i.e. the constant sheaf $\mathbb{Z}$ on D_0-C extended by 0 over D_0;

(ii) Denote $i_{j,q}:\tilde{C}^{(q+1)}\to\tilde{C}^{(q)}$ $(q>0, 0\leqq j\leqq q)$ the map which has the inclusions

$$C_{i_0}\cap\cdots\cap C_{i_q}\to C_{i_0}\cap\cdots\cap C_{i_{j-1}}\cap C_{i_{j+1}}\cap\cdots\cap C_{i_q}$$

as its components. Let $A^{\cdot}(Y_0)_{\mathbb{Q}}$ be the object in the filtered derived category $D^+F(D_0,\mathbb{Q})$ represented by the complex of sheaves

$$\mathbb{Q}_{D_0}\xrightarrow{d_0''}(c_1)_*\mathbb{Q}_{\tilde{C}}\xrightarrow{d_1''}(c_2)_*\mathbb{Q}_{\tilde{C}^{(2)}}\longrightarrow\cdots$$

with $d_q''=\sum_{j=0}^{q+1}(-1)^{q+j}(i_{j,q+1})^*$ and with the filtration W defined by

$$W_{-q}A^{\cdot}(Y_0)_{\mathbb{Q}}=\sigma_{\geqq q}A^{\cdot}(Y_0)_{\mathbb{Q}}.$$

(iii) Let $A^{pq}(Y_0)=(c_q)_*\Omega^p_{\tilde{C}^{(q)}}$; define $d': A^{pq}\to A^{p+1,q}$ to be the differentiation in the complex $(c_q)_*\Omega^{\cdot}_{\tilde{C}^{(q)}}$ and define $d'': A^{pq}\to A^{p,q+1}$ by $d''=\sum_{j=0}^{q+1}(-1)^{q+j}(i_{j,q+1})^*$. In this way one obtains a double complex $A^{\cdot\cdot}(Y_0)$. Let $A^{\cdot}(Y_0)_{\mathbb{C}}$ be its associated single complex. One defines filtrations F and W on $A^{\cdot}(Y_0)_{\mathbb{C}}$ by

$$F^pA^{\cdot}(Y_0)_{\mathbb{C}}=\bigoplus_{r\geqq p}A^{r\cdot}(Y_0);$$

$$W_qA^{\cdot}(Y_0)_{\mathbb{C}}=\bigoplus_{s\geqq -q}A^{\cdot s}(Y_0).$$

For $q>0$ one has $Gr^W_qA^{\cdot}(Y_0)=0$; if $q\geqq 0$ then $Gr^W_{-q}A^{\cdot}(Y_0)$ is equal to $((c_q)_*\mathbb{Q}_{\tilde{C}^{(q)}}[-q], ((c_q)_*\Omega^{\cdot}_{\tilde{C}^{(q)}}[-q], F))$ which is indeed a cohomological Hodge complex of weight $-q$.

(3.5) Corollary. *The spectral sequence of hypercohomology for the filtered complex $(A^{\cdot}(Y_0)_{\mathbb{Q}}, W)$:*

$$E_1^{pq}=H^q(\tilde{C}^{(p)},\mathbb{Q})\Rightarrow H^{p+q}(Y_0,\mathbb{Q})$$

degenerates at E_2, i.e. $Gr^W_qH^{p+q}(Y_0,\mathbb{Q})=E_2^{pq}$ is isomorphic to

$$H(H^q(\tilde{C}^{(p-1)},\mathbb{Q})\to H^q(\tilde{C}^{(p)},\mathbb{Q})\to H^q(\tilde{C}^{(p+1)},\mathbb{Q})).$$

(3.6) Remark. The cohomological mixed Hodge complex $A^{\cdot}(Y_0)$ can be considered as a subcomplex of the complex $A^{\cdot}$ as defined in (2.8) in the following way. The inclusion $D_0\subset D$ gives inclusions $\tilde{C}^{(i)}\subset\tilde{D}^{(i+1)}$ for all $i\leqq 0$. In this way the complex $A^{\cdot}(Y_0)_{\mathbb{Q}}$ is a direct factor of the complex

$$a_*\mathbb{Q}_{\tilde{D}}\to(a_2)_*\mathbb{Q}_{\tilde{D}^{(2)}}\to(a_3)_*\mathbb{Q}_{\tilde{D}^{(3)}}\to\cdots$$

which is a subcomplex of $A_{\mathbb{Q}}$. Moreover $A^{pq}(Y_0)$ is a direct factor of $(a_{q+1})_*\Omega^p_{\tilde{D}^{(q+1)}}=Gr^W_{q+1}\Omega^{p+q+1}_{\tilde{X}}(\log D)\subset A^{pq}$. These inclusions are compatible with the filtrations W and F. They define a morphism of cohomological mixed Hodge complexes

$$A^{\cdot}(Y_0)\to A^{\cdot}$$

which induces morphisms of mixed Hodge structures

$$H^i(Y_0)\to H^i(X_\infty).$$

These are the same as those induced by the contraction map $Y_t\to Y_0$.

(3.7) Definition. The cohomological mixed Hodge complex $A^{\cdot}(B_\infty)$ is given by

$$A^{\cdot}(B_\infty)=A^{\cdot}/A^{\cdot}(Y_0).$$

(3.8) Theorem. $\boldsymbol{H}^q(D, A^{\cdot}(B_\infty)_{\mathbb{Q}})=H^q(B_\infty,\mathbb{Q})$ *for* $q\geqq 0$.

Proof. We may suppose that S is so small that X_t intersects ∂B transversally for all $t\in S$. Then $B_\infty=B\times_S H$. Denote $\tilde{g}: B_\infty\to\tilde{X}$ the natural map. Then $A^{\cdot}_{\mathbb{Q}|\rho^{-1}(B)\cap D}=i^{\cdot}\boldsymbol{R}\tilde{g}_*\mathbb{Q}_{B_\infty}$, because $A^{\cdot}_{\mathbb{Q}}=i^{\cdot}\boldsymbol{R}\tilde{k}_*\mathbb{C}_{X_\infty}$. This implies that

$$\boldsymbol{H}^q(D\cap\rho^{-1}(B), A^{\cdot}_{\mathbb{Q}|\rho^{-1}(B)\cap D})=H^q(B_\infty,\mathbb{Q})$$

(compare [19], Lemmas (2.4) and (2.5)). Because $\rho^{-1}(B)\cap D/D_1\cup\cdots\cup D_m=Y_0\cap B$ is a cone, the restriction map $\boldsymbol{H}^q(D\cap\rho^{-1}(B), A^{\cdot}_{\mathbb{Q}|D\cap\rho^{-1}(B)})\to\boldsymbol{H}^q(D_1\cup\cdots\cup D_m, A^{\cdot}_{\mathbb{Q}|D_1\cup\cdots\cup D_m})$ is an isomorphism for all $q\geqq 0$. The latter is isomorphic to $\boldsymbol{H}^q(D, A^{\cdot}(B_\infty)_{\mathbb{Q}})$ because $A^{\cdot}(B_\infty)_{\mathbb{Q}}$ has support on $D_1\cup\cdots\cup D_m$ and because $A^{\cdot}_{\mathbb{Q}|D_1\cup\cdots\cup D_m}=A^{\cdot}(B_\infty)_{\mathbb{Q}|D_1\cup\cdots\cup D_m}$.

(3.9) Corollary. *$H^n(B_\infty,\mathbb{Q})$ carries a mixed Hodge structure such that the exact sequence from (3.3) is an exact sequence of mixed Hodge structures.*

Proof. Use the exact sequence

$$0\to A^{\cdot}(Y_0)\to A^{\cdot}\to A^{\cdot}(B_\infty)\to 0.$$

(3.10) Corollary. *The weight spectral sequence*

$$E_1^{-r,q+r}=\bigoplus_{k\geqq 0}H^{q-r-2k}(\tilde{D}^{(2k+r+1)})(-r-k)\qquad(r>0)$$

$$=\bigoplus_{k\geqq 1-r}H^{q-r-2k}(\tilde{D}^{(2k+r+1)})(-r-k)\oplus H^{q+r}(\tilde{D}^{(1-r)}-\tilde{C}^{(-r)})\qquad(r\leqq 0)$$

abutting to $H^q(B_\infty)$, *degenerates at* E_2, i.e. $E_2^{-r,q+r} = Gr^W_{q+r}H^q(B_\infty)$. *Moreover* $E_2^{-r,q+r} = 0$ *for* $q \neq n$ *(except* $E_2^{0,0} = \mathbb{Q}$*) because* $H^q(B_\infty) = 0$ *for* $q \neq 0, n$. *This enables one to compute the weights on* $H^n(B_\infty)$ *from the* E_1*-terms.*

(3.11) REMARK. The actions of $N = \log(T_0)$ and γ_s = semisimple part of T_0 can both be lifted to actions on $A_{\mathbb{C}}^{\cdot}$ which are 0 resp. I on the subcomplex $A^{\cdot}(Y_0)_{\mathbb{C}}$ (cf. [19], (4.22) for the lifting $\tilde{\nu}$ of N and (2.13) for the lifting λ^* of γ_s). Hence they act on $H^*(B_\infty)$ in such a way that the sequence of (3.3) is both N- and γ_s-equivariant, and such that N and γ_s are morphisms of Hodge structures of type $(-1, -1)$ and $(0, 0)$ respectively. The map T_0 is a limit of conjugates of the monodromy T.

(3.12) EXAMPLE. Let $P \in \mathbb{C}[z_0, \cdots, z_n]$ be a homogeneous polynomial with an isolated singular point at 0. Let $\rho: X \to \mathbb{C}^{n+1}$ be the blowing up with center $0 \in \mathbb{C}^{n+1}$. The map $f = P\rho$ satisfies our hypotheses. One has $f^{-1}(0) = E_0 \cup E_1$ where $E_1 \cong \mathbb{P}^n$, E_0 is a desingularization of $P^{-1}(0)$ and $E_0 \cap E_1$ is the smooth projective hypersurface given by $P(z) = 0$. Here D_1 is a covering of $\mathbb{P}^n$ of degree d = degree of P, which is ramified along $E_0 \cap E_1$. This implies that D_1 is isomorphic to the hypersurface in $\mathbb{P}^{n+1}$ with equation $P(z) - z_{n+1}^d = 0$ and $\pi: D_1 \to E_1$ is the projection from the point $(1, 0, \cdots, 0)$. One concludes that the cohomology of B_∞ inherites the mixed Hodge structure of the cohomology of the affine hypersurface $D_1 - D_0 = \{z \in \mathbb{C}^{n+1} \mid P(z) = 1\}$. A similar statement holds if P is quasi-homogeneous. See [20] for a detailed description of these mixed Hodge structures.

(3.13) EXAMPLE. Families of curves. Suppose $P: \mathbb{C}^2 \to \mathbb{C}$ has an isolated critical point 0. Let $\rho: X \to \mathbb{C}^2$ be a resolution of the singularities of P. It can be obtained by successive blowing ups of points. Let $f = P\rho$. Denote $f^{-1}(0) = E_0 \cup E_1 \cup \cdots \cup E_m$ with $\rho^{-1}(0) = E_1 \cup \cdots \cup E_m$. Then E_i is a non-singular complete rational curve for $i > 0$. Denote e_i its multiplicity. If $i, j > 0$ and $i \neq j$, the curves E_i and E_j intersect in at most one point P_{ij}, hence if $E_i \cap E_j \neq \varnothing$, then D_i and D_j intersect in $m_{ij} = (e_i, e_j)$ points. Denote m_{i0} = Card $E_0 \cap E_i$ and k_i = Card $E_i \cap \bigcup_{j \neq i} E_j$. Let $\mathring{E}_i = E_i - \bigcup_{j \neq i} E_j$, $\mathring{D}_i = D_i - \bigcup_{j \neq i} D_j$. Then $\mathring{D}_i \to \mathring{E}_i$ is an étale covering with cyclic covering group generated by $\lambda_i = \lambda_{|D_i}$ (cf. the proof of (2.13)), which is of order e_i. The fundamental group $\pi_1(\mathring{E}_i, *)$ is a free group of rank $k_i - 1$, generated by loops $l_1, \cdots, l_{k_i}$ around the intersection points $E_i \cap \bigcup_{j \neq i} E_j$ with the relation

$$l_{k_i} l_{k_i - 1} \cdots l_2 l_1 = 1.$$

If l_j is a loop going around the intersection point of E_i with a curve of multiplicity $e(j)$, then the action of l_j on D_i is just $\lambda^{e_i/(e_i, e(j))}$. Therefore the index of the image of $\pi_1(\mathring{E}_i, *)$ in the covering group is equal to $r_i = gcd(e_j \mid E_i \cap E_j \neq \varnothing)$. This is also equal to the number of connected components of D_i because D_i is normal.

One has the well-known relation

$$\prod_{q \geq 0} \det(I - t\lambda_i; H^q(\mathring{D}_i))^{(-1)^{q+1}} = (1 - t^{e_i})^{\chi(\mathring{E}_i)}$$

so

$$\det(\lambda_i - tI; H^1(\mathring{D}_i)) = (t^{e_i} - 1)^{k_i - 2}(t^{r_i} - 1)$$

From the action of λ_i on the exact sequence

$$0 \to H^1(D_i) \to H^1(\mathring{D}_i) \to H^0(D_i - \mathring{D}_i)(-1) \to H^2(D_i) \to 0$$

one gets

$$\det(\lambda_i - tI; H^1(D_i)) = \frac{(t^{e_i} - 1)^{k_i - 2}(t^{r_i} - 1)}{(t-1)^{m_{i0}} \prod_{j \neq i, 0} (t^{m_{ij}} - 1)}$$

Remark that numerator and denominator contain an equal number of factors $t - 1$. The spectral sequence (3.10) gives exact sequences

$$0 \to \mathbb{Q} \to \bigoplus_{i > 0} H^0(D_i) \to \bigoplus_{i > j > 0} H^0(D_i \cap D_j) \to Gr_0^W H^1(B_\infty) \to 0$$

and

$$0 \to Gr_2^W H^1(B_\infty) \to \bigoplus_{i > j \geq 0} H^0(D_i \cap D_j)(-1) \to \bigoplus_{i > 0} H^2(D_i) \to 0;$$

Moreover $Gr_1^W H^1(B_\infty) = \bigoplus_{i>0} H^1(D_i)$. So the characteristic polynomial of λ^* on $Gr_r^W H^1(B_\infty)$ is given by

$$(t-1) \prod_{i > j > 0} (t^{m_{ij}} - 1) \Big/ \prod_{i > 0} (t^{r_i} - 1) \quad \text{if} \quad r = 0;$$

$$\prod_{i > 0} (t^{e_i} - 1)^{k_i - 2}(t^{r_i} - 1)^2 \Big/ (t-1)^{\Sigma_{i>0} m_{i0}} \prod_{i > j > 0} (t^{m_{ij}} - 1)^2 \quad \text{if} \quad r = 1;$$

$$(t-1)^{\Sigma_{i>0} m_{i0}} \prod_{i > j > 0} (t^{m_{ij}} - 1) \Big/ \prod_{i > 0} (t^{r_i} - 1) \quad \text{if} \quad r = 2.$$

Multiplication of these gives the characteristic polynomial of the monodromy and the Milnor number:

$$\Delta(t) = (t-1) \prod_{i > 0} (t^{e_i} - 1)^{k_i - 2};$$

$$\mu = 1 + \Sigma_{i>0} e_i (k_i - 2).$$

These are the formulas which follow from A'Campo's formula for the zeta function of the monodromy [1], which can also be proved using (3.10).

Durfee [6] has introduced the notion of the number of cycles in the fiber X_0 and uses this to give a criterion for finiteness of the monodromy. This number is equal to dim $Gr_0^W H^1(B_\infty)$. In particular the monodromy has finite order if and only if $W_0 H^1(B_\infty) = 0$. In theorem (4.4) we will give a generalization of this fact to higher dimensions.

The following lemma is useful to determine the relation between the Hodge filtration on $H^1(B_\infty)$ (or $H^n(B_\infty)$ in higher dimensions) and the eigenvalues of λ^*.

(3.14) LEMMA. *With notations as before, the action of λ_j leads to a splitting*

$$\pi_* \mathcal{O}_{D_j} = \bigoplus_{k=0}^{e_j-1} L_k$$

where λ_j acts on L_k by multiplication with $\exp(2\pi i k/e_j)$. *One has*

$$L_k \cong \mathcal{O}_{E_j}\left(\Sigma_{i\neq j}\left(-\frac{ke_i}{e_j}+\left[\frac{ke_i}{e_j}\right]\right)E_i\cap E_j\right).$$

PROOF. One uses remark (2.3) and the fact that $\mathcal{O}_X(\Sigma_{i=0}^m e_i E_i)$ is isomorphic to $\mathcal{O}_X$ (by multiplication with a parameter on S). The divisor associated to the normal bundle of E_j in X is $-E_j\cdot E_j \sim \Sigma_{j\neq i}(e_i/e_j)E_j\cdot E_i$, hence the latter indeed defines an element of $\operatorname{Pic}(E_j)$ and not merely an element of $\operatorname{Pic}(E_j)\otimes\mathbb{Q}$.

(3.15) EXAMPLE. $P(x,y,z)=x^4+y^2+z^2$. Twice blowing up a point gives the fiber $E=E_0\cup E_1\cup E_2$; E_0 is the non-compact component, $e_1=4$ and $e_2=2$; $E_1=\mathbb{P}^2$, E_2 is isomorphic to $\mathbb{P}^2$ with one point blown up, $E_1\cap E_2$ is the exceptional curve in E_2, $E_0\cap E_1$ is a quadric in E_1 and $E_0\cap E_2$ is the disjoint union of two lines, each intersecting $E_1\cap E_2$ in one point. Hence D_1 is the disjoint union of two quadrics in $\mathbb{P}^3$, D_2 is isomorphic to a quadric in $\mathbb{P}^3$ with 2 points blown up and $D_1\cap D_2$ is the union of the two exceptional curves on D_2. One obtains that $H^2(B_\infty)=\mathbb{Q}(-1)^2$ is purely of type $(1,1)$.

(3.16) EXAMPLE. $P(x,y,z)=x^8+y^8+z^8+(xyz)^2$ (Malgrange) We will determine the following data:

(i) the dimensions of the spaces $H^{pq}=Gr_F^p Gr_{p+q}^W H^2(B_\infty)$;

(ii) the eigenvalues of λ^* on each of these.

We first describe a resolution for P, following A'Campo [1]. Blow up $0\in\mathbb{C}^3$. The exceptional divisor intersects the strict transform of $P^{-1}(0)$ in three double lines which are in general position. After blowing up each of these lines one obtains a resolution $\rho: X\to\mathbb{C}^3$ for P. Here $f^{-1}(0)=\bigcup_{i=0}^4 E_i$ with $e_1=6$, $e_2=e_3=e_4=8$. One has $E_1=\mathbb{P}^2$, E_4 is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ whose zero section $E_1\cap E_4=a$ has self-intersection -4 and E_2 and E_3 are obtained form E_4 by blowing up 2 resp. 1 points which are not on the zero section. This implies that D_2 and D_3 are obtained from D_4 by blowing up 16 resp. 8 points. D_1 is the disjoint union of two components, each of which is a 3-fold covering of $\mathbb{P}^2$, ramified along three lines in general position. Hence $H^i(D_1)=0$ for $i=1,3$ and $H^2(D_1)=\mathbb{Q}(-1)^2$. Moreover $E_0\cap E_1=\varnothing$, $E_0\cap E_i$ is a curve of genus 3 for $i=2,3,4$ and $E_i\cap E_j$ is a rational curve if $1\leqq i<j\leqq 4$.

We first show that $H^1(D_4,\mathbb{Q})=0$. The composition of π with the projection map $p: E_4\to\mathbb{P}^1$ gives a flat projective morphism $\tau: D_4\to\mathbb{P}^1$. Each fiber $\tau^{-1}(z)$ is an 8-fold covering of $p^{-1}(z)=\mathbb{P}^1$, ramified above $p^{-1}(z)\cap(E_0\cup E_1)$. The critical points of τ correspond to the ramification points of the map $p_{|E_0\cap E_4}$, which is a double covering of $\mathbb{P}^1$ with 8 ramification points. Hence τ has 8 singular fibers, each of which is the union of two rational singular curves which in their intersection point are of type u^6-x^8 and which are smooth everywhere else. Denote C_{λ_i}, $i=1,\cdots,8$ these singular fibres and let $U=D_4-\bigcup_{i=1}^8 C_{\lambda_i}$. Let z be a regular value of τ. The local monodromy around each of the λ_i acts on $H^1(C_z)$ with eigenvalues $\neq 1$. In particular $H^0(\mathbb{P}^1-\{\lambda_1,\cdots,\lambda_8\}, R^1(\tau_{|U})_*\mathbb{Q})=0$. It follows from [4], (4.1.1) that $H^1(U,\mathbb{Q})=H^1(\mathbb{P}^1-\{\lambda_1,\cdots,\lambda_8\},\mathbb{Q})=\mathbb{Q}(-1)^7$ is purely of weight 2, hence $H^1(D_4,\mathbb{Q})=Gr_1^W H^1(U,\mathbb{Q})=0$. Consequently $H^1(D_4,\mathcal{O}_{D_4})=0$. Denote $b\in H^2(E_4,\mathbb{Z})$ the cohomology class of $E_2\cap E_4$. Then a and b form a basis for $H^2(E_4,\mathbb{Z})$ and $a^2=-4$, $ab=1$ and $b^2=0$. The canonical divisor on D_4 is $-2a-6b$ and the cohomology class of $E_0\cap E_4$ is $2a+8b$. Using lemma (3.14) one obtains for the cohomology class of L_k $(k=0,\cdots,7)$: $[L_0]=0$, $[L_1]=-a-b$, $[L_2]=-a-2b$, $[L_3]=-a-3b$, $[L_4]=-a-4b$, $[L_5]=-2a-5b$, $[L_6]=-2a-6b$, $[L_7]=-2a-7b$. The Riemann-Roch theorem on E_4 gives that $\chi(\mathcal{O}_{E_4}(ka+lb))=(k+1)(l+1-2k)$ so $\chi(L_0)=1=\dim H^0(E_4,\mathcal{O}_{E_4})$, $\chi(L_k)=0$ for $i=1,\cdots,5$, $\chi(L_6)=1$ and $\chi(L_7)=2$. This implies that $\dim H^2(D_4,\mathcal{O}_{D_4})=3$ and that λ^* acts on it with the eigenvalues ξ^6,ξ^7,ξ^7 where $\xi=\exp(\pi i/4)$. Consequently one has the following picture:

space	dimension	eigenvalues of λ	multiplicity
$H^{0,0}$	1	-1	1
$H^{0,1}$	9	ξ^5,ξ^6,ξ^7	3
$H^{0,2}$	9	ξ^6	3
		ξ^7	6
$H^{1,1}$	137	1	4
		-1	25
		ξ,ξ^7	15
		ξ^2,ξ^6	18
		ξ^3,ξ^5	21
$H^{2,2}$	5	1	4
		-1	1
$H^{1,2}$	18	ξ^5,ξ^6,ξ^7	3
		1	9

To show this one uses again (3.10). The eigenvalues of λ^* on the other spaces H^{pq} can be deduced from $H^{pq}=\bar{H}^{qp}$ and the fact that λ^* is defined over $\mathbb{Q}$.

In (4.4) we will show that these data give the minimal polynomial of T_0: in this example it equals $(t^8-1)^2(t+1)$ while the characteristic polynomial of T_0 equals $(t^8-1)^{27}(t-1)^{-1}$.

4. Applications

In this chapter we investigate the relations between the mixed Hodge structure on the vanishing cohomology, the monodromy, the intersection form and the local cohomology groups of an isolated hypersurface singularity. We keep the notations of the preceding chapter.

(4.1) THEOREM. *The map $N=\log \bar{T}_0: H^n(B_\infty)\to H^n(B_\infty)$ is a morphism of mixed Hodge structures of type $(-1,-1)$. The map $\gamma_s=\lambda^*: H^n(B_\infty)\to H^n(B_\infty)$ is an automorphism of mixed Hodge structures.*

PROOF. See (3.11). Remark that it is no longer true in general that $N^r: Gr^W_{n+r}H^n(B_\infty)\to Gr^W_{n-r}H^n(B_\infty)$ is an isomorphism.

(4.2) LEMMA. *If $r\geqq 1$ the map $Gr^W_{n+r}H^n(X_\infty)\to Gr^W_{n+r}H^n(B_\infty)$ is injective; for $r\geqq 2$ it is an isomorphism.*

PROOF. This follows from the fact that $Gr^W_{n+r}H^n(Y_0)=0$ for $r\geqq 1$ and $Gr^W_{n+r}H^{n+1}(Y_0)=0$ for $r\geqq 2$ because Y_0 is complete, and from corollary (3.9).

(4.3) For $d\in\mathbb{N}$ let $\Phi_d(t)$ be the dth cyclotomic polynomial and

$$H^n(X_\infty)_d=\{x\in H^n(X_\infty)\mid \Phi_d(\gamma_s)(x)=0\};$$
$$H^n(B_\infty)_d=\{x\in H^n(B_\infty)\mid \Phi_d(\gamma_s)(x)=0\}.$$

These are the subspaces on which the monodromy acts with primitive dth roots of unity as eigenvalues. Then clearly $H^n(X_\infty)_d=H^n(B_\infty)_d$ for all $d>1$. Because N and γ_s commute, one has

$$N^r: Gr^W_{n+r}H^n(B_\infty)_d \xrightarrow{\sim} Gr^W_{n-r}H^n(B_\infty)_d \quad \text{for} \quad r\geqq 0,\ d>1.$$

Consequently for $d>1$ the exponent of $\Phi_d(t)$ as a factor of the minimal polynomial $\delta(t)$ of the monodromy equals

$$k_d=1+\max(r\mid Gr^W_{n+r}H^n(B_\infty)_d\neq 0) \quad \text{if} \quad H^n(B_\infty)_d\neq 0;$$
$$0 \quad \text{if} \quad H^n(B_\infty)_d=0.$$

To determine k_1 we use the exact sequence

$$0\longrightarrow H^n(Y_0)\longrightarrow H^n(X_\infty)_1\xrightarrow{u} H^n(B_\infty)_1\xrightarrow{v} H^{n+1}(Y_0)\longrightarrow H^{n+1}(X_\infty)\longrightarrow 0.$$

Suppose that $H^n(B_\infty)_1\neq 0$. Claim: $\delta(t)$ contains $t-1$ as a factor of multiplicity $k_1=\max(1,\max(r\mid Gr^W_{n+r}H^n(B_\infty)_1\neq 0))$.

PROOF. If $Gr^W_{n+r}H^n(B_\infty)_1=0$ for all $r>0$, then $Gr^W_{n+r}H^n(X_\infty)_1=0$ for all $r>0$ by lemma (4.3). Consequently $T_0=I$ on $H^n(X_\infty)_1$ hence $H^n(Y_0)=H^n(X_\infty)_1$ by the invariant cycle theorem (cf. [19], (5.12)). So v is injective and $T_0=I$ on $H^n(B_\infty)_1$. Hence $k_1=1$. If r is maximal such that $Gr^W_{n+r}H^n(B_\infty)_1\neq 0$ and $r>0$, then let $x\in Gr^W_{n+r}H^n(B_\infty)_1$ with $x\neq 0$. Then $Nv(x)=0=vN(x)$ so $N(x)=u(y)$ for some $y\in Gr^W_{n+r-2}H^n(X_\infty)_1$. If $r=1$ then again lemma (4.3) says that $Gr^W_{n+k}H^n(X_\infty)_1=0$ for $k\geqq 2$ so $Gr^W_{n-1}H^n(Y_0)=Gr^W_{n-1}H^n(X_\infty)_1$ by the invariant cycle theorem. Hence $N(x)=u(y)=0$. If $r>1$ then r is also maximal such that $Gr^W_{n+r}H^n(X_\infty)_1\neq 0$. Write $x=u(z)$ for $z\in Gr^W_{n+r}H^n(X_\infty)_1$. Then $N^r(z)\in Gr^W_{n-r}H^n(X_\infty)_1=Gr^W_{n-r}H^n(Y_0)$ so $N^r(x)=uN^r(z)=0$. Moreover $N^{r-1}(z)\notin Gr^W_{n-r+2}H^n(Y_0)$ so $N^{r-1}(x)\neq 0$. Putting this all together we find:

(4.4) THEOREM. *With notations as in (4.3) we have*

$$\delta(t)=\prod_{d\geqq 1}\Phi_d(t)^{k_d}.$$

(4.5) EXAMPLE. Consider $P(x,y,z)=x^8+y^8+z^8+(xyz)^2$. It follows from (3.16) that $k_1=2$, $k_2=3$, $k_4=k_8=2$ and $k_i=0$ if $i\neq 1,2,4,8$. Hence $\delta(t)=(t+1)(t^8-1)^2$ which shows that the monodromy is not quasi-unipotent of degree 2, i.e. $N^2\neq 0$.

(4.6) EXAMPLE. Suppose that we know that the monodromy has finite order. Then $H^n(B_\infty)_d$ is purely of weight n if $d\geqq 2$ and is of weight $\leqq n+1$ if $d=1$. If moreover $n=1$, we even know that $H^1(B_\infty)_1$ is purely of weight 2 because $H^2(Y_0)$ is purely of weight 2. Hence in the case $n=1$ the double factors of $\delta(t)$ are precisely the $(t-\nu)^2$ where ν is an eigenvalue of γ_s acting on $W_0H^1(B_\infty)$.

We can treat the general case too because of

(4.7) LEMMA. *$H^{n+1}(Y_0)$ is purely of weight $n+1$.*

PROOF. In the case $n=2$, one may use the fact that the curve $C_1\cup\cdots\cup C_m$ on D_0 can be blown down to a point, hence, by a criterion of Grauert [8], the intersection matrix $(C_i\cdot C_j)$ is negative definite. In particular the cohomology classes $[C_1],\cdots,[C_m]\in H^2(D_0)$ are linearly independent, which implies that the map $H^2(D_0)\to H^2(\tilde{C})$ is surjective.

In the general case the collapsing map $X_0=E_0\cup\cdots\cup E_m\to Y_0$ induces an exact sequence of mixed Hodge structures.

$$\cdots\to H^k(Y_0)\to H^k(X_0)\to H^k(E_1\cup\cdots\cup E_m)\to H^{k+1}(Y_0)\to\cdots.$$

Because $E_1 \cup \cdots \cup E_m$ can be blown down to a point in $\mathbb{P}^{n+1}$ which is smooth, $H^k(E_1 \cup \cdots \cup E_m)$ is purely of weight k and $H^k(X_0) \to H^k(E_1 \cup \cdots \cup E_m)$ is surjective for all $k \geq 0$.

Hence it suffices to show that $H^{n+1}(X_0)$ is purely of weight $n+1$. We have the exact sequence

$$\longrightarrow H^{n+1}_{X_0}(X) \longrightarrow H^{n+1}(X_0) \longrightarrow H^{n+1}(X_\infty) \xrightarrow{N} H^{n+1}(X_\infty).$$

Because the monodromy on $H^{n+1}(X_\infty)$ is the identity, $H^{n+1}(X_\infty)$ is purely of weight $n+1$. Moreover $Gr^W_k H^{n+1}_{X_0}(X) = 0$ for $k \leq n$, hence $H^{n+1}(X_0)$ is purely of weight $n+1$.

(4.8) COROLLARY. *If T_0 has finite order on $H^n(B_\infty)$, then $H^n(B_\infty)_1$ is purely of weight $n+1$.*

(4.9) COROLLARY. *N^k defines for all $k \geq 0$ an isomorphism between $Gr^W_{n+1+k} H^n(B_\infty)_1$ and $Gr^W_{n+1-k} H^n(B_\infty)_1$.*

PROOF. Use (4.7), (2.13)(2) and the invariant cycle theorem.

(4.10) We look for relations between the intersection form on $H^n_c(B_\infty)$ and the mixed Hodge structure. The tools are the exact sequence (3.3), theorem (2.13) and theorem (2.18). We consider $H^n_c(B_\infty)$ as the dual space of $H^n(B_\infty)$, more precisely there is a perfect pairing $\langle\, ,\rangle$:

$$H^n_c(B_\infty) \otimes_{\mathbb{Q}} H^n(B_\infty) \to \mathbb{Q}(-n)$$

and we give $H^n_c(B_\infty)$ the mixed Hodge structure of $\mathrm{Hom}_{\mathbb{Q}}(H^n(B_\infty), \mathbb{Q}(-n))$. Dual to the restriction map $k: H^n(X_\infty) \to H^n(B_\infty)$ we have ${}^t k: H^n_c(B_\infty) \to H^n(X_\infty)$ and for $\omega \in H^n_c(B_\infty)$, $\omega' \in H^n(B_\infty)$ one has

$$\langle \omega, \omega' \rangle = \langle {}^t k(\omega), k(\omega') \rangle.$$

Denote $j = k\,{}^t k: H^n_c(B_\infty) \to H^n(B_\infty)$. The intersection form S on $H^n_c(B_\infty)$ is defined by $S(x, y) = \langle x, j(y) \rangle$. We want to express its invariants as a real bilinear form in terms of the Hodge numbers $h^{pq} = \dim_{\mathbb{C}} H^{pq}(B_\infty)$. Let $h^{pq}_1 = \dim_{\mathbb{C}} H^{pq}(B_\infty) \cap H^n(B_\infty)_1$ and let $h^{pq}_{\neq 1} = \dim_{\mathbb{C}} H^{pq}(B_\infty) \cap \bigoplus_{d>1} H^n(B_\infty)_d$. Then $h^{pq} = h^{pq}_1 + h^{pq}_{\neq 1}$. Let μ be the Milnor number of P. Let μ_0 be the dimension of the null-space of S. The rank of S is the number $\mu - \mu_0$. This is the only real invariant besides μ if n is odd, because in that case S is antisymmetric. If n is even, S is symmetric and we can diagonalize S. Let μ_+ and μ_- be the number of positive resp. negative diagonal entries of S. Then the numbers μ_0, μ_+, and μ_- form a complete set of invariants for S as a real form. Particularly important is the signature $\mu_+ - \mu_-$.

(4.11) THEOREM. *With notations as above one has:*

(i)
$$\mu_0 = \sum_{p+q \leq n+1} h^{pq}_1 - \sum_{p+q \geq n+3} h^{pq}_1;$$

(ii) If n is even then

$$\mu_+ = \sum_{\substack{p+q=n+2 \\ q \text{ even}}} h^{pq}_1 + 2 \sum_{\substack{p+q \geq n+3 \\ q \text{ even}}} h^{pq}_1 + \sum_{q \text{ even}} h^{pq}_{\neq 1};$$

$$\mu_- = \sum_{\substack{p+q=n+2 \\ q \text{ odd}}} h^{pq}_1 + 2 \sum_{\substack{p+q \geq n+3 \\ q \text{ odd}}} h^{pq}_1 + \sum_{q \text{ odd}} h^{pq}_{\neq 1}.$$

PROOF. Because the sequence (3.3) is monodromy equivariant, the map k induces an isomorphism $H^n(X_\infty)_{\neq 1} \cong H^n(B_\infty)_{\neq 1} \cong H^n_c(B_\infty)_{\neq 1}$. Hence the intersection form on $H^n_c(B_\infty)_{\neq 1}$ is non-degenerate. Because the monodromy acts trivially on $H^q(X_\infty)$ for $q \neq n$, one knows that $H^n(X_\infty)_{\neq 1}$ is a direct factor of the primitive cohomology $P^n(X_\infty)$. This implies that for n even, the numbers μ_+ and μ_- for the restriction of S to $H^n_c(B_\infty)$ equal $\sum_{q \text{ even}} h^{pq}_{\neq 1}$ resp. $\sum_{q \text{ odd}} h^{pq}_{\neq 1}$, as follows from theorem (2.18).

Next consider the restriction of S to $H^n_c(B_\infty)_1$. Clearly Ker (j) is contained in its null-space. Denote $U = \mathrm{Im}\,({}^t k) \cap H^n(X_\infty)_1$ and $V = \mathrm{Ker}\,(k) \cap U$. Then $H^n_c(B_\infty)_1/\mathrm{Ker}\,(j) = U/V$. Moreover using cup product on $H^n(X_\infty)_1$ one has $V = U \cap \mathrm{Ker}\,(N)$ and $U = \mathrm{Im}\,(N)$ by the invariant cycle theorem and its dual version. Hence $V = U \cap U^\perp$ and cup product induces a non-degenerate bilinear form on U/V. This shows that the null-space of S coincides with Ker (j). We calculate μ_0 by computing the Hodge numbers of U/V. Clearly $N^r: Gr^W_{n+r}(U/V) \to Gr^W_{n-r}(U/V)$ is an isomorphism for all $r \geq 0$. Then $Gr^W_{n+r}(\mathrm{Ker}\,(N) \cap \mathrm{Im}\,(N)) = 0$ so $Gr^W_{n+r}(U/V) \cong Gr^W_{n+r}(\mathrm{Im}\,(N)_1) \cong Gr^W_{n+r+2} H^n(X_\infty)_1$. This gives $h^{pq}(U/V) = h^{p+1,q+1}_1$ for $p+q \geq n$ in view of lemma (4.2). One deduces from these the numbers $h^{pq}(U/V)$ for all p, q, using $h^{n-p,n-q}(U/V) = h^{pq}(U/V)$. Hence $\dim U/V = \sum_{p+q=n+2} h^{pq}_1 + 2\sum_{p+q \geq n+3} h^{pq}_1$. One obtains the formula for μ_0 by the relation $\dim \mathrm{Ker}\,(j) = \dim H^n_c(B_\infty)_1 - \dim\,(U/V)$. Suppose that n is even. Denote b^{pq} the Hodge numbers of $P^n(X_\infty)$ and a^{pq} those of $P^n(X_\infty) \cap H^n(Y_0)$. Then the Hodge numbers c^{pq} of $P^n(Y_t)$ $(t \neq 0)$ with its pure Hodge structure are given by $c^{pq} = 0$ for $p+q \neq n$ and $c^{p,n-p} = \sum_q b^{pq}$. Hence the invariants of the intersection form on $P^n(X_\infty)$ are $\mu_+(P^n(X_\infty)) = \sum_{q \text{ even}} b^{pq}$ and $\mu_-(P^n(X_\infty)) = \sum_{q \text{ odd}} b^{pq}$. For its subspace $A = P^n(X_\infty) \cap H^n(Y_0)$ these are $\mu_0(A) = \sum_{p+q<n} a^{pq}$, $\mu_+(A) = \sum_{q \text{ even}} a^{n-q,q}$ and $\mu_-(A) = \sum_{q \text{ odd}} a^{n-q,q}$. Further

$H^n_c(B_\infty)_1/\mathrm{Ker}\ (j)=(A+A^\perp)/A$ and $\dim\ (A+A^\perp)=\sum b^{pq}-\sum_{p+q<n} a^{pq}$. Consequently:

$$\mu_+=\sum_{q \text{ even}} b^{pq}-\sum_{q \text{ even}} a^{n-q,q}-\sum_{p+q<n} a^{pq};$$

$$\mu_-=\sum_{q \text{ odd}} b^{pq}-\sum_{q \text{ odd}} a^{n-q,q}-\sum_{p+q<n} a^{pq}.$$

Moreover:

$$\sum_{q \text{ even}} b^{pq}=\sum_{\substack{p+q=n\\ q \text{ even}}} b_1^{pq}+2\sum_{\substack{p+q=n+1\\ q \text{ even}}} h_1^{pq}+2\sum_{\substack{p+q=n+2\\ q \text{ even}}} h_1^{pq}+2\sum_{\substack{p+q\geqq n+3\\ q \text{ even}}} h_1^{pq}+\sum_{q \text{ even}} h_{\neq}^{pq}\ ;$$

$$\sum_{q \text{ even}} a^{n-q,q}=\sum_{\substack{p+q=n\\ q \text{ even}}} b_1^{pq}-\sum_{\substack{p+q=n+2\\ q \text{ odd}}} h_1^{pq} \text{ because}$$

$$a^{n-q,q}=b_1^{n-q,q}-b_1^{n-q+1,q+1} \quad \text{and in the same way:}$$

$$\sum_{p+q<n} a^{pq}=\sum_{p+q<n} b_1^{pq}-\sum_{p+q<n-2} b_1^{pq}=\sum_{p+q=n+1} h_1^{pq}+\sum_{p+q=n+2} h_1^{pq}.$$

The formula for μ_+ follows, for $\sum_{p+q=n+1} h_1^{pq}=2\sum_{\substack{p+q=n+1\\ q \text{ odd}}} h_1^{pq}$ because n is even and $h_1^{pq}=h_1^{qp}$ for all p, q. The formula for μ_- is proved analogously.

(4.12) Example. The singularities $x^p+y^q+z^r+\lambda xyz(\lambda\neq 0, (1/p)+(1/q)+(1/r)<1)$ are called the hyperbolic singularities. The intersection form on $H_c^2(B_\infty)$ has been calculated by Gabrielov [7]. One obtains $\mu_0=1$, $\mu_+=1$ and $\mu=p+q+r-1$. Hence the only non-zero Hodge numbers are $h_1^{2,2}=h_1^{1,1}=1$ and $h_{\neq 1}^{1,1}=p+q+r-3$. This implies that the minimal polynomial of T_0 has the factor $(t-1)^2$ which implies that the monodromy has infinite order.

(4.13) Example. For the singularity $x^8+y^8+z^8+(xyz)^2$ the Hodge numbers of $H^2(B_\infty)_1$ and $H^2(B_\infty)_{\neq 1}$ are given by the matrices

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & 4 & 9\\ 0 & 9 & 4\end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 1 & 9 & 9\\ 9 & 133 & 9\\ 9 & 9 & 1\end{pmatrix}$$

Hence $\mu_0=22$, $\mu_+=42$, $\mu_-=151$ so the signature equals -109.

(4.14) Proposition. *If n is odd then $\mu-\mu_0$ is even;*

if $n\equiv 2 \bmod 4$ then $\mu-\mu_-$ is even;

if $n\equiv 0 \bmod 4$ then $\mu-\mu_+$ is even.

Proof. First remark that it follows from (4.3) and (4.9) that $h_{\neq 1}^{pq}=h_{\neq 1}^{qp}=h_{\neq 1}^{n-p,n-q}$ and $h_1^{pq}=h_1^{qp}=h_1^{n+1-p,n+1-q}$ for all $p, q\geqq 0$. This implies that for n odd we have

$$\mu-\mu_0=\sum_{p+q\neq n,n+1} h_1^{pq}+\sum h_{\neq 1}^{pq}$$

$$=2\Bigg(\sum_{p+q\geqq n+2} h_1^{pq}+\sum_{p+q\geqq n+1} h_{\neq 1}^{pq}+\sum_{\substack{p+q=n\\ p<q}} (h_{\neq 1}^{pq}-h_1^{pq})\Bigg).$$

The other statements are proved analogously. This proves the conjecture of V.I. Arnol'd [2] that $\mu_++\mu_0$ is even if $n=2$.

(4.15) Example. Suppose that $n=2$ and that S is negative semi-definite. Then $\mu_+=0$ so either $H^2(B_\infty)_1=0$ and S is negative definite (this gives the simple singularities A_k, D_k, E_6, E_7, E_8) in which case $H^2(B_\infty)$ is purely of type $(1, 1)$, or $H^2(B_\infty)$ is of mixed type $\{(1, 1), (1, 2), (2, 1)\}$ with $\mu_0=2h^{1,2}>0$ (this gives the simple elliptic or parabolic singularities $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$).

(4.16) Example. Suppose that $n=2$, the monodromy has finite order and $\mu_0=0$. This occurs in the case of Arnol'd's exceptional singularities. Then $h_1^{pq}=0$ for all p, q and $h^{pq}=0$ if $p+q\neq 2$. Then $\mu_+=2h^{2,0}$ and $\mu_-=h^{1,1}$.

(4.17) Local cohomology

The local cohomology groups of an isolated singular point x_0 of a projective variety Y carry a mixed Hodge structure ([5], Example (8.3.8)) namely

$$H^q_{\{x_0\}}(Y)=H^q(Y \bmod Y-\{x_0\}).$$

We want to compute these for a hypersurface singularity x_0 and relate them to the mixed Hodge structure on $H^n(B_\infty)$. Let Y be a projective variety with only one singular point x_0. Let $\rho:\tilde{Y}\to Y$ be a resolution of singularities for Y and let $\rho^{-1}(x_0)=C=C_1\cup\cdots\cup C_m$ be a union of smooth divisors C_i with normal crossings on $\tilde{Y}$.

The mixed Hodge structure on $H^*(Y-\{x_0\})$ is computed using the logarithmic De Rham complex $\Omega^{\cdot}_{\tilde{Y}}(\log C)$ and the restriction map $H^*(Y)\to H^*(Y-\{x_0\})$ is induced by the inclusion

$$\Omega^{\cdot}_{\tilde{Y}}=W_0\Omega^{\cdot}_{\tilde{Y}}(\log C)\subset\Omega^{\cdot}_{\tilde{Y}}(\log C).$$

The mixed Hodge structure on $H^*(Y)$ has been computed in (3.4) and the pull-back map $H^*(Y)\to H^*(\tilde{Y})$ is induced by the surjective map $A^\cdot(Y)_{\mathbb{C}}\to A^\cdot(Y)_{\mathbb{C}}/W_{-1}A^\cdot(Y)_{\mathbb{C}}=\Omega^\cdot_{\tilde{Y}}$. Hence we have a morphism ϕ of cohomological mixed Hodge complexes which on $\mathbb{C}$-level is the composed map $A^\cdot(Y)_{\mathbb{C}}\to \Omega^\cdot_{\tilde{Y}}\to\Omega^\cdot_{\tilde{Y}}(\log C)$.

The mixed Hodge structure on $H^*_{\{x_0\}}(Y)$ is obtained by taking the hypercohomology of the mapping cone of ϕ. One has the exact sequence of cohomological mixed Hodge complexes, which on $\mathbb{C}$-level is

$$0\to\Omega^\cdot_{\tilde{Y}}(\log C)[-1]\to A^\cdot_{\{x_0\}}(Y)_{\mathbb{C}}\to A^\cdot(Y)_{\mathbb{C}}\to 0.$$

It gives the exact sequence of mixed Hodge structures

$$\cdots\to H^k_{\{x_0\}}(Y)\to H^k(Y)\to H^k(Y-\{x_0\})\to H^{k+1}_{\{x_0\}}(Y)\to\cdots$$

dual to the classical sequence of relative homology. Let's go back to the case where Y is the singular fiber Y_0 of a one-parameter family as before. One has the isomorphism

$$H^k_{\{x_0\}}(Y_0)\to H^{k-1}(K)$$

where $K=Y_0\cap\partial B$ is the common boundary of all fibers F of the Milnor fibration. The well-known exact sequence

$$0\to H_n(K)\to H_n(\bar{F})\to H_n(\bar{F},K)\to H_{n-1}(K)\to 0$$

gives an exact sequence of mixed Hodge structures

$$0\longrightarrow H^n_{\{x_0\}}(Y_0)\longrightarrow H^n_c(B_\infty)\xrightarrow{\ j\ } H^n(B_\infty)\longrightarrow H^{n+1}_{\{x_0\}}(Y_0)\longrightarrow 0$$

and $H^k_{\{x_0\}}(Y_0)=0$ for $k\neq n, n+1, 2n$. The mixed Hodge structures on $H^n_{\{x_0\}}(Y_0)$ and $\operatorname{Hom}(H^{n+1}_{\{x_0\}}(Y_0),\mathbb{Q}(-n))$ are isomorphic.

(4.18) LEMMA. $Gr^W_n H^n(Y_0)\cong Gr^W_n H^n(Y_0-\{x_0\})$.

PROOF. Cup product on $H^n(X_\infty)$ induces a non-degenerate bilinear form on $Gr^W_n H^n(Y_0)=H^n(Y_0)/H^n(Y_0)\cap H^n(Y_0)^\perp$. This implies that $Gr^W_n H^n(Y_0)$ is isomorphic to its dual $Gr^W_n H^n(Y_0-\{x_0\})$.

(4.19) EXAMPLE. Let $P(x,y,z)=x^4+y^4+z^4+xyz$. Once blowing up $0\in\mathbb{C}^3$ resolves the singularity of Y_0 (not the one of P). One obtains $\rho^{-1}(0)=C_1\cup C_2\cup C_3$, a union of 3 lines in $\mathbb{P}^2$ in general position. Here $H^2_{\{x_0\}}(Y_0)=\mathbb{Q}$, $H^3_{\{x_0\}}(Y_0)=\mathbb{Q}(-2)$ hence $H^1(Y_0-\{x_0\})=0$.

5. Open problems and conjectures

(5.1) '*Thom-Sebastiani*'. Let $f\in\mathbb{C}[z_0,\cdots,z_n]$ and $g\in\mathbb{C}[z_{n+1},\cdots,z_{n+m+1}]$ have an isolated singularity at the origin. Then the polynomial $f+g\in\mathbb{C}[z_0,\cdots,z_{n+m+1}]$ also has an isolated singularity and, denoting $B_{\infty,f}$ and $B_{\infty,g}$ the respective Milnor fibers and T_f, T_g the respective monodromy transformations, on has [17]:

$$H^{n+m+1}(B_{\infty,f+g})\cong H^n(B_{\infty,f})\otimes H^m(B_{\infty,g});$$
$$T_{f+g}=T_f\otimes T_g.$$

(5.2) PROBLEM. What is the relation between the mixed Hodge structures on $H^n(B_{\infty,f})$, $H^m(B_{\infty,g})$ and $H^{n+m+1}(B_{\infty,f+g})$?

(5.3) The quasi-homogeneous case suggests the following possibility. To the mixed Hodge structure on $H^n(B_{\infty,f})$ together with its automorphism of finite order γ_s we associate a sequence $(u_1,w_1),\cdots,(u_\mu,w_\mu)$ of pairs of rational numbers as follows: for $\lambda\in\mathbb{C}^*$ and $q\in\mathbb{Z}$ let $\ell_q(\lambda)\in\mathbb{C}$ be determined by $\exp 2\pi i\ell_q(\lambda)=\lambda$ and $q\leqq\operatorname{Re}\ell_q(\lambda)<q+1$; to any eigenvalue λ of γ_s acting on $H^{p,q}(B_\infty)$ associate the pair $(\ell_q(\lambda),p+q)$ if $\lambda\neq 1$ and $(q,p+q-1)$ if $\lambda=1$. Doing this with all eigenvalues of γ_s on all $H^{p,q}(B_\infty)$, one obtains an unordered μ-tuple (u_i,w_i), $i=1,\cdots,\mu$, which we shall call the *characteristic pairs* of $H^n(B_\infty)$.

Conversely one recovers the discrete invariants h_1^{pq}, $h_{\neq 1}^{pq}$ and the eigenvalues of γ_s on each $H^{pq}(B_\infty)$ from the characteristic pairs in the obvious way.

(5.4) CONJECTURE. Let f, g have isolated singularities at 0. Let (u_i,w_i), $i=1,\cdots,\mu_f$ and (u'_j,w'_j), $j=1,\cdots,\mu_g$ be the characteristic pairs of $H^n(B_{\infty,f})$ resp. $H^m(B_{\infty,g})$. Then the characteristic pairs for

$$H^{n+m+1}(B_{\infty,f+g})\quad\text{are}\quad (u_i+u'_j,\ w_i+w'_j+1),\ i=1,\cdots,\mu_f,\ j=1,\cdots,\mu_g.$$

This conjecture is true if f and g (and hence $f+g$) are quasi-homogeneous, as follows from the explicit calculations in [20].

(5.5) EXAMPLE. Let $f(x,y)=x^5+x^2y^2+y^4$, $g(z)=z^2$. The characteristic pairs for f are:

$$(\tfrac12,0),(\tfrac{7}{10},1),(\tfrac{9}{10},1),(\tfrac34,1),(1,1),(1,1),(\tfrac54,1),(\tfrac{11}{10},1),(\tfrac{13}{10},1),(\tfrac32,2)$$

as one computes by resolving f and applying (3.14). The characteristic pair of g is $(\frac12,0)$. According to the conjecture the characteristic pairs for $f+g$ would be

$$(1,1),(\tfrac65,2),(\tfrac75,2),(\tfrac54,2),(\tfrac32,2),(\tfrac32,2),(\tfrac74,2),(\tfrac85,2),(\tfrac95,2),(2,3)$$

hence $H^2(B_{\infty,f+g})$ has $h_{\neq 1}^{pq}=0$ unless $(p,q)=(1,1)$, $h_{\neq 1}^{11}=8$, $h_1^{1,1}=h_1^{2,2}=1$. This is correct because $f+g$ is equivalent to a hyperbolic germ $x^4+y^5+z^2+\lambda xyz$ (see example (4.12)).

More evidence that the conjecture should be true is obtained from the conjectural method of computing the characteristic pairs from the Newton diagram. See (5.8).

(5.6) Mixed Hodge structure and Newton diagram

We refer to [21] for the notions of Newton diagrams and non-degenerate principal parts of polynomials. Let $f \in \mathbb{C}[z_0, \cdots, z_n]$ be a polynomial with a non-degenerate principle part. Denote A the graded ring associated to the Newton filtration on $\mathbb{C}[z_0, \cdots, z_n]$. Denote $f_0, F_0, \cdots, F_n$ the principle parts of $f, z_0\partial f/\partial z_0, \cdots, z_n\partial f/\partial z_n$. The degree on A takes rational values and is normalized in such a way that

$$\deg(f_0) = \deg(F_0) = \cdots = \deg(F_n) = 1.$$

We use the method of Poincaré series to deduce the characteristic pairs of $H^n(B_{\infty,f})$ from the Newton diagram Γ of f. Write $\tau \leqq \sigma$ for 'τ is a face of σ'.

For σ a face of Γ denote A_σ the corresponding graded subring of A. We consider $\{0\}$ as a common face of every face of Γ and put $A_{\{0\}} = \mathbb{C}$, with degree 0. The Poincaré series of any A_σ is defined by

$$p_{A_\sigma}(t) = \sum_{\alpha \in \mathbb{Q}} \dim(A_\sigma)_\alpha \cdot t^\alpha.$$

It follows from [21], p. 15 that

$$p_{A_\sigma/(F_{0,\sigma}, \cdots, F_{n,\sigma})}(t) = (1-t)^{d(\sigma)} p_{A_\sigma}(t)$$

where $d(\sigma) = \dim A_\sigma = \dim \text{Cone}(\sigma)$.

Let $q_\sigma(t)$ be the Poincaré polynomial for the 'interior' of $A_\sigma/(F_{0,\sigma}, \cdots, F_{n,\sigma})$ i.e.

$$q_\sigma(t) = \sum_{\tau \leqq \sigma} (-1)^{d(\sigma)-d(\tau)} (1-t)^{d(\tau)} p_{A_\tau}(t).$$

Then clearly

$$(1-t)^{d(\sigma)} p_{A_\sigma}(t) = \sum_{\tau \leqq \sigma} q_\sigma(t).$$

For σ a face of Γ, define $k(\sigma) = \min\{k \in \mathbb{Z} \mid \exists i_1, \cdots, i_k \in \{0, \cdots, n\}$ such that $\sigma \subset \mathbb{R} e_{i_1} + \cdots + \mathbb{R} e_{i_k}\}$. Then [21], Prop. (2.6) implies:

$$p_A(t) = \sum_{\sigma : k(\sigma) = n+1} (-1)^{n+1-d(\sigma)} p_{A_\sigma}(t).$$

In fact we are interested in the Poincaré polynomial of

$$H = (z_0 \cdots z_n)A/(F_0, \cdots, F_n)A \cong Gr\mathbb{C}[[z_0, \cdots, z_n]]/(\partial f/\partial z_0, \cdots, \partial f/\partial z_n)$$

where $\mathbb{C}[[z_0, \cdots, z_n]]$ has been filtered such that the monomial $z_0^{a_0} \cdots z_n^{a_n}$ gets the same degree as its image $z_0^{a_0+1} \cdots z_n^{a_n+1}$ in A. This can be computed by calculating the Poincaré polynomials of $A/(F_0, \cdots, F_n)$ and of all its quotients corresponding to intersections of coordinate hyperplanes in $\mathbb{R}_+^{n+1}$.

By [21], Theorem 2.8 one obtains

$$P_{A/(F_0, \cdots, F_n)}(t) = (1-t)^{n+1} p_A(t)$$

hence

$$p_H(t) = \sum_\sigma (-1)^{n+1-d(\sigma)} \cdot (1-t)^{k(\sigma)} p_{A_\sigma}(t)$$

so we have proved:

(5.7) Theorem. $p_H(t) = \sum_{\tau \leqq \sigma} (-1)^{n+1-d(\sigma)} (1-t)^{k(\sigma)-d(\sigma)} q_\sigma(t)$.

Let G be the free abelian group on pairs (u, v) with $u, v \in \mathbb{Q}$ as generators. The characteristic pairs (u_i, w_i) of f correspond to the element $cp(f) = \sum_{i=1}^{\mu} (u_i, w_i)$ of G.

(5.8) Conjecture. For any simplex σ write $q_\sigma(t) = \sum_{\alpha \in \mathbb{Q}} \varepsilon_{\sigma,\alpha} t^\alpha$ (because f_0 is non-degenerate this is a finite sum); then the characteristic pairs of $H^n(B_{\infty,f})$ are represented by the element $cp(f)$ of G where

$$cp(f) = \sum_{\substack{\tau \leqq \sigma \\ \alpha \in \mathbb{Q}}} \Sigma_{j=0}^{k(\tau)-d(\tau)} (-1)^{n+1-d(\tau)+j} \binom{k(\tau)-d(\tau)}{j} \varepsilon_{\sigma,\alpha} \cdot (\alpha + j, 2k(\sigma) - d(\sigma) - 2j - 1).$$

(5.9) Example. Let $f(x, y) = x^5 + x^2y^2 + y^4$. Then the faces of Γ are $\sigma_1 = \{0\}$, $\sigma_2 = \langle(5, 0)\rangle$, $\sigma_3 = \langle(0, 4)\rangle$, $\sigma_4 = \langle(2, 2)\rangle$, $\sigma_5 = \langle(5, 0), (2, 2)\rangle$ and $\sigma_6 = \langle(2, 2), (0, 4)\rangle$.

Thus

$$q_{\sigma_1} = 1;$$
$$q_{\sigma_2} = t^{\frac{1}{5}} + t^{\frac{2}{5}} + t^{\frac{3}{5}} + t^{\frac{4}{5}};$$
$$q_{\sigma_3} = t^{\frac{1}{4}} + t^{\frac{1}{2}} + t^{\frac{3}{4}};$$
$$q_{\sigma_4} = t^{\frac{1}{2}};$$
$$q_{\sigma_5} = t^{\frac{7}{10}} + t^{\frac{9}{10}} + t^{\frac{11}{10}} + t^{\frac{13}{10}};$$
$$q_{\sigma_6} = t^{\frac{3}{4}} + t + t^{\frac{5}{4}}.$$

Moreover $p_H(t) = t \cdot q_1 + (1+t)q_{\sigma_4} + q_{\sigma_5} + q_{\sigma_6}$, so applying the conjecture one obtains precisely the pairs as listed in example (5.5).

(5.10) Remark. If we know a priori that the monodromy has finite order, the characteristic pairs are all of the form (u, n); hence if

$$p_H(t) = \sum_{\alpha \in \mathbb{Q}} c_\alpha t^\alpha, \quad \text{then} \quad cp(f) = \Sigma_\alpha c_\alpha (\alpha, n).$$

In general we first have to split up $p_H(t)$ as in (5.7) in order to determine the weights.

(5.11) EXAMPLE. If f is a quasi-homogeneous germ with weights $w_0, \cdots, w_n$, and isolated singularity at 0, i.e. $f = \Sigma_\alpha a_\alpha z^\alpha$ and $a_\alpha \neq 0 \Rightarrow \Sigma_{i=0}^n w_i \alpha_i = 1$, then the above computation shows that

$$p_H(t) = \prod_{i=0}^{n} \frac{t^{w_i} - t}{1 - t^{w_i}}.$$

In particular the Hodge numbers only depend on the weights.

(5.12) EXAMPLE. $f(x, y) = x^2 y + y^4$; we have $w_0 = \frac{3}{8}$, $w_1 = \frac{1}{4}$,

$$p_H(t) = (t^{\frac{3}{8}} - t)(t^{\frac{1}{4}} - t)(1 - t^{\frac{3}{8}})^{-1}(1 - t^{\frac{1}{4}})^{-1}$$
$$= t^{\frac{5}{8}} + t^{\frac{7}{8}} + t + t^{\frac{9}{8}} + t^{\frac{11}{8}}.$$

(5.13) REMARK. Assuming (5.8), one can use (5.7) and (5.8) to show that conjecture (5.4) is true if f and g both have non-degenerate principle parts.

(5.14) We shall sketch a proof that conjecture (5.8) is true in the case $n = 1$.

Let $f \in \mathbb{C}[z_0, z_1]$ be a non-degenerate function. Let $(a_0, b_0), \cdots, (a_k, b_k)$ be the vertices of its Newton diagram Γ, ordered in such a way that $a_i b_{i+1} - a_{i+1} b_i > 0$ for $i = 0, \cdots, k-1$. Moreover we suppose that $b_0 = a_k = 0$.

One obtains a resolution of f as follows. Let Γ^* be the dual diagram of Γ and let Σ be a subdivision of Γ^* such that the corresponding torus-embedding X_Σ is smooth. Let $(0, 1) = (\alpha_0, \beta_0), (\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r) = (1, 0)$ be the vertices of Σ, ordered such that $\alpha_i \beta_{i+1} - \alpha_{i+1} \beta_i = -1$ for $i = 0, 1, \cdots, r$. There exists a unique increasing function $p: \{1, \cdots, k\} \to \{1, \cdots, r-1\}$ with the property that

$$\alpha_{p(i)}(a_i - a_{i+1}) + \beta_{p(i)}(b_i - b_{i+1}) = 0.$$

One can show the following facts:

(i) $E = E_0 \cup E_1 \cup \cdots \cup E_{r-1}$ where E_i is the canonical divisor of X_Σ corresponding to (α_i, β_i) for $i = 1, \cdots, r-1$ and E_0 is the union of all components of E with multiplicity 1;

(ii) D_i is a curve of genus $> 0 \Leftrightarrow i = p(j)$ for some $j \in \{1, \cdots, k\}$; in that case the multiplicity e_i of E_i equals $\alpha_i a_j + \beta_i b_j$;

(iii) Moreover E_i intersects the curves E_{i-1} of multiplicity

$$\alpha_{i-1} a_{j+1} + \beta_{i-1} b_{j+1} = e_{i-1}, \quad E_{i+1} \quad \text{with} \quad e_{i+1} = \alpha_{i+1} a_j + \beta_{i+1} b_j$$

and precisely $m_i = (a_j b_{j+1} - a_{j+1} b_j)/e_i$ components of E_0;

(iv) The eigenvalues of γ_s on $H^{0,1}(B_\infty) = \bigoplus_{j=1}^k H^1(D_{p(j)}, \mathcal{O}_{D_{p(j)}})$ are computed using (3.14).

One obtains if $i = p(j)$:

$$\pi_* \mathcal{O}_{D_i} = \bigoplus_{s=0}^{e_i - 1} \mathcal{O}_{E_i}\left(-\frac{se_{i+1}}{e_i} - \frac{se_{i-1}}{e_i} + \left[\frac{se_{i+1}}{e_i}\right] + \left[\frac{se_{i-1}}{e_i}\right] - \frac{m_i s}{e_i}\right) = \bigoplus_{s=0}^{e_i - 1} \mathcal{O}_{E_i}(n_{s,i})$$

where

$$n_{s,i} = \left[\frac{se_{i+1}}{e_i}\right] + \left[\frac{se_{i-1}}{e_i}\right] - \frac{s(m_i + e_{i-1} + e_{i+1})}{e_i} \leqq 0.$$

Hence the divisor of characteristic pairs for $H^{0,1}(B_\infty)$ is

$$\sum_{j=1}^{k} \sum_{s=1}^{e_{p(j)}-1} (-1 - n_{s,p(j)})(s/e_{p(j)}, 1).$$

The divisor associated to $H^{1,0}(B_\infty)$ is obtained by duality:

$$\sum_{j=1}^{k} \sum_{s=1}^{e_{p(j)}-1} (-1 - n_{s,p(j)})(2 - s/e_{p(j)}, 1).$$

The remaining eigenvalues of γ_s on $H^1(B_\infty)$ either are equal to 1, giving the pair (1, 1), or different from 1; in that case they occur in pairs, corresponding to $H^{0,0}(B_\infty)_{\neq 1} \cong H^{1,1}(B_\infty)_{\neq 1}$.

Thus the computation above gives complete information on the characteristic pairs.

Remains to prove that if σ is the simplex spanned by (a_j, b_j) and (a_{j+1}, b_{j+1}) in Γ, then for $i = p(j)$:

$$q_\sigma(t) = \sum_{s=1}^{e_i - 1} n_{s,i}(t^{s/e_i} + t^{2 - s/e_i}) + m_i t.$$

This is left to the reader as an exercise.

Remark that in this case the monodromy has finite order if and only if $gcd(a_j, b_j) = 1$ for $j = 1, \cdots, k-1$.

(5.15) Behavior under deformations

One may ask for a relation between the mixed Hodge structures on $H^n(B_{\infty,f})$ and $H^n(B_{\infty,g})$ if g is a germ in the universal unfolding of f. One possible question is how the discrete invariants of the mixed Hodge structures are related.

Suppose that the numbers h^{pq} remain constant under some deformation of f. Then the deformation gives rise to a 'variation of mixed Hodge structure' on the parameter space. This occurs for example when the deformation is equisingular in the following sense: there exists a simultaneous resolution of all functions in the deformation such that all components of the exceptional divisor and all their intersections are smooth over the parameter space. In that case one should compare the moduli of the singularity with the moduli of the triple consisting of $H^n(B_{\infty,f})$ with its mixed Hodge structure (or its associated graded Hodge structure) and its endomorphisms N and γ_s. In the quasi-homogeneous case one may consider the residue of the Gauss-Manin connection instead of γ_s.

(5.16) EXAMPLE. Let $P(x, y) = x^2 - y^3$. Because an elliptic curve with an automorphism of order 6 is rigid, the Hodge structure on $H^1(B_\infty)$ has no moduli in the sense of (5.15). In the same way the polynomial $x^3 + y^3$ gives rise to the Hodge structure of weight 1 with an automorphism of order 4, hence it is also rigid.

(5.17) EXAMPLE. Let $P(x, y, z) = x^3 + y^3 + z^3 + \lambda xyz$. The moduli of P are reflected by the Hodge structure $Gr_3^W H^2(B_{\infty,P}) = H^1(E_P)(-1)$ with E_P the curve in $\mathbb{P}^2$ with homogeneous equation $P = 0$.

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