APPENDIX Q

On the group completion of a simplicial monoid

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To any small category $\mathcal{A}$ endowed with a coherent unitary associative and commutative operation there is associated a generalized cohomology theory $K^*(?;\mathcal{A})$, the K-theory with coefficients in $\mathcal{A}$ (see Segal [**8**], Anderson [**1**]). In the case of the category $\mathcal{P}_R$ of finitely generated projective R-modules and their isomorphisms endowed with the direct sum operation, the groups $K^{-n}(pt;\mathcal{P}_R)$ are isomorphic to the algebraic K-groups K_nR for $n = 0, 1, 2$ of Bass and Milnor, and to their generalizations which I have been able to compute for a finite field [**6**]. In order to establish these isomorphisms and thereby compute the generalized K-groups in many interesting cases, one uses a theorem calculating the homology of the loop space of the classifying space of a topological monoid. When translated into the (semi-)simplicial framework for homotopy theory, this amounts to a theorem describing the behavior of the homology of a simplicial monoid under the process of group completion. It is the purpose of this paper to present such a theorem; a closely related result has been proved by Barratt and Priddy [**2**].

To state the theorem, let M be a simplicial monoid and $\overline{M}$ its group completion. The canonical map from M to $\overline{M}$ induces a ring homomorphism

$$H_*(M,k) \to H_*(\overline{M},k)$$

for any commutative ring k. Elements of $\pi_0 M$ are carried into invertible elements by this homomorphism, hence it is natural to ask if the homology of $\overline{M}$ is obtained by localizing the homology of M with respect to the multiplicative system $\pi_0 M$.

Theorem. *Assume i) $\pi_0 M$ is contained in the center of $H_*(M,k)$ and ii) M_n*

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is a free monoid for all n. Then the homology of $\overline{M}$ is obtained by localization:

$$H_*(M,k)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\overline{M},k).$$

Consequently if M_s, $s \in \pi_0 M$, are the connected components of M and $\overline{M}_e$ is the identity component of $\overline{M}$, then

$$\lim_{s}\text{ind.}\, H_*(M_s,k) \xrightarrow{\sim} H_*(\overline{M}_e,k).$$

This is proved by applying the Zeeman comparison theorem to the Eilenberg-Moore spectral sequences of M and $\overline{M}$. Actually the argument shows a bit more, and in Section 6 one will find a slightly more general version of the theorem in which hypothesis i) is replaced by the condition that the localization of $H_*(M,k)$ with respect to $\pi_0 M$ admit calculation by right fractions, while ii) becomes the requirement that M be good for group completion. These notions are discussed in the first and fifth sections respectively.

The applications of the theorem to K-theory, in particular the isomorphism of $K^{-n}(pt;\mathcal{P}_R)$ and $K_n R$, are given in Section 7. Anderson's version of K-theory has been used because it goes with the methods of this paper, but there is no difficulty in working in the more flexible and more geometric framework provided by Segal's special simplicial spaces, once the group completion theorem is formulated in this framework as I have indicated in Section 9.

In Section 8 I show how a theorem of John Mather about the group of diffeomorphisms with compact support of the line can be obtained from the results of this paper.

Q.1. Rings of fractions

Let R be a ring (always supposed with identity, but not necessarily commutative) and let S be a multiplicative system in R, i.e. a subset closed under multiplication containing the identity. Denote by $\gamma : R \to R[S^{-1}]$ the localization of R with respect to S; γ is a universal ring homomorphism carrying elements of S into invertible elements. If S lies in the center of R, one knows that $R[S^{-1}]$ is a ring of fractions of the form rs^{-1}. The same holds trivially if all the elements of S are already invertible in R. It is the purpose of this section to develop a common generalization of these two cases.

Let $\mathcal{C}$ be the category whose objects are the elements of S, and in which a morphism from s_1 to s_2 is an element t of S such that $s_1 t = s_2$. We assume

i) *$\mathcal{C}$ is a filtering category,* which means that the following conditions hold:

i′) For every s_1, $s_2 \in S$ there exists t_1, $t_2 \in S$ with $s_1 t_1 = s_2 t_2$.

i″) Given s, s_1, $s_2 \in S$ such that $ss_1 = ss_2$, there exists $t \in S$ with $s_1 t = s_2 t$.

Set

$$RS^{-1} = \lim_{\mathcal{C}}\text{ind.}\, R$$

where the right side denotes the inductive limit of the functor from $\mathcal{C}$ to the category of R-modules sending each object to R and sending the arrow $t : s_1 \to s_2$ into right multiplication by t. Elements of RS^{-1} may be identified with fractions rs^{-1}, where $r_1 s_1^{-1} = r_2 s_2^{-1}$ if and only if there exist t_1, t_2 in S with $r_1 t_1 = r_2 t_2$, $s_1 t_1 = s_2 t_2$. There is an R-module homomorphism

$$u : RS^{-1} \to R[S^{-1}] \qquad rs^{-1} \mapsto \gamma(r)\gamma(s)^{-1}$$

and to make u an isomorphism we assume

ii) *For each $s \in S$, left multiplication by s on RS^{-1} is bijective.* This means the following hold:

ii′) Given $r \in R$ and $s \in S$ with $sr = 0$, there exists $t \in S$ with $rt = 0$.

ii″) Given $r \in R$ and $s \in S$, there exist $r' \in R$ and $t \in S$ such that $rt = sr'$. This condition implies that the ring homomorphism from R to the ring endomorphisms of RS^{-1} as an abelian group given by left multiplication carries elements of S into invertible elements, hence it factors through γ, showing that RS^{-1} has a unique module structure over $R[S^{-1}]$ extending its R-module structure. Since every element of $R[S^{-1}]$ is a finite product of elements of the form $\gamma(r)$ or $\gamma(s)^{-1}$, it is clear that u is a homomorphism of $R[S^{-1}]$-modules. As RS^{-1} is generated by the fraction $1(1)^{-1}$ over $R[S^{-1}]$ and u carries this fraction to the identity of $R[S^{-1}]$, it follows that u is an isomorphism as claimed.

When R and S satisfy the conditions i) and ii), we shall say that the localization of R with respect to S *admits calculation by right fractions.*

Q.2. Grading of RS^{-1}

Suppose in addition to the assumptions of the preceding section that the ring R admits a grading

$$R = \bigoplus_{s \in S} R_s$$

with respect to S, i.e. $R_s R_{s'} \subset R_{ss'}$. Let $\bar{S}$ be the group completion of S, i.e. the target of a universal homomorphism from S to a group. In virtue of i), elements of $\bar{S}$ may be identified with fractions $s's^{-1}$. It is easy to see that RS^{-1} admits a grading

$$RS^{-1} = \bigoplus_{\bar{s} \in \bar{S}} \bar{R}_{\bar{s}}$$

where $\bar{R}_{\bar{s}}$ is the subgroup of fractions of the form rs^{-1} with $r \in R_t$ and $ts^{-1} = \bar{s}$. We claim this grading is compatible with the ring structure. Indeed, given $rs^{-1} \in \bar{R}_{\bar{s}}$ with $r \in R_t$ and $ts^{-1} = \bar{s}$, and similarly with primes, we have

$$(rs^{-1})(r's'^{-1}) = rr''(s's'')^{-1}$$

where r'' and s'' are chosen so that $r's'' = sr''$. Right multiplying s'' and r'' by the same element of S if necessary, we can suppose that $r'' \in R_{t''}$ with

$t's'' = st''$. Then the above product lies in $\bar{R}_{\bar{s}\bar{s}'}$, because $rr'' \in R_{tt''}$ and $ss' = (ts^{-1})(t's'^{-1}) = tt''(s's'')^{-1}$.

It follows from the fact that RS^{-1} is graded as a ring with respect to $\bar{S}$ that $\bar{R}_e$ (e = identity of $\bar{S}$) is a subring and that the group $\bar{S}$ acts on this subring by conjugation. Furthermore, if k is any commutative ring over which R is an algebra, there is an algebra isomorphism

$$k[\bar{S}] \,\tilde{\otimes}_k\, \bar{R}_e \xrightarrow{\sim} RS^{-1}$$

where on the left is the semi-tensor product of the group algebra $k[-\bar{S}]$ acting on $\bar{R}_e$ by the conjugation action.

It will be useful later to note here that elements of $\bar{R}_e$ may be identified with fractions of the form rs^{-1} with $r \in R_s$, or equivalently

$$\bar{R}_e \xrightarrow{\sim} \text{lim.ind.}\, R_s$$

where on the right is the inductive limit of the functor from $\mathcal{C}$ to abelian groups sending s to R_s and sending the arrow $t : s \to s'$ into right multiplication by t from R_s to $R_{s'}$.

Q.3. The Eilenberg-Moore spectral sequence

Let M be a simplicial monoid and let $\text{Nerv}(M) : (p,q) \mapsto (M_q)^p$ be the bisimplicial set which for fixed q is the nerve of the monoid M_q. Define the classifying "space" of M, denoted BM, to be the diagonal simplicial set of $\text{Nerv}(M)$. (One can use instead the simplicial set $\overline{W}M$, as there is a weak equivalence[2] $BM \to \overline{W}M$.) Let $S = \pi_0 M$ be the monoid of connected components of M. As $M_0 = BM_1$ and BM is reduced, there is a map $S \to \pi_1 BM$; one sees easily that this is a monoid homomorphism and that it allows one to identify the fundamental group with the group completion of S:

$$\bar{S} = \pi_1 BM.$$

If L is an $\bar{S}$-module, then one of the two spectral sequences associated to the bisimplicical abelian group $(p,q) \mapsto \mathbb{Z}[M_q^p] \otimes L$ of chains on $\text{Nerv}(M)$ with coefficients in L is of the form

$$E^1_{pq} = H_q(M^p, L) \implies H_{p+q}(BM, L).$$

Let k be a field and $R = H_*(M,k)$ the homology ring of M. If L is a $k[\bar{S}]$-module, the Kunneth formula

$$H_*(M^p, L) = R^{\otimes p} \otimes L$$

shows that the E^1 term is the bar resolution of k over R tensored L over R, hence we obtain the Eilenberg-Moore spectral sequence:

$$E^2_{pq} = \text{Tor}^R_p(k,L)_q \implies H_{p+q}(BM, L).$$

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[2] A weak equivalence is a map inducing isomorphisms on homotopy groups.

Suppose now that the localization $\bar{R} = R[S^{-1}]$ of R with respect to S admits calculation by right fractions (§1). Then $\bar{R}$ is a flat left R-module as it is a filtered inductive limit of free S-modules, hence if P is a resolution of k by projective right R-modules, $P \otimes_R \bar{R}$ is a resolution of $k \otimes_R \bar{R}$ by projective right $\bar{R}$-modules. Passing to homology on both sides of the canonical isomorphism

$$P \otimes_R L = (P \otimes_R \bar{R}) \otimes_{\bar{R}} L$$

yields an isomorphism

$$\mathrm{Tor}_p^R(k, L) \simeq \mathrm{Tor}_p^{\bar{R}}(k \otimes_R \bar{R}, L).$$

Note that $k \otimes_R \bar{R} \xrightarrow{\sim} k$ as the elements of S go into 1 under the augmentation map from R to k.

Since R is graded with respect to S:

$$R = \bigoplus_{s \in S} H_*(M_s)$$

where M_s denotes the connected component indexed by s, we have an isomorphism (§2)

$$\bar{R} \simeq k[\bar{S}] \,\tilde{\otimes}_k\, \bar{R}_e, \qquad \bar{R}_e = \lim_{\substack{\longrightarrow \\ s}}\mathrm{ind.}\, H_*(M_s).$$

Consequently R is a free right $\bar{R}_e$-module, hence arguing as before, there is a canonical isomorphism

$$\mathrm{Tor}_p^{\bar{R}_e}(k, k) \simeq \mathrm{Tor}_p^{\bar{R}}(k, k[\bar{S}])$$

since $\bar{R} \otimes_{\bar{R}_e} k \xrightarrow{\sim} k[\bar{S}]$.

Putting these two Tor isomorphisms together, we conclude that when the localization of R with respect to S admits calculation by right fractions, then the Eilenberg-Moore spectral sequence of M with $L = k[\bar{S}]$ takes the form

$$E^2_{pq} = \mathrm{Tor}_p^{\bar{R}_e}(k, k)_q \implies H_{p+q}(BM, k[\bar{S}]).$$

The abutment of the spectral sequence is isomorphic to the homology of the universal covering $\widetilde{BM}$ of BM

$$H_*(BM, k[\bar{S}]) \simeq H_*(\widetilde{BM}, k),$$

because the spectral sequence for the map $\widetilde{BM} \to BM$ is degenerate.

Q.4. A comparison lemma

Let $u : M \to M'$ be a homomorphism of simplicial monoids. We continue with the notations of the preceding section, denoting with primes the corresponding objects for M'.

LEMMA. Suppose the localizations of R with respect to S and R' with respect to S' both admit calculation by right fractions, and assume that u induces isomorphisms $\pi_1 BM \xrightarrow{\sim} \pi_1 BM'$, $H_*(\widetilde{BM}, k) \xrightarrow{\sim} H_*(\widetilde{BM}', k)$, (e.g. if $BM \to BM'$ is a weak equivalence). Then u induces an isomorphism

$$R[S^{-1}] \xrightarrow{\sim} R'[S'^{-1}].$$

PROOF. The hypotheses imply that $R[S^{-1}] = k[\bar{S}] \otimes_k \bar{R}_e$ (resp. with primes), and that $\bar{S} \xrightarrow{\sim} \bar{S}'$, hence it suffices to show that the homomorphism $\bar{R}_e \to \bar{R}'_e$ induced by u is an isomorphism. To save writing denote this homomorphism by $A \to A'$. We apply the Zeeman comparison theorem to the homomorphism of Eilenberg-Moore spectral sequences induced by u:

$$\begin{array}{ccc} E^2_{pq} = \mathrm{Tor}^A_p(k,k)_q & \Longrightarrow & H_{p+q}(\widetilde{BM}, k) \\ \downarrow & & \downarrow \simeq \\ {}'E^2_{pq} = \mathrm{Tor}^{A'}_p(k,k)_q & \Longrightarrow & H_{p+q}(\widetilde{BM'}, k) \end{array}$$

in the range $p \geq 1$, $q \geq 1$ (this includes everything but the trivial copy of k in degree 0,0). We argue by induction that $A_q \xrightarrow{\sim} A'_q$ for $q < n$, this being clear for $n = 1$ as both A and A' are connected. The induction hypothesis implies that $E^2_{pq} \xrightarrow{\sim} {}'E^2_{pq}$ for all p and all $q < n$. Since the map on abutments is an isomorphism, the comparison theorem tells us that $E^2_{1n} \xrightarrow{\sim} {}'E^2_{1n}$ and that E^2_{2n} maps onto ${}'E^2_{1n}$ (compare [5], 3.8). Computing $\mathrm{Tor}^A_*(k,k)$ using the bar resolution gives exact sequences

$$0 \to Z_n \to \bigoplus_{\substack{i+j=n \\ i,j>0}} A_i \otimes A_j \to A_n \to E^2_{1n} \to 0$$

$$\bigoplus_{\substack{i+j+k=n \\ i,j,k>0}} A_i \otimes A_j \otimes A_k \to Z_n \to E^2_{2n} \to 0$$

where Z_n is defined so the top row is exact. Applying the five lemma to the homomorphism of the lower exact sequence to the similar one with primes, we see that Z_n maps onto Z'_n; using this in the case of the upper exact sequence shows that $A_n \xrightarrow{\sim} A'_n$, establishing the induction hypothesis for $n+1$ and concluding the proof of the lemma. □

Q.5. Good simplicial monoids

Let M be a simplicial monoid and $\overline{M}$ its group completion, i.e. the simplicial group with $\overline{M}_n$ = the group completion of M_n for all n. We say that M is *good* (for group completion) if the canonical homomorphism $M \to \overline{M}$ induces a weak equivalence $BM \to B\overline{M}$. Since $\pi_1 BM \xrightarrow{\sim} \pi_1 B\overline{M} \xrightarrow{\sim} \pi_0 \overline{M}$, the Whitehead theorem tells us that M is good if and only if

$$H_*(BM, L) \xrightarrow{\sim} H_*(B\overline{M}, L)$$

for all $\pi_0\overline{M}$-modules L (it suffices in fact to have this only for $L = \mathbb{Z}[\pi_0\overline{M}]$). We say that a monoid S is good if the associated constant simplicial monoid is good. As there is a canonical isomorphism in this case

$$H_*(BS, L) = \operatorname{Tor}_*^{\mathbb{Z}[S]}(\mathbb{Z}, L),$$

one sees that S is good if and only if

$$\operatorname{Tor}_*^{\mathbb{Z}[S]}(\mathbb{Z}, L) \xrightarrow{\sim} \operatorname{Tor}_*^{\mathbb{Z}[\bar{S}]}(\mathbb{Z}, L)$$

for any $\bar{S}$-module L.

PROPOSITION Q.1. *If S is a free monoid, then S is good. If $S \to \bar{S}$ admits calculation by right fractions in the sense that condition i) of §1 holds, then S is good.*

The first assertion is proved by using the standard resolutions for computing the homology of free monoids and free groups ([**3**], X, §5). In the second situation $\mathbb{Z}[\bar{S}] = \mathbb{Z}[S]S^{-1}$ is a flat $\mathbb{Z}[S]$-module, so the map of Tor's is an isomorphism as in section 3.

PROPOSITION Q.2. *If M_n is good for all n, then M is good.*

In effect there is a spectral sequence

$$E^1_{pq} = \operatorname{Tor}_q^{\mathbb{Z}[M_p]}(\mathbb{Z}, L) \implies H_{p+q}(BM, L)$$

which is the other spectral sequence associated to the bisimplicial abelian group of chains on $\operatorname{Nerv}(M)$ with coefficients in L (compare §3). Considering the morphism of such spectral sequences induced by the map $M \to \overline{M}$, the proposition follows.

Although not necessary for the sequel, it is perhaps worthwhile to give the relation between the notion of goodness and the left derived functor of the group completion functor from simplicial monoids to simplicial groups. The derived functor is defined by the formula

$$\overline{M}^{\mathbf{L}} = \bar{P}$$

where P is a free simplicial monoid resolution of M, that is, a free simplicial monoid endowed with a map $P \to M$, which as a map of simplicial sets is a fibration with contractible fibres. One knows (e.g. [9], §4) that up to homotopy such a free resolution is unique and that it depends functorially on M, hence $M \mapsto \overline{M}^{\mathbf{L}}$ is a functor from simplicial monoids to the homotopy category of simplicial groups. There is a canonical natural transformation $\overline{M}^{\mathbf{L}} \to \overline{M}$ represented by the completion $\bar{P} \to \overline{M}$ of the given map $P \to M$.

PROPOSITION Q.3. *A simplicial monoid M is good if and only if the canonical map* $\overline{M}^{\mathbf{L}} \to \overline{M}$ *is a weak equivalence.*

PROOF. We consider the square of classifying spaces

$$\begin{array}{ccc} BP & \xrightarrow{v} & BM \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B\bar{P} & \xrightarrow{\bar{v}} & B\overline{M} \end{array}$$

where v is induced by the resolution map $P \to M$, $\bar{v}$ by its completion, and the vertical maps by the canonical homomorphism to the completion. As P is free, it is good by propositions 1 and 2, hence u' is a weak equivalence. We show v is a weak equivalence using the Whitehead theorem. First of all, $\pi_0 P \xrightarrow{\sim} \pi_0 M$ and $\pi_1 BP$ is the group completion of $\pi_0 P$ (resp. for M), so v induces an isomorphism of fundamental groups. On the other hand, the map from P to M is a homotopy of equivalence of the underlying simplicial sets, hence induces isomorphisms of the homology of M^p and P^p with arbitrary coefficients; using the spectral sequence

$$E^1_{pq} = H_q(M^p, L) \implies H_{p+q}(BM, L)$$

and the similar one for P, one sees that v induces isomorphisms on homology with coefficients in any $\pi_1 BM$-module L. Therefore u' and v are weak equivalences, hence u is a weak equivalence if and only if $\bar{v}$ is, proving the proposition. □

Remark: Instead of group completion one can consider the functor $M \mapsto G\overline{W}(M)$ from simplicial monoids to simplicial groups, where $\overline{W}$ is the classifying "space" functor of MacLane and G is the loop "space" functor of Kan [4]. there is a natural transformation

$$G\overline{W}(M) \to \overline{M}$$

obtained by composing the map $G\overline{W}(M) \to G\overline{W}(\overline{M})$ induced by canonical homomorphism $M \to \overline{M}$ with the adjunction morphism $G\overline{W}(\overline{M})$. When M is good, the former map is a weak equivalence, because $\overline{W}(M) \to \overline{W}(\overline{M})$ is a weak equivalence by definition, and because G preserves weak equivalences. As the adjunction morphism is always a weak equivalence, it follows that the above natural transformation is a weak equivalence for good M. Therefore the group completion functor for good simplicial monoids corresponds geometrically to the functor sending a topological monoid into the loop space of its classifying space.

Q.6. Homology of the group completion

THEOREM Q.4. *Let M be a simplicial monoid, $\overline{M}$ its group completion, and let k be a commutative ring. Assume:*

i) The localization of $H_*(M,k)$ with respect to $\pi_0 M$ admits calculation by right fractions (e.g. if $\pi_0 M$ is in the center of the ring $H_*(M,k)$.)

ii) M is good for group completion (e,g, if M_n is free for all n).

Then the canonical map $M \to \overline{M}$ induces an isomorphism

$$H_*(M,k)[\pi_0 M^{-1}] \xrightarrow{\sim} H_*(\overline{M},k).$$

Moreover if M_s, $s \in \pi_0 M$, are the connected components of M and $\overline{M}_e$ is the identity component of $\bar{M}$, then

$$\lim_{\substack{s}}\text{ind.}\, H_*(M_s,k) \xrightarrow{\sim} H_*(\overline{M}_e,k),$$

where the inductive limit is taken with respect to the right multiplication action of S (end of §2).

PROOF. When k is a field this follows by applying the comparison lemma to the map $M \to \overline{M}$. The general case may then be deduced by dévissage as follows. Hypothesis i) implies that $S = \pi_0 M$ satisfies condition i) of §1, whence the limit with respect to the right S-action

$$F_q(A) = \lim_{\substack{s}}\text{ind.}\, H_q(M,A)$$

is a homological functor of the abelian group A. Hypothesis i) also implies that the left multiplication operator L_s acts bijectively on $F_*(k)$. One shows by induction that L_s is bijective on $F_*(A)$ for all k-modules A; assuming this to be so in degrees $< n$ and writing $A = P/K$ with P free, it follows by applying the five lemma to

$$F_n(K) \longrightarrow F_n(P) \longrightarrow F_n(A) \longrightarrow F_{n-1}(K) \longrightarrow F_{n-1}(P)$$

that L_s is surjective on $F_n(A)$ first of all, then that it is bijective. Thus if k' is any field such that there exists a homomorphism from k to k', one has that hypothesis i) holds for $H_*(M,k')$, hence applying the theorem for field coefficients, one finds that the canonical map

$$(*) \qquad F_*(A) \longrightarrow H_*(\overline{M},A)$$

is an isomorphism for $A = k'$.

This means $(*)$ is an isomorphism for $A = k \otimes_{\mathbb{Z}} \mathbf{Q}$, for if this is non-zero, then $(*)$ is an isomorphism for some field of characteristic zero, hence also for $\mathbf{Q}$, and hence for any $\mathbf{Q}$-module. Similarly it is an isomorphism for k/pk, p prime, and arguing again by induction on the degree, one sees it is an isomorphism for all k/pk-modules. By induction on n, $(*)$ is an isomorphism for all $k/p^n k$-modules, hence by passage to the limit, for all k-modules which are torsion abelian groups. Thus $(*)$ is an isomorphism for three of the terms in the exact sequence

$$0 \longrightarrow T \longrightarrow k \longrightarrow k \otimes_{\mathbb{Z}} \mathbf{Q} \longrightarrow T' \longrightarrow 0$$

so it is an isomorphism for $A = k$, proving the theorem. □

Q.7. Applications to K-theory

Let $\mathcal{A}$ be a small category endowed with a coherent unitary associative and commutative operation. In [**3**] Graeme Segal showed how to construct an Ω-spectrum of iterated classifying spaces for $\mathcal{A}$, thereby defining a generalized cohomology theory which will be denoted $K^*(?;\mathcal{A})$ and called the K-theory with coefficients in $\mathcal{A}$. Segal's procedure also yields a double classifying space for category $\mathcal{A}$ with only a coherent unitary associative operation, hence it furnishes a connected sequence of K-functors $K^n(?;\mathcal{A})$ for $n \leq 1$ in this case. In this and the next section we give some applications of the preceding theorem to this K-theory. We work in the alternative framework of Anderson [**1**], as it goes with the simplicial setup of this paper.

Given simplicial sets X and Y, let $[X, Y]$ denote the set of morphisms from X to Y in the homotopy category; such a morphism may be identified with a homotopy class of maps between the geometric realizations. Let $\mathcal{A}$ be a category endowed with a coherent unitary associative operation $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. If $B\mathcal{A}$ denotes the nerve of $\mathcal{A}$, then this operation induces a map $B\mathcal{A} \times B\mathcal{A} \to B\mathcal{A}$, which in turn provides a monoid structure on the set $[X, B\mathcal{A}]$ for any X. One way of thinking of the functor $K^0(?;\mathcal{A})$ is as the result of converting the monoid-valued functor $[?, B\mathcal{A}]$ to a representable group-valued functor in an intelligent way. In Anderson's framework, one chooses a monoid category $\mathcal{A}'$, that is, a category with an operation making the set of arrows into a monoid, such that $\mathcal{A}'$ is equivalent to $\mathcal{A}$ in a fashion compatible with the operations. The nerve $B\mathcal{A}'$ is a simplicial monoid; one chooses a free simplicial monoid resolution $M \to B\mathcal{A}'$, which is unique up to homotopy ([**9**], §4), and defines K^0 to be the functor represented by the group completion

$$K^0(X;\mathcal{A}) = [X, \overline{M}].$$

There is a canonical natural transformation of monoid-valued functors

$$[X, B\mathcal{A}] \to K^0(X;\mathcal{A})$$

represented by the canonical homomorphism from M to $\overline{M}$. As pointed out at the end of section 5, $\overline{M}$ has the same homotopy type as the loop space of the classifying space $B(B\mathcal{A}')$; in general one has

$$K^n(X;\mathcal{A}) = [X, \Omega^{1-n}B(B\mathcal{A}')], \qquad n \leq 1.$$

The integral homology ring $H_*(M)$ is isomorphic to $H_*(B\mathcal{A})$ with product induced by the operation $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$. If this operation is commutative in the sense that there is a natural isomorphism $X \oplus Y \simeq Y \oplus X$, then the ring $H_*(M)$ is commutative, hence Theorem 1 applies to M yielding

THEOREM Q.5. *Assume $\pi_0 B\mathcal{A}$ is in the center of $H_*(B\mathcal{A})$, e.g. if the operation $\oplus$ is commutative. Then the homology of the representing space for $K^0(?;\mathcal{A})$ is obtained by localization:*

$$H_*(B\mathcal{A})[\pi_0 B\mathcal{A}^{-1}] \xrightarrow{\sim} H_*(\overline{M}).$$

We now consider the case where $\mathcal{A}$ is the category $\mathcal{P}_R$ of finitely generated projective modules and their isomorphisms over a ring R with identity, and where the operation on $\mathcal{P}_R$ is direct sum. Let $S = \pi_0 B\mathcal{P}_R$ be the monoid of isomorphism classes of $\mathcal{P}_R$ and let P_s denote an object in the class s. Then there is an isomorphism

$$H_*(B\mathcal{P}_R) \simeq \bigoplus_{s\in S} H_*(B\,\mathrm{Aut}(P_s))$$

where BG denotes the nerve of the group G. The ring structure on this homology is induced by the direct sum homomorphism

$$\mathrm{Aut}(P_s) \times \mathrm{Aut}(P_{s'}) \to \mathrm{Aut}(P_s \oplus P_{s'})$$

together with the isomorphism of the last group with $\mathrm{Aut}(P_{s+s'})$ furnished by any isomorphism of $P_s \oplus P_{s'}$ and $P_{s+s'}$ (which one does not matter, since inner automorphisms of a group act trivially on homology). As the direct sum operation is commutative, Theorem 2 implies that the representing space $\overline{M}$ for $K^0(?;\mathcal{P}_R)$ has for its homology the localization of $H_*(B\mathcal{P}_R)$ with respect to S. Since the group completion of S is the projective class group K_0R, this localization takes the form (§2)

$$H_*(B\mathcal{P}_R)[\pi_0 B\mathcal{P}_R^{-1}] \simeq \mathbb{Z}[K_0R] \otimes \operatorname*{lim.ind.}_{s} H_*(B\,\mathrm{Aut}(P_s)),$$

where the limit is taken over the category $\mathcal{C}$ having elements of S for its objects, and for morphisms from s to s' the elements t of S with $s+t = s'$, and where the map $H_*(B\,\mathrm{Aut}(P_s)) \to H_*(B\,\mathrm{Aut}(P_{s'}))$ assigned to t is induced by the injection

$$\mathrm{Aut}(P_s) \to \mathrm{Aut}(P_s \oplus P_t), \qquad \alpha \mapsto \alpha \oplus id$$

followed by the isomorphism of the latter group with $\mathrm{Aut}(P_{s'})$ furnished by any isomorphism $P_s \oplus P_t \simeq P_{s'}$. As the functor from the ordered set $\mathbb{N}$ of natural numbers to $\mathcal{C}$ sending n to the free module R^n is cofinal, the limit can be taken over $\mathbb{N}$ yielding isomorphisms

$$\lim_{s}\text{ind.}\, H_*(B\,\text{Aut}(P_s)) \xrightarrow{\sim} \lim_{n}\text{ind.}\, H_*(BGL_nR) \xrightarrow{\sim} H_*(BGL(R)).$$

Therefore from Theorem 2 we obtain isomorphisms

$$H_*(\overline{M}) \simeq \mathbb{Z}[K_0R] \otimes H_*(BGL(R))$$
$$H_*(\overline{M}_e) \simeq H_*(BGL(R))$$

for the homology of the space representing K^0 and its identity component.

We are now going to make the second isomorphism explicit and show that it is induced by a map. Following Anderson, let $\mathcal{A}'$ be the monoid category equivalent to $\mathcal{A}$ consisting of pairs (n, u), with $n \in \mathbb{N}$ and u a projection operator on R^n, in which a morphism from (n, u) to (n', u') is an isomorphism of the images of u and u'; the operation on $\mathcal{A}'$ is given by $(n', u') + (n'', u'') = (n' + n'', u)$ where u corresponds to the operator $u' \oplus u''$ on $R^{n'} \oplus R^{n''}$ under the obvious isomorphism with R^n. Let b be the vertex of $B\mathcal{A}'$ represented by the object $(1, id)$ of $\mathcal{A}'$, and let c be a vertex of M lying over b with respect to the resolution map $p : M \to B\mathcal{A}'$, which we recall is a fibration with contractible fibres. Consider the composite homomorphism

$$H_*(BGL_nR) \xrightarrow{j_{n*}} H_*(B\mathcal{A}') \xrightarrow{p_*^{-1}} H_*(M) \to H_*(\overline{M})$$

where $j_n : BGL_nR \to B\mathcal{A}'$ denotes the map of nerves induced by the inclusion of the full subcategory consisting of the object (n, id), and where the last map is induced by the canonical homomorphism $M \to \overline{M}$ followed by right multiplication by the vertex c^{-n} of $\overline{M}$. This homomorphism has its image contained in the homology of $\overline{M}_e$, and the family of these homomorphisms is compatible with the maps $i_n : BGL_nR \to BGL_{n+1}R$ induced by the standard inclusion of GL_n in GL_{n+1}. The limit homomorphism $H_*(BGL(R)) \to H_*(\overline{M}_e)$ is clearly the isomorphism described above.

To show this isomorphism is induced by a map, we construct a sequence of maps $f_n : BGL_nR \to M$ such that $pf_n = j_n$ and $(.c)f_n = f_{n+1}i_n$, where $(.c)$ denotes right multiplication by the vertex c. We start with f_0 equal to the map sending the unique vertex of BGL_0R to the identity vertex of M; the covering homotopy extension theorem guarantees that $(.c)f_n$ compatible family of maps $BGL_nR \to \overline{M}_e$, $z \mapsto f_nz.c^{-n}$, hence to a map $f : BGL(R) \to \overline{M}_e$ in the limit. It is clear that f is the desired map, hence we have proved

Proposition Q.6. *There exists a map $f : BGL(R) \to \overline{M}_e$ inducing isomorphisms on homology.*

As the fundamental group of a simplicial monoid is abelian, $\pi_1(f) : GL(R) \to \pi_1\overline{M}_e$ kills the subgroup $E(R) = (GL(R), GL(R))$ generated by elementary matrices. In the version of K-theory for a ring announced in [**6**], it is shown that there is a map

$$g : BGL(R) \to BGL(R)^+$$

in the homotopy category of pointed spaces which is universal subject to the requirement that $\pi_1(g)$ kill $E(R)$, and that moreover g induces isomorphisms on homology with coefficients in any $\pi_1 BGL(R)^+$-module. Thus f factors into g followed by a map

$$BGL(F)^+ \to \overline{M}_e$$

inducing isomorphisms on homology. As $BGL(R)^+$ and $\overline{M}_e$ are both simple, the Whitehead theorem implies this map is a homotopy equivalence, proving

COROLLARY 4. *The identity component of the representing space for the functor $K^0(?; \mathcal{P}_R)$ is homotopy equivalent to $BGL(R)^+$.*

COROLLARY 5. *The groups $K^{-n}(pt; \mathcal{P}_R)$ for $n = 0, 1, 2$ are respectively isomorphic to the projective class group K_0R, the K_1R of Bass, and the K_2R of Milnor.*

The assertion for $n = 0$ follows from the fact that K_0R and $\pi_0\overline{M}$ are both group completions of $\pi_0 B\mathcal{P}_R \simeq \pi_0 M$. The assertions for $n = 1, 2$ follow either from Corollary 1 using the theory of the space $BGL(R)^+$, or as Anderson points out, directly from Proposition 4 in the following way.

First of all, $\pi_1\overline{M}_e$ is abelian, so

$$\pi_1\overline{M}_e \xrightarrow{\sim} M_1(\overline{M}_e) \xrightarrow{\sim} H_1(BGL(R))$$

establishing the isomorphism for $n = 1$. Next, consider the map of fibrations

$$\begin{array}{ccccc} BE(R) & \longrightarrow & BGL(R) & \longrightarrow & BK_1R \\ \downarrow{\scriptstyle f^{\sim}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} \\ \overline{M}_e^{\sim} & \longrightarrow & \overline{M}_e & \longrightarrow & BK_1R. \end{array}$$

The comparison theorem for spectral sequences can be used to show that $f^{\sim}$ induces isomorphisms on homology, provided it is known that K_1R acts trivially on the homology of $BE(R)$. Given $\alpha \in K_1R$ and $\beta \in H_*(BE(R))$, α and β come from finitely generated subgroups $H \subset GL(R)$ and $H' \subset E(R)$. Replacing H' by a conjugate subgroup, one can suppose that elements of H centralize H', whence α acts trivially on β. Thus $f^{\sim}$ induces isomorphisms on homology, so

$$\pi_2\overline{M}_e \simeq \pi_2(\overline{M}_e^{\sim}) \simeq H_2(\overline{M}_e^{\sim}) \simeq (BE(R)),$$

proving the $n = 2$ part of the corollary.

Q.8. A theorem of Mather

Denote by G the group of diffeomorphisms with compact support of the line R. John Mather has shown that its classifying space BG has a classifying space BBG and that the canonical map $BG \to \Omega BBG$ induces isomorphisms on homology. We will show how his results can be obtained from Theorem 2.

It will be convenient to replace G by the isomorphic group, denoted by the same letter, consisting of diffeomorphisms of the unit interval $[0,1]$ with support in the interior. Denote by $\mathcal{G}$ the category with one object defined by G. Given $u, v \in G$, let $u+v$ denote the diffeomorphism of $[0,2]$ such that $(u+v)(x) = u(x)$ for $0 \leq x \leq 1$ and $(u+v)(x) = v(x-1)+1$ for $1 \leq x \leq 2$. Choosing a diffeomorphism $h : [0,2] \to [0,1]$ such that $h(x) = x$ for x near 0 and $h(x) = x-1$ for x near 1, we obtain a homomorphism $(u,v) \mapsto h(u+v)h^{-1}$ from $G \times G$ to G, whence a functor $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$. As h is unique up to left multiplication by an element of G, this functor is independent of the choice of h up to canonical isomorphism. This means that the category $\mathcal{G}$ has an intrinsic internal operation, and it is with respect to this operation that Mather's double classifying space BBG is constructed. Observe that the operation is associative up to canonical isomorphisms, but it is not unitary, because there is no element of G conjugating $h(u+id)h^{-1}$ to u for all u in G. Thus it will be necessary to adjoin an identity object to $\mathcal{G}$ before obtaining a situation to which Theorem 2 applies.

Let $\mathcal{A}'$ denote the category whose objects are the intervals $I_n = [0,n]$ for each integer $n \geq 0$, in which a morphism $h : I_n \to I_{n'}$ is a diffeomorphism between these intervals such that $h(x) = x$ for x near 0 and $h(x) = x - n + n'$ for x near n. We endow $\mathcal{A}'$ with the operation $+ : \mathcal{A}' \times \mathcal{A}' \to \mathcal{A}'$ sending $(I_n, I_m) \mapsto I_{n+m}$ and $(u,v) \mapsto u+v$, where if $u : I_n \to I_{n'}$ and $v : I_m \to I_{m'}$ are morphisms, then $u+v : I_{n+m} \to I_{n'+m'}$ is the diffeomorphism given by

$$(u+v)(x) = \begin{cases} u(x), & 0 \leq x \leq n \\ v(x-n)+n', & n \leq x \leq n+m. \end{cases}$$

It is clear that $\mathcal{A}'$ is a monoid category equivalent to its full subcategory consisting of I_0 and I_1, and that this subcategory is the result of adjoining an identity object to $\mathcal{G}$.

As the nerve $B\mathcal{A}'$ is a simplicial monoid, it has a classifying "space" $BB\mathcal{A}'$ as in Section 3. There is a canonical map in the homotopy category

$$B\mathcal{A}' \longrightarrow \Omega BB\mathcal{A}'$$

represented in the simplicial setup we have been using by the canonical map $M \to \overline{M}$, where M is a free simplicial monoid resolution of $B\mathcal{A}'$. We are going to apply Theorem 2 to determine the homology of this loop space.

First of all, $H_*(BG)$ has an associative product induced by the operation on $\mathcal{G}$, that is, by the homomorphism $(u,v) \to h(u+v)h^{-1}$. We show that $H_*(BG)$ has an identity element, which is not evident a priori, because the operation of $\mathcal{G}$ is not unitary. If c denotes the unique element of $\pi_0 BG$, then for c to be a right identity in $H_*(BG)$ means that the homomorphism $u \to h(u+id)h^{-1}$ induces the identity on this homology. To show this, it suffices, as G is the union of the subgroups G_a consisting of the diffeomorphisms with support in the interior of $[0,a]$ for $0 < a < 1$, to show that the inclusion homomorphism and the homomorphism $u \to h(u+id)h^{-1}$ from G_a to G induce the same map on homology. But this is clear, as these two homomorphisms are conjugate in G. Consequently c is a right identity and similarly it is a left identity.

Clearly

$$H_*(B\mathcal{A}') = \mathbb{Z} \oplus H_*(BG)$$

is the ring obtained by adjoining to the ring $H_*(BG)$ an identity element 1 representing the component of $B\mathcal{A}'$ corresponding to the identity object I_0. By what has been shown, $\pi_0 B\mathcal{A}' = \{1, c\}$ is in the center of $H_*(B\mathcal{A}')$, so Theorem 2 shows that the homology of $\Omega BB\mathcal{A}'$ is obtained by localizing that of $B\mathcal{A}'$. The localization is the image of the idempotent c, hence we obtain Mather's theorem

$$H_*(BG) \xrightarrow{\sim} H_*(\Omega BB\mathcal{A}').$$

As with $GL(R)$ in section 7, this implies that $\Omega BB\mathcal{A}'$ has the same homotopy type as the space BG^+ obtained by killing the commutator subgroup of G.

Q.9. The group completion theorem in Segal's setup

The operation of passing from a simplicial monoid to its group completion corresponds geometrically to going from a topological monoid, or more generally a special simplicial space in the sense of G. Segal [**8**], to the loop space of its classifying space. In this section we state the theorem for special simplicial spaces corresponding to the group completion theorem of §6, and indicate how it can be proved by essentially the same method.

Let $M_\bullet$ be a simplicial space which is special, i.e. it is reduced ($M_0 = pt$) and for each n, one has a weak homotopy equivalence $M_n \to M_1^n$, where the map has for its components the face operators associated to the simplices $\{i-1, i\}$ of $\{0, \dots, n\}$. Such a thing is a generalization of a topological monoid, which is the case where $M_n = M_1^n$ for all n. The space M_1, is a homotopy associative H-space in a natural way, in particular, the singular homology $H_*(M_1, k)$ with coefficients in any commutative ring k has a ring structure, and $\pi_0 M_1$ is a multiplicative system in this homology ring.

As a functor from reduced simplicial spaces to pointed connected spaces, the realization $X_\bullet \mapsto |X_\bullet|$ has a right adjoint $Y \mapsto \Omega_\bullet Y$, where $\Omega_n Y$ is the space of maps of the standard n-simplex to Y which carry all the vertices to the basepoint.

It is easy to see that $\Omega_\bullet Y$ is a special simplicial space, whence one has adjoint functors $M_\bullet \mapsto |M_\bullet|$ and $Y \mapsto \Omega_\bullet Y$ between special simplicial spaces and pointed connected spaces. One thinks of the former as the classifying space functor, for it gives the classifying space for a topological monoid, and of the latter functor as a more precise version of the loop space, as $\Omega_1 Y = \Omega Y$. The adjunction arrow $M_\bullet \mapsto \Omega_\bullet |M_\bullet|$ is the analogue of group completion, hence the group completion theorem (Theorem 1) in this setup reads as follows:

THEOREM. *Assume that the localization of the ring $H_*(M_1, k)$ with respect to the multiplicative system $\pi_0 M_1$ admits calculation by right fractions. Then the canonical map $M_1 \longrightarrow \Omega|M_\bullet|$ induces an isomorphism*

$$H_*(M_1, k)[\pi_0 M_1^{-1}] \xrightarrow{\sim} H_*(\Omega|M_\bullet|, k).$$

As in section 6 it suffices to prove this when k is a field, in which case the spectral sequence of the simplicial space $M_\bullet$ [7] takes the Eilenberg-Moore form

$$E^2_{pq} = \mathrm{Tor}_p^{H_*(M_1)}(k, L)_q \implies H_{p+q}(|M_\bullet|, L)$$

for any local coefficient system on $|M_\bullet|$. We consider the map of these spectral sequences induced by the adjunction arrow $M_\bullet \rightarrow \Omega_\bullet|M_\bullet|$. The induced map of realizations $|M_\bullet| \rightarrow |\Omega_\bullet|M_\bullet||$ is a weak homotopy equivalence, because its composition with the other adjunction map $|\Omega_\bullet|M_\bullet|| \rightarrow |M_\bullet|$ is the identity, and because the adjunction map $|\Omega_\bullet Y| \rightarrow Y$ is a weak homotopy equivalence for connected Y by a theorem of Segal [8]. Thus the map on abutments is an isomorphism, and from this point one can argue with the spectral sequences as in sections 3 and 4 to prove the theorem.

References for Appendix Q

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