Isolated singularities of algebraic surfaces with $\mathbf{C}^*$ action

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Introduction

Let V be an algebraic surface in $\mathbf{C}^{n+1}$ with an isolated singularity at the origin. The main result of this paper is to find the resolution of this singularity for those V which admit a suitable action of $\mathbf{C}^*$, the multiplicative group of non-zero complex numbers.

Our method is topological. We consider the intersection $K = V \cap S^{2n+1}$ of V with a small sphere in $\mathbf{C}^{n+1}$. Then K is a smooth, orientable, closed 3-manifold. Since V admits an action of $\mathbf{C}^*$, if S^{2n+1} is invariant under the action of the subgroup $U(1) \subset \mathbf{C}^*$, then so is K. Identify $U(1) \simeq SO(2)$. Such actions were classified in [12], and K together with the action is described by a set of orbit invariants. We investigate the connection between the resolution of the singularity at the origin and the orbit invariants of K. This connection was anticipated by work of F. Hirzebruch [6, 7], F. Hirzebruch and K. Jänich [8], R. von Randow [13], and E. Brieskorn [2].

In § 1 the algebraic preliminaries are introduced and a canonical equivariant resolution is constructed. We also discuss the singular (Seifert) fibration of $V - \{0\}$. In § 2 we use equivariant plumbing to show that the canonical equivariant resolution is star shaped with at most one non-rational curve (the center). We prove the main result that the orbit invariants of K determine the canonical equivariant resolution.

In § 3 these results are applied to weighted homogeneous polynomials in $\mathbf{C}^3$ with an isolated singularity. First we show that up to equivalence of equivariant resolutions there are only six classes to consider and then proceed to compute the orbit invariants, and thereby the resolution, for these. Each section has its own introduction.

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1. In this section we study certain algebraic aspects of singularities with $\mathbf{C}^*$ action. In (1.1) we recall some results about embeddings of these

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singularities in complex affine space. Then in (1.2) we associate to every variety $V \subset \mathbf{C}^{n+1}$ with $\mathbf{C}^*$ action a homogeneous variety that covers V. We use this to construct the Seifert fibration F associated to V. Finally in (1.3) we consider the case when V is a surface with an isolated singularity. We use F to construct a *canonical equivariant resolution* of the singularity of V.

1.1. *Weighted homogeneous varieties.*

Definition (1.1.1). Suppose $(w_0, \cdots, w_n)$ are fixed non-zero rational numbers. A polynomial $h(Z_0, \cdots, Z_n)$ is weighted homogeneous of type $(w_0, \cdots, w_n)$ if it can be expressed as a linear combination of monomials $Z_0^{i_0} \cdots Z_n^{i_n}$ for which

$$i_0/w_0 + i_1/w_1 + \cdots + i_n/w_n = 1 .$$

This is equivalent to requiring that there exist non-zero integers $q_0, \cdots, q_n$ and a positive integer d so that $h(t^{q_0}Z_0, \cdots, t^{q_n}Z_n) = t^d h(Z_0, \cdots, Z_n)$. In fact if h is weighted homogeneous of type $(w_0, \cdots, w_n)$ then let $\langle w_0, \cdots, w_n \rangle$ denote the smallest positive integer d such that there exists, for each i, an integer q_i so that $q_i w_i = d$. These are the q_i and d above.

Let V be a variety defined by weighted homogeneous polynomials $h_1, \cdots, h_r$, each with exponents $(q_0, \cdots, q_n)$. Then V is invariant under the $\mathbf{C}^*$ action

$$\sigma(t, (z_0, \cdots, z_n)) = (t^{q_0}z_0, \cdots, t^{q_n}z_n) .$$

Now consider the converse.

PROPOSITION (1.1.2). *Suppose $V \subset \mathbf{C}^{n+1}$ is an irreducible analytic variety, σ is a $\mathbf{C}^*$ action leaving V invariant,*

$$\sigma(t, (z_0, \cdots, z_n)) = (t^{q_0}z_0, \cdots, t^{q_n}z_n)$$

and $q_i > 0$ for all i. Then V is algebraic and the ideal of polynomials in $\mathbf{C}[Z_0, \cdots, Z_n]$ vanishing on V is generated by weighted homogeneous polynomials.

Proof. Suppose $f \in \mathbf{C}\{Z_0, \cdots, Z_n\}$, the ring of convergent power series. We let f_i denote the unique *polynomial* such that

$$f(t^{q_0}Z_0, \cdots, t^{q_n}Z_n) = \textstyle\sum_{i=0}^{\infty} t^i f_i(Z_0, \cdots, Z_n) .$$

The power series on the right converges for sufficiently small $t \in \mathbf{C}$ and $z \in \mathbf{C}^{n+1}$. Now suppose f vanishes on V near o. Then $v \in V$ implies $\sum_{i=0}^{\infty} t^i f_i(v) = 0$ for all sufficiently small t. Hence $f_i(v) = 0$ for all i and all $v \in V$ near o. Let $f^{(1)}, \cdots, f^{(r)}$ generate the ideal $I(V)$ of all functions in

$\mathbf{C}\{Z_0, \cdots, Z_n\}$ vanishing on V. Let J be the ideal generated by $\{(f^{(j)})_i\}$. Clearly $J \subset I(V)$. Now if $v \notin V$ is within the radius of convergence of $f^{(j)}$ for all j, there is some $f_i^{(j)}$ so that $f_i^{(j)}(v) \neq 0$. Hence the locus of zeros of J is V and hence by the Nullstellensatz the radical of J is $I(V)$. Let J' be the ideal generated by $\{(f^{(j)})_i\}$ in $\mathbf{C}[Z_0, \cdots, Z_n]$ and let I' be the radical of J'. Then $I'\mathbf{C}\{Z_0, \cdots, Z_n\} =$ the radical of $J = I(V)$. Therefore $I(V)$ is generated by polynomials. Clearly the algebraic variety defined by $I(V)$ equals V.

Now let $I'(V)$ be the ideal of V in $\mathbf{C}[Z_0, \cdots, Z_n]$. If $f \in I'(V)$ then $f_i \in I'(V)$. If f is a polynomial, there are only a finite number of integers i so that $f_i \neq 0$. Therefore if $f^{(1)}, \cdots, f^{(r)}$ generate $I'(V)$, then the weighted homogeneous polynomials $\{f_i^{(j)}\}$ generate $I'(V)$.

Remark. If V is a hypersurface then the ideal of V is principal and hence V is defined by a weighted homogeneous polynomial.

PROPOSITION (1.1.3). *If $V \subset \mathbf{C}^m$ is an algebraic variety and there is a $\mathbf{C}^*$ action on V defined by a morphism $\sigma\colon \mathbf{C}^* \times V \to V$ of algebraic varieties then*

(i) *there is an embedding $j\colon V \to \mathbf{C}^{n+1}$ for some n and a $\mathbf{C}^*$ action $\tilde{\sigma}$ on $\mathbf{C}^{n+1}$ such that $j(V)$ is invariant and $\tilde{\sigma}$ induces σ on V,*

(ii) *by a suitable choice of coordinates in $\mathbf{C}^{n+1}$ we may write $\tilde{\sigma}(t, (z_0, \cdots, z_n)) = (t^{q_0}z_0, \cdots, t^{q_n}z_n)$ where $q_i \in \mathbf{Z}$.*

Proof. (i) is a special case of [14, Lem. 2], (ii) is proven in [3, exposé 4, séminaire 1, 1956/58]. We do not know if the analogue is true if V is a Stein space.

1.2. *The cone over a variety with good $\mathbf{C}^*$ action.* Henceforth we shall assume $V \subset \mathbf{C}^{n+1}$, V spans $\mathbf{C}^{n+1}$ and σ is a $\mathbf{C}^*$ action leaving V invariant, defined by

$$\sigma(t, (z_0, \cdots, z_n)) = (t^{q_0}z_0, \cdots, t^{q_n}z_n) .$$

If $q_i > 0$ for all i and g.c.d. $(q_0, \cdots, q_n) = 1$ we say that σ is a *good* $\mathbf{C}^*$ *action.* It will follow from (3.2) that for any weighted homogeneous polynomial $h(Z_0, Z_1, Z_2)$ whose variety V has an isolated singularity, σ may be chosen to be a good action.

Definition (1.2.1). Let $\varphi\colon \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ be defined by $\varphi(z_0, \cdots, z_n) = (z_0^{q_0}, \cdots, z_n^{q_n})$ and let $V' = \varphi^{-1}(V)$. Then V' has a natural $\mathbf{C}^*$ action defined by $\tau(t, (z_0, \cdots, z_n)) = (tz_0, \cdots, tz_n)$ and the induced map $\varphi\colon V' \to V$ commutes with the $\mathbf{C}^*$ action. We call (φ, V') *the cone over* V.

Remarks (1.2.2). (i) Let $U = \{z = (z_0, \cdots, z_n) \mid z \in V$ is a simple point and $z_i \neq 0, \forall i\}$. Since φ is unramified off the coordinate axes, every point of

$\varphi^{-1}(U)$ is simple on V'.

(ii) Identify $\mathbf{Z}_{q_i}$ with the group of q_i^{th} roots of 1. Then V is the quotient of V' by $G = \mathbf{Z}_{q_0} \oplus \cdots \oplus \mathbf{Z}_{q_n}$ acting on V' by coordinatewise multiplication.

The cone V' above V is defined by homogeneous polynomials which define a projective variety $X' = (V' - \{0\})/\mathbf{C}^* \subset \mathbf{P}^n$. Let $\eta'\colon (V' - \{0\}) \to X'$ be the quotient map. There is a well-known way of adding a zero section to this $\mathbf{C}^*$ bundle to get a $\mathbf{C}$ bundle. Let $\Gamma_{\eta'} \subset (V' - \{0\}) \times X'$ be the graph of η', let F' be the closure of $\Gamma_{\eta'}$ in $V' \times X'$, and let $\tau'\colon F' \to X'$ be induced by projection on the second factor. The induced map $\gamma'\colon F' \to V'$ is just the monoidal transform with center $o \in V'$, and (τ', F') is the dual of the hyperplane bundle on $X' \subset \mathbf{P}^n$. Clearly $\mu'\colon X' \to F'$ given by $\mu'(x') = (0, x')$ defines the zero section of (τ', F'). The actions of $\mathbf{C}^*$ and G on V' commute, hence G acts on X' and defining $X = X'/G$ we see that $X = (V - \{0\})/\mathbf{C}^*$. Let $\eta\colon (V - \{0\}) \to X$ be the quotient map. As above, we would like to add a zero section to this map to get a map with fibers $\mathbf{C}$. The action of G extends to F' and we define $F = F'/G$. Then F is just the closure of Γ_η in $V \times X$. Now we have a commutative diagram

$$\begin{array}{ccc} F' & \xrightarrow{\varphi} & F \\ \tau' \big\downarrow \big) \mu' & & \big\downarrow \tau \big) \mu \\ X' & \xrightarrow{\psi} & X \end{array}$$

where φ and ψ are the quotient maps, μ is the map induced by μ' and τ is induced by τ'. Let $\gamma\colon F \to V$ be the natural map. We call (τ, F) the *singular (Seifert) fibration associated to* (V, σ).

1.3. *Resolution of singularities.* Henceforth we shall assume that $V \subset \mathbf{C}^{n+1}$ is a normal complex algebraic surface (and hence it has only isolated singularities). If D and D' are divisors (or 2-cycles) with compact support on a non-singular complex surface we let $(D.\ D') \in \mathbf{Z}$ denote the intersection product [11].

Definition (1.3.1). A resolution $\pi\colon \tilde{V} \to V$ of an isolated singularity $v \in V$ is called *minimal* if for any resolution $\pi_1\colon V_1 \to V$ there is a unique map $\chi\colon V_1 \to \tilde{V}$ so that $\pi \circ \chi = \pi_1$. Of course the minimal resolution is unique. Brieskorn [1] has shown that the minimal resolution exists if V is a surface.

Remark (1.3.2). There is a simple criterion for a resolution of a surface to be minimal. Suppose V_0 is a non-singular surface and $X \subset V_0$ is a compact irreducible curve. Then there is a *non-singular* surface V_1 and a proper morphism $\pi: V_0 \to V_1$ so that $\pi(X) = v \in V_1$ and π induces an isomorphism between $V_0 - X$ and $V_1 - \{v\}$ if and only if X is isomorphic to $\mathbf{P}^1_{\mathbf{C}}$ and $(X \cdot X) = -1$. This is known as *Castelnuovo's criterion*. A curve X satisfying the above is called *exceptional of the first kind*. A resolution $\pi: \tilde{V} \to V$ of an isolated singularity $v \in V$ is minimal if and only if no component of $\pi^{-1}(v)$ is exceptional of the first kind. Note that in general if π is the minimal resolution, the components of $\pi^{-1}(v)$ may have singularities, may have non-normal crossings, etc.

Suppose $\pi: \tilde{V} \to V$ is a resolution of a normal singularity $v \in V$ and $\pi^{-1}(v) = X_1 \cup \cdots \cup X_r$, where the X_i are irreducible curves. Then the matrix $A = ((X_i \cdot X_j))$ is an important invariant of π. One can see without difficulty that A is negative definite, the diagonal entries are negative and the off diagonals are $\geqq 0$ [11].

Definition (1.3.3). Suppose V is a complex surface and $v \in V$ is an isolated singular point. We say that $\pi: V_1 \to V$ is a *good* resolution of the singularity at v if

(i) π is a proper morphism, V_1 is non-singular in a neighborhood U of $\pi^{-1}(v)$ and π induces an isomorphism $\pi: U - \pi^{-1}(v) \xrightarrow{\sim} \pi(U) - \{v\}$;

(ii) if $\pi^{-1}(v) = \bigcup_{i=1}^{r} X_i$, where each X_i is an irreducible curve, then X_i is non-singular for each i;

(iii) X_i meets X_j at most in one point and they meet normally there;

(iv) $X_i \cap X_j \cap X_k = \emptyset$ for i, j, and k distinct.

It is a well-known classical result that a good resolution exists [18]. The fact that V is normal implies that $\pi^{-1}(o)$ is connected (Zariski's connectedness theorem [19]).

Consider the case where V has a good $\mathbf{C}^*$-action. Recalling the notation of (1.2) the map $\varphi: F' \to F$ is ramified only along a finite number of fibers of τ'. Hence there is an open subset $U \subset X$ so that $\tau^{-1}(U)$ is non-singular. But $F - \mu(X)$ is non-singular since o is an isolated singular point of V, hence F has only a finite number of singularities. These singularities are quotient singularities, hence they are rational singularities [2] and therefore they can be resolved by a sequence of monoidal transforms with centers at isolated singular points (σ-transforms) [16]. Let $\rho_1: V_1 \to F$ be such a resolution. Isolated singular points of a surface with $\mathbf{C}^*$-action must be invariant under the action, hence the action extends to V_1. Let $\tilde{\rho}: \tilde{V} \to F$ be the minimal resolution of the singularities of F. Then there is an induced $\mathbf{C}^*$ action on $\tilde{V}$. The composite map $\rho = \gamma\tilde{\rho}: \tilde{V} \to V$ will be called the *canonical equivariant resolution* of V.

For $f\colon W \to V$ a birational map of surfaces and X a curve in V we let $f^{\#}(X)$ denote the unique irreducible curve in W for which $f(f^{\#}(X)) = X$. This curve is called the proper transform of X.

Note that the induced $\mathbf{C}^*$ action on $\tilde{X} = \tilde{\rho}^{\#}(X)$ is trivial and it can be shown that the other curves of the resolution have trivial stability groups.

2. In this section we shall describe a pasting process for manifolds known as plumbing [6]. The building blocks are in our case D^2 bundles over closed, orientable 2-manifolds. We first define plumbing according to a weighted graph. Next we let $SO(2)$ act on the building blocks and define equivariant plumbing. In Theorem (2.2.1) we describe the restrictions imposed on the graph by requiring the plumbing to be equivariant.

The result of an equivariant plumbing is a compact orientable 4-manifold with $SO(2)$ action. Its boundary is a closed orientable 3-manifold with $SO(2)$ action. These were classified in [12]. The manifolds in question were first treated by Seifert [15]. The orbit invariants of the $SO(2)$ action coincide with the Seifert invariants, as computed by Hirzebruch [6] and von Randow [13] from the weighted graph of the plumbing.

Let V be an algebraic surface in $\mathbf{C}^{n+1}$ defined by a weighted homogeneous polynomial. Restrict the natural $\mathbf{C}^*$ action to the $U(1) \subset \mathbf{C}^*$ action and consider $K_\varepsilon(V) = V \cap S_\varepsilon^{2n+1}$, the intersection of V with the sphere of radius ε in $\mathbf{C}^{n+1}$. Clearly $K_\varepsilon(V)$ is a closed, orientable 3-manifold with $U(1) \simeq SO(2)$ action.

The existence of a canonical equivariant resolution and the equivariant Plumbing Theorem (2.2.1) together give the main result(2 .6.1) showing how the resolution of the isolated singularity of V is obtained from the orbit invariants of $K_\varepsilon(V)$.

2.1. *Plumbing*. The principal $SO(2)$ bundles over a closed, orientable 2-manifold M are classified by $H^2(M; \mathbf{Z}) = \mathbf{Z}$.

Denote the associated D^2 bundles indexed by $m \in \mathbf{Z}$ as $\eta = (Y_m, \pi, M)$. The compact 4-manifold Y_m has the homotopy type of M and if we let the zero cross-section $\nu\colon M \to Y_m$ represent the positive generator $g \in H_2(Y_m, \mathbf{Z})$ then its self-intersection number, $g \cdot g = m$, is the Euler class of Y_m.

It is customary to let the bundle with Euler class $m = -1$ over S^2, $\eta = (Y_{-1}, \pi, S^2)$, be the disc bundle whose boundary, S^3, has the Hopf fibration. This specifies orientations.

Define plumbing as follows. Suppose we have two D^2 bundles, $\eta_i = (Y_{m_i}, \pi_i, M_i)$ $i = 1, 2$. Choose a 2-disc B^2_{ij} in the base space of η_i and let $\pi_i^{-1}(B^2_{ij}) = Y_{ij}$. Since $\eta_i \,|\, B^2_{ij}$ is trivial, there is a homeomorphism $\mu_{ij}\colon D^2 \times D^2 \to Y_{ij}$ whose first component gives base coordinates and second fiber coordinates.

Let $t: D^2 \times D^2 \to D^2 \times D^2$ be the reflection $t(x, y) = (y, x)$. Then there is a homeomorphism (with $j = i + 1 \bmod 2$) $f_{ji}: Y_{ji} \to Y_{ij}$ given by $f_{ji} = \mu_{ij} t \mu_{ji}^{-1}$. Since $Y_{ij} \subset Y_{m_i}$ we may paste Y_{m_2} and Y_{m_1} together along Y_{21} and Y_{12} by f_{21} to obtain a topological 4-manifold with corners. It may be smoothed according to [6]. Note that the resulting manifold is independent (up to diffeomorphism) of the choices involved.

A *graph* is a finite one-dimensional simplicial complex. (We shall always assume that graphs are connected.) Let $A_1, \cdots, A_n$ denote its vertices.

A *star* is a contractible graph where at most one vertex is connected with more than two other vertices. If there is such a vertex, call it the center. A *weighted graph* is a graph where each vertex A_i has associated with it a non-negative integer g_i (the genus of A_i) and an integer m_i (the weight of A_i).

Given a weighted graph G we define a compact 4-manifold $P(G)$ as follows. For each vertex (A_i, g_i, m_i) take the D^2 bundle $\eta_i = (Y_{m_i}, \pi_i, M_i)$, where M_i is a closed, orientable 2-manifold of genus g_i. If an edge connects A_i and A_j then perform plumbing on η_i and η_j. If A_i is connected with more than one other vertex, choose pairwise disjoint discs on M_i and perform the plumbing over each. Finally smooth the resulting manifold to obtain $P(G)$.

2.2. *Equivariant plumbing*. Now let us define an action of $SO(2)$ on $\eta = (Y_m, \pi, M)$.

If $g > 0$, let $SO(2)$ act trivially on the base space M and by rotation in each fiber.

If $g = 0$, we define linear actions on $\eta = (Y_m, \pi, S^2)$. Let the base space be the union of two discs $S^2 = B_1^2 \cup B_2^2$ then $Y_m = B_1^2 \times D_1^2 \cup B_2^2 \times D_2^2$. We parametrize the discs in polar coordinates, radii λ_i, ρ_i, $0 \leqq \lambda_i, \rho_i \leqq 1$, and angles γ_i, δ_i, $0 \leqq \gamma_i, \delta_i < 2\pi$, $i = 1, 2$. The actions of $SO(2)$ on D^2 are equivalent to linear actions and we shall think of them as addition of angles. Let $\theta \in SO(2)$, $0 \leqq \theta < 2\pi$.

Define

$$B_1^2 \times D_1^2 \xrightarrow{\theta} B_1^2 \times D_1^2$$

$$(\lambda_1, \gamma_1, \rho_1, \delta_1) \xrightarrow{\theta} (\lambda_1, \gamma_1 + u_1\theta, \rho_1, \delta_1 + v_1\theta)$$

$$B_2^2 \times D_2^2 \xrightarrow{\theta} B_2^2 \times D_2^2$$

$$(\lambda_2, \gamma_2, \rho_2, \delta_2) \xrightarrow{\theta} (\lambda_2, \gamma_2 + u_2\theta, \rho_2, \delta_2 + v_2\theta) .$$

Now Y_m is obtained by an equivariant sewing

$$h: \partial B_1^2 \times D_1^2 \longrightarrow \partial B_2^2 \times D_2^2 .$$

Since the action is linear, h is completely determined by

$$h'\colon \partial B_1^2 \times \partial D_1^2 \longrightarrow \partial B_2^2 \times \partial D_2^2$$

which in turn is isotopic to a linear map of the torus. Let h' be

$$h'(\gamma_1, \delta_1) = (x\gamma_1 + y\delta_1, z\gamma_1 + t\delta_1)\ .$$

In order that h' be equivariant we need $u_1x + v_1y = u_2$, $u_1z + v_1t = v_2$. In order that h be equivariant on $\partial B_1^2 \times 0 \longrightarrow \partial B_2^2 \times 0$ we need $u_1x = u_2$. Thus we must have $y = 0$. Since the determinant of h' is -1 and the sewing results in a total space with Euler class m, we need $x = -1$, $t = 1$, $z = -m$. Thus $u_2 = -u_1$, $v_2 = -mu_1 + v_1$. The action is effective if and only if $(u_1, v_1) = 1$. Note that this action is in general different from the action where $SO(2)$ operates on each fiber of the disc bundle. The latter corresponds to $u_1 = 0$, $v_1 = \pm 1$ ($u_2 = 0$, $v_2 = \pm 1$).

A plumbing is *equivariant* if the identifying map f_{ji} and the trivializing maps μ_{ij} are equivariant. Given a weighted graph G we say that $P(G)$ is equivariant if each plumbing involved is equivariant.

THEOREM (2.2.1). *Let G be a weighted graph and assume that $P(G)$ is equivariant. If*

(a) *G has a vertex (A_0, g_0, m_0) where the action is trivial in the base,*

(b) *for each vertex (A_i, g_i, m_i) we have $m_i \leqq -1$, and*

(c) *for each vertex $(A_i, 0, -1)$ connected with (A_j, g_j, m_j) we have $g_j > 0$ or $m_j \leqq -2$ (or both), then*

(i) *$g_i = 0$ for all vertices $i > 0$,*

(ii) *G is a weighted star with center A_0,*

(iii) *the action is non-trivial on the base for $i > 0$.*

Proof. First note that we plumb about a fixed point ($0 \times 0 \in D^2 \times D^2$) of the action. Thus, if a vertex is connected with more than two vertices, then its base must have trivial action.

Let the action at A_0 be defined $u_{01} = 0$, $v_{01} = 1$, $u_{02} = 0$, $v_{02} = 1$. Note that the action is independent of m_0.

Now suppose A_1 is connected to A_0. Then the action in the base of Y_{m_1} is non-trivial, hence $g_1 = 0$ and $u_{11} = v_{02}$, $v_{11} = u_{02}$, $u_{12} = -v_{02} = -1$, $v_{12} = -m_1v_{02} + u_{02} = -m_1$.

Define inductively

$$p_0 = -u_{12} = 1\ ,\quad p_1 = v_{12} = -m_1\ ,\quad p_2 = -m_2p_1 - p_0\ ,$$
$$p_j = -m_jp_{j-1} - p_{j-2} \qquad j = 2, \cdots, r\ .$$

Then the action is as follows. At A_1 we have $u_{12} = -p_0$, $v_{12} = p_1$. If A_2 is connected to A_1 then $g_2 = 0$ and $u_{22} = -p_1$, $v_{22} = p_2$. Since the action has only two fixed points, no further vertices are connected with A_1. Similarly, if A_3 is connected with A_2 then $g_3 = 0$, $u_{32} = -p_2$, $v_{32} = p_3$, etc.

Define the auxiliary parameters

$$q_0 = 0 \ , \quad q_1 = 1 \ , \quad q_2 = -m_2 \ , \quad q_3 = -m_3 q_2 - q_1 \ ,$$
$$q_j = -m_j q_{j-1} - q_{j-2} \qquad j = 2, \cdots, r \ .$$

The following statements are easy to prove by induction [13].

(1) $p_j q_{j-1} - p_{j-1} q_j = -1$ for $0 < j \leqq r$.

(2) $(p_j, q_j) = 1$, $(p_j, p_{j-1}) = 1$, $(q_j, q_{j-1}) = 1$ for $0 < j \leqq r$.

(3) If $-m_j \geqq 1$ for $0 < j \leqq r$ and if $-m_j = 1$ implies $-m_{j\pm 1} > 1$, then for $0 < j \leqq r$ we have $p_j \neq 0$ and $0 < q_j < p_j$.

This proves the theorem.

2.3. *The weighted graph associated to a resolution.* Suppose $V \subset \mathbf{C}^{n+1}$ is a complex surface with an isolated singularity $o \in V$ and $\rho_0 \colon V_0 \to V$ is a good resolution of the singularity so that $\rho_0^{-1}(o) = X_0 \cup \cdots \cup X_r$ where the X_i are irreducible curves.

We associate a weighted graph G to ρ_0 in the following way. To each X_i there corresponds a weighted vertex (A_i, g_i, m_i) where g_i is the genus of X_i and $m_i = (X_i \cdot X_i)$. We join A_i to A_j by an edge if X_i meets X_j.

Let S_ε be a small $2n+1$ sphere around o and let $K = V \cap S_\varepsilon$. Now $\rho_0^{-1}(K)$ is homeomorphic to K and is the boundary of a tubular neighborhood of $\rho_0^{-1}(o)$. In fact, it is obtained by plumbing according to the graph associated to ρ_0 [11].

Now assume that σ is a good $\mathbf{C}^*$ action. Let ρ be the canonical equivariant resolution. Then K is obtained by an *equivariant* plumbing.

THEOREM (2.3.1). *In the above situation*

(1) *ρ is a good resolution,*

(2) *the action is trivial on $X_0 = \widetilde{X}$,*

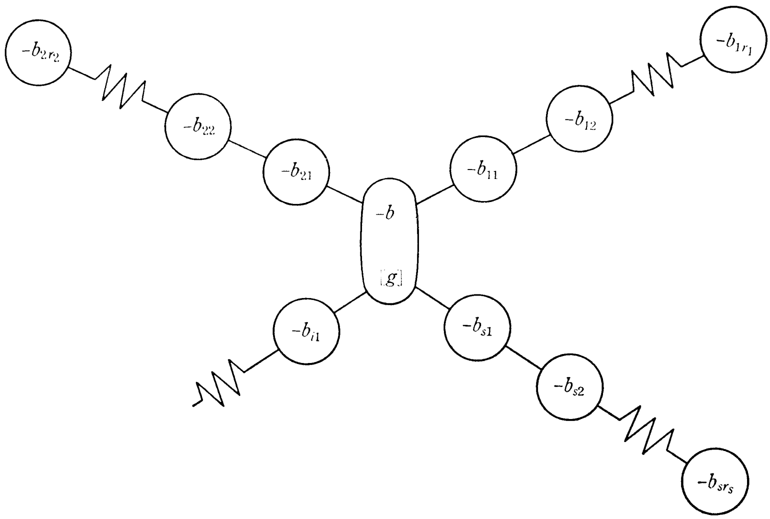
(3) *the action is non-trivial on X_i, $i > 0$, and $g_i = 0$, $i > 0$,*

(4) *G is a weighted star with center A_0,*

(5) *$m_i \leqq -2$, for all $i > 0$.*

Proof. Let $\chi \colon W \to F$ be a resolution of F such that $\gamma\chi \colon W \to V$ is a good resolution of V. Let G' be the graph associated to $\gamma\chi$. The action of σ on $\chi^{\#}(X)$ is trivial and the intersection matrix of the resolution is negative definite, hence (a)–(c) of (2.2.1) are satisfied. Thus (i)–(iii) are satisfied for G'. Contracting exceptional curves of the first kind (other than the proper transform of X) we see that (1)–(5) holds for ρ.

2.4. *The star S.* Let S denote the weighted star below



where $g \geqq 0$, $b \geqq 1$ and all other vertices have genus zero and $b_{ij} \geqq 2$. Let

$$\alpha_i/\beta_i = b_{i1} - \cfrac{1}{b_{i2} - \cfrac{1}{\ddots - \cfrac{1}{b_{ir_i}}}} = [b_{i1}, \cdots, b_{ir_i}]$$

for $i = 1, \cdots, s$.

It is easily seen that $(\alpha_i, \beta_i) = 1$ and $0 < \beta_i < \alpha_i$ for all i.

2.5. *Actions of $SO(2)$ on 3-manifolds.* The equivariant plumbing of S, $P(S)$ is a compact 4-manifold with boundary admitting an $SO(2)$ action. Let $K(S) = \partial P(S)$.

According to [12] a 3-manifold with $SO(2)$ action, K, may be described by the orbit invariants

$$K = \{\beta; (\varepsilon, g, \bar{h}, t), (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n)\} .$$

Here is a brief explanation of the meaning of these orbit invariants following [12]. The orbit space K^* is a weighted 2-manifold whose orientability is given by $\varepsilon = o$ or n and genus by g. The number of boundary components of K^* is $t + \bar{h} \geqq 0$. Of these $\bar{h}$ represent components of the fixed point set and t components of orbits with stability group Z_2 which acts on the slice D^2 by reflection. Note that along these orbits the local orientation is reversed. The ordered pair of integers (α_j, β_j), $0 \leqq \beta_j < \alpha_j$, $(\alpha_j, \beta_j) = 1$, corresponds to an orbit with finite stability group Z_{α_j} and representation $Z_{\alpha_j} \to SO(2)$ given by β_j. Orbits of type (1, 0) are principal. If these are omitted from the expression of K, then the pairs (α_j, β_j) are unique up to order [12]. Finally remove a small disk D_j^* around the image of each orbit (α_j, β_j) from K^*. Let $K_0^* = K^* - \bigcup_j \operatorname{int} D_j^*$. We can specify a cross-section to the orbit map on ∂K_0^*. The obstruction to extending this cross-section to all of K_0^* is the integer β.

Clearly $K(S)$ is orientable and the action has no fixed points, hence $\varepsilon = o$, $t = \bar{h} = 0$.

The following information will be needed about $K(S)$ (see [12], [15]).

Let a_i, b_i, $i = 1, \cdots, g$ generate $\pi_1(K(S)^*)$ and q_j, $j = 1, \cdots, n$ be the additional generators of $\pi_1(K(S)_0^*)$. If we let h be a typical orbit, then

$$\pi_1(K(S)) = (a_i, b_i, q_j, h \mid \pi_* h^{-\beta}, [a_i, h], [b_i, h], [q_j, h], q_j^{\alpha_j} h^{\beta_j})$$

where $i = 1, \cdots, g$; $j = 1, \cdots, n$ and $\pi_* = q_1 \cdots q_n [a_1, b_1] \cdots [a_g, b_g]$.

It follows that $H_1(K(S))$ has $2g$ free generators, $\{a_1, b_1, \cdots, a_g, b_g\}$, and a subgroup $T(K(S))$ generated by $\{h, q_1, \cdots, q_n\}$ with relations

$$-\beta h + q_1 + \cdots + q_n = 0$$
$$\beta_j h + \alpha_j q_j = 0 \qquad j = 1, \cdots, n .$$

Let $b = -\beta$ and let R denote the coefficient matrix of the above relations. Let

$$p = \det R = b\alpha_1 \cdots \alpha_n - \beta_1 \alpha_2 \cdots \alpha_n - \cdots - \alpha_1 \cdots \alpha_{n-1} \beta_n .$$

$$\text{Then } T(K(S)) = \begin{cases} \text{torsion} & \text{if } p \neq 0 , \\ Z + \text{torsion} & \text{if } p = 0 . \end{cases}$$

THEOREM (2.5.1). *Let S be the weighted star of* (2.4). *Then*

$$K(S) = \{-b; (o, g, 0, 0); (\alpha_1, \beta_1), \cdots, (\alpha_s, \beta_s)\}$$

where

$$\alpha_i/\beta_i = [b_{i1}, \cdots, b_{ir_i}] \qquad \textit{for } i = 1, \cdots, s .$$

This result is due to Hirzebruch [6] and von Randow [13] when $g = 0$. For $g > 0$ the proof is the same.

THEOREM (2.5.2). *Let*

$$K = \{\beta; (o, g, 0, 0); (\alpha_1, \beta_1), \cdots, (\alpha_s, \beta_s)\}$$

be the orbit invariants of a 3-manifold with $SO(2)$ action and $\alpha_j > 1$ for $j = 1, \cdots, s$. Then K determines a unique weighted star $S(K)$ with the property that the center has genus g and weight $-b = \beta$. There are s arms. If

$$\alpha_i/\beta_i = [b_{i1}, \cdots, b_{ir_i}] , \qquad i = 1, \cdots, s$$

with $b_{ij} \geqq 2$ *for all* i, j*, then the vertices on the* i*-th arm have genus* 0 *and weights* $-b_{i1}, -b_{i2}, \cdots, -b_{ir_i}$ *starting from the center. Furthermore*

$$K(S(K)) = K$$

equivariantly.

The only part of this theorem that is not obvious is the uniqueness of the continued fraction decomposition of α_i/β_i, but this follows from the assumption that $b_{ij} \geqq 2$ for all i, j.

2.6. *The main theorem.* Now let V be an algebraic surface in $\mathbf{C}^{n+1}$ with an isolated singularity at the origin. Suppose V has a good $\mathbf{C}^*$ action.

Consider the $U(1) \subset \mathbf{C}^*$ action restricted to the invariant intersection

$$K = V \cap S_\varepsilon^{2n+1} .$$

Our results now yield the following.

THEOREM (2.6.1). *The weighted graph associated to the canonical resolution of the isolated singularity at the origin of* V *is the star of* K, $S(K)$.

In particular we may obtain this resolution by computing the orbit invariants of the $U(1) = SO(2)$ action on K.

Remark (2.6.2). Since the intersection matrix of the resolution is negative definite, we see that the determinant of the relation matrix for $H_1(K)$, $p > 0$ and therefore the rank of $H_1(K)$ equals $2g$.

Note also that the orbit space K^* of the $U(1)$ action on K coincides with the orbit space X of the C^* action on $V - \{0\}$.

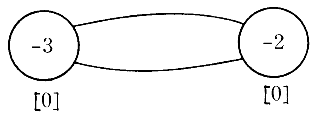
3. In this section we apply our results to surfaces in $\mathbf{C}^3$. More precisely, let V be defined as the locus of the zeros of a weighted homogeneous polynomial, with an isolated singularity at the origin. We shall find the resolution of this singularity by computing the orbit invariants of the natural $SO(2)$ action on $K = V \cap S_\varepsilon^5$. We noted in the remark after (1.1.2) that if an algebraic surface with an isolated singularity is invariant under a good $\mathbf{C}^*$ action on $\mathbf{C}^3$, then it is defined by a weighted homogeneous polynomial.

We first show that up to equivariant resolution weighted homogeneous polynomials divide into six classes, one being the Brieskorn varieties. Next we compute the weights for these classes. Then we proceed to find the orbit invariants of K. We determine the orbits with non-trivial stability groups: their number and the orders α_j of the stability groups. The slice representation is used to determine the corresponding β_j. Finally we use covering arguments to compute the genus of the "central curve" $\tilde{X}$ and its self intersection, $-b$.

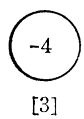
It should be pointed out that not all algebraic surfaces with an isolated singularity in $\mathbf{C}^3$ admit good $\mathbf{C}^*$ actions. Here is an example:

$$V = \{z_0^2 + z_1^3 - 3z_1z_2^4 + z_1z_2^5 + 2z_2^6 - z_2^7 = 0\} .$$

This is an elliptic singularity with the following graph (cf. [17]):



No resolution of this singularity can have a star-shaped graph. Moreover $V = \{z_0^4 + z_1^4 + z_2^4 + z_0^2z_1^2z_2 = 0\}$ has graph



but V has no $\mathbf{C}^*$ action.

3.1. *The six classes.* Consider weighted homogeneous polynomials $h(Z_0, Z_1, Z_2)$ with the property that the variety $\{h(z_0, z_1, z_2) = 0\}$ in $\mathbf{C}^3$ has an isolated singularity. We show that all such polynomials fall into six classes. A variety in one of these classes is diffeomorphic to a variety having a certain simple normal form.

Definition (3.1.1). A weighted homogeneous polynomial $h(Z_0, Z_1, Z_2)$ is said to be of class I (resp. II, III, $\cdots$) if there is a permutation π of $\{0, 1, 2\}$ and non-zero complex numbers $\alpha_0, \alpha_1, \alpha_2$ such that $h(\alpha_0 Z_{\pi(0)}, \alpha_1 Z_{\pi(1)}, \alpha_2 Z_{\pi(2)})$ is equal to

(I) $Z_0^{a_0} + Z_1^{a_1} + Z_2^{a_2}$,

(II) $Z_0^{a_0} + Z_1^{a_1} + Z_1 Z_2^{a_2}$, $\quad a_1 > 1$,

(III) $Z_0^{a_0} + Z_1^{a_1} Z_2 + Z_2^{a_2} Z_1$, $\quad a_1 > 1$, $a_2 > 1$,

(IV) $Z_0^{a_0} + Z_0 Z_1^{a_1} + Z_1 Z_2^{a_2}$, $\quad a_0 > 1$,

(V) $Z_0^{a_0} Z_1 + Z_1^{a_1} Z_2 + Z_0 Z_2^{a_2}$,

(VI) $Z_0^{a_0} + Z_1 Z_2$.

PROPOSITION (3.1.2). *Suppose $h(Z_0, Z_1, Z_2)$ is a polynomial and the locus V of h has an isolated singularity. Then $h(Z_0, Z_1, Z_2) = f(Z_0, Z_1, Z_2) + g(Z_0, Z_1, Z_2)$ where f is in one of the six classes above and f and g have no monomial in* *common. If h is weighted homogeneous of type (w_0, w_1, w_2) then so are f and g.*

Proof. Let $h(Z_0, Z_1, Z_2) = \sum \alpha_{(i)} Z_0^{i_0} Z_1^{i_1} Z_2^{i_2}$. If $i_0 + i_1 > 1$ for all monomials of h then the line $Z_0 = Z_1 = 0$ is a subset of V and $(\partial h/\partial Z_i) = 0$ for all i at every point on this line. This contradicts the fact that the singularity is isolated. Hence there must be a monomial of the form $Z_2^{a_2}$, $Z_1 Z_2^{a_2}$ or $Z_0 Z_2^{a_2}$ in h. The same reasoning implies that $Z_1^{a_1}$, $Z_0 Z_1^{a_1}$ or $Z_2 Z_1^{a_1}$ *and* $Z_0^{a_0}$, $Z_1 Z_0^{a_0}$ or $Z_2 Z_0^{a_0}$ must appear. Putting these three facts together one can easily see that we must get a polynomial f in one of the six classes above.

Remarks. (1) It should be noted that f is not unique.

(2) An analogous theorem holds for polynomials in more variables.

Now we want to show that if h is weighted homogeneous then the variety of h is diffeomorphic to the variety of f. The crucial fact we need is that $K_\varepsilon = V \cap S_\varepsilon$ is independent of ε.

PROPOSITION (3.1.3). *Suppose $V \subset \mathbf{C}^{n+1}$, $\sigma(t, z_0, \cdots, z_n) = (t^{q_0} z_0, \cdots, t^{q_n} z_n)$ is a $\mathbf{C}^*$ action on V, $q_i > 0$ for all i. Let S_ε be the real $2n+1$ sphere of radius ε about the origin and $K_\varepsilon = V \cap S_\varepsilon$. Then for any ε, $\varepsilon' > 0$ K_ε is equivariantly homeomorphic to $K_{\varepsilon'}$.*

Proof. Suppose $\varepsilon \leqq \varepsilon'$. We define a homeomorphism $f: K_\varepsilon \to K_{\varepsilon'}$ by letting $f(z)$ be the unique point $z' \in K_{\varepsilon'}$ so that there is a positive $t \in \mathbf{R}$ where $\sigma(t, z) = z'$. Suppose $s \in \mathbf{C}^*$ and $\| s \| = 1$. If $\sigma(t, z) = z'$ then $\sigma(t, \sigma(s, z)) = \sigma(s, z') \in K_{\varepsilon'}$, hence f is an equivariant map. Clearly we can define f^{-1} similarly.

THEOREM (3.1.4). *Suppose $h(Z_0, Z_1, Z_2)$ is weighted homogeneous of type (w_0, w_1, w_2), the variety V of h has an isolated singularity and $h = f + g$ where f belongs to one of the six classes and no monomial appears in both f and g. Let V_o be the variety of f and let*

$$K = V \cap S^5 , \quad K_o = V_o \cap S^5$$

where S^5 is a sphere around the origin. Then K is equivariantly diffeomorphic to K_o.

Proof. Let $g(Z_0, Z_1, Z_2) = \sum_{i=1}^r \alpha_i M_i$ where M_i is a monomial. Let $x = (x_1, \cdots, x_r)$ and let V_x be the locus of

$$f(Z_0, Z_1, Z_2) + \sum_{i=1}^r x_i M_i .$$

Let $K_x = V_x \cap S^5$. Then if $\alpha = (\alpha_1, \cdots, \alpha_r)$, $V = V_\alpha$. Now it is sufficient to find a manifold M with $SO(2)$ action, an open set $U \subset \mathbf{C}^r$ and a map $\varphi: M \to U$ such that $o \in U$, $\alpha \in U$, the action leaves $\varphi^{-1}(x)$ invariant for all $x \in U$,

$\varphi^{-1}(x) = K_x$ equivariantly and φ is a locally trivial fiber space.

Let $k(Z_0, Z_1, Z_2, X_1, \cdots, X_r) = f(Z_0, Z_1, Z_2) + \sum_{i=1}^{r} X_i M_i(Z_0, Z_1, Z_2)$, let $N \subset \mathbf{C}^{r+3}$ be the locus of k, let $C = \{(z_0, z_1, z_2, x_1, \cdots, x_r) \in \mathbf{C}^{r+3} \mid |z_0|^2 + |z_1|^2 + |z_2|^2 = 1\}$, let $\varphi_0 \colon \mathbf{C}^{r+3} \to \mathbf{C}^r$ be defined by $\varphi_0(z_0, z_1, z_2, x_1, \cdots, x_r) = (x_1, \cdots, x_r)$ and let $\varphi_1 = \varphi_0 \mid N$. We denote by U the (open) set of $x \in \mathbf{C}^r$ such that $\varphi_1^{-1}(x)$ has an isolated singularity at o. Finally define $M = N \cap C \cap \varphi_0^{-1}(U)$ and $\varphi = \varphi_0 \mid M$. Clearly o and $\alpha \in U$. Now let $q_i = \langle w_0, w_1, w_2 \rangle / w_i \in \mathbf{Z}$. Then

$$\sigma\big(t, (z_0, z_1, z_2, x_1, \cdots, x_r)\big) = (t^{q_0} z_0, t^{q_1} z_1, t^{q_2} z_2, x_1, \cdots, x_r)$$

induces an $SO(2)$ action on M leaving the fibers of φ invariant. It is sufficient to show now that φ has no critical points. Suppose $(z_0, z_1, z_2, x_1, \cdots, x_r) = m \in M$. Let T_M, T_N and T_C denote the tangent planes at m to M, N and C respectively. Now T_N is the complex plane perpendicular to

$$v = \left(\frac{\partial k}{\partial z_0}, \frac{\partial k}{\partial z_1}, \frac{\partial k}{\partial z_2}, \frac{\partial k}{\partial x_1}, \cdots, \frac{\partial k}{\partial x_r}\right)_m$$

and T_C is the real plane perpendicular to $v' = (z_0, z_1, z_2, 0, \cdots, 0)$. We must show that (kernel φ_0) $+ T_M = \mathbf{C}^{r+3}$. But $T_M = T_N \cap T_C$ and $v' \in$ kernel φ_0. Hence it is sufficient to show that (kernel φ_0) $+ T_N = \mathbf{C}^{r+3}$ or equivalently that $T_N \not\supset$ kernel φ_0. Now suppose kernel $\varphi_0 \subset T_N$. Then $(y_0, y_1, y_2, 0, \cdots, 0)$ is perpendicular to v for all $(y_0, y_1, y_2) \in \mathbf{C}^3$. But then

$$\frac{\partial k}{\partial z_2}(m) = \frac{\partial k}{\partial z_1}(m) = \frac{\partial k}{\partial z_0}(m) = 0 \ .$$

But $\varphi_1^{-1}(x_1, \cdots, x_r)$ has an isolated singularity at o. Hence, we get a contradiction.

Remarks. (1) An analogous theorem holds for polynomials in more variables.

(2) It should be noted that we have constructed a complex analytic deformation between V and V_o.

Definition (3.1.5). Let

$$V(a_0, a_1, a_2; \mathrm{I}) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\} \ ,$$

and

$$K(a_0, a_1, a_2; \mathrm{I}) = V(a_0, a_1, a_2; \mathrm{I}) \cap S^5$$

where S^5 is the unit sphere in $\mathbf{C}^3$. Use similar definitions in the other classes.

Remark (3.1.6). Although it is convenient to discuss these classes separately, it is clear from the above that the *weights* form a complete set of invariants for the variety. Thus either (w_0, w_1, w_2) determine the class of a polynomial or, if more than one class is possible, the corresponding varieties are diffeomorphic.

3.2. *Weights.* Let $h(Z_0, Z_1, Z_2)$ be weighted homogeneous with weights $w_i = u_i/v_i$, $i = 0, 1, 2$ in reduced form. For integers $a_1, a_2, \cdots, a_k$ let $(a_1, a_2, \cdots, a_k)$ denote their greatest common divisor. Define

$$c = (u_0, u_1, u_2);\ c_0 = (u_1, u_2)/c;\ c_1 = (u_0, u_2)/c;\ c_2 = (u_0, u_1)/c\ .$$

Then for some positive integers $\gamma_0, \gamma_1, \gamma_2$ we have

$$u_0 = cc_1c_2\gamma_0,\ u_1 = cc_0c_2\gamma_1,\ u_2 = cc_0c_1\gamma_2\ .$$

Note that c_0, c_1, c_2 are pairwise relatively prime, $\gamma_0, \gamma_1, \gamma_2$ likewise and $(c_i, \gamma_i) = 1$ for $i = 0, 1, 2$. Thus we have

$$d = \langle w_0, w_1, w_2\rangle = cc_0c_1c_2\gamma_0\gamma_1\gamma_2$$

and

$$q_0 = v_0c_0\gamma_1\gamma_2\ ,\ q_1 = v_1c_1\gamma_0\gamma_2\ ,\ q_2 = v_2c_2\gamma_0\gamma_1\ .$$

In the six classes we note the following.

Class I. $w_i = a_i$, so $v_i = 1$ for $i = 0, 1, 2$.

Class II. $w_0 = a_0$, $w_1 = a_1$, $w_2 = a_1a_2/(a_1 - 1)$, so $v_0 = v_1 = c_2 = \gamma_1 = 1$.

Class III. $w_0 = a_0$, $w_1 = (a_1a_2 - 1)/(a_2 - 1)$, $w_2 = (a_1a_2 - 1)/(a_1 - 1)$, so $v_0 = c_1 = c_2 = \gamma_1 = \gamma_2 = 1$.

Class IV. $w_0 = a_0$, $w_1 = a_0a_1/(a_0 - 1)$, $w_2 = a_0a_1a_2/(a_0a_1 - a_0 + 1)$, so $v_0 = c_1 = c_2 = \gamma_0 = \gamma_1 = 1$.

Class V. $w_0 = (a_0a_1a_2 + 1)/(a_1a_2 - a_2 + 1)$, $w_1 = (a_0a_1a_2 + 1)/(a_0a_2 - a_0 + 1)$, $w_2 = (a_0a_1a_2 + 1)/(a_0a_1 - a_1 + 1)$, so $c_0 = c_1 = c_2 = \gamma_0 = \gamma_1 = \gamma_2 = 1$.

Class VI. The polynomial $Z_0^{a_0} + Z_1Z_2$ is analytically isomorphic to the polynomial $Z_0^{a_0} + Z_1^2 + Z_2^2$ so it may be treated as a subclass of I.

3.3. *Orbits with non-trivial stability groups.* Recall that the $U(1)$ action on K is defined by

$$t(z_0, z_1, z_2) = (t^{q_0}z_0, t^{q_1}z_1, t^{q_2}z_2)\ , \qquad t \in U(1)\ .$$

If $z_0 \neq 0$, $z_1 \neq 0$, $z_2 \neq 0$ then the orbit of (z_0, z_1, z_2) has trivial stability group. On the other hand it is clear that for example

$$K(a_0, a_1, a_2; \mathrm{I}) \cap \{z_0 = 0\} = \{z_1^{a_1} + z_2^{a_2} = 0;\ |z_1|^2 + |z_2|^2 = 1\}$$

is fixed by the subgroup of $U(1)$ that fixes z_1 and z_2 pointwise; that is $\mathbf{Z}_{(q_1, q_2)} = \mathbf{Z}_{\gamma_0}$. This set consists of a collection of linked, knotted circles whose number equals the number of irreducible factors in the factorization of $Z_1^{a_1} + Z_2^{a_2}$ over the complex numbers, $n_0 = (a_1, a_2) = cc_0$.

In each class there are at most three non-trivial stability groups. If we call their orders α_0, α_1, α_2 and the number of orbits n_0, n_1 and n_2 respectively, then the following table arises.

	α_0	n_0	α_1	n_1	α_2	n_2
I	γ_0	cc_0	γ_1	cc_1	γ_2	cc_2
II	γ_0	$(cc_0 - 1)/v_2$	$v_2\gamma_0$	1	γ_2	c
III	γ_0	$(cc_0 - v_1 - v_2)/v_1v_2$	$v_2\gamma_0$	1	$v_1\gamma_0$	1
IV	γ_2	$(c - 1)/v_1$	v_2	1	$v_1\gamma_2$	1
V	v_0	1	v_1	1	v_2	1

3.4. *Slice representation.* In this section we compute the β_j determining the representation $\mathbf{Z}_{\alpha_j} \to SO(2)$, $0 \leqq \beta_j < \alpha_j$ for the stability groups of (3.3), see [12].

Consider the n_0 orbits with stability group $\mathbf{Z}_{\alpha_0}$ in $K(a_0, a_1, a_2; \mathrm{I})$. Since all of $\{z_0 = 0\} \cap S^5$ has the same stability group, all n_0 orbits are of the same type, (α_0, β_0) for some β_0. If we let $\xi = \exp(2\pi i/\alpha_0)$ then the action in the slice is described by $\xi(z_0, z_1, z_2) = (\xi^{q_0}z_0, z_1, z_2)$ and by definition [12]*, $q_0\beta_0 \equiv -1 \pmod{\alpha_0}$.

The orbit with stability group $\mathbf{Z}_{\alpha_1}$ in $K(a_0, a_1, a_2; \mathrm{II})$ is $\{z_0 = z_1 = 0, |z_2|^2 = 1\}$. At $z_2 = 1$ the slice is $\{z_0^{a_0} + z_1^{a_1} + z_1 = 0\} \cap S^5$. Very near $(0, 0, 1)$ we may approximate it by $\{z_0^{a_0} + z_1 = 0\} \cap S^5$, hence the action in the slice is determined by the projection into the z_0 plane, thus $q_0\beta_1 \equiv -1 \pmod{\alpha_1}$.

Similar considerations result in the table below, where each entry is congruent to -1 modulo the α_j on the top of its column. This determines the β_j since $0 \leqq \beta_j < \alpha_j$.

	α_0	α_1	α_2
I	$q_0\beta_0$	$q_1\beta_1$	$q_2\beta_2$
II	$q_0\beta_0$	$q_0\beta_1$	$q_2\beta_2$
III	$q_0\beta_0$	$q_0\beta_1$	$q_0\beta_2$
IV	$q_2\beta_0$	$q_0\beta_1$	$q_2\beta_2$
V	$q_2\beta_0$	$q_0\beta_1$	$q_1\beta_2$

* In [12] we defined β by the equation $q\beta \equiv 1 \pmod{\alpha}$. Reversing orientation sends (α, β) into $(\alpha', \beta') = (\alpha, \alpha - \beta)$ and hence $q\beta' \equiv -1 \pmod{\alpha}$. This turns out to be the induced orientation in our case.

3.5. *The genus of $\tilde{X}$.*

PROPOSITION (3.5.1). *With the notation above*

$$2g = \frac{c^2c_0c_1c_2 - c(c_0v_0 + c_1v_1 + c_2v_2) + v_0v_1 + v_1v_2 + v_2v_0 - v_0v_1v_2}{v_0v_1v_2} .$$

For a proof consider the normalization Y' of X' and compute its genus using classical formulas. The map $Y' \to X$ allows one to find the genus of X and note (1.3) that $\tilde{X}$ is isomorphic to X. We shall carry out the verification of the above formula in Class II and leave the remaining cases to the reader. First we need a definition and a lemma.

Definition (3.5.2). Suppose p_0 and p_1 are positive integers, $p_1 \leqq p_0$. Then there are unique integers $p_2, \cdots, p_t, r_1, \cdots, r_t$ such that

$$p_{i-1} - r_ip_i = p_{i+1} \quad 0 < p_{i+1} \leqq p_i ,$$

$i = 1, 2, \cdots, t-1$ and $r_tp_t = p_{t-1}$. Then $p_t = (p_0, p_1)$ and we define $| p_0, p_1 |$ to be

$$| p_0, p_1 | = \sum_{i=1}^{t} r_ip_i(p_i - 1) = p_0p_1 - p_0 - p_1 + (p_0, p_1) .$$

If $X \subset \mathbf{P}^2_{\mathbf{C}}$ is defined by a homogeneous polynomial of degree d and Y is the desingularization of X then the genus of Y is equal to $(1/2)(d-1)(d-2) - \sum_{x \in X}\delta_x$. Suppose X is a curve, $x \in X$ is the only singular point of X and $X \subset V$ where V is a non-singular surface. Define inductively $\pi_i: V_i \to V_{i-1}$ where $V_0 = V$, $X_0 = X$, π_i is the monoidal transform with center at the singular points of X_{i-1} and X_i is the proper transform of X_{i-1} by π_i. There is an integer n such that X_n is non-singular. By the classical Plücker formula [5, Th. 1]

$$2\delta_x = \sum_{i=0}^{n} \sum_{y \in X_i} m_y(m_y - 1)$$

where m_y is the multiplicity of the singular point y.

LEMMA (3.5.3). *Suppose X is the curve in $\mathbf{C}^2$ given by*

$$z_0^{p_0} + z_1^{p_1} + z_0^az_1^b = 0$$

where $p_0 \geqq p_1$, $a + b \geqq p_0$, $b \geqq p_1$. Then the singularity of X at the origin has order

$$\delta_0 = \frac{1}{2}| p_0, p_1 | .$$

Moreover if $f: Y \to X$ is the desingularization of X then $f^{-1}(o)$ consists of (p_0, p_1) points.

Proof. Blow up and proceed inductively.

We shall also need the classical Hurwitz formula. Let $\varphi: X' \to X$ be a finite morphism of compact non-singular complex analytic curves. Then

$$(2 - 2g_{X'}) = (\text{degree } \varphi)(2 - 2g_X) - \sum_{x' \in X'} \big(e(x') - 1\big)$$

where g denotes genus and e denotes ramification index.

Proof of (3.5.1) *for class* II. $z_0^{a_0} + z_1^{a_1} + z_1 z_2^{a_2} = 0$. With the notation of (1.2) and (3.3), X' is defined by the homogeneous equation

$$z_0^d + z_1^d + z_1^{q_1} z_2^{q_2 a_2} = 0 .$$

It has precisely one singular point, at (0, 0, 1). In affine coordinates this singularity is given by

$$z_0^d + z_1^d + z_1^{q_1} = 0 .$$

By (3.5.3) $\delta = d\big((q_1 - 1)/2\big)$. Let f: $Y' \to X'$ be the desingularization of X'. Then the genus of Y' is $g_{Y'} = (1/2)(d-1)(d-2) - d\big((q_1 - 1)/2\big)$. Now we apply the Hurwitz formula to the covering $Y' \to X$ to get the genus of X. It is sufficient to find the ramification indices of the map. Consider the map φ: $V' - \{0\} \to V - \{0\}$. It is ramified along the three sets $z_0 = 0$, $z_1 = 0$, and $z_2 = 0$. Above the n_0 orbits given by $z_0 = 0$, $z_1^{a_1 - 1} + z_2^{a_2} = 0$ (cf. 3.3) there are $q_2 a_2$ orbits given by $z_0 = 0$, $z_1^{q_2 a_2} + z_2^{q_2 a_2} = 0$. Hence we get a contribution to the ramification of $q_0 q_1 q_2 n_0 - q_2 a_2$. Similarly we get a contribution of $q_0 q_1 q_2 n_2 - d$ from the n_2 orbits given by $z_2 = 0$, $z_0^{a_0} + z_1^{a_1} = 0$. Finally there is one orbit defined by $z_0 = z_1 = 0$ and there is one orbit in V' lying above this defined by $z_0 = z_1 = 0$. This orbit corresponds to the singular point of X'. There are $(q_1, d) = q_1$ points of Y' lying above this point (by 3.5.3). Hence we get a contribution of $q_0 q_1 q_2 - q_1$ to the ramification. We conclude that

$$2g_X = \frac{d(d - q_1)}{q_0 q_1 q_2} - n_0 - n_2 + 1 .$$

Substituting from (3.3) gives

$$2g = \frac{c^2 c_0 c_1 - c(c_0 + c_1 + v_2) + v_2 + 1}{v_2}$$

as required.

3.6. *Computation of b.*

THEOREM (3.6.1). *Suppose V is an algebraic surface in* $\mathbf{C}^{n+1}$ *with an isolated singularity at o and V is invariant under a good* $\mathbf{C}^*$*-action σ of the form*

$$\sigma\big(t, (z_0, \cdots, z_n)\big) = (t^{q_0} z_0, \cdots, t^{q_n} z_n) .$$

Assume that V is not contained in any coordinate hyperplane. The manifold $K = V \cap S^{2n+1}$ has an induced circle action with exceptional orbits $O_1, \cdots, O_r$ of type $(\alpha_1, \beta_1), \cdots, (\alpha_r, \beta_r)$ respectively. Then

$$b = \frac{d}{q_0 \cdots q_n} + \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i}$$

where d is the degree of the cone V' over V.

Remark (3.6.2). Suppose U and U' are non-singular surfaces, X is a non-singular curve on U and $f\colon U' \to U$ is a proper birational map. Then by a theorem of Zariski [18], f is a composite of σ-transforms $f_i\colon U_i \to U_{i-1}$, $U = U_0$, $U' = U_n$ where f_i has center $x_{i-1} \in U_{i-1}$. Let $X_0 = X$, and $X_i = f_i^{\#}(X_{i-1})$ the proper transform. Then

$$\big(f^{\#}(X)\cdot f^{\#}(X)\big) + m = (X\cdot X)$$

where m is the number of indices such that $x_i \in X_i$. We shall say that "*f has m centers along X*". If $f\colon U' \to U$ is a finite covering of degree n then $n(X.\, X) = \big(f^*(X).\, f^*(X)\big)$ where $f^*(X)$ is the inverse image of the divisor (or cohomology) class determined by X.

Proof of (3.6.1). The idea of the proof is as follows. Let $\varphi\colon V' \to V$ be the covering map. Recall that $V = V'/G$ where $G = \mathbf{Z}_{q_0} \oplus \cdots \oplus \mathbf{Z}_{q_n}$. Now let F, X, F', X' be as in (1.2) and let $h\colon Y' \to X'$ be the desingularization of X'. Define $F_0 = F' \times_{X'} Y'$. Clearly F' is a line bundle of degree $-d$ over X', hence F_0 is a line bundle of degree $-d$ over Y' and $(Y'\cdot Y')_{F_0} = -d$. Now we shall construct non-singular varieties W_0 and V_1 and birational maps $\tau\colon W_0 \to F_0$ and $\rho_1\colon V_1 \to \tilde{V}$ and a map $\eta\colon W_0 \to V_1$ so that $\pi\tau = \tilde{\rho}\rho_1\eta$.

$$\begin{array}{ccccc} W_0 & & \xrightarrow{\quad\tau\quad} & & F_0 \\ \eta\big\downarrow & & & & \big\downarrow\pi \\ V_1 & \xrightarrow{\rho_1} & \tilde{V} & \xrightarrow{\tilde{\rho}} & F\,. \end{array}$$

Let $Y_0 = \tau^{\#}(Y')$, $\tilde{X} = \tilde{\rho}^{\#}(X)$ and $X_1 = \rho_1^{\#}(\tilde{X})$. We want to compute $-b = (\tilde{X}\cdot\tilde{X})_{\tilde{V}}$. It is easy to see that degree $\eta =$ degree $\eta \mid Y_0$ and hence η is unramified along Y_0 and therefore we get:

$$\big(\textstyle\prod_{i=0}^{n} q_i\big)(X_1.\, X_1)_{V_1} = \big(\eta^*(X_1)\cdot\eta^*(X_1)\big)_{W_0} = (Y_0.\, Y_0)_{W_0}\,.$$

For the desired result it is sufficient to show that ρ_1 has no centers along $\tilde{X}$ and τ has

$$\big(\textstyle\prod_{i=0}^{n} q_i\big)\sum_{j=1}^{r}\frac{\beta_j}{\alpha_j}$$

centers along Y'.

The construction of W_0 will be a local process, i.e., it will be a composite of monoidal transforms with centers over the finite number of fixed points of G on F_0. We consider the following general situation.

LEMMA (3.6.3). *Let G be a finite abelian group acting analytically on* $\mathbf{C}^2$ *leaving o fixed. Let Y be a non-singular curve passing through o. Assume that Y is invariant under G and the action of G on $\mathbf{C}^2$ is effective. Let* $U = \mathbf{C}^2/G$, $\pi\colon \mathbf{C}^2 \to U$, $u = \pi(o)$ *and* $Z = \pi(Y)$. *Then for some relatively prime integers, α, β, $0 \leqq \beta < \alpha$ we have that $u \in U$ is the quotient singularity of type $\langle\alpha, \beta\rangle$,* [cf. 2]. *There is a canonical resolution* $\gamma\colon U''_{\alpha,\beta} \to U$ *depending only on α and β (not on G) so that there is a proper birational morphism of non-singular surfaces* $\sigma_{\alpha,\beta}\colon \tilde{S}_{\alpha,\beta} \to \mathbf{C}^2$, *an extension of the action of G to $\tilde{S}_{\alpha,\beta}$ and a map* $\tilde{S}/G \to U''_{\alpha,\beta}$ *which is an isomorphism in a neighborhood of* $Z'' = \gamma^{\#}(Z)$ *and σ has (order G)(β/α) centers along Y.*

Proof. We may choose coordinates (z_0, z_1) in $\mathbf{C}^2$ so that $Y = \{z_1 = 0\}$, and G acts linearly [2a]. Then it is easy to show that $G = G' \oplus H$ where G' is cyclic generated by $\begin{pmatrix} \xi & 0 \\ 0 & \xi^{\beta} \end{pmatrix}$, H is cyclic generated by $\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$, $\zeta = e^{2\pi i/k}$, $\xi = e^{2\pi i/\alpha}$, $0 \leqq \beta < \alpha$ and $(\alpha, \beta) = 1$. Suppose first that $H = (0)$. Let $T_0 = \mathbf{C}^2$, $Y_0 = Y$ and $y_0 = o$. Define inductively for $i = 1, \cdots, \beta$ $\tau_i\colon T_i \to T_{i-1}$ the σ transform with center y_{i-1}, $Y_i = \tau_i^{\#}(Y_{i-1})$, $y_i =$ the point of intersection of Y_i and $\tau_i^{-1}(y_{i-1})$. The action of G extends to T_β. Let $U' = T_\beta/G$, $\pi_\beta\colon T_\beta \to U'$. Then one can easily compute the action of G on T_β to see that U' has no singularities along $\pi_\beta(Y_\beta)$. Let U'' be the minimal resolution of U'. In this case $\bigl(H = (o)\bigr)$ we construct $\tilde{S}$ to be the minimal resolution of $T_\beta \times_{U'} U''$. The composite map $\sigma\colon \tilde{S} \to \mathbf{C}^2$ has β centers along Y and this is the desired result since order $G = \alpha$. The general case follows from the following fact. If $\mu\colon \mathbf{C}^2 \to \mathbf{C}^2$, $\mu(z_0, z_1) = (z_0^k, z_1)$, and $\tau_1\colon T_1 \to \mathbf{C}^2$ is the σ transform with center o then we must perform a composite of k monoidal transforms centered at o, $\tau'_1\colon T'_1 \to \mathbf{C}^2$ to get a commutative diagram

$$\begin{array}{ccc} T'_1 & \longrightarrow & \mathbf{C}^2 \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ T_1 & \longrightarrow & \mathbf{C}^2 . \end{array}$$

Now we return to the proof of the theorem. If $y \in Y'$ is any fixed point of the action of G there is a neighborhood of y satisfying the hypotheses of Lemma (3.6.3). Performing the composite of monoidal transforms $\sigma_{\alpha,\beta}$ described in the lemma, at each fixed point, we get $\tau\colon W_0 \to F_0$ and $\rho_1\tilde{\rho}\colon V_1 \to F$. Let $\tilde{\gamma}_{\alpha,\beta}\colon \tilde{U}_{\alpha,\beta} \to U_{\alpha,\beta}$ be the minimal resolution of $U_{\alpha,\beta}$, $\tilde{Z} = \tilde{\gamma}^{\#}_{\alpha,\beta}(Z)$ and let $\delta(\alpha, \beta)$ be the number of centers of the canonical map $\gamma''_{\alpha,\beta}\colon U''_{\alpha,\beta} \to \tilde{U}_{\alpha,\beta}$ on $\tilde{Z}$. Locally ρ_1 is of the form $\gamma''_{\alpha,\beta}$ hence

$$b = \frac{d}{q_0 \cdots q_n} + \sum_{i=1}^{r}\left(\frac{\beta_i}{\alpha_i} - \delta(\alpha_i, \beta_i)\right).$$

The proof will be completed by showing that $\delta(\alpha_i, \beta_i) = 0$ for all i.

LEMMA (3.6.4). *If V is the variety defined by*

$$z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0$$

with $(a_0, a_1, a_2) = 1$ then the theorem holds for V. Moreover this implies $\delta(\alpha, \beta) = 0$, for all pairs (α, β) with $0 < \beta < \alpha$, $(\alpha, \beta) = 1$.

Proof. Recall that the intersection matrix of a resolution is negative definite. From this we concluded (2.6.2) that the determinant p of the relation matrix of $H_1(K)$ is positive and by (2.5)

$$p = \alpha_1 \cdots \alpha_n\left(b - \sum_{i=1}^n \beta_i/\alpha_i\right) .$$

In the above case we have c_i orbits with stability group $\mathbf{Z}_{\alpha_i}$ and hence

$$p = \alpha_0^{c_0-1}\alpha_1^{c_1-1}\alpha_2^{c_2-1}(b\alpha_0\alpha_1\alpha_2 - c_0\beta_0\alpha_1\alpha_2 - c_1\beta_1\alpha_0\alpha_2 - c_2\beta_2\alpha_0\alpha_1) .$$

Now p is a positive integer hence

$$b \geqq \frac{1}{\alpha_0\alpha_1\alpha_2} + \sum c_i \frac{\beta_i}{\alpha_i} .$$

But $b = d/(q_0q_1q_2) + \sum c_i(\beta_i/\alpha_i) - \sum \delta(\alpha_i, \beta_i)$ and by (3.3), $d/(q_0q_1q_2) = 1/(\alpha_0\alpha_1\alpha_2)$ thus we must have $\delta(\alpha_i, \beta_i) = 0$, $i = 0, 1, 2$.

For a given orbit type (α, β), $0 < \beta < \alpha$, $(\alpha, \beta) = 1$ find ξ so that $0 < \xi < \alpha$ and $\xi\beta \equiv -1 \pmod{\alpha}$. Apply the above lemma to the variety of $\{z_0^\alpha + z_1^\xi + z_2^\xi = 0\}$ which has one orbit of type (α, β). This shows that $\delta(\alpha, \beta) = 0$ for all orbit types and completes the proof of Theorem (3.6.1).

Remark (3.6.5). According to Milnor [9] there is a fibration $F \to S^5 - K \to S^1$, where the fiber F is an open 4-manifold. Let $h: F \to F$ be the characteristic map of this fibration and I the identity map of F. It is proved in [9, § 8] that

$$0 \longrightarrow H_2K \longrightarrow H_2F \xrightarrow{I_*-h_*} H_2F \longrightarrow H_1K \longrightarrow 0$$

is exact. It follows from (2.4) that the group H_1K has rank $2g$ and the order of its torsion equals p. Let $\Delta(t) = \det(tI_* - h_*)$ denote the characteristic polynomial of h_* and let κ be the exponent of $(t-1)$ in $\Delta(t)$.

In [10] $\Delta(t)$ and κ are computed for weighted homogeneous polynomials and it is noted that the minimal polynomial of h_* has no multiple roots. Thus $\kappa = \text{rank } H_1(K)$, providing an alternate way of computing $2g$. It would be of interest to obtain p also using [10].

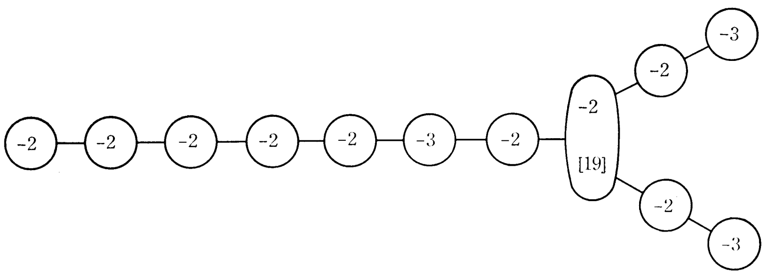
We conclude with an example. The weighted homogeneous polynomial $Z_0^{105} + Z_1^9 + Z_1Z_2^{14}$ is of class II. Its variety has an isolated singularity at the origin, whose resolution may be found as follows.

From (3.2) the weights are (105, 9, 63/4), hence $c = 3$, $c_0 = 3$, $c_1 = 7$, $\gamma_0 = 5$, $v_2 = 4$, $c_2 = \gamma_1 = \gamma_2 = v_0 = v_1 = 1$ and $d = 315$, $q_0 = 3$, $q_1 = 35$, $q_2 = 20$. From (3.3) $\alpha_0 = 5$, $n_0 = 2$; $\alpha_1 = 20$, $n_1 = 1$; $\alpha_2 = 1$, $n_2 = 3$. From (3.4) $\beta_0 = 3$, $\beta_1 = 13$, $\beta_2 = 0$. From (3.5) $2g = 38$ and from (3.6) $b = 2$.

Since orbits of type (1, 0) are principal and do not appear in the resolution, they may be omitted. Thus

$$K(105, 9, 14; \mathrm{II}) = \{-2; (o, 19, 0, 0); (5, 3), (5, 3), (20, 13)\} .$$

We note that $5/3 = [2, 3]$ and $20/13 = [2, 3, 2, 2, 2, 2, 2]$ and apply Theorem (2.6.1) to conclude that the graph of the resolution is as below.



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