S.P. Novikov (Ed.)

Topology I

General Survey

With 78 Figures

Springer

Encyclopaedia of Mathematical Sciences

Volume 12

Editor-in-Chief: R.V. Gamkrelidze

Topology

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Introduction

In the present essay, we attempt to convey some idea of the skeleton of topology, and of various topological concepts. It must be said at once that, apart from the necessary minimum, the subject-matter of this survey does not include that subdiscipline known as "general topology" — the theory of general spaces and maps considered in the context of set theory and general category theory. (Doubtless this subject will be surveyed in detail by others.) With this qualification, it may be claimed that the "topology" dealt with in the present survey is that mathematical subject which in the late 19th century was called *Analysis Situs*, and at various later periods separated out into various subdisciplines: "Combinatorial topology", "Algebraic topology", "Differential (or smooth) topology", "Homotopy theory", "Geometric topology".

With the growth, over a long period of time, in applications of topology to other areas of mathematics, the following further subdisciplines crystallized out: the global calculus of variations, global geometry, the topology of Lie groups and homogeneous spaces, the topology of complex manifolds and algebraic varieties, the qualitative (topological) theory of dynamical systems and foliations, the topology of elliptic and hyperbolic partial differential equations. Finally, in the 1970s and 80s, a whole complex of applications of topological methods was made to problems of modern physics; in fact in several instances it would have been impossible to understand the essence of the real physical phenomena in question without the aid of concepts from topology.

Since it is not possible to include treatments of all of these topics in our survey, we shall have to content ourselves here with the following general remark: Topology has found impressive applications to a very wide range of problems concerning qualitative and stability properties of both mathematical and physical objects, and the algebraic apparatus that has evolved along with it has led to the reorientation of the whole of modern algebra.

The achievements of recent years have shown that the modern theory of Lie groups and their representations, along with algebraic geometry, which subjects have attained their present level of development on the basis of an ensemble of deep algebraic ideas originating in topology, play a quite different role in applications: they are applied for the most part to the exact formulaic investigation of systems possessing a deep internal algebraic symmetry. In fact this had already been apparent earlier in connexion with the exact solution of problems of classical mechanics and mathematical physics; however it became unequivocally clear only in modern investigations of systems that are, in a certain well-defined sense, integrable. It suffices to recall for instance the method of inverse scattering and the (algebro-geometric) finite-gap integration of non-linear field systems, the celebrated solutions of models of statistical physics and quantum field theory, self-dual gauge fields, and string theory. (One particular aspect of this situation is, however, worthy of note, namely the need for a serious "effectivization" of modern algebraic geometry, which would return the subject in spirit to the algebraic geometry of the 19th century, when it was regarded as a part of formulaic analysis.

This survey constitutes the introduction to a series of essays on topology, in which the development of its various subdisciplines will be expounded in greater detail.

Introduction to the English Translation

This survey was written over the period 1983-84, and published (in Russian) in 1986. The English translation was begun in 1993. In view of the appearance in topology over the past decade of several important new ideas, I have added an appendix summarizing some of these ideas, and several footnotes, in order to bring the survey more up-to-date.

I am grateful to several people for valuable contributions to the book: to M. Stanko, who performed a huge editorial task in connexion with the Russian edition; to B. Botvinnik for his painstaking work as scientific editor of the English edition, in particular as regards its modernization; to R. Burns for making a very good English translation at high speed; and to C. Shochet for advice and help with the translation and modernization of the text at the University of Maryland. I am grateful also to other colleagues for their help with modernizing the text.

Sergei P. Novikov,
November, 1995

Chapter 1
The Simplest Topological Properties

Topology is the study of topological properties or *topological invariants* of various kinds of mathematical objects, starting with rather general geometrical figures. From the topological point of view the name "geometrical figures" signifies: general polyhedra (polytopes) of various dimensions (*complexes*); or continuous or smooth "surfaces" of any dimension situated in some Euclidean space or regarded as existing independently (*manifolds*); or sometimes subsets of a more general nature of a Euclidean space or manifold, or even of an infinite-dimensional space of functions. Although it is not possible to give a precise general definition of "topological property" ("topological invariant") of a geometrical figure (or more general geometrical structure), we may describe such a property intuitively as one which is, generally speaking, "stable" in some well-defined sense, i.e. remains unaltered under small changes or deformations (*homotopies*) of the geometrical object, no matter how this is given to us. For instance for a general polytope (*complex*) the manner in which the polytope is given may be, and often is, changed by means of an operation of subdivision, whereby each face of whatever dimension is subdivided into smaller parts, and so converted into a more complex polyhedron, the subdivision being carried out in such a way as to be compatible on that portion of their boundaries shared by each pair of faces. In this way the whole polyhedron becomes transformed formally into a more complicated one with a larger number of faces of each dimension. The various topological properties, or numerical or algebraic invariants, should be the same for the subdivided complex as for the original.

The simplest examples. 1) Everyone is familiar with the elementary result called "Euler's Theorem", which, so we are told, was in fact known prior to Euler:

For any closed, convex polyhedron in 3-dimensional Euclidean space $\mathbb{R}^3$, *the number of vertices less the number of edges plus the number of (2-dimensional) faces, is* 2.

Thus the quantity $V - E + F$ is a topological invariant in that it is the same for any subdivision of a convex polyhedron in $\mathbb{R}^3$.

2) Another elementary observation of a topological nature, also dating back to Euler, is the so-called "problem of the three pipelines and three wells". Here one is given three points a_1, a_2, a_3 in the plane $\mathbb{R}^2$ (three "houses") and three other points A_1, A_2, A_3 ("wells"), and it turns out that it is not possible to join each house a_i to each well A_j by means of a non-self-intersecting path ("pipeline") in such a way that no two of the 9 paths intersect in the plane. (Of course, this *is* possible in $\mathbb{R}^3$.) In topological language this conclusion may be rephrased as follows: Consider the one-dimensional complex (or *graph*) consisting of 6 vertices a_i, A_j, and 9 edges x_{ij}, $i, j = 1, 2, 3$, where the "boundary" of each edge, denoted by ∂x_{ij}, is given by $\partial x_{ij} = \{a_i, A_j\}$. The conclusion is that this one-dimensional complex cannot be situated in the plane $\mathbb{R}^2$ without incurring self-intersections. This represents a topological property of the given complex. □

These two observations of Euler may be considered as the archetypes of the basic ideas of combinatorial topology, i.e. of the topological theory of polyhedra and complexes established much later by Poincaré. It is important to bear in mind that the use of combinatorial methods to define and investigate topological properties of geometrical figures represents just one interpretation of such properties, providing a convenient and rigorous approach to the formulation of these concepts at the first stage of topology, though of course remaining useful for certain applications. However those same topological properties admit of alternative formulations in various different situations, for instance in the contexts of differential geometry and mathematical analysis. For an example, let us return to the general convex polyhedron of Example 1 above. By smoothing off its corners and edges a little, we obtain a general smooth, closed, convex surface in $\mathbb{R}^3$, the boundary of a convex solid. Denote this surface by M^2. At each point x of this surface the Gaussian curvature $K(x)$ is defined, as also the area-element $d\sigma(x)$, and we have the following formula of Gauss:

$$\frac{1}{2\pi}\iint\limits_{M^2} K(x)d\sigma(x) = 2. \tag{0.1}$$

In the sequel it will emerge that this formula reflects the same topological property as does Euler's theorem concerning convex polyhedra. (Euler's theorem can be deduced quickly from the Gauss formula (0.1) by continuously deforming a suitable surface into the given convex polyhedron and taking into account the relationship between the integral of the Gaussian curvature and the solid angles at the vertices.) Note that the formula (0.1) holds also for nonconvex closed surfaces "without holes". A third interpretation, as it turns out, of the same general topological property (which we have still not formulated!) lies hidden in the following observation, attributed to Maxwell: Consider an island with shore sloping steeply away from the island's edge into the sea, and whose surface has no perfectly planar or linear features; then the number of peaks plus the number of pits less the number of passes is exactly 1. This may be easily transformed into an assertion about closed surfaces in $\mathbb{R}^3$ by formally extending the island's surface underneath so that it is convex everywhere under the water (i.e. by imagining the island to be "floating", with a convex underside satisfying the same assumption as the surface). The resulting floating island then has one further pit, namely the deepest point on it. We conclude that for a closed surface in $\mathbb{R}^3$ satisfying the above assumption, the number of peaks (points of locally maximum height) plus the number of pits (local minimum points) less the number of passes (saddle points) is equal to 2, the same number as appears in both Euler's theorem and the Gauss formula (0.1) for surfaces without holes.

What if the polyhedron or closed surface in $\mathbb{R}^3$ or floating island is more complicated? With an arbitrary closed surface M^2 in $\mathbb{R}^3$ we may associate an integer, its "genus" $g \geq 0$, naively interpreted as the "number of holes". Here we have the Gauss-Bonnet formula

$$\frac{1}{2\pi}\iint\limits_{M^2} K(x)d\sigma(x) = 2 - 2g, \tag{0.2}$$

and the theorems of Euler and Maxwell become modified in exactly the same way: the number 2 is replaced by $2 - 2g$. Since Poincaré it has become clear that these results prefigure general relationships holding for a very wide class of geometrical figures of arbitrary dimension.

Gauss also discovered certain topological properties of non-self-intersecting (i.e. simple) closed curves in $\mathbb{R}^3$. It is well known that a simple, closed, continuous (or if you like smooth, or piecewise smooth, or even piecewise linear) curve separates the plane $\mathbb{R}^2$ into two parts with the property that it is impossible to get from one part to the other by means of a continuous path avoiding the given curve. The ideally rigorous formulation of this intuitively obvious fact in the context of an explicit system of axioms for geometry and analysis carries the title "The Jordan Curve Theorem" (although of course in fact it is, in somewhat simplified form, already included in the axiom system; if one is not concerned with economy in the axiom system, then it might just as well be included as one of the axioms). The same conclusion (as for a simple, closed, continuous curve) holds also for any "complete" curve in $\mathbb{R}^2$, i.e. a simple, continuous, unboundedly extended, non-closed curve both of those ends go off to infinity, without nontrivial limit points in the finite plane. This principle generalizes in the obvious way to n-dimensional space: a closed hypersurface in $\mathbb{R}^n$ separates it into two parts. In fact a local version of this principle is basic to the general topological definition of dimension (by induction on n).

There is however another less obvious generalization of this principle, having its most familiar manifestation in 3-dimensional space $\mathbb{R}^3$. Consider two continuous (or smooth) simple closed curves (loops) in $\mathbb{R}^3$ which do not intersect:

$$\gamma_1(t) = \left(x_1^1(t), x_1^2(t), x_1^3(t)\right), \qquad \gamma_1(t+2\pi) = \gamma_1(t),$$

$$\gamma_2(\tau) = \left(x_2^1(\tau), x_2^2(\tau), x_2^3(\tau)\right), \quad \gamma_2(\tau+2\pi) = \gamma_2(\tau).$$

Consider a "singular disc" D_i bounded by the curve γ_i, i.e. a continuous map of the unit disc into $\mathbb{R}^3$: $x_i^\alpha = x_i^\alpha(r, \phi)$, $i = 1, 2$, $\alpha = 1, 2, 3$, where $0 \le r \le 1$, $0 \le \phi \le 2\pi$, sending the boundary of the unit disc onto γ_i:

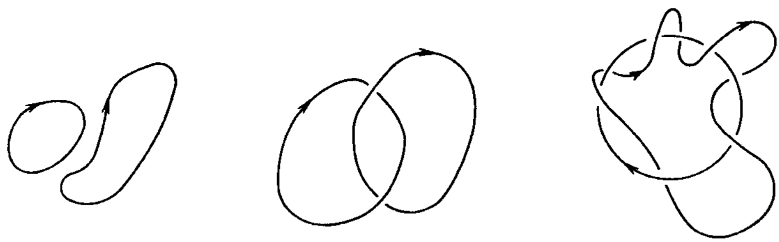
$$x_i^\alpha(r, \phi)|_{r=1} = x_i^\alpha(\phi), \quad \alpha = 1, 2, 3,$$

where $\phi = t$ for $i = 1$, and $\phi = \tau$ for $i = 2$.

Definition 0.1 Two curves γ_1 and γ_2 in $\mathbb{R}^3$ are said to be *nontrivially linked* if the curve γ_2 meets every singular disc D_1 with boundary γ_1 (or, equivalently, if the curve γ_1 meets every singular disc D_2 with boundary γ_2).

Simple examples are shown in Figure 1.1. In n-dimensional space $\mathbb{R}^n$ certain pairs of closed surfaces may be linked, namely submanifolds of dimensions p and q where $p + q = n - 1$. In particular a closed curve in $\mathbb{R}^2$ may be linked with a pair of points (a "zero-dimensional surface") – this is just the original principle that a simple closed curve separates the plane.

Gauss introduced an invariant of a link consisting of two simple closed curves γ_1, γ_2 in $\mathbb{R}^3$, namely the signed number of turns of one of the curves around the other, the *linking coefficient* $\{\gamma_1, \gamma_2\}$ of the link. His formula for this is



(a) Unlinked curves (b) Linking coefficient 1 (c) Linking coefficient 4

Fig. 1.1

$$N = \{\gamma_1, \gamma_2\} = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{([d\gamma_1(t), d\gamma_2(t)], \gamma_1 - \gamma_2)}{|\gamma_1(t) - \gamma_2(t)|^3}, \tag{0.3}$$

where [,] denotes the vector (or cross) product of vectors in $\mathbb{R}^3$ and (,) the Euclidean scalar product. Thus this integral always has an integer value N. If we take one of the curves to be the z–axis in $\mathbb{R}^3$ and the other to lie in the (x, y)–plane, then the formula (0.3) gives the net number of turns of the plane curve around the z–axis.

It is interesting to note that the linking coefficient (0.3) may be zero even though the curves are nontrivially linked (see Figure 1.2). Thus its having non-zero value represents only a sufficient condition for nontrivial linkage of the loops.

Elementary topological properties of paths and homotopies between them played an important role in complex analysis right from the very beginning of that subject in the 19th century. They without doubt represent one of

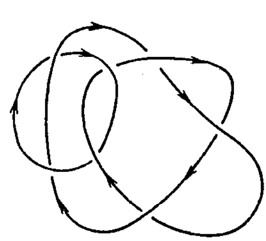


Fig. 1.2. The linking coefficient = 0, yet the curves are non-trivially linked

the most important features of the theory of functions of a complex variable, instrumental to the effectiveness and success of that theory in all of its applications. A complex analytic function $f(z)$ is often defined and single-valued only in a part of the complex plane, i.e. in some region $U \subset \mathbb{R}^2$ free of poles, branch points, etc. The Cauchy integral around each closed contour $\gamma \subset U$ yields a "topological" functional of the contour:

$$I_f(\gamma) = \oint_\gamma f(z)dz, \tag{0.4}$$

in the sense that the integral remains unchanged under continuous homotopies (deformations) of the curve γ within the region U, i.e. by deformations of γ avoiding the singular points of the function. It is this very latitude – the possibility of deforming the closed contour without affecting the integral – which opens up enormous opportunities for varied application.

More complicated topological phenomena appeared in the 19th century – in essence beginning with Abel and Riemann – in connexion with the investigation of functions $f(z)$ of a complex variable, given only implicitly by an equation

$$F(z,w) = 0, \quad w = f(z), \tag{0.5}$$

or else by means of analytic continuation throughout the plane, of a function originally given as analytic and single–valued only in some portion of the plane. The former situation arises in especially sharp form, as became clear after Riemann and Poincaré, in the context of Abel's resolution of the well-known problem of the insolubility of general algebraic equations by radicals, where the function $F(z,w)$ is a polynomial in two variables:

$$F(z,w) = w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0. \tag{0.6}$$

Such a polynomial equation has, in general, finitely many isolated branch points $z_1, \cdots, z_m$ in the plane, away from which it has exactly n distinct roots $w_j(z)$, $z \neq z_k$ $(k = 1, \ldots, m)$. Here the region U is just the plane $\mathbb{R}^2$ with the m branch points removed:

$$U = \mathbb{R}^2 \setminus \{z_1, \ldots, z_m\}.$$

It turns out that in general the branch points cannot be merely ignored, for the following reason. In some neighborhood of each point z_0 that is not a branch point, the equation (0.6) determines exactly n distinct functions $w_j(z)$ such that $F(z, w_j(z)) = 0$. If, however, we attempt to continue any one w_j of these functions analytically outside that neighborhood, we encounter a difficulty of the following sort: if we continue w_j along a path which goes round some of the branch points and back to the point z_0, it may happen that we obtain

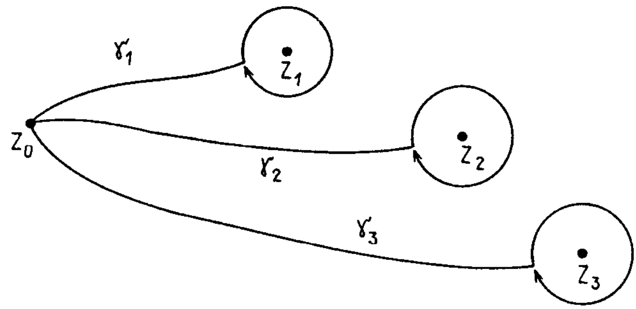


Fig. 1.3

nontrivial "monodromy", i.e. that we arrive at one of the other solutions at z_0:

$$w_s(z_0) \neq w_j(z_0), \quad s \neq j.$$

Proceeding more systematically, consider all possible loops $\gamma(t)$, $a \leq t \leq b$, in the region $U = \mathbb{R}_2 \setminus \{z_1, \cdots, z_m\}$, with $\gamma(a) = \gamma(b) = z_0$. Each such loop determines a permutation of the branches of the function $w(z)$: if we start at the branch given by $w_j(z)$ and continue around the loop from a to b, then we arrive when $t = b$ at the branch defined by w_s, so that the loop $\gamma(t)$ determines a permutation $j \to s$ of the branches (or sheets) above z_0:

$$\gamma \to \sigma_\gamma, \quad \sigma_\gamma(j) = s.$$

The inverse path γ^{-1} (i.e. the path traced backwards from b to a) yields the inverse permutation ${\sigma_\gamma}^{-1} : s \to j$, and the superposition $\gamma_1 \cdot \gamma_2$ of two paths γ_1 (traced out from time a to time b) and γ_2 (from b to c), i.e. the path obtained by following γ_1 by γ_2, corresponds to the product of the corresponding permutations:

$$\sigma_{\gamma_1 \cdot \gamma_2} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1}, \qquad \sigma_{\gamma^{-1}} = (\sigma_\gamma)^{-1}. \tag{0.7}$$

In the general, non-degenerate, situation the permutations of the form σ_γ generate the full symmetric group of permutations of n symbols. (This is the underlying reason for the general insolubility by radicals of the algebraic equation (0.6) for $n \geq 5$.) To see this, note that the "basic" path γ_j, $j = 1, \cdots, m$, which starts from z_0, encircles the single branch point z_j, and then proceeds back to z_0 along the same initial segment (see Figure 1.3) corresponds, in the typical situation of maximally non-degenerate branch points, to the interchange of two sheets (i.e. σ_{γ_j} is just a transposition of two indices). The claim then follows from the fact that the transpositions generate all permutations.

It is noteworthy that the permutation σ_γ is unaffected if the loop γ is subjected to a continuous homotopy within U, throughout which its beginning and end remain fixed at z_0. This is analogous to the preservation of the Cauchy integral under homotopies (see (0.4) above), but is algebraically more complicated: the dependence of the permutation σ_γ on the path γ is non-commutative, in contrast with the Cauchy integral:

$$\sigma_{\gamma_1 \cdot \gamma_2} = \sigma_{\gamma_2} \circ \sigma_{\gamma_1} \neq \sigma_{\gamma_2 \cdot \gamma_1}, \quad I_f(\gamma_1 \cdot \gamma_2) = I_f(\gamma_1) + I_f(\gamma_2). \tag{0.8}$$

This sort of consideration leads naturally to a group with elements the *homotopy classes* of continuous loops $\gamma(t)$ beginning and ending at a particular point $z_0 \in U$, for any region, or indeed any manifold, complex or *topological space* U. This group is called the *fundamental group* of U (with base point z_0) and is denoted by $\pi_1(U, z_0)$. The *Riemann surface* defined by $F(z, w) = 0$ thus gives rise to a homomorphism – monodromy – from the fundamental group to the group of permutations of its "sheets", i.e. the branches of the function $w(z)$ in a neighborhood of $z = z_0$:

$$\sigma : \pi_1(U, z_0) \to S_n, \tag{0.9}$$

where S_n denote the symmetric group on n symbols, and U is as before – a region of $\mathbb{R}^2$.

For transcendental functions F, on the other hand, the equation $F(z, w) = 0$ may determine a many-valued function $w(z)$ with infinitely many sheets ($n = \infty$). Here the simplest example is

$$F(z, w) = \exp w - z = 0, \quad U = \mathbb{R}^2 \setminus 0, \quad w = \ln z.$$

In this example the sheets are numbered in a natural way by means of the integers: taking $z_0 = 1$, we have $w_k = \ln z_0 = 2\pi i k$, where k ranges over the integers. The path $\gamma(t)$ with $|\gamma| = 1$, $\gamma(0) = \gamma(2\pi) = 1$, going round the point $z = 0$ in the clockwise direction exactly once, yields the monodromy $\gamma \to \sigma_\gamma$, $\sigma_\gamma(k) = k - 1$.

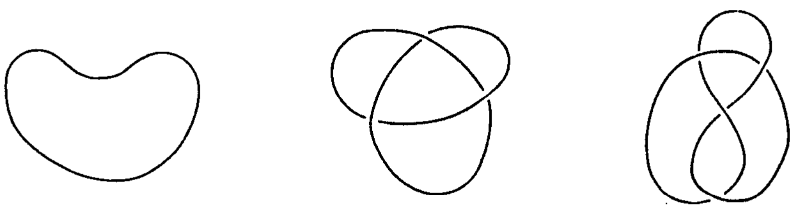
An interesting topological theory where the non-abelianness of the fundamental group $\pi(U, z_0)$ plays an important role is that of *knots*, i.e. smooth (or, if preferred, piecewise smooth, or piecewise linear) simple, closed curves $\gamma(t) \subset \mathbb{R}^3$, $\gamma(t + 2\pi) = \gamma(t)$, or, more generally, the theory of *links*, as introduced above, a link being a finite collection of simple, closed, non-intersecting curves $\gamma_1, \ldots, \gamma_k \subset \mathbb{R}^3$. For $k > 1$, one has the matrix with entries the linking coefficients $\{\gamma_i, \gamma_j\}$, $i \neq j$, given by the formula (0.3), which however does not determine all of the topological invariants of the link. In the case $k = 1$, that of a knot, there is no such coefficient available. Let γ be a knot and U the complementary region of $\mathbb{R}^3$:

$$U = \mathbb{R}^3 \setminus \gamma. \tag{0.10}$$

It turns out that the fundamental group $\pi_1(U, z_0)$, where z_0 is any point of U, is abelian precisely when the given knot γ can be deformed by means of a smooth homotopy-of-knots (i.e. by an "isotopy", as it is called) into the trivial knot, i.e. into the unknotted circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$, where the circle S^1 lies in for instance the (x, y)–plane (see Chapter 4, §5). Elementary knot theory (see Figure 1.5) shows that the abelianized fundamental group

$$H_1(U) = \pi_1/[\pi_1, \pi_1],$$

where $[\pi_1, \pi_1]$ denotes the commutator subgroup of π_1, and U is, as before, the knot-complement in $\mathbb{R}^3$, is for every knot infinite cyclic. More generally, for a link $\{\gamma_1, \ldots, \gamma_k\}$ the group $H_1(U)$ is the direct sum of k infinite cyclic groups. For *any* topological space U, the abelianized fundamental group $\pi_1/[\pi_1, \pi_1]$ is called the *one-dimensional integral homology group* , denoted by $H_1(U)$ as above, or by $H_1(U; \mathbb{Z})$. The group operation in H_1 is always written additively.



(a) Unknotted curve (the "trivial" knot)

(b) The simplest nontrivial knot (the "trefoil" knot)

(c) The "figure-eight" knot

Fig. 1.4

We saw earlier that in planar regions $U \subset \mathbb{R}^2$ the Cauchy integral of a single–valued function analytic in U (without poles!) around closed contours γ lying in U,

$$I_f(\gamma) = \oint_\gamma f(z)dz,$$

determines a complex-valued linear form on the one–dimensional homology group $H_1(U; \mathbb{Z})$ (see (0.8) in particular).

It is appropriate to round off our collection of elementary topological observations with a more modern example, dating from the 1930s, namely the theory of singular integral equations on a circle or arc, which originated as one of several important boundary-value problems in the 2-dimensional theory of elasticity (Noether, Muskhelishvili). Subsequently this theory came to have much greater significance by virtue of its considerable role in the development of the theory of *elliptic linear differential operators* and *pseudo-differential operators.* Let H_1, H_2 be Hilbert spaces, and let $A : H_1 \longrightarrow H_2$ be a *Noetherian operator* (*Fredholm* in modern terminology), i.e. a closed (and bounded) linear operator with finite-dimensional kernel Ker $A = \{h \mid A(h) = 0\}$ (not to

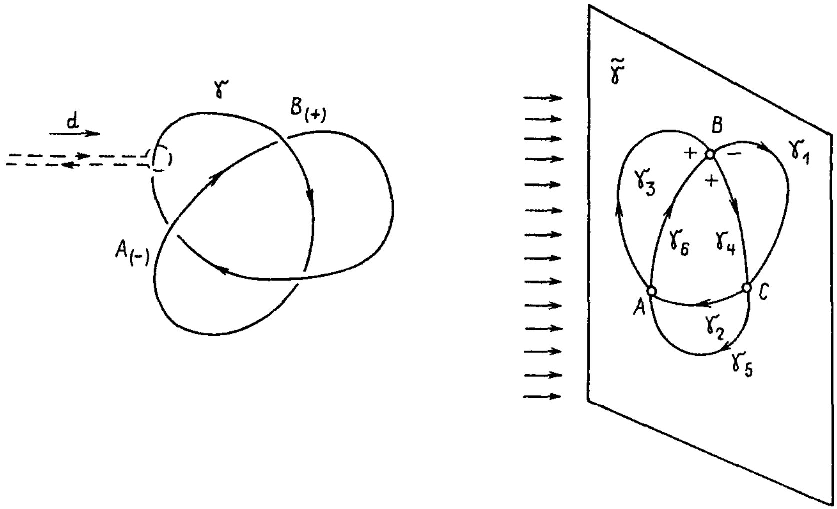


Fig. 1.5

be confused with the kernel of an integral operator!), and finite-dimensional "cokernel", i.e. kernel of the adjoint operator $A^* : H_2 \longrightarrow H_1$. It turns out that the *index* $i(A)$ of such an operator A, defined by

$$i(A) = \dim(\text{Ker } A) - \dim(\text{Ker } A^*), \tag{0.11}$$

i.e. the difference in the respective numbers of linearly independent solutions of the equations $A(h) = 0$ and $A^*(h^*) = 0$, is a homotopy invariant. This means simply that the index $i(A)$ remains unchanged under continuous deformations of the operator A, although the individual dimensions on the right-hand side of the equation (0.11) may change.

In the simplest case of nonsingular kernels $K(x, y)$ of integral operators $\widehat{K}$, the "Fredholm alternative" was discovered at the beginning of this century. In the language of functional analysis this is in effect just the assertion that $i(A) = 0$ for operators A of the form $A = \widehat{1} + \widehat{K}$, where $\widehat{K}$ is a "compact operator", i.e. $\widehat{K}(M)$ is a compact subset of H_2 for every bounded subset M of H_1, and the operator $\widehat{1}$ is an isomorphism of the Hilbert spaces H_1 and H_2. In fact the addition of a compact operator to any Noetherian operator A preserves the Noetherian property, so that the simplest deformation in the class of Noetherian operators has the form $A_t = A_0 + t\widehat{K}$ (with $A_0 = \widehat{1}$ in the classical Fredholm situation).

For singular integral operators, on the other hand, the index is a rather more complicated topological characteristic. Much classical work of the 1920s and 1930s was devoted to explicit calculation of the index via the kernel of the operator. Far-reaching generalizations of this theory to higher-dimensional manifolds, culminating in the Atiyah-Singer Index Theorem, have come to be of exceptional significance for topology and its applications.

This example shows how topological properties arise not only in connexion with geometrical figures in the naive sense, but also in mathematical contexts of a quite different nature.

Chapter 2
Topological Spaces. Fibrations. Homotopies

§1. Observations from general topology. Terminology

Although topological properties are sometimes hidden behind a combinatorial – algebraic mask, they nonetheless all partake organically of the concept of continuity. The most general definition of a continuous map or function between sets requires very little in the way of structure on the sets. As introduced by Fréchet, this structure or *topology* on a set X, making it a *topological space*, consists merely in the designation of certain subsets U of X (among all subsets of X) as the *open sets* of X, subject only to the requirements that the empty set and the whole space X be open, and that the collection of all the open sets be closed under the following operations: the intersection of any finite subcollection of open sets should again be open, and the union of any collection, infinite or finite, of open sets should likewise be open. The complement $X \setminus U$ of any open set U is called a *closed set*. The *closure* of any subset $V \subset X$, denoted by $\overline{V}$, is the smallest closed set containing V. A *continuous map* (or, briefly, *map*) $f : X \longrightarrow Y$ between topological spaces X and Y, is then one for which the complete inverse image $f^{-1}(U)$ of every open set U of Y is open in X. (The *complete inverse image* $f^{-1}(D)$ of any subset $D \subset Y$, is just the set of all $x \in X$ such that $f(x) \in D$.) A *compact* space is a topological space X with the property that, given any covering of X by open sets U_α, i.e. $\bigcup U_\alpha = X$, there always exist a finite set of indices $\alpha_1, \dots, \alpha_N$ such that the open sets $U_{\alpha_1}, \dots, U_{\alpha_N}$ already cover X, i.e. there is always a finite subcover. It can be shown that in a compact space X every sequence of points x_i, $i = 1, 2, \dots$, has a limit point x_∞ in X, i.e. a point such that every open set containing it contains also terms x_i of the sequence for infinitely many i.

A *Hausdorff* topological space is one with the property that for every two distinct points x_1, x_2 there are disjoint open sets U_1, U_2 containing them: $U_1 \cap U_2 = \emptyset$, $x_1 \in U_1$, $x_2 \in U_2$. A topological space X is called a *metric space* if there is a real-valued "distance" $\rho(x_1, x_2)$ defined for each pair of points $x_1, x_2 \in X$, continuous in x_1, x_2 (with respect to the "product topology" on $X \times X$ – see below), satisfying

$$\rho(x_1, x_2) > 0 \quad \textit{if} \quad x_1 \neq x_2,$$

$$\rho(x, x) = 0, \quad \rho(x_1, x_2) = \rho(x_2, x_1),$$

$$\rho(x_1, x_2) + \rho(x_2, x_3) \geq \rho(x_1, x_3).$$

Metric spaces are always Hausdorff. A *path-connected* space X is one in which each pair of points x_1, x_2 can be joined by a continuous path in X, a *path* being a map $I \longrightarrow X$ of an interval I to X. Every topological space decomposes into pairwise disjoint, maximal path-connected components (*path-components*). The set of path-components of X is denoted by $\pi_0(X)$ and is called the "zero–dimensional analogue of the homotopy groups" of X; in general this set does not come with a natural group structure, except in certain special (and important) cases (see below).

Given topological spaces X, Y, we can form their *direct product* $X \times Y$, with points the ordered pairs (x, y), $x \in X$, $y \in Y$, and as a "basis" for its topology the subsets of the form $U \times V$ where U is open X and V in Y. (An arbitrary open set of $X \times Y$ is then obtained by taking the union of any collection of basic ones.)

Given topological spaces X, Y, one usually endows the set Y^X of all continuous maps $f : X \longrightarrow Y$ with a topology called the *compact–open* topology. This is defined as follows: take any compact set $K \subset X$, and any open set V of Y; the set of all maps f sending K to V, $f(K) \subset V$, is then a typical basic open set of Y^X. If the space X itself is compact and Y is a metric space with metric ρ, then Y^X is a metric space with metric $\widetilde{\rho}$ given by

$$\widetilde{\rho}(f, g) = \max_{x \in X} \rho(f(x), g(x)), \quad f,\ g : X \longrightarrow Y.$$

There is yet another simple but important construction from a pair of topological spaces X, Y, namely their *bouquet* $X \vee Y$. Strictly speaking the bouquet is defined for *pointed* spaces X, Y, i.e. spaces with specified points $x_0 \in X$, $y_0 \in Y$; their bouquet $X \vee Y$ is the space resulting from identification of x_0 and y_0, $x_0 \approx y_0$, in the formal disjoint union of X and Y:

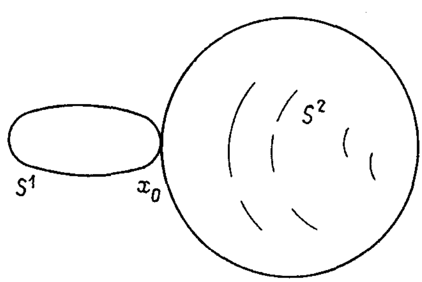


Fig. 2.1. The bouquet $X \vee Y$

$$X \vee Y = X\dot{\cup}Y/x_0 \approx y_0. \tag{1.1}$$

More generally, given any closed subset $A \subset X$ and map $f : A \longrightarrow Y$, one may by means of identification form the following analogue of the bouquet:

$$X \vee (A, f)Y = X \dot{\cup} Y \ / \ x \approx f(x), \quad x \in A \ .$$

An important case of this is the *mapping cylinder* C_f of a map $f : Z \longrightarrow Y$. Consider the product of Z with an interval $I = [a, b]$, and form the identification space

$$C_f = (Z \times I)\dot{\cup}Y/(z, b) \approx f(z), \quad z \in Z. \tag{1.2}$$

(Here $Z \times I$ plays the role of X and $Z \times \{b\}$ that of A.) The topology on the space C_f is defined in the natural way: a subset of C_f is taken as open precisely if its complete inverse images under the natural maps $Z \times I \longrightarrow C_f$ and $Y \longrightarrow C_f$, are both open.

On any subset A of a topological space X the *subspace* or *induced* topology is defined by taking the open sets of A to be simply the intersections with A of the open sets of X.

A sequence of points x_i of a topological space X is said to *converge* in X, if it has a limit in X , i.e. a point x_∞ of X with the property that every open set U containing x_∞ contains the x_i for almost all i (that is for all but finitely many i). The topology on X may be recovered from the knowledge of its convergent sequences.

A *homeomorphism* between topological spaces X and Y is a continuous, one-to-one surjection $f : X \longrightarrow Y$, such that the inverse $f^{-1} : Y \longrightarrow X$ is also continuous. Here the continuity of the inverse function f^{-1} does not in general follow from the other defining conditions; it does follow, however, if X is compact and Y Hausdorff.

Functional analysis provides many examples of continuous bijections with discontinuous inverses. In particular, for spaces of real-valued smooth functions there are various natural kinds of convergence definable in terms of different numbers of derivatives, so that the existence of continuous bijections with discontinuous inverses is to be expected even in such relatively concrete contexts.

One often encounters topological spaces which carry at the same time some algebraic structure, compatible with the topological structure in the sense that the various algebraic operations are continuous when considered as maps; thus one has *topological groups, topological vector spaces, topological rings*, etc.

From the purely abstract point of view, it is very natural to consider topological spaces which have the property of being locally Euclidean, although in fact most naturally occurring examples of such entities come with some additional smooth or piecewise linear structure (PL-structure).

Definition 1.1 A *topological manifold* (of dimension n) is a Hausdorff topological space X with the property that each of its points x has an open neighbourhood U (i.e. open set, or "region", containing x) which is homeomorphic to an open set of n-dimensional Euclidean space $\mathbb{R}^n$ (for some fixed n).

Thus an n-manifold is covered by open sets U_α each homeomorphic to $\mathbb{R}^n$, and therefore each having induced on it via some specific homeomorphism $\varphi : U_\alpha \longrightarrow \mathbb{R}^n$, *local co-ordinates* $x_\alpha^1, \dots, x_\alpha^n$. On each region of overlap $U_\alpha \cap U_\beta$ there will then be defined two systems (or more) of *local co-ordinates*, and hence a co-ordinate transformation from each of these to the other:

$$\begin{array}{ccc} & (x_\alpha^1(x), \dots, x_\alpha^n(x)) & \\ & \nearrow^{\phi_\alpha} & \\ x & & \\ & \searrow_{\phi_\beta} & \\ & (x_\beta^1(x), \dots, x_\beta^n(x)) & \end{array} \qquad \begin{array}{c} x_\alpha^j = f_\alpha^j(x_\beta^1, \dots, x_\beta^n), \\ x_\beta^k = g_\beta^k(x_\alpha^1, \dots, x_\alpha^n), \\ f \circ g = 1. \end{array} \tag{1.3}$$

§2. Homotopies. Homotopy type

A *continuous homotopy* (or briefly *homotopy* or *deformation*) of a map $f : X \longrightarrow Y$, is a continuous map of the cylinder $X \times I$ to Y:

$$F = F(x,t) : X \times I \longrightarrow Y, \quad x \in X, \quad a \le t \le b,$$

(I an interval $[a,b]$) for which

$$F(x,a) = f(x) \ \text{ for all } \ x \in X.$$

Two maps $f, g : X \longrightarrow Y$ are *homotopic* if there is a continuous homotopy F such that

$$F(x,a) = f(x), \quad F(x,b) = g(x), \quad x \in X.$$

One often needs to consider in this context *pointed spaces* X, Y, i.e. with particular points $x_0 \in X$, $y_0 \in Y$ specified. For such spaces maps $f : X \longrightarrow Y$ are usually also required to be "pointed", i.e. to satisfy $f(x_0) = y_0$, and homotopies between "pointed" maps are then also normally "pointed", in the sense that one requires $F(x_0, t) = y_0$ for all t.

Each equivalence class of homotopic maps $f : X \longrightarrow Y$ constitutes a path-component of the function space Y^X, and is called a *homotopy class* of maps $X \longrightarrow Y$ (or of pointed maps, as the case may be). Thus the set $\pi_0(Y^X)$ is comprised of homotopy classes.

Sometimes one has to deal with pairs of spaces $A \subset X$, $B \subset Y$, where the appropriate maps $f : X \longrightarrow Y$ are those for which $f(A) \subset B$. Such a map of pairs is denoted by

$$f : (X, A) \longrightarrow (Y, B) ,$$

and the space of all such maps of pairs has as its path-components the analogous homotopy classes of maps of pairs. There are several important reasons, as we shall see below, for considering also the category of triples $(x_0 \in A \subset X)$ for which the appropriate maps $f : X \longrightarrow Y$ are those for which both $f(A) \subset B$ and $f(x_0) = y_0$, where $(y_0 \in B \subset Y)$ is another such triple. Here one has homotopy classes of pointed maps of pairs.

Definition 2.1 A continuous map $f : X \longrightarrow Y$ is called a *homotopy equivalence* if there is a "homotopy inverse" map $g : Y \longrightarrow X$, i.e. such that the two composites $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ are homotopic to the respective identity maps

$$1_X : X \longrightarrow X \quad (1_X(x) \equiv x), \quad 1_Y : Y \longrightarrow Y \quad (1_Y(y) \equiv y).$$

(For pointed spaces one modifies this definition in the obvious way to yield the concept of a *pointed* homotopy equivalence.) The spaces X, Y are then said to be *homotopy equivalent*, $X \sim Y$, or to have the *same homotopy type*.

Suppose now that the following conditions hold: X is a subspace of Y (or embedded in Y); $f : X \longrightarrow Y$ is the inclusion map; $g : Y \longrightarrow X$ restricted to X is the identity map on X; and, finally, throughout the homotopy F deforming $f \circ g : Y \longrightarrow Y$ to the identity map 1_Y, we have $F(x,t) = x$ for all $x \in X \subset Y$. In this situation the subspace X is called a *deformation retract* of Y. For instance any open region Y of $\mathbb{R}^n$ has a deformation retract of lower dimension. The whole Euclidean space $\mathbb{R}^n$ has any point as a deformation retract. Spaces Y with the latter property are said to be *contractible* (over themselves) or *homotopically trivial*: $Y \sim 0$.

A *retraction* of a space Y onto a subspace $X \subset Y$ is a map $f : Y \longrightarrow X$ with the property that the restriction of f to X is the identity map on X: $f|_X = 1_X$. The space X is then called a *retract* of Y.

§3. Covering homotopies. Fibrations

Consider a (continuous) map $p : X \longrightarrow Y$. We say that an arbitrary mapping $f : Z \longrightarrow Y$ is *covered* (via p) if there is a mapping $g : Z \longrightarrow X$ such that $f = p \circ g$.

Suppose now that we have a homotopy $F : Z \times I \longrightarrow Y$, where $I = [a, b]$, and that at the initial time $t = a$ the map $f(z) = F(z, a)$ is covered by some map $g : Z \longrightarrow X$.

Definition 3.1 The map $p : X \longrightarrow Y$ is called a *fibration* if given any space Z and any homotopy $F : Z \times I \longrightarrow Y$ whose initial map $f(z) = F(z, a) : Z \longrightarrow Y$ is covered (by $g(z)$, say), the whole homotopy F "down below" in Y is covered "up above" in X by some homotopy $G : Z \times I \longrightarrow X$, i.e.

$p \circ G(z,t) = F(z,t)$. The homotopy G is called a *covering homotopy for F with initial map g*.

For various technical reasons a weakened form of this definition is often employed in situations where the space Z has one or another condition imposed on it (for example, cellularity – see Chapter 3). However the essential character of the concept of fibration is unaffected by such changes.

Usually the following additional condition is imposed in the above definition, namely that each point $z_1 \in Z$ remaining fixed under the homotopy $F(z,t)$ for all t in any subinterval of $[a,b]$, should likewise remain fixed on that subinterval under $G(z,t)$.

In the most important situations the construction of a covering homotopy is carried out by means of a "homotopy connexion". Roughly speaking a *homotopy connexion* is a recipe for obtaining from a given path in Y beginning at $y_0 \in Y$ and any prescribed point $x_0 \in X$ above y_0, a unique covering path in X beginning at x_0. Furthermore this covering path should depend continuously on both the given path in Y and the initial point $x_0 \in X$ at which the covering path is to begin; this secures the covering-homotopy property for all reasonably well-behaved spaces Z.

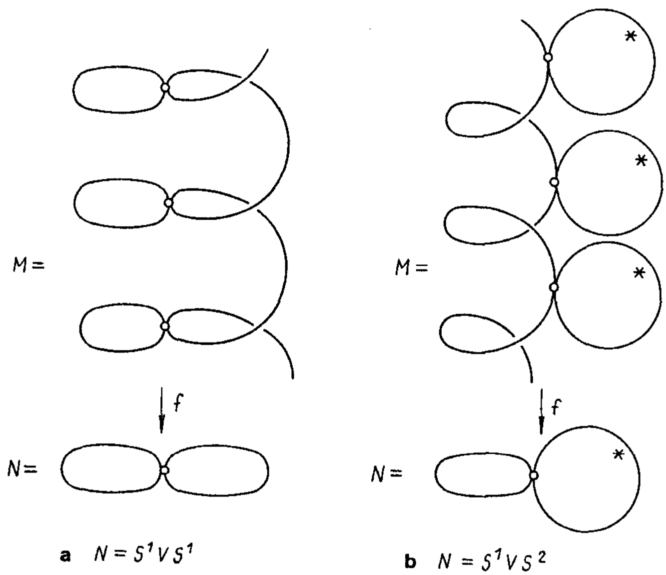


Fig. 2.2

Some more terminology: given a fibration $p : X \longrightarrow Y$, we call p a *projection*, X the *total space*, Y the *base*, and each space $F_y = p^{-1}(y)$, $y \in Y$, a *fiber* of the fibration.

Take Z to be a fiber F_{y_0} of a fibration $p : X \longrightarrow Y$, $g : Z \longrightarrow X$ the inclusion and $f : Z \longrightarrow Y$ to be the projection of $Z = F_{y_0}$ to the single point

$y_0 \in Y$. Let $\gamma(t)$ be a path in Y joining y_0 to any other point y_1. Using the covering–homotopy property (in the form of the above-mentioned "homotopy connexion", which we always presuppose) it is straightforward to establish the following important fact:

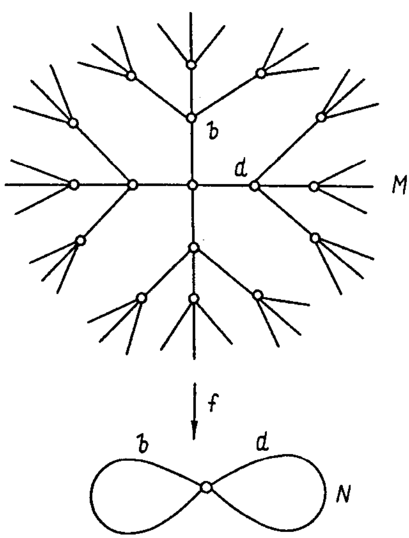
All fibers F_y over a path-connected component of Y are homotopy equivalent. We shall in fact assume henceforth that the base Y of a fibration is path-connected.

Here are the simplest examples.

1. Covering spaces. A map $p : X \longrightarrow Y$ is a *covering map*, and X is a *covering space* of Y, if it is a fibration with a discrete fiber F, i.e a space all of those subsets are open (so that its points, which may be infinite in number, are all isolated from one another), such that for each point $y \in Y$ there is an open neighbourhood U of y ($y \in U \subset Y$) whose complete inverse image $p^{-1}(U)$ is homeomorphic to the direct product $U \times F$ with $F = \bigcup_\alpha \{x_\alpha\}$:

$$p^{-1}(U) = \bigcup_\alpha U_\alpha \;\cong\; U \times F,$$

where each $U_\alpha \subset X$ is a homeomorphic copy of U. The sets U_α are open in X and the restriction $p|_{U_\alpha} : U_\alpha \longrightarrow U$ of p to each of them is a homeomorphism with U. The existence of a homotopy connexion is easily established for covering spaces.



Here the covering space is an infinite tree of "crosses" (without cycles and therefore contractible). Each vertex has four edges incident with it

Fig. 2.3

Concrete examples of covering spaces over regions of $\mathbb{R}^2$ were discussed in Chapter 1 in connexion with Riemann surfaces. In those examples the homotopy connexion is constructed in the obvious way. Other examples are depicted in Figures 2.2 and 2.3.

2. Serre fibrations. Let $B \subset Y$ be any pair of spaces, and denote by X the space consisting of all paths $\gamma : [a, b] \longrightarrow Y$, in Y beginning in B:

$$\gamma(a) \in B, \quad \gamma(b) \in Y, \quad \gamma \in X .$$

Consider the evaluation map $p : X \longrightarrow Y$ defined by

$$p(\gamma) = \gamma(b).$$

This defines a *Serre fibration* $p : X \longrightarrow Y$.

To see that $p : X \longrightarrow Y$ is indeed a fibration, consider any path $y(\tau)$, $b \leq \tau \leq c$, in the base Y beginning at the end of a given path $\gamma \in X$, $y(b) = \gamma(b)$. By associating with each point y of the curve $y(\tau)$ the path γ_τ obtained by adjoining to γ the segment from $y(b)$ to $y(\tau) = y$, we trivially obtain a homotopy connexion, i.e. recipe for covering each path $y(\tau)$ in the base Y by a path γ_τ in the total space X with $\gamma_b = \gamma$ (see Figure 2.4). In

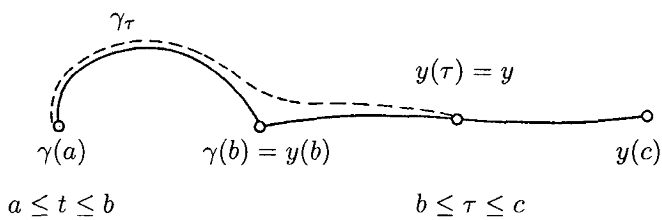


Fig. 2.4

this example, the total space X contains an embedded copy of the space B, consisting of the constant paths $\gamma(t) \equiv$ const., which we may therefore identify with B. It is easy to see that this subspace B is a deformation retract of X. Of especial importance is the situation where B is a single point $\{y_0\}$, $y_0 \in Y$; the space X is then contractible. In this case X is usually denoted by E_{y_0}, and the fiber F_y over $y \in Y$ by $\Omega(y_0, y)$. (As noted earlier in the more general context, the fibers $\Omega(y_0, y)$ are all of the same homotopy type provided the base Y is path-connected.) For $y = y_0$ the space $\Omega(y_0, y_0) = \Omega$ is the *loop space* of Y based at y_0.

3. For *locally trivial fibrations* (or *fiber bundles*) the covering-homotopy property is somewhat more difficult to establish. For these fibrations $p : X \longrightarrow Y$ all the fibers F_y are actually homeomorphic to a space F. The defining conditions are as follows: it is required that, analogously to covering spaces, each point y of the base Y should have a neighbourhood U, $y \in U \subset Y$, whose inverse image $p^{-1}(U) \subset X$, is homeomorphic to the direct product $U \times F$ via a homeomorphism $q : p^{-1}(U) \longrightarrow U \times F$ "compatible" with the projection p. (Here the fiber F need not be discrete, as in the case of covering spaces.) Let $\{U_\alpha\}$ be a covering of Y by such open sets U_α. Then given any two sets U_α, U_β of the covering, we have on the complete inverse image of their intersection $U_\alpha \cap U_\beta = U_{\alpha,\beta}$ two homeomorphisms defined:

$$\phi_\alpha : p^{-1}(U_{\alpha,\beta}) \longrightarrow U_{\alpha,\beta} \times F ,$$

$$\phi_\beta : p^{-1}(U_{\alpha,\beta}) \longrightarrow U_{\alpha,\beta} \times F .$$

The map $\lambda_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha,\beta} \times F \longrightarrow U_{\alpha,\beta} \times F$ then respects fibers and induces the identity map of the base $U_{\alpha,\beta}$. It therefore has the form

$$\lambda_{\alpha,\beta}(w, f) = \left(w, \widehat{\lambda}_{\alpha,\beta}(w)(f)\right),$$

where $\widehat{\lambda}_{\alpha,\beta}(w) : F \longrightarrow F$ is a homeomorphism of the fiber over w, varying continuously with w. On intersections $U_{\alpha,\beta,\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ we require

$$\lambda_{\alpha,\beta} \circ \lambda_{\beta,\gamma} \circ \lambda_{\gamma,\alpha} = 1. \tag{3.1}$$

The maps $\lambda_{\alpha,\beta}$ are called *glueing* or *transition functions*. If there is such a covering of Y by open regions U_α satisfying the above requirements, notably that for each α there exists a homeomorphism

$$\phi_{U_\alpha} : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F,$$

compatible with p, and these homeomorphisms collectively satisfy (3.1), then the covering-homotopy property of a fibration follows, and moreover these data characterize the fiber bundle uniquely.

The concept of a fiber bundle, as just described, turns out to be fundamental in the theory of manifolds, in differential topology and geometry, and in the major applications of these theories. We shall encounter the concept repeatedly in the sequel. In the most important examples of fiber bundles the homotopy connexions will be determined by "differential-geometric connexions" on the total spaces of the bundles. It is pertinent to note that for certain bundles such "differential-geometric connexions", when expressed in term of local co-ordinates, turn out to be what are termed by physicists "Maxwell-Yang-Mills fields".

§4. Homotopy groups and fibrations. Exact sequences. Examples

The *homotopy groups* of a topological space, which we shall now define, are the most important invariants, and play a fundamental role in topology. It has turned out that they are of crucial importance also in the applications of topological methods to modern physics, determining for instance the structure of singularities ("declinations") in liquid crystals. However we shall not in this section be in a position to embark on a description of the more serious methods of computation of homotopy groups; this will have to be postponed until we consider manifolds and homology theory.

Let S^n denote the n-dimensional sphere in $\mathbb{R}^{n+1}$, i.e. the subspace consisting of the points $(x^0, \ldots, x^n)$ satisfying

$$\sum_{\alpha=0}^{n} (x^\alpha)^2 \leq 1.$$

Definition 4.1 The set of pointed homotopy classes of maps $(S^n, s_0) \longrightarrow (X, x_0)$, is called the *n-dimensional homotopy group* of (X, x_0), and is denoted by $\pi_n(X, x_0)$.

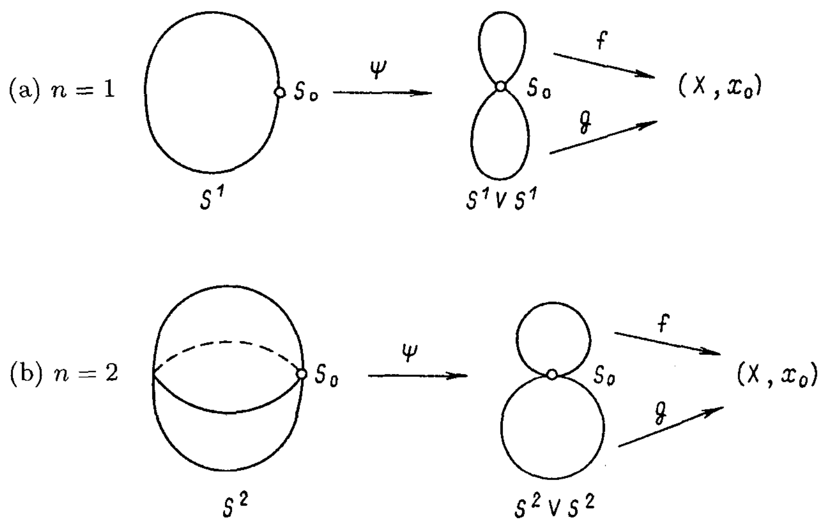


Fig. 2.5. $f + g = (f \vee g) \circ \psi$

We have yet to specify the group operation on the elements of this set, i.e. on the homotopy classes. Let f, g be two maps $(S^n, s_0) \longrightarrow (X, x_0)$. Their *sum* is then defined as the composite first of the map ψ from S^n onto the bouquet $S^n \vee S^n$ which identifies the equator of S^n to the point s_0 on it, followed by the map of the bouquet to the space (X, x_0) which coincides with f on the first sphere of the bouquet and with g on the second (see Figure 2.5).

Extended to homotopy classes of maps this sum is well-defined, and yields a group operation on $\pi_n(X, x_0)$ (see Figures 2.6, 2.7). For $n > 1$ this group operation is abelian; this is a consequence of the fact that for $n > 1$ the n-sphere may be continuously rotated, while keeping the point s_0 fixed, so as

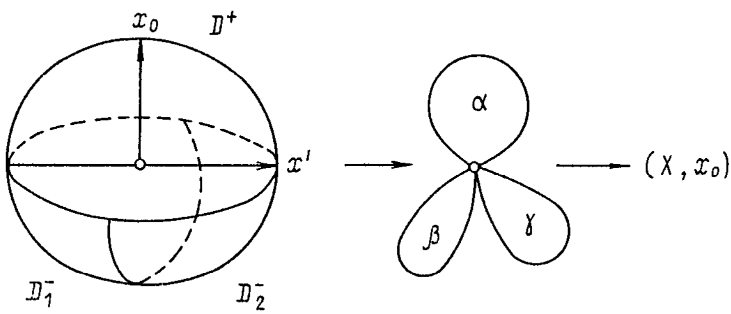


Fig. 2.6

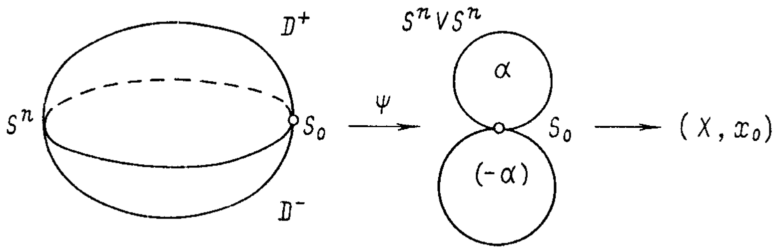


Fig. 2.7

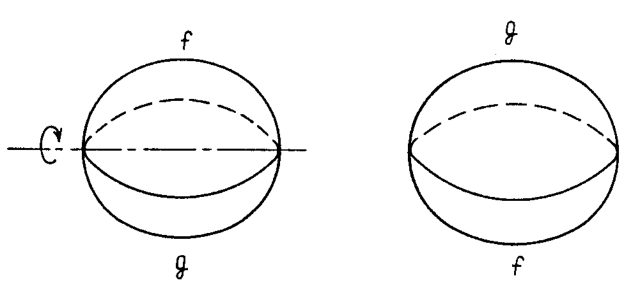


Fig. 2.8

to interchange its upper and lower hemispheres (see Figure 2.8). Hence for $n > 1$ the additive notation is used for the group operation on $\pi_n(X, x_0)$.

Observe that the elements of $\pi_n(X, x_0)$ may also be realized as homotopy classes of maps f of the disc D^n sending the boundary $\partial D^n = S^{n-1}$ to the point x_0:

$$f : (D^n, S^{n-1}) \longrightarrow (X, x_0) .$$

Any path $\gamma(t)$, $1 \le t \le 2$, in X, beginning at $\gamma(1) = x_0$ and ending at $\gamma(2) = x_1$, yields a map of the cylinder $S^{n-1} \times I$ to X:

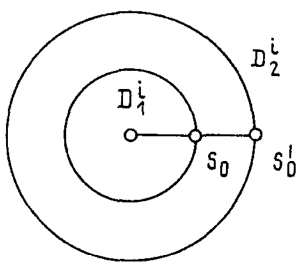
$$S^{n-1} \times I \longrightarrow I \xrightarrow{\gamma} X,$$

which depends only on the coordinate t. If we take the union of this map with any map $f : (D^n, S^{n-1}) \longrightarrow (X, x_0)$, then we obtain an extension of the latter along the path γ:

$$(f, \gamma) : (D^n \cup (S^{n-1} \times I), S^{n-1} \times \{2\}) \longrightarrow (X, x_1) .$$

By means of this construction we arrive at a natural isomorphism between the n-th homotopy groups with different base points x_0, x_1 (see Figures 2.9, 2.10, 2.11):

$$\gamma_*^{(n)} : \pi_n(X, x_0) \longrightarrow \pi_n(X, x_1).$$



$$D_1^i : \sum_{j=1}^{i} (x^j)^2 \le 1$$

$$D_2^i : \sum_{j=1}^{i} (x^j)^2 \le 2$$

$$s_0 = (1, 0, \dots, 0)$$

$$s_0' = (2, 0, \dots, 0)$$

Fig. 2.9

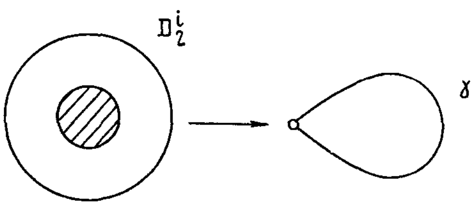


Fig. 2.10

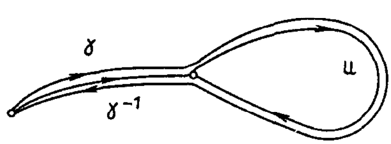


Fig. 2.11

The isomorphism $\gamma_*^{(n)}$ depends only on the homotopy class of the path γ in the class of all paths joining x_0 to x_1. In the case $x_0 = x_1$ each such homotopy class is a homotopy class of loops based at x_0, and is therefore the element of the first homotopy group $\pi_1(X, x_0)$ (the *fundamental group* of (X, x_0)). We conclude that the fundamental group $\pi_1(X, x_0)$ acts naturally as a group of operators on each of the groups $\pi_n(X, x_0)$. For $n = 1$ this action is just the conjugating action inducing inner automorphisms of the group π_1 (see Figures 2.12, 2.13):

$$\gamma_*^{(1)}(u) = \gamma \cdot u \cdot \gamma^{-1}, \quad u \in \pi_1, \quad \gamma \in \pi_1 .$$

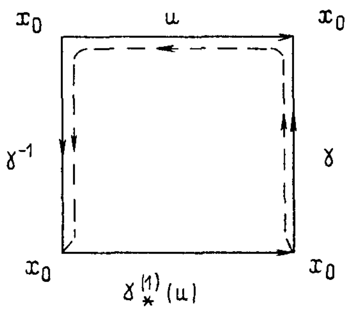


Fig. 2.12

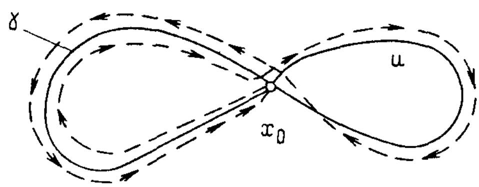


Fig. 2.13

A very important class of topological spaces is that of *simply-connected* ones, i.e. those with trivial fundamental group $\pi_1(X, x_0)$. In this case it follows from the preceding discussion that the homotopy groups $\pi_n(X, x_0)$ are essentially independent of the choice of base point x_0 (for path-connected spaces X); $\pi_n(X, x_0)$ may simply be defined as consisting of the *free* homotopy classes of maps $S^n \longrightarrow X$, without specifying any base point. In the general case (of connected but not necessary simply-connected spaces) the free homotopy classes of maps $S^n \longrightarrow X$ (i.e. unpointed) are determined algebraically as the orbits under the action of $\pi_1 = \pi_1(X, x_0)$ on the group $\pi_n(X, x_0)$. In the case $n = 1$, these are just the conjugacy classes of the group π_1.

It follows easily from its definition, that the n-th homotopy group of a product of two spaces is just the direct product of the n-th homotopy groups of those factor spaces:

$$\pi_n (X \times Y, x_0 \times y_0) = \pi_n (X, x_0) \times \pi_n (Y, y_0) .$$

There is also the *smash* (or *tensor*) product of spaces:

$$X \wedge Y = X \times Y/(X \times y_0) \vee (x_0 \times Y) ,$$

where the relationship between the homotopy groups of X, Y and $X \wedge Y$ is given by a *bilinear pairing of homotopy groups*:

$$\Phi : \pi_l(X) \otimes \pi_m(Y) \longrightarrow \pi_{l+m}(X \wedge Y) ,$$

where Φ has the form

$$\Phi = (f, g), \quad f : S^l \longrightarrow X, \quad g : S^m \longrightarrow Y, \quad S^{l+m} \cong S^l \wedge S^m .$$

Given any group π, we define the *integral group ring* $\mathbb{Z}[\pi]$ of π to have as elements the formal finite linear combinations of the elements of π with integer coefficients:

$$a = \sum_{i=1}^{N} \lambda_i u_i, \quad \lambda_i \in \mathbb{Z}, \quad u_i \in \pi .$$

Addition is as usual for linear combinations, and the multiplication is defined as follows: given $a = \sum \lambda_i u_i$, $b = \sum \mu_j v_j$, we set

$$a \cdot b = \left(\sum \lambda_i u_i\right) \cdot \left(\sum \mu_j v_j\right) = \sum_{i,j} \lambda_i \mu_j (u_i \cdot v_j). \tag{4.1}$$

There is an alternative, equivalent, definition of the (integral) group ring more usual in the context of analysis: the elements are taken to be the functions $a : \pi \longrightarrow \mathbb{Z}$ $(u_i \rightarrow \lambda_i)$ with finite support, added in the usual way for integer-valued functions, and multiplied via "convolution" as follows: given two such functions a, b their product is given by

$$a \cdot b(u) = \sum_{v \cdot w = u} a(v)b(w) = \sum_{v \in \pi} a(v)b(v^{-1}u) . \tag{4.2}$$

As already noted, this is the same ring $\mathbb{Z}[\pi]$. This concept can be generalized to the situation of a continuous (i.e. topological) group by replacing the sums by integrals.

The point for us here is that, to put it in algebraic language, every homotopy group $\pi_n(X, x_0)$, $n > 1$, is in a natural way a $\mathbb{Z}[\pi]$-module, with $\pi = \pi_1(X, x_0)$, i.e has the structure of a linear space with ring of "scalars" $\mathbb{Z}[\pi]$.

The fundamental group π_1 was introduced by Poincaré. It was only in the 1930s that the higher homotopy groups π_n, $n > 1$, were defined (by Hurewicz). Initially the $\mathbb{Z}[\pi]$-module structure of the latter groups went unnoticed, and the abelianness of the π_n for $n > 1$ created a false impression of their formal simplicity in comparison with π_1. It was only somewhat later (beginning with Whitehead sometime in the 1940s) that their module structure came to be used — although rather unsystematically. The active and systematic exploitation of the $\mathbb{Z}[\pi]$-module structure of the higher homotopy groups occurred from the 1960s on, in the course of intensive development of the theory of non-simply-connected manifolds. It is curious that in certain particular problems of homology theory, module-theoretic considerations played a significant role already in the 1930s.

We now turn to the category of triples $(x_0 \in A \subset X)$.

Definition 4.2 The n-dimensional *relative homotopy group* $\pi_n(X, A, x_0)$, $n > 1$, has as its elements the homotopy classes of maps

$$f : \left(D^n, S^{n-1}, s_0\right) \longrightarrow (X, A, x_0),$$

where $f(S^n) \subset A$, $f(s_0) = x_0$. The group structure is defined on this set in a similar manner to that of the corresponding absolute homotopy group (see Figure 2.14).

It turns out that for $n > 2$ the groups $\pi_n(X, A, x_0)$ are abelian. Here it is the group $\pi_1(A, x_0)$ which acts as a group of operators on the $\pi_n(X, A, x_0)$, $n > 2$, so that they may be considered as $\mathbb{Z}[\pi]$-modules with $\pi = \pi_1(A, x_0)$.

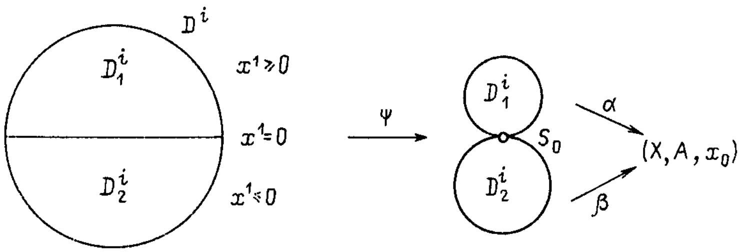


Fig. 2.14

In the "relative" context there arise three natural homomorphisms:

$$j_* : \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0),$$

$$\partial : \pi_n(X, A, x_0) \longrightarrow \pi_{n-1}(A, x_0), \tag{4.3}$$

$$i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0).$$

The definition of the homomorphism j_* depends on the observation that a map $f : (D^n, S^{n-1}) \longrightarrow (X, x_0)$ may be regarded as a map g of triples:

$$f = g : \left(D^n, S^{n-1}, s_0\right) \longrightarrow (X, A, x_0),$$

in view of the fact that $x_0 \in A$.

The homomorphism ∂ is defined by restricting maps $f : \left(D^n, S^{n-1}\right) \longrightarrow (X, A, x_0)$ to the boundary $\partial D^n = S^{n-1}$ of D^n:

$$g = f|_{S^{n-1}} : \left(S^{n-1}, s_0\right) \longrightarrow (A, x_0).$$

The "inclusion homomorphism" i_* comes directly from the inclusion $A \subset X$.

It is almost obvious also that the composition of any two "neighbouring" homomorphisms i_*, j_*, ∂ yields the zero homomorphism:

$$j_* \circ i_* = 0, \quad \partial \circ j_* = 0, \quad i_* \circ \partial = 0,$$

whence we infer that

$$\operatorname{Im} i_* \subset \operatorname{Ker} j_*, \quad \operatorname{Im} j_* \subset \operatorname{Ker} \partial, \quad \operatorname{Im} \partial \subset \operatorname{Ker} i_*.$$

It turns out (after a little calculation) that these subgroup inclusions are in fact equalities (called "exactness conditions"):

$$\operatorname{Im} i_* = \operatorname{Ker} j_*, \quad \operatorname{Im} j_* = \operatorname{Ker} \partial, \quad \operatorname{Im} \partial = \operatorname{Ker} i_*. \tag{4.4}$$

This is normally expressed by the statement that the following sequence of groups and homomorphisms is "exact":

$$\cdots \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \tag{4.5}$$

This is called the *exact homotopy sequence of the pair* (X, A).

The construction of the homotopy groups — the absolute ones $\pi_n(X, x_0)$ on the one hand and the relative ones $\pi_n(X, A, x_0)$ on the other — may be regarded as determining *covariant functors* from the category of pointed topological spaces (or the category of triples, as the case may be) to the category of (abelian) groups. This means simply that maps

$$f : X \longrightarrow Y, \quad x_0 \to y_0, \quad (A \longrightarrow B)$$

of pointed spaces (X, x_0) (or triples (X, A, x_0)) determine homomorphisms of the corresponding homotopy groups

$$f_* : \pi_n(X, x_0) \longrightarrow \pi_n(Y, y_0),$$

$$f_* : \pi_n(X, A, x_0) \longrightarrow \pi_n(Y, B, y_0).$$

These homomorphisms are obtained essentially in the following way: to each map $g : (D^n, S^{n-1}) \longrightarrow (X, x_0)$ representing an element of $\pi_n(X, x_0)$ we associate the map

$$f \circ g : \left(D^n, S^{n-1}\right) \longrightarrow (Y, y_0),$$

and similarly for triples.

In the particular examples above we had

$$i : A \longrightarrow X, \quad i(x_0) = x_0,$$

$$i_* : \pi_n(A, x_0) \longrightarrow \pi_n(X, x_0),$$

(the inclusion homomorphism), and

$$j : \begin{cases} X \longrightarrow X \\ x_0 \longrightarrow A \\ x_0 \longrightarrow x_0 \end{cases},$$

$$j_* : \pi_n(X, x_0, x_0) = \pi_n(X, x_0) \longrightarrow \pi_n(X, A, x_0).$$

The above collection of elementary properties of the homotopy (and likewise homology) groups — especially their functoriality and the exact sequence of a pair (X, A) — leads, after very substantial further development, to an elaborate algebraic apparatus for topology, as we shall see below.

In one case, of extreme importance in connexion with algebraic methods for computing the homotopy groups, the relative homotopy groups reduce to the absolute ones. Consider an arbitrary fibration $p : X \longrightarrow Y$, with the covering–homotopy property (as usual with respect to any prescribed initial condition at $t = 0$). Let $y_0 \in Y$, and $x_0 \in p^{-1}(y_0) = F_0$. An important, albeit not especially difficult, theorem asserts that there is an isomorphism

$$\pi_n(X, F_0, x_0) \cong \pi_n(Y, y_0). \tag{4.6}$$

This is established using the projection homomorphism p_* (arising from the projection p of X to Y sending the fiber F_0 to the point y_0). Starting with the covering-homotopy property, we may see this isomorphism intuitively as follows. Each map $f : D^n \longrightarrow Y$, representing an element of $\pi_n(Y, y_0)$, may

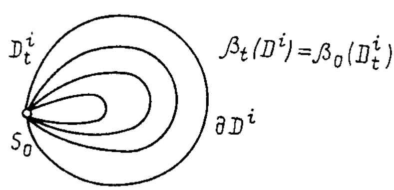


Fig. 2.15

be lifted to X in view of the contractibility of the disc D^n to the point s_0 on its boundary S^{n-1} (see Figure 2.15). By lifting this map to X we obtain a covering map $D^n \longrightarrow X$ which maps the boundary S^{n-1} not necessarily to a point, but to the fiber F_0 over y_0. A straightforward argument now yields the desired isomorphism (4.6).

As a consequence of this isomorphism, i.e. ultimately of the covering–homotopy property, one obtains the following *exact sequence of the fibration* (rather than of the pair (X, F_0)):

$$\begin{aligned}\cdots \longrightarrow \pi_n(F_0, x_0) \longrightarrow \pi_n(X, x_0) \xrightarrow{p_*} \pi_n(Y, y_0) \xrightarrow{\partial} \pi_{n-1}(F_0, x_0)\\ \longrightarrow \pi_{n-1}(X, x_0) \rightarrow \cdots \quad .\end{aligned} \tag{4.7}$$

Important cases. 1. Let $p : X \longrightarrow Y$ be a covering–space projection (see above) with (discrete) fiber F_0. Then since $\pi_i(F_0, x_0) = 0$ for $i \geq 1$ (0 denoting the trivial group), the exact sequence (4.7) becomes in this situation:

$$\begin{aligned}&\text{for } n > 1, \quad 0 \longrightarrow \pi_n(X, x_0) \xrightarrow{p_*} \pi_n(Y, y_0) \longrightarrow 0;\\ &\text{for } n = 1, \quad 0 \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0) \xrightarrow{\partial} \pi_0(F_0, x_0) \longrightarrow 0.\end{aligned}$$

Thus from exactness we obtain:

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0), \quad n > 1, \tag{4.8}$$

and in the case $n = 1$ that $\pi_1(X, x_0)$ is isomorphic to a subgroup of $\pi_1(Y, y_0)$, furthermore in such way that the set of cosets is "isomorphic" to (i.e. in one-to-one correspondence with) the number of components of the fiber; in other words each coset corresponds to a sheet of the covering. □

2. Consider the Serre fibration over any space Y; thus here (as before) the total space $X = E_{y_0}$ is the space of all paths $\gamma(t)$, $a \leq t \leq b$, beginning at the point $y_0 = \gamma(a)$. Clearly X is contractible. The fiber $F_0 = p^{-1}(y_0)$ over the chosen point y_0 is the loop space $\Omega(y_0)$, consisting of all closed paths beginning and ending at y_0. For this fibration the exact sequence (4.7) yields:

$$0 \longrightarrow \pi_n(Y, y_0) \xrightarrow{\partial} \pi_{n-1}(\Omega(y_0), x_0) \longrightarrow 0, \quad n \geq 1,$$

whence we infer the isomorphism;

$$\pi_n(Y, y_0) \cong \pi_{n-1}(\Omega(y_0), x_0), \tag{4.9}$$

where x_0 denotes the trivial loop with image the point y_0 for all t. This isomorphism was in fact used by Hurewicz to define the homotopy groups recursively. □

Returning now to an arbitrary fibration $p : X \longrightarrow Y$, we recall the basic assumption that the fibration comes with a "homotopy connexion". In particular this allows us to translate a fiber along a path in the base; i.e. to each path $\gamma(t)$, $a \leq t \leq b$, in the base there corresponds a map of fibers

$$\begin{gathered}\tilde{\gamma} : F_0 \longrightarrow F_1, \quad F_0 = p^{-1}(y_0), \quad F_1 = p^{-1}(y_1),\\ y_0 = \gamma(a), \quad y_1 = \gamma(b),\end{gathered}$$

which depends continuously on the path γ. A closed path γ beginning and ending at y_0 yields in this way a homotopy equivalence

$$\tilde{\gamma} : F_0 \longrightarrow F_0,$$

which in turn induces "monodromy" isomorphisms between the n-th homotopy groups:

$$\widetilde{\gamma}_* : \pi_n(F_0, x_0) \longrightarrow \pi_n(F_0, \widetilde{\gamma}(x_0)).$$

We met with a particular case of this in the discussion of Riemann surfaces in Chapter 1, where we had $n = 0$, the base was a region of $\mathbb{R}^2$, and the monodromy, denoted by σ_γ, was a permutation of the discrete fiber.

It is worthwhile distinguishing, from among the classes of all covering spaces, the *regular* ones, namely those where the image $p_*\pi_1(X) \subset \pi_1(Y)$ is a normal subgroup of $\pi_1(Y)$. For such a covering space the quotient group $\pi_1(Y)/p_*\pi_1(X) = \Gamma$ acts naturally as a discrete group of transformations (homeomorphisms) of the space X, the action being defined via the transport of fibers along closed paths γ in the base Y, i.e. via monodromy; the paths belonging to those homotopy classes contained in the image $p_*\pi_1(X)$ yield trivial monodromy.

The largest covering space X of a given space Y is called a *universal covering space* of Y. It can be shown to exist uniquely for a large class of spaces Y. The space X is simply-connected: $\pi_1(X)$ is trivial, so that, in view of the above, it is a regular covering, and $\Gamma = \pi_1(Y)$ has a natural discrete action on it, with orbits the fibers $p^{-1}(y)$. Since X is simply–connected, its homotopy groups are independent of the base point x_0. The group Γ (acting discretely and freely on X, i.e. only the identity element fixes any point) induces homomorphisms

$$\gamma : \pi_n(X) \longrightarrow \pi_n(X), \quad n > 1, \quad \gamma \in \Gamma.$$

Taking into account that $\Gamma = \pi_1(Y, y_0)$, we thus have actions of π_1 on all π_n with the natural geometric interpretation.

Example 1. Let $U \subset \mathbb{R}^3$ be the region obtained by removing from $\mathbb{R}^3$ a line and any point off that line. The region U then has as deformation retract the bouquet of the circle and 2–sphere:

$$U \sim S^2 \vee S^1 = Y.$$

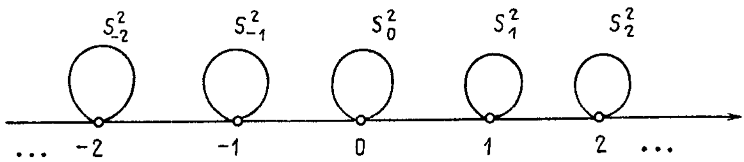
One easily deduces that $\pi_1(U) = \pi_1(S^2 \vee S^1) \cong \mathbb{Z} \cong \pi_1(S^1)$, where $\mathbb{Z}$ is the infinite cyclic group, with generator t, say. Consider the universal cover of U, or, rather, the homotopically equivalent universal cover X of the bouquet $S^2 \vee S^1 = Y$:

$$X \longrightarrow Y = S^2 \vee S^1.$$

The space X turns out to be representable as the line $\mathbb{R}$ (coordinatized by λ, say) with 2–spheres S^2_j attached at the integer points j (see Figure 2.16). The action of $\pi_1(Y)$ on X is here given by

$$t(\lambda) = \lambda + 1, \quad t\left(S^2_j\right) = S^2_{j+1}. \tag{4.10}$$

The space X has as its 2–dimensional homotopy group $\pi_2(X)$ the direct sum of a countably infinite collection of copies of $\mathbb{Z}$, with basis $\{d_j\}$ say, each d_j



$$t(\lambda) = \lambda + 1, \quad t\left(S^2_n\right) = S^2_{n+1}$$

Fig. 2.16

defined geometrically by the obvious map $S^2 \longrightarrow S^2_j$. The action of $\pi_1(Y)$ on $\pi_2(X)$ is then obviously given by

$$t(d_j) = d_{j+1}. \tag{4.11}$$

In view of the isomorphism (4.8), this describes also the structure of $\pi_2(S^2 \vee S^1) \cong \pi_2(U)$, inclusive of the action on it of $\pi_1(S^2 \vee S^1) \cong \mathbb{Z}$. In module language, $\pi_2(U)$ is thus a free $\mathbb{Z}[\pi]$–module ($\pi = \pi_1(U)$) on the one generator $d = d_0$, since each element a of $\pi_2(U)$ has the unique form

$$a = \sum_{j=-N}^{+M} \lambda_j t^j(d),$$

where the λ_j are integers, $\sum \lambda_j t^j \in \mathbb{Z}[\pi]$, and $\pi = \pi_1(Y, y_0)$. □

Example 2. The real projective plane $\mathbb{R}P^2$ has as its points the equivalence classes of non-zero real vectors $\left(\lambda^0, \lambda^1, \lambda^2\right)$ where two triples are equivalent if one is a nonzero scalar multiple of the other:

$$\left(\lambda\lambda^0, \lambda\lambda^1, \lambda\lambda^2\right) \sim \left(\lambda^0, \lambda^1, \lambda^2\right), \quad \lambda \neq 0;$$

in other words the points of $\mathbb{R}P^2$ are the straight lines through the origin in $\mathbb{R}^3$, with the origin removed. One obtains exactly two representatives of each equivalence class by imposing the requirement of unit length:

$$\sum_{j=0}^{2} (\lambda^j)^2 = 1.$$

From this it is clear that $\mathbb{R}P^2$ may be considered as the orbit space of the action on the 2–sphere S^2 of the discrete group $\mathbb{Z}/2$ of order 2:

$$\left(\lambda^0, \lambda^1, \lambda^2\right) \sim \left(-\lambda^0, -\lambda^1, -\lambda^2\right).$$

Since the group $\pi_1(S^2)$ is trivial this covering is universal for $\mathbb{R}P^2$. From (4.8) we have

$$\pi_2(S^2) \cong \pi_2(\mathbb{R}P^2) \cong \mathbb{Z}. \tag{4.12}$$

Denote by d a generator of $\pi_2(\mathbb{R}P^2)$. We also have $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2$ in view of the above description of $\mathbb{R}P^2$ as the orbit space of its universal cover under the action of $\mathbb{Z}/2$; write t for the generator of $\pi_1(\mathbb{R}P^2)$, $t^2 = 1$, which can be regarded as acting appropriately on the sphere S^2. The structure of $\pi_2(\mathbb{R}P^2)$ as $\mathbb{Z}[\pi_1]$–module is then given by

$$t(d) = -d, \quad t^2 = 1. \tag{4.13}$$

It is readily established that $\pi_i(S^n) = 0$ for $i < n$, and that $\pi_n(S^n) = \mathbb{Z}$. It follows much as in the case $n = 2$ just treated that for the *n-dimensional real projective space* $\mathbb{R}P^n$, one has

$$\pi_i(\mathbb{R}P^n) \cong \pi_i(S^n), \quad i > 1; \quad \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2. \tag{4.14}$$

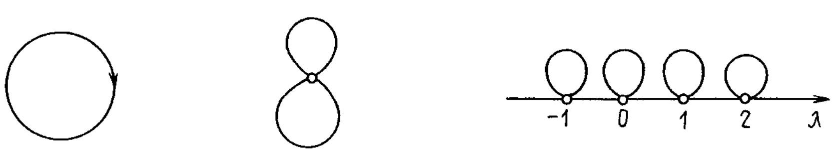
Moreover the generator t of the group $\pi_1(\mathbb{R}P^n)$ acts on the basis element d of $\pi_n(\mathbb{R}P^n)$ (regarded as $\mathbb{Z}[\pi_1]$–module) according to the formula:

$$t(d) = (-1)^{n+1}d, \tag{4.15}$$

where the factor $(-1)^{n+1}$ arises from the reflection $x \to -x$ of $\mathbb{R}^{n+1}$ restricted to the sphere $S^n \subset \mathbb{R}^{n+1}$, as it affects orientation.

We remark in conclusion that for the space $\mathbb{R}P^\infty$, defined as the direct limit of the sequence of spaces $\mathbb{R}P^n$, $n = 1, 2, \ldots$, each embedded naturally in its successor, the fundamental group is again $\mathbb{Z}/2$, while the groups $\pi_i(\mathbb{R}P^\infty)$, $i > 1$, are all trivial. □

Example 3. Every connected planar region (or, more generally, every open 2-dimensional manifold) has as deformation retract a one-dimensional complex. Every one-dimensional complex is homotopy equivalent to a bouquet of finitely or countably infinitely many circles. (The latter case is conveniently realized as the complex depicted in Figure 2.17 (c).)



(a) Circle (b) Bouquet (c) Circles attached at the integer points $\lambda = n$ of the λ-line

Fig. 2.17

It is easy to construct covering spaces for these 1-complexes that are trees, and so simply-connected. (A *tree* is a connected graph without cycles.) The bouquet $S^1 \vee S^1$, for instance, has the tree shown in Figure 2.18 as its universal cover. To see this, observe that every vertex in the covering graph lies above the single vertex of the bouquet and therefore must have a neighbourhood homeomorphic to some neighbourhood of the vertex of the bouquet, i.e. to a "cross". On this tree the free group on two generators acts freely in a natural way, with orbit space the bouquet $S^1 \vee S^1$. It follows by this sort of argument that $\pi_1(K)$ is a free group for every 1-complex (or graph) K. □

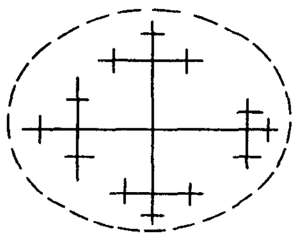


Fig. 2.18. The universal covering tree of $S^1 \vee S^1$

Example 4. The n-dimensional torus T^n is obtained from $\mathbb{R}^n$ by means of the identification

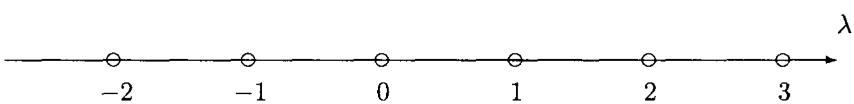
$$(\lambda^1, \ldots, \lambda^n) \sim (\bar{\lambda}^1, \ldots, \bar{\lambda}^n)$$

whenever $(\lambda^1 - \bar{\lambda}^1, \ldots, \lambda^n - \bar{\lambda}^n)$ has all components integers. We infer immediately a discrete, free action of the integer lattice $\mathbb{Z}^n$ on $\mathbb{R}^n$, and the orbit space is the quotient group:

$$T^n = \mathbb{R}^n/\mathbb{Z}^n, \quad T^1 = S^1 = \mathbb{R}^1/\mathbb{Z}. \tag{4.16}$$

It follows that

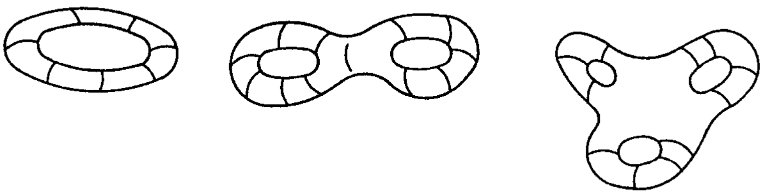
$$\pi_1(T^n) \cong \mathbb{Z}^n, \qquad \pi_i(T^n) = 0 \quad \text{for} \quad i > 1.$$ □



$\lambda \to \lambda + 1, \;\; \pi_1(S^1) = \mathbb{Z}$

Fig. 2.19

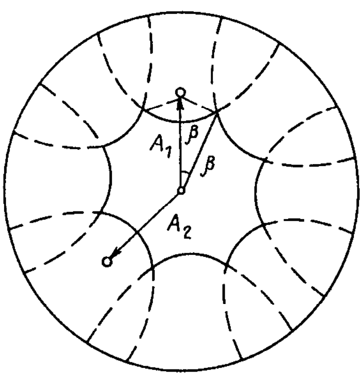
Example 5. Up to homeomorphism every closed, orientable surface is a sphere–with–handles (see Figure 2.20). This fact was established by Möbius. We shall denote the sphere with g handles attached by M_g^2; the integer g is called the *genus*. For $g \geq 2$ each of the surfaces M_g^2 may be obtained as an orbit space of the Lobachevskian plane L^2 under the discrete, free action of a group Γ of (metric-preserving) transformations (see Figure 2.21, where L^2 is represented as the interior of the unit disc). For the upper-half plane model,



(a) $g = 1$ (b) $g = 2$ (c) $g = 3$

Fig. 2.20

Im $z > 0$, of L^2, the group Γ may be taken as that generated by the following



$$\beta = \frac{\pi}{4g}$$

The transformations A_1 and A_2 move the center of the octagon to the points indicated by the arrows

Fig. 2.21

$2g$ linear–fractional transformations $A_1, \ldots, A_{2g} : L^2 \longrightarrow L^2$ (all preserving the half-plane):

$$A_1 = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}, \quad A_k = B_g^{-k+1} A_1 B_g^{k-1},$$

$$B_g = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}, \quad l = \ln \frac{\cos\beta + \sqrt{\cos 2\beta}}{\sin\beta}, \tag{4.17}$$

$$\alpha = \pi\frac{2g-1}{4g}, \quad \beta = \frac{\pi}{4g}.$$

(Here a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents the linear-fractional transformation

$$z \longrightarrow \frac{az+b}{cz+d}.$$

Figure 2.21 shows some of this transferred over to the unit-disc model of L^2.) It can be checked that

$$A_1 \cdots A_{2g} A_1^{-1} \cdots A_{2g}^{-1} = 1, \tag{4.18}$$

and shown that all relations among $A_1, \ldots, A_{2g}$ are consequences of (4.18). Thus $\pi_1(M_g^2)$ is isomorphic to the group defined by the generators $A_1, \ldots, A_{2g}$ with the single defining relation (4.18). One may find other generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$ for the group $\pi_1(M_g^2)$ with the single defining relation:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1. \tag{4.19}$$

The effect of taking the orbit space under the action of this group of transformations is, essentially, to identify appropriate sides of a $4g$-gon in L^2 (in $\mathbb{R}^2$ for $g = 1$) — see Figure 2.22. □

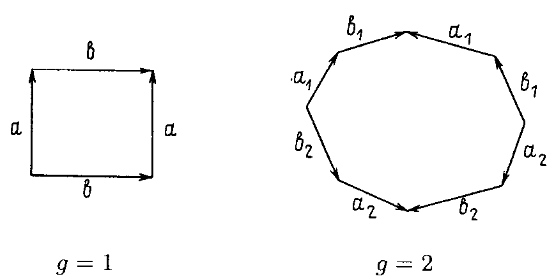


Fig. 2.22

Example 6. The closed non-orientable surfaces may be constructed from the orientable ones by taking the orbit spaces of the latter under a suitable discrete, free action of $\mathbb{Z}/2$, analogously to the construction of $\mathbb{R}P^2$ from S^2, described earlier. There arise two families of nonorientable closed surfaces: $N_{g,1}^2$ and $N_{g,2}^2$.

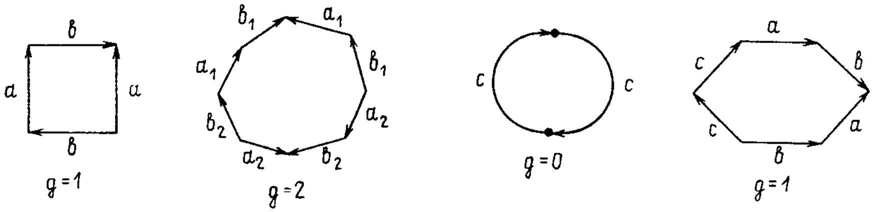
These may also be obtained from polygons of $4g$ and $4g+2$ sides respectively by suitable identification of edges — see Figure 2.23. One can find generators $a_1, \ldots, a_g, b_1, \ldots, b_g$ for $\pi_1(N_{g,1}^2)$ with the single defining relation:

$$N_{g,1}^2: \quad a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_{g-1} b_{g-1} a_{g-1}^{-1} b_{g-1}^{-1} a_g b_g a_g^{-1} b_g = 1. \tag{4.20}$$

For $N_{g,2}^2$, on the other hand, the group $\pi_1(N_{g,2}^2)$ is given by generators $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, c, and the single defining relation

$$N_{g,2}^2: \quad \left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) c^2 = 1. \tag{4.21}$$

That these two families exhaust the closed, non-orientable surfaces was first established by Poincaré. □



$N_{1,1}^2$ (the Klein bottle)

(a) $N_{g,1}^2$, $g \geq 1$

$N_{0,2}^2$ (the projective plane)

(b) $N_{g,2}^2$, $g \geq 0$

Fig. 2.23

It can be shown that for all 2-dimensional manifolds (i.e. surfaces), open or closed, with the exception of $\mathbb{R}P^2$ and S^2, all of the higher homotopy groups are trivial:

$$\pi_i(M^2) = 0 \quad \text{for } i > 1, \quad M^2 \neq \mathbb{R}P^2,\ S^2. \tag{4.22}$$

A space X with the property that all but one of its homotopy groups are trivial is called an *Eilenberg-MacLane space*, denoted by $K(\pi, n)$ if

$$\pi_n(X) \cong \pi, \pi_i(X) = 0 \quad \text{for } i \neq n. \tag{4.23}$$

(In fact, $K(\pi, n)$ may be realized as a "CW-complex" — see below.) We have already encountered spaces of the type $K(\pi, 1)$. Note in particular that the space $\mathbb{R}P^\infty$ mentioned at the end of the Example 2 is of type $K(\mathbb{Z}/2, 1)$. □

Example 7. The Hopf fibration. The *complex projective space* $\mathbb{C}P^n$ of n complex dimensions is defined, analogously to $\mathbb{R}P^n$, by identifying non-zero vectors in $\mathbb{C}^{n+1}$ if they are non-zero multiples of one another by complex scalars:

$$(\lambda^0, \ldots, \lambda^n) \sim (\lambda\lambda^0, \ldots, \lambda\lambda^n), \quad \lambda \in \mathbb{C}, \quad \lambda \neq 0.$$

Restricting to just those points of $\mathbb{C}^{n+1}$ satisfying

$$\sum_j |\lambda^j|^2 = 1,$$

we infer that $\mathbb{C}P^n$ may be realized as the orbit space under the free action of the circle group $S^1 \cong U(1) \cong SO_2$ on S^{2n+1} given by

$$\left(\lambda^0, \ldots, \lambda^n\right) \longrightarrow \left(e^{i\alpha}\lambda^0, \ldots, e^{i\alpha}\lambda^n\right).$$

This defines the *Hopf fibration*: $p : S^{2n+1} \longrightarrow \mathbb{C}P^n$, with fiber $S^1 \cong U(1) \cong SO_2$. Since $\pi_1(S^1) \cong \mathbb{Z}$, $\pi_i(S^1) = 0$ for $i > 1$ (see above), we obtain from the exact sequence of the fibration (see (4.7)) the following isomorphisms:

$$\begin{gathered}
\pi_i\left(S^{2n+1}\right) \cong \pi_i\left(\mathbb{C}P^n\right), \quad i \neq 2, \\
\pi_{2n+1}\left(S^{2n+1}\right) \cong \pi_{2n+1}\left(\mathbb{C}P^n\right) \cong \mathbb{Z}, \\
\pi_2\left(\mathbb{C}P^2\right) \cong \mathbb{Z} \cong \pi_1\left(S^1\right).
\end{gathered} \tag{4.24}$$

It follows by taking the direct limit as $n \longrightarrow \infty$, that the infinite-dimensional complex projective space $\mathbb{C}P^\infty$ has the type of the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$. From (4.24) and the fact that $\mathbb{C}P^1 \cong S^2$ we deduce also *Hopf's theorem*:

$$\pi_3\left(S^2\right) \cong \mathbb{Z}. \quad \square \tag{4.25}$$

Example 8. Consider a knot, i.e. an embedded circle S^1 in the 3-sphere S^3, and its complement $S^3 \setminus S^1 = U$, say. (Earlier we considered knots in $\mathbb{R}^3$; here we have adjoined a point at infinity to obtain the compact space S^3.) A theorem of Papakyriakopoulos asserts that $\pi_2(U) = 0$, whence it can be shown without difficulty that $\pi_j(U) = 0$ for $j > 2$. Hence knot-complements are all Eilenberg-MacLane spaces of type $K(\pi, 1)$. $\square$

In conclusion we note that of the simple, well-known examples of spaces of type $K(\pi, n)$, there is only one with $n > 1$, namely $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. As far as the apparatus of algebraic topology is concerned, for the most part the existence of the spaces $K(\pi, n)$ suffices; in only one important paper (by E. Cartan) is knowledge of a concrete algebraic model made significant use of.

Chapter 3
Simplicial Complexes and CW-complexes. Homology and Cohomology. Their Relation to Homotopy Theory. Obstructions

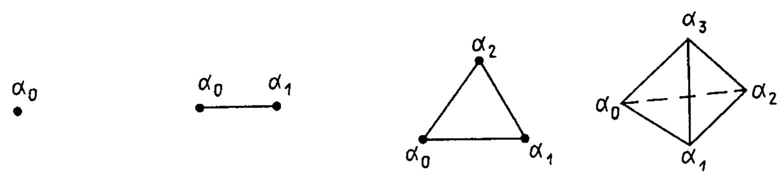
§1. Simplicial complexes

Simplicial complexes (first introduced by Poincaré) furnish the most elementary, convenient, most accessible to a rigorous treatment, and cleanest means for defining the homology and cohomology groups and investigating their formal properties, as we shall shortly see; however they are inconvenient when it comes to particular concrete calculation of these groups.

An *n-dimensional simplex* σ_n is defined as the convex hull in $\mathbb{R}^n$ of any $n+1$ points $\alpha_0, \ldots, \alpha_n$ not contained in any $(n-1)$-dimensional hyperplane (see Figure 3.1). Thus the points of $\sigma^n = (\alpha_0, \ldots, \alpha_n)$ are the linear combinations of the vertices $\alpha_0, \ldots, \alpha_n$ (regarded as n-vectors) of the following form:

$$x \in \sigma^n, \quad x = \sum_{j=0}^{n} x^j \alpha_j, \quad \sum_{j=0}^{n} x^j = 1, \quad x^j \geq 0. \tag{1.1}$$

A *face* of a simplex σ^n of any dimension n is the simplex determined by any



(a) 0-dimensional (b) 1-dimensional (c) 2-dimensional (d) 3-dimensional

Fig. 3.1

proper subset of the vertices, i.e. the convex hull in $\mathbb{R}^n$ of a proper subset of $\{\alpha_0, \ldots, \alpha_n\}$. In particular the faces of dimension $n-1$ are just the simplexes

$$\sigma_j{}^{n-1} = (\alpha_0, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n), \tag{1.2}$$

(where the hat indicates that a symbol is to be considered omitted). Thus σ^n has exactly $n+1$ faces of dimension $n-1$.

A *simplicial complex* K is then an arbitrary (finite or countably infinite) collection of simplexes, with the following properties:

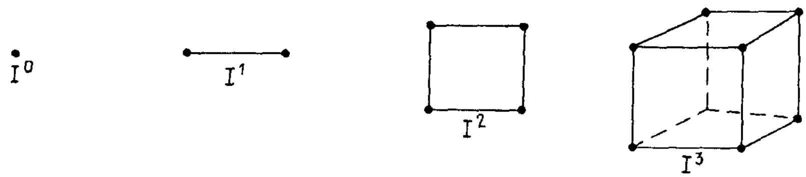
1. Together with each simplex in the collection, all of its faces of all dimensions should also be in the collection.

2. Any two simplexes in the collection that intersect should either coincide (i.e. be the same simplex) or intersect precisely in a common face.

A simplicial complex is most conveniently given by indicating its vertices $\alpha_0, \ldots, \alpha_n, \ldots$, together with those (finite) subsets of these that determine simplexes of the complex.

One might consider more general complexes composed not just of simplexes but also of other general convex polyhedra (or rather polytopes), still satisfying the conditions 1 and 2 above; and we shall in fact often be considering such complexes. However it is easy to show that by means of subdivision such complexes may be refined to simplicial ones. In investigations into the homology of concrete complexes, an operation of "consolidation" is often employed, involving the union of appropriate sets of simplexes into convex polytopes.

Remark. Besides simplicial complexes, *cubic complexes* are of interest. Such a complex is made from cubes I^n of all dimensions (see Figure 3.2), fitting together into a cubic complex under the analogues of the conditions 1, 2 above. Every simplicial complex may, by means of appropriate subdivision, be given the form of a cubic complex, and vice versa. The convenience of cubic complexes consists in the fact that $I^{n+m} = I^n \times I^m$, so that a product of cubic complexes is again a cubic complex, and they derive interest also from the fact that $\mathbb{R}^n$ has regular tesselations into cubes (regular cubic lattices). In modern statistical mechanics several questions arise in the theory of lattice models concerning cubic subcomplexes occurring as images of submanifolds of $\mathbb{R}^n$ with prescribed regular cubic tesselations. In the case $n = 3$, in particular, it turns out to be necessary to find a description of the maps of surfaces M^2 into the regular cubic lattice in $\mathbb{R}^3$, where M^2 is subdivided into 2-cubes (squares), and the map preserves this structure. Refinements of the respective subdivisions must be such as to preserve the regularity of the lattice *in* $\mathbb{R}^3$. [1]



(a) (b) (c) (d)

Fig. 3.2

Returning to our arbitrary simplicial (or convex polyhedral) complex K, we note that if K is finite, i.e. consists of only finitely many simplexes (or convex polytopes), then the *Euler-Poincaré characteristic*, already mentioned in Chapter 1, may be defined for it:

$$\chi(K) = \sum_{j \geq 0} (-1)^j \gamma_j, \tag{1.3}$$

[1] "Novikov's problem", considered recently by N.P. Dolbilin, M.A. Stanko and M.I. Shtogrin.

where γ_j is the number of j-dimensional simplexes (or j-dimensional convex polytopes) of K. If every simplex of K has dimension $\leq n$ and there are some simplexes of dimension n, then we say that the complex K is *n-dimensional.* The *m-dimensional skeleton* of K is the subcomplex made up of all simplexes of K of dimension $\leq m$.

Given two simplicial complexes K and K', we say that K' is a *subdivision* of K if each simplex of K is a union of finitely many simplexes of K' and the simplexes of K' are contained linearly in the simplexes of K. There is a standard subdivision of any simplicial complex, called the *barycentric subdivision,* defined inductively as follows: A point (or zero-dimensional simplex) does not subdivide. The barycentric subdivision of a one-dimensional simplex $\sigma^1 = (\alpha_0, \alpha_1)$ is carried out by introducing a new vertex at the centre (midpoint) of σ^1, so that σ^1 becomes the complex made up 3 vertices and 2 one-simplexes (the edges). To barycentrically subdivide an n-simplex

(a) (b)

Fig. 3.3. The barycentric subdivision of a simplex

$\sigma^n \subset \mathbb{R}^n$, one first barycentrically subdivides the faces of σ^n, then introduces a further new vertex $\alpha' \in \sigma^n$ at the centre (centroid) of σ^n, and admits as new simplexes (of the barycentric subdivision) all those of the form $(\beta_0, \ldots, \beta_k, \alpha')$ where $(\beta_0, \ldots, \beta_k)$ is an arbitrary simplex of the barycentric subdivision of a face of σ^n. The totality of new simplexes then comprises the barycentric subdivision of σ^n (see Figure 3.3). The barycentric subdivision of a simplicial complex is then obtained by barycentrically subdividing all of the simplexes of the complex, ensuring that on faces common to two or more simplexes the subdivisions coincide.

Definition 1.1 Two simplicial complexes K_1, K_2 are said to be *combinatorially equivalent* if there exists a simplicial complex K isomorphic to a subdivision of each of them.

At a certain stage in the development of topology, when all of the known topological invariants were defined in combinatorial terms, the conjecture known as the *Hauptvermutung der Topologie,* regarded then as of the very first importance, was proposed. This is the conjecture that any two homeomorphic complexes are combinatorially equivalent. It was shown to be valid in dimensions ≤ 3 by means of direct, elementary methods (by Moise in the

1950s – at least for manifolds), but turned out to be false in dimensions ≥ 6 (Milnor in the early 1960s). Then in the late 1970s it was shown (by Edwards) that the double suspension of any 3-dimensional manifold that is a "homology 3-sphere" (i.e. has the same homology as the 3-sphere) is actually homeomorphic to the 5-sphere, and it is certainly the case that the associated triangulation of S^5 is combinatorially inequivalent to the trivial one, namely as the boundary of a simplex σ^6. [2]

Notwithstanding the truth or otherwise of the *Hauptvermutung* for particular spaces, the simplest topological invariants depending for their definition on a combinatorial structure imposed on a space — for example the Euler-Poincaré characteristic and the homology groups (see below) — turn out to be invariant not only under homeomorphisms but even under homotopy equivalences (Alexander in the 1920s). (It would appear that this was in fact the source of the concept of homotopy equivalence.) Deeper invariants — the integrals of "Pontryagin classes" over cycles, which admit of a combinatorial definition by the Thom–Rohlin–Schwarz theorem, and "Reidemeister–Whitehead torsion" — may also be defined initially combinatorially. While it has long been known (since the middle 1950s for the Pontryagin classes, and since the 1930s for Reidemeister–Whitehead torsion) that these quantities are not homotopy invariants, it turns out that they are at least topological invariants, i.e. invariant under homeomorphisms (Novikov in the mid–1960s for the Pontryagin classes, and Edwards–Chapman in the 1970s for Reidemeister–Whitehead torsion; note that for $n = 3$ this follows from Moise's theorem of the 1950s, mentioned above). Reidemeister–Whitehead torsion constitutes an essentially self–contained chapter of topology. We shall subsequently return to later developments in connexion with this invariant (occuring in the 1960s and 70s), which have come to comprise a unique and highly complex theory. In the mid 1960s Sullivan outlined a beautiful theory having as a consequence that the *Hauptvermutung* holds for simply connected PL-manifolds M provided that the group $H_3(M;\mathbb{Z})$ does not have 2-torsion.[3] Kirby and Siebenmann have developed a theory clarifying completely the relationship between PL- and topological structures on manifolds in dimensions $n \neq 4$.[4] In any case for

[2]The results of Freedman and Donaldson (early 1980s) show that the *Hauptvermutung* is false also for four-dimensional manifolds.

[3]In the first version of his work Sullivan placed no restrictions on the group $H_3(M)$; the need for the above–mentioned requirement emerged in a discussion with Browder and Novikov (Spring, 1967). An exposition of Sullivan's theory has not yet appeared in the literature (at least as of 1994, the time of writing).

[4]The book by R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations, Princeton University Press, 1977, Annals of mathematics studies, no. 88,* is devoted to this theory. In this book the authors have apparently attempted to give an independent validation of Sullivan's theory. However there are several crucial statements in the book whose proofs require Sullivan's results. For example, in the present author's opinion the proofs on the pages 268-296 do not seem to be complete independently of references to certain of Sullivan's results. It is of course important that these fundamental results be established in full rigour and published. It should be noted here that some of difficulties of the theory outlined by Sullivan have been overcome

each $n \geq 5$ there exist altogether only finitely many homeomorphism classes of closed, smooth or piecewise linear (PL-) manifolds (Novikov, in the mid-1960s, in the simply-connected case). More detailed results, dating from the late 1960s and the 1970s, will be indicated at the conclusion of §5 of Chapter 4.

Returning to our exposition of the elementary theory of simplicial complexes, we define a *map of simplexes* $\sigma_1^n \longrightarrow \sigma_2^m$ to be any mapping of the vertices of σ_1^n to those of σ_2^m, extended linearly to the whole of σ_1^n. A *simplicial map* $f : K_1 \longrightarrow K_2$ of complexes is then a map whose restriction to each simplex is a map of simplexes; thus a simplicial map between complexes is also determined by the images $f(\alpha_j) = \beta_j$ of the vertices α_j of K_1, where the β_j are vertices of K_2. A *piecewise linear* (PL–)*map* $K_1 \longrightarrow K_2$ is one that is simplicial between some suitable subdivisions $\bar{K}_1, \bar{K}_2$ of K_1 and K_2 (not necessarily barycentric).

The fairly straightforward "Theorem on Simplicial Approximation" asserts that every continuous map $f : K_1 \longrightarrow K_2$ of complexes, can be abitrarily closely approximated by a PL–map $g : \bar{K}_1 \longrightarrow \bar{K}_2$ of suitable subdivisions of K_1 and K_2, which is homotopic to f.

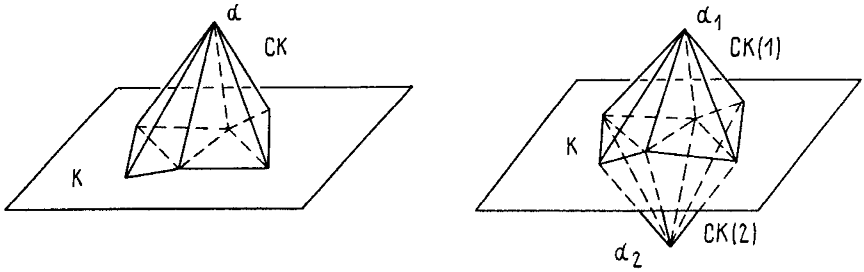
Sometimes a variant of this theorem is used, according to which, given as before any map $f : K_1 \longrightarrow K_2$, there is a simplicial map $g : \bar{K}_1 \longrightarrow K_2$ now of some subdivision of K_1 only, no longer necessarily close to f, but homotopic to f by means of a homotopy throughout which the motion of each image point is confined to a single simplex of K_2.

Definition 1.2 A simplicial complex K is an *n–dimensional PL–manifold* if, after application of a sequence of barycentric subdivisions, the *combinatorial neighbourhood* of each simplex of K — i.e. the complex made up of the simplexes, together with their boundaries, having that simplex as a face — is a complex combinatorially equivalent to σ^n.

A PL–manifold of dimension n is said to be *orientable* if its n–simplexes may all be so oriented that the orientations induced on each $(n-1)$–simplex from the orientations of the two n–simplexes of which it is a common face, are opposite. Each particular way of orienting the n–simplexes of an (orientable) PL–manifold so as to satisfy this criterion, is called an *orientation* of the PL–manifold.

Every PL–manifold is, of course, a topological manifold. However not every triangulation of a topological manifold yields a simplicial complex which is a PL–manifold: we have already mentioned Edwards' theorem (from the 1970s) to the effect that the triangulation of the sphere S^5 arising from the double suspension $\Sigma\Sigma M^3$ of any homology 3–sphere M^3 (i.e. a 3–manifold with $H_1 =$

by other topologists; for instance all required results of homotopy theory have been established by Milgram and Madsen, in their book *Classifying Spaces in Surgery and Cobordism of manifolds, University Press, 1979, Annals of mathematics studies, no. 92.* This work uses several technical tools and ideas not yet developed in 1967.



(a) The cone on K (b) The suspension $\Sigma K = (CK)_1 \cup (CK)_2$

Fig. 3.4

$H_2 = 0$ but with $\pi_1 \neq 0$) is not combinatorially equivalent to σ^5 yet $\Sigma\Sigma M^3 \cong S^5$ (where $\cong$ means "is homeomorphic to").

We shall now define some of these terms. The *cone* on a simplicial complex K, denoted by CK, is the complex formed by joining every point of K by means of an interval to a new vertex α outside K; the triangulation of CK making it a simplicial complex is carried out in the obvious manner (see Figure 3.4(a)). The *suspension* ΣK of K is then the union of two cones on K with their common base K identified:

$$\Sigma K = (CK)_1 \cup_K (CK)_2. \tag{1.4}$$

The suspension can then be triangulated in the obvious way (see Figure 3.4(b))

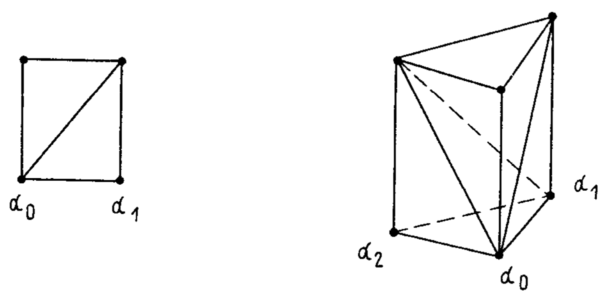


Fig. 3.5. Subdivisions of cylinders on simplexes

The basic operations on topological spaces introduced in §1 of Chapter 2, namely the bouquet $K_1 \vee K_2$, the (simplicial) mapping cylinder $C_f, f : A \longrightarrow K_2, A \subset K_1$, and the product $K_1 \times K_2$, all go over to the class of simplicial complexes upon following each operation by some (standard) subdivision.

(Thus although the product of two simplexes is not a simplex, it is readily triangulated — see Figure 3.5.)

§2. The homology and cohomology groups. Poincaré duality

We now turn to the definition of the homology and cohomology groups of simplicial complexes. The *algebraic boundary* of any simplex $\sigma^n = (\alpha_0, \ldots, \alpha_n)$ is defined as the following formal linear combination of its faces of dimension $n-1$

$$\partial\sigma^n = \sum_{i=0}^{n}(-1)^i\sigma_i^{n-1}, \tag{2.1}$$

where $\sigma_i{}^{n-1} = (\alpha_0, \ldots, \widehat{\alpha}_i, \ldots, \alpha_n)$, the hat over α_i indicating its omission. Observe that $\partial\circ\partial = 0$. For any complex K the *group* $C_n(K)$ *of* n*–dimensional integral chains* of K is the free abelian group consisting of all finite formal integral linear combinations of the n–simplexes of K:

$$c \in C_n(K), \quad c = \sum_\alpha \lambda_\alpha\sigma_\alpha^n, \quad \sigma_\alpha^n \in K;$$

here α indexes the n–simplexes of K, and the coefficients λ_α are integers. From the defining formula (2.1) for the algebraic boundary of a simplex we obtain the *boundary operator* ∂ on the integral n–chains of K:

$$\partial : C_n(K) \longrightarrow C_{n-1}(K), \quad \partial\left(\sum_\alpha \lambda_\alpha\sigma_\alpha^n\right) = \sum_\alpha \lambda_\alpha\left(\partial\sigma_\alpha^n\right),$$

and thence the following *chain complex* $C_*(K)$:

$$\cdots \xrightarrow{\partial} C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} C_{n-2}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} 0, \tag{2.2}$$

where $\partial\circ\partial = 0$. We now define for each $n \geq 0$ the n*–dimensional integral homology group* of K by

$$H_n(K;\mathbb{Z}) = \text{Ker }\partial/\text{Im }\partial, \tag{2.3}$$

noting that this makes sense, i.e. Ker $\partial \supset$ Im ∂, since $\partial\circ\partial = 0$. The elements of the subgroup Ker $\partial \subset C_n(K)$ are called *cycles* of K, and of the subgroup Im $\partial \subset C_n$ *boundaries.*

The operations of forming $G_1 \otimes G_2$ and $\text{Hom}(G_1, G_2)$ from arbitrary pairs G_1, G_2 of (additively written) abelian groups, will be important for us; they are defined as follows:

1. The abelian group $G_1 \otimes G_2$ is obtained from the group of all finite formal sums of expressions $g_1 \otimes g_2$, $g_1 \in G_1$, $g_2 \in G_2$, by imposing the bilinearity relations:

$$\begin{aligned}(g_1' + g_1'') \otimes g_2 &= g_1' \otimes g_2 + g_1'' \otimes g_2, \\ g_1 \otimes (g_2' + g_2'') &= g_1 \otimes g_2' + g_1 \otimes g_2''.\end{aligned} \tag{2.4}$$

The resulting group $G_1 \otimes G_2$ is called the *tensor product* of G_1 with G_2.

2. The abelian group $\mathrm{Hom}(G_1, G_2)$ has as its elements the homomorphisms $h : G_1 \to G_2$ from the abelian group G_1 to the abelian group G_2, two such homomorphisms being added in the natural way:

$$(h_1 + h_2)(g) = h_1(g) + h_2(g). \tag{2.5}$$

In the case where G_2 is the additive group of a field, one denotes $\mathrm{Hom}(G_1, G_2)$ by G_1^*. (Cf. the notation V^* for the dual space of a vector space V, consisting of all vector–space homomorphisms from V to the one–dimensional vector space, i.e. scalar–valued linear functions on V.) Note in particular that $G \otimes \mathbb{Z} \cong G$ and $\mathrm{Hom}(\mathbb{Z}, G) \cong G$.

Equipped with these two operations we can construct from a chain complex $C(K)$ and any abelian group G the following *chain* and *cochain complexes over* G:

(i) $$C(K) \otimes G : \quad \ldots \xrightarrow{\partial} C_n(K) \otimes G \xrightarrow{\partial} C_{n-1}(K) \otimes G \xrightarrow{\partial} \ldots,$$

(ii) $$\mathrm{Hom}(C(K), G) : \quad \ldots \xleftarrow{\partial^*} C^n(K; G) \xleftarrow{\partial^*} C^{n-1}(K; G) \xleftarrow{\partial^*} \ldots,$$

where $C^i(K; G) = \mathrm{Hom}(C_i(K), G)$ and ∂^* is the formal dual operator of ∂. We call the groups $C_n(K; G) = C_n(K) \otimes G$ the *chain groups of K with coefficients from G*, and the $C^n(K; G)$ the *cochain groups of K over G*. The *homology groups H_n* and *cohomology groups H^n with coefficients from G* are defined in terms of the above two algebraic complexes by

$$\begin{aligned} H_n(K; G) &= \mathrm{Ker}\ \partial / \mathrm{Im}\ \partial, \qquad \mathrm{Ker}\ \partial \subset C_n(K) \otimes G; \\ H^n(K; G) &= \mathrm{Ker}\ \partial^* / \mathrm{Im}\ \partial^*, \qquad \mathrm{Ker}\ \partial^* \subset C^n(K; G). \end{aligned}$$

The elements of the group $\mathrm{Ker}\ \partial^*$ are called *cocycles*, and of the group $\mathrm{Im}\ \partial^*$ *coboundaries*.

Remark. Chains and cochains may of course be defined also with coefficients from a non-abelian group (written multiplicatively in this case). However then the boundaries, coboundaries and cohomology groups admit of natural definitions only in dimensions 0 and 1. The zero-dimensional cocycles (which are actually also the elements of the zero-dimensional cohomology group $H^0(K; G)$) are just functions g defined on the 0-simplexes of K, the vertices, taking the same value at the two ends of each edge:

$$\begin{aligned} \partial^* g(\sigma_\alpha^1) &= g(\sigma_{\alpha,1}^0) g(\sigma_{\alpha,0}^0)^{-1}, \\ \partial^* g \equiv 1 &\Longleftrightarrow g(\sigma_{\alpha,1}^0) = g(\sigma_{\alpha,0}^0). \end{aligned} \tag{2.6}$$

Hence $H^0(K;G) \cong G \times \cdots \times G$, where the number of direct factors G is the number of connected components of K. In dimension one we have:

$$\begin{aligned} \partial^* g(\sigma_\alpha^2) &= g(\sigma_{\alpha,2}^1) g(\sigma_{\alpha,1}^1)^{-1} g(\sigma_{\alpha,0}^1), \\ g = \partial^* f &\Longleftrightarrow g(\sigma_{\alpha,1}^1) = g(\sigma_{\alpha,0}^1) g(\sigma_{\alpha,0}^2). \end{aligned} \tag{2.7}$$

For nonabelian G, $H^1(K;G)$ will in general be simply a set without a natural group structure, since the coboundaries need not constitute a normal subgroup of the group of cocycles.

Generalizations to the case $n = 2$ have been suggested although there is some doubt as to the naturalness of the definition in this case, and they have so far not found significant use. One may for now at least consider such generalizations merely arbitrary.

Later on, when we come to the theory of sheaves and fiber spaces, we shall see definitions of cohomology groups $H^0(K;\mathcal{F})$ and $H^1(K;\mathcal{F})$ with coefficients in a "sheaf" $\mathcal{F}$ of non-abelian groups, whose naturality will be very clear. □

Returning to the standard situation of additive abelian groups G, we introduce next the *scalar product* $\langle f, c\rangle \in G$ of an n-cochain f and an n-chain c:

$$\langle f, c\rangle = \sum_\alpha \lambda_\alpha f(\sigma_\alpha^n), \quad c = \sum_\alpha \lambda_\alpha \sigma_\alpha^n. \tag{2.8}$$

This scalar product then induces a natural scalar product between the elements of the n-th cohomology and homology groups.

Example 1. If G is the additive group of a field (for instance $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}/p$), then the homology and cohomology groups have the supplementary structure of vector spaces over G, one has

$$H^n(K;G) \cong (H_n(K;G))^*, \tag{2.9}$$

and the scalar product (2.8) is non-degenerate.

Example 2. Suppose $G = \mathbb{Z}$ and the complex K is finite. For appropriate maximal free abelian direct factors $\widetilde{H}^n$ and $\widetilde{H}_n$ of $H^n(K;\mathbb{Z})$ and $H_n(K;\mathbb{Z})$ respectively, one has $\widetilde{H}^n = \left(\widetilde{H}_n\right)^*$. The scalar product (2.8) between $\widetilde{H}^n$ and $\widetilde{H}_n$ is non-degenerate and, relative to naturally corresponding free bases, has determinant ± 1. For the torsion subgroups there is an isomorphism (in the case K finite)

$$\operatorname{Tor} H^n(K;\mathbb{Z}) \cong \operatorname{Tor} H_{n-1}(K;\mathbb{Z}). \tag{2.10}$$

Example 3. Again suppose G is the additive group of any field. Then it follows from the definitions that

$$H^n(K;G) = \text{Hom }(H_n(K;\mathbb{Z}),G),$$

$$H_n(K;G) = H_n(K;\mathbb{Z}) \otimes G.$$

If the field G has characteristic 0 (for instance $G = \mathbb{Q}$) then

$$H^n(K;\mathbb{Q}) \cong \widetilde{H}^n \otimes \mathbb{Q} \cong \text{Hom }\left(\widetilde{H}_n,\mathbb{Q}\right). \tag{2.11}$$

If the field G has finite characteristic $p < \infty$ (for instance $G = \mathbb{Z}/p$) then for finite complexes K one has

$$H^n(K;\mathbb{Z}/p) = \text{Hom }\left(\widetilde{H}_n(K;\mathbb{Z}),\mathbb{Z}/p\right) \oplus \text{Hom }(\text{Tor } H_{n-1}(K;\mathbb{Z}),\mathbb{Z}/p),$$
$$H_n(K;\mathbb{Z}/p) = \left(\widetilde{H}_n(K;\mathbb{Z}) \otimes \mathbb{Z}/p\right) \oplus ((\text{Tor } H_{n-1}(K;\mathbb{Z})) \otimes \mathbb{Z}/p). \tag{2.12}$$

From (2.11) and (2.12) we see that the ranks of the homology groups over $\mathbb{Z}/p$ are at least equal to their ranks over $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. In standard notation, the rank of the group $H_n(K;\mathbb{Q})$ (i.e. dimension as a vector space over $\mathbb{Q}$) is denoted by b_n (the n-th *Betti number*) and the rank (dimension) of $H_n(K;\mathbb{Z}/p)$ by $b_n^{(p)} \geq b_n$.

Example 4 (Poincaré's Theorem). For any finite complex K one has

$$\chi(K) = \sum_{j\geq 0}(-1)^j b_j. \tag{2.13}$$

The analogue of this turns out to be valid for the $b_j^{(p)}$:

$$\chi(K) = \sum_{j\geq 0}(-1)^j b_j^{(p)}. \tag{2.14}$$

As noted earlier, the Euler-Poincaré characteristic $\chi(K)$ is defined as the alternating sum $\sum_{j\geq 0}(-1)^j\gamma_j$, where γ_j is the number of j-dimensional simplexes of K. The name "Betti numbers" was introduced by Poincaré, however not as the torsion-free rank of the groups $H_j(K;\mathbb{Z})$. (It is easy to see that the group $H_0(K;\mathbb{Z})$ is always free abelian of rank equal to the number of connected components of K.) A many-sided investigation of the homology groups over various coefficient groups began in the late 1920s after E. Noether gave the formal definition of them, thereby bringing algebraic order into the non-algebraic homology theory created by the topologists. □

It is not difficult to calculate the homology and cohomology of a product $K_1 \times K_2$ refined to a simplicial complex:

$$H_m(K_1 \times K_2;\mathbb{Z}) \cong \sum_{i+j=m} H_i(K_1;H_j(K_2;\mathbb{Z})),$$
$$H^m(K_1 \times K_2;\mathbb{Z}) \cong \sum_{i+j=m} H^i(K_1;H^j(K_2;\mathbb{Z})). \tag{2.15}$$

Over a field G these formulae simplify to the following:

$$\begin{aligned} H_m(K_1 \times K_2; G) &\cong \sum_{i+j=m} H_i(K_1; G) \otimes H_j(K_2; G), \\ H^m(K_1 \times K_2; G) &\cong \sum_{i+j=m} H^i(K_1; G) \otimes H^j(K_2; G). \end{aligned} \tag{2.16}$$

Note that a simplicial map between complexes induces homomorphisms between the respective homology groups (and, likewise, though in the opposite direction, between the cohomology groups), in view of the fact that such maps commute with the boundary operator ∂:

$$\begin{gathered} f : K_1 \longrightarrow K_2, \qquad f_* : H_n(K_1; G) \longrightarrow H_n(K_2; G), \\ f^* : H^n(K_2; G) \longrightarrow H^n(K_1; G). \end{gathered} \tag{2.17}$$

Thus, as for the homotopy groups, the operations of forming the homology and cohomology groups are, as they say, functorial (covariantly and contravariantly respectively) on the category of simplicial complexes and simplicial maps.

A *simplicial homotopy* is a simplicial map $F : K_1 \times I \longrightarrow K_2$, where $K_1 \times I$ is subdivided (barycentrically or a refinement thereof — see Figure 3.5) so as to turn it into a simplicial complex. To each simplex σ^n of K_1 there corresponds the $(n+1)$-dimensional chain representing $\sigma^n \times I$ in $K_1 \times I$, and the image chain $F(\sigma^n \times I)$ in K_2. Denote by f_a and f_b respectively the maps of the base and lid of the cylinder $K_1 \times I$ ($I = [a, b]$):

$$\begin{aligned} f_a : K_1 \times \{a\} \longrightarrow K_2, \qquad f_a(x) = F(x, a), \\ f_b : K_1 \times \{b\} \longrightarrow K_2, \qquad f_b(x) = F(x, b), \end{aligned}$$

and denote the chain $F(\sigma^n \times I)$ by $D(\sigma^n)$. The following formula:

$$D(\partial\sigma^n) \pm \partial D(\sigma^n) = f_a(\sigma^n) - f_b(\sigma^n), \tag{2.18}$$

which has a simple geometric interpetation, is a consequence of the obvious decomposition $\partial(\sigma^n \times I) = ((\partial\sigma^n) \times I) \cup (\sigma^n \times \{a\}) \cup (\sigma^n \times \{b\})$, and the appropriate attaching of signs. The formula (2.18) extends by linearity to arbitrary n-chains. In particular if z is a cycle, $\partial z = 0$, then the formula gives

$$f_a(z) - f_b(z) = \pm\partial D(z), \tag{2.19}$$

from which it follows that homotopic maps induce the same homomorphisms of the homology groups. The analogous conclusion is similarly valid for cohomology.

The homotopy invariance (i.e. invariance under homotopy equivalences) of the homology and cohomology groups, is thus ultimately a consequence of their invariance under barycentric subdivision, which was known, it would appear, to Poincaré. The invariance of the homology and cohomology of finite complexes under homeomorphisms then follows by invoking the additional fact that any self-map $f : K \longrightarrow K$ of a finite complex, having the property that every point x is sufficiently close to its image $f(x)$, is homotopic to the identity map: $f \sim 1_K$. For, given any homeomorphism $h : K_1 \longrightarrow K_2$ of finite complexes, then after carrying out sufficiently fine subdivisions of the complexes, we can (by the Simplicial Approximation Theorem — see above) find simplicial maps $\widetilde{f}$, $\widetilde{g}$ approximating h and h^{-1} respectively such that the composites $\widetilde{f} \circ \widetilde{g}$ and $\widetilde{g} \circ \widetilde{f}$ are as close as desired to the identity maps 1_{K_2} and 1_{K_1}, and therefore homotopic to them.

In the same way one infers via simplicial approximation that any (continuous) map of simplicial complexes induces homomorphisms of the homology and cohomology groups, that are unchanged by (continuous) homotopies. Thus the operation of taking the homology and cohomology groups of a simplicial complex is a functor with values depending only on the homotopy type of the complexes and the homotopy classes of maps between them.

Consider now the diagonal map

$$\Delta(x) = (x, x), \quad \Delta : K \longrightarrow K \times K,$$

K a simplicial complex.

Definition 2.1 Given any ring G, we define the *product of two cohomology classes* $a \in H^j(K;G)$ and $b \in H^q(K;G)$ to be the quantity

$$ab = \Delta^* (a \otimes b) \in H^m(K;G), \quad m = j + q, \tag{2.20}$$

where

$$\Delta^* : H^m(K \times K; G) \longrightarrow H^m(K;G)$$

is the homomorphism induced by the diagonal map Δ.

Here $a \otimes b$ denotes the cohomology class of the product $K \times K$ naturally determined by a and b as follows: The chain complex $C_*(K_1 \times K_2)$ of a product may be identified naturally with the appropriate sum of tensor products:

$$\begin{gathered} C_m(K_1 \times K_2) = \sum_{j+q=m} C_j(K_1) \otimes C_q(K_2), \\ \partial(a \otimes b) = (\partial a) \otimes b + (-1)^j a \otimes \partial b, \end{gathered} \tag{2.21}$$

where we take as a basis for $C_*(K_1 \times K_2)$ the simplexes of some simplicial subdivision of the products $\sigma_1^j \times \sigma_2^q$ of simplexes σ_1^j, σ_2^q of K_1, K_2. (For CW-complexes (see below) one may simply take the $\sigma_1^j \times \sigma_2^q$ as basis elements.) It follows that a pair z_1, z_2 of cocycles of K_1, K_2 respectively, of dimensions

j, q, yields a cocycle $z_1 \times z_2$ say, in $H^m(K_1 \times K_2)$, $m = j + q$, and it is this "product of cocycles" that defines the monomorphism

$$H^j(K_1) \otimes H^q(K_2) \longrightarrow H^{j+q}(K_1 \times K_2)$$

used in the definition (2.20).

Equipped with this multiplication, the direct sum of the cohomology groups becomes a graded skew-commutative associative ring[5] with identity element $1 \in H^0(K;G)$, called the *cohomology ring* of K (with coefficients from an arbitrary commutative ring G with a multiplicative identity):

$$H^*(K;G) = \sum_{j \geq 0} H^j(K;G). \tag{2.22}$$

Here "skew-commutativity" signifies that for a, b as in the preceding definition one has

$$ab = (-1)^{jq} ba, \quad j = \dim a, \quad q = \dim b. \tag{2.23}$$

The formation of the cohomology ring is functorial in the (usual) sense that the mapping $f^* : H^*(K;G) \longrightarrow H^*(L;G)$ induced by any map $f : L \longrightarrow K$, is a ring homomorphism.

It was Kolmogorov and Alexander who, in the mid–1930s, in the course of carrying out an algebraic analysis of certain topological work (of Van Kampen and Pontryagin, for instance) where cohomological ideas were being used in embryo, defined the cohomology groups explicitly and introduced the multiplication of cohomology classes. It seems that they were led to the cohomological product partly by the analogy with tensor analysis where cohomology groups with coefficients in $\mathbb{R}$, defined in terms of differential forms and so having an obvious multiplication, had already been discovered (by E. Cartan), and partly by the analogy with Lefschetz' ring of intersections of cycles on closed manifolds, dual to the cohomology ring (see below).

In this connexion it is pertinent to mention "Pontryagin duality", which for finite complexes K is the isomorphism

$$H^j(K; \text{Char } G) \cong \text{Char } H_j(K;G),$$

where G is any abelian group and Char G is the group of *characters* of G, i.e. of arbitrary homomorphisms

$$G \longrightarrow S^1 \cong U_1 \cong SO_2.$$

For finite abelian groups it is not difficult to see that $G \cong \text{Char } G$, for $G = \mathbb{Z}$ we clearly have that Char $\mathbb{Z} \cong S^1$, and for $G = S^1$, Char $S^1 \cong \mathbb{Z}$. In general one has (Pontryagin)

$$\text{Char Char } G \cong G.$$

[5] In modern terminology, a "super-ring".

In the early 1930s, as a result of investigating the algebraic aspects of earlier work of Poincaré and Alexander on "duality laws", Pontryagin established the following isomorphisms:

$$H_j(M^n; \text{Char } G) \cong \text{Char } H_{n-j}(M^n; G),$$

for any orientable PL-manifold M^n, and

$$H_j(S^n \setminus K; \text{Char } G) \cong \text{Char } H_{n-j-1}(K; G), \quad 0 < j < n-1,$$

for any simplicial complex K embedded in the n-sphere S^n. Translated into cohomological notation these isomorphisms become respectively

$$\begin{aligned} H_j(M^n; G) &\cong H^{n-j}(M^n; G) \\ H_j(S^n \setminus K; G) &\cong H^{n-j-1}(K; G), \quad 0 < j < n-1. \end{aligned} \tag{2.24}$$

The first of these isomorphisms is referred to as "Poincaré duality", and the second as "Alexander duality".

The multiplication of integral cohomology classes may be defined alternatively in terms of multiplication of cochains. Let the vertices of K be $\alpha_0, \alpha_1 \ldots$, and let f and g be cochains of dimensions j and q respectively, in other words integer–valued functions on the sets of simplexes of these dimensions. For each oriented simplex $\sigma^m = (\alpha_{i_0}, \ldots, \alpha_{i_m})$ of K with $m = j + q$, we set

$$\begin{aligned} fg\,(\sigma^m) &= f(\sigma_1^j)g(\sigma_2^q), \\ \sigma_1^j = (\alpha_{i_0}, \ldots, \alpha_{i_j}), &\quad \sigma_2^q = (\alpha_{i_j}, \ldots, \alpha_{i_m}). \end{aligned} \tag{2.25}$$

(Note that σ_1^j and σ_2^q meet in a single vertex.) The multiplication (2.25) is not skew-commutative (in fact not even well-defined) at the level of cochains, but does determine a well-defined skew-commutative product of elements in the cohomology groups, in view of the equations

$$\begin{aligned} \partial^*(fg) &= (\partial^* f)g + (-1)^j f(\partial^* g), \\ fg - (-1)^{jq} gf &= \partial^* \phi. \end{aligned} \tag{2.26}$$

The second equation 2.26 is valid for cocycles only.

Multiplication of cochains yields another important operation: the *cap product*. Let c be a chain of dimension m and a, b cochains of dimensions j, q respectively, where $m = j+q$. The cap product $a \cap b$ is then the chain determined via the scalar product of chains with cochains by

$$\langle a \cap c, b\rangle = \langle c, ab\rangle. \tag{2.27}$$

The cap product of a cochain with a chain then determines the *cap product between cohomology and homology classes*:

$$a \cap c \in H_{m-j}(K;\mathbb{Z}),\qquad a \in H^j(K;\mathbb{Z}),\quad c \in H_m(K;\mathbb{Z}). \tag{2.28}$$

Under maps $f : K_1 \longrightarrow K_2$, the cap product obeys the following rule:

$$f_*(f^*(a) \cap c) = a \cap f_*(c),\qquad a \in H^j(K;\mathbb{Z}),\quad c \in H_m(K;\mathbb{Z}). \tag{2.29}$$

Each integral homology class $c \in H_m(K;\mathbb{Z})$ determines, via the cap product, the following duality operator:

$$H^j(K;\mathbb{Z}) \longrightarrow H_{m-j}(K;\mathbb{Z}),\qquad a \mapsto a \cap c. \tag{2.30}$$

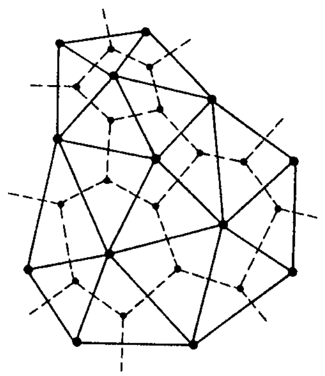
It turns out that in the case where K is a closed m-dimensional orientable manifold M^m, and c is the fundamental class $[M^m] \in H_m(M^m;\mathbb{Z})$ (i.e. the class $[M^m]$ represented by the manifold M itself with a chosen orientation; if M is connected then $H^m(M^m;\mathbb{Z}) \cong \mathbb{Z}$ with generator $[M^m]$), then the operation (2.30) is just the isomorphism of Poincaré duality:

$$D : H^j(M^m;\mathbb{Z}) \xrightarrow{\cong} H_{m-j}(M^m;\mathbb{Z}),\quad D(a) = a \cap [M^m]. \tag{2.31}$$

The operation D determines a dual operation to cohomological multiplication, called the *intersection of cycles* (for orientable manifolds M):

$$(D(a)) \circ (D(b)) = D(ab),\qquad H_{m-j}(M^m;\mathbb{Z}) \circ H_{m-q}(M^m;\mathbb{Z}) \subset H_{m-q-j}(M^m;\mathbb{Z}). \tag{2.32}$$

We shall give below a simple geometric interpretation of this operation. For non-orientable manifolds, Poincaré duality and the intersection of cycles are defined in mod 2 homology, i.e. with $G = \mathbb{Z}/2$. For piecewise linear (*PL*-) manifolds $K = M^m$, the Poincaré duality isomorphism has a simple geometrical interpretation. This involves the *Poincaré dual complex* DK of the complex K, originating in the idea of the dual of a graph in the plane or on a two-dimensional surface. Let K' be the barycentric subdivision of K. For each simplex σ^m of K of largest dimension, we define its dual $D\sigma^m$ to be the point of K' at the center of σ^m; the points $\{D\sigma^m\}$ are then to be the vertices of the dual complex DK. At the other extreme, the dual $D\sigma^0$ of a vertex σ^0 of K is taken to be the convex m-dimensional polytope obtained as the union of all m-simplexes of the barycentric subdivision K' of K, having σ^0 as a vertex (see Figure 3.6). Each simplex σ^k of K, $1 \leq k \leq m$, has at its center a point of the barycentric subdivision K'; the dual $(m-k)$-dimensional "simplex" $D\sigma^k$ of DK is then obtained by consolidating the $(m-k)$-simplexes of K' that are transverse to the k-simplex σ^k, and have the center of σ^k as a vertex (see Figure 3.6). Thus the complex DK is obtained by consolidation from K', and as a topological space it coincides with the complexes K and K' which are just different subdivisions of the same PL-manifold M^m.



Here $m = 2$. The simplexes of the complex K are indicated by the solid line-segments, those of the dual complex DK by dotted line segments

Fig. 3.6

We now define the *intersection index* of each simplex σ^j_α with each polytope $D\sigma^j_\beta$ of DK (where σ^j_α and σ^j_β are simplexes of K of the same dimension) to be 1 if σ^j_α meets $D\sigma^j_\beta$, and 0 otherwise:

$$\sigma^j_\alpha \circ D\sigma^j_\beta = \delta_{\alpha\beta}. \tag{2.33}$$

This extends by linearity to yield the *intersection index* of arbitrary pairs of cycles of complementary dimensions:

$$\begin{gathered}\left(\sum_\alpha \lambda^\alpha \sigma^j_\alpha\right) \circ \left(\sum_\beta \mu^\beta D\sigma^j_\beta\right) = \delta_{\alpha\beta}\lambda^\alpha\mu^\beta, \\ c \circ Dc' = \delta_{\alpha\beta}\lambda^\alpha\mu^\beta\end{gathered} \tag{2.34}$$

(where on the right-hand sides the Einstein summation convention is being used).

In the orientable case the boundary operator commutes with the dual in the sense that

$$D\partial(c) = \partial^* D(c). \tag{2.35}$$

(In the case of non-orientable PL-manifolds this formula remains valid provided the coefficients are from $\mathbb{Z}/2$.) The Poincaré–duality isomorphism (2.31) now follows from (2.35) since the homology of the given manifold M^m coincides on the one hand with that of K, a simplicial subdivision of it, and on the other hand with that of DK, a subdivision of M^m into convex polytopes.

The intersection of cycles of noncomplementary dimensions of K and DK respectively, also clearly makes geometrical sense. Thus the operation of "intersection of cycles" (and thence of homology classes) can indeed be seen to arise from actual geometrical intersections.

§3. Relative homology. The exact sequence of a pair. Axioms for homology theory. CW-complexes

As in the case of the homotopy groups, one can define in a natural way the *relative* homology and cohomology groups of a pair (K, L), where K is a simplicial complex and L a subcomplex.

A *relative chain* of the pair (K, L) is simply an equivalence class of chains of K whose pairwise differences are chains of L; i.e. two chains are now to be considered the same if they are congruent modulo the chains of L. Hence the *relative chain complexes*

$$C_j(K, L) = C_j(K)/C_j(L). \tag{3.1}$$

The boundary operator ∂ on $C(K, L)$ is that induced by the usual boundary operator on K-chains; since $\partial C_j(L) \subset C_{j-1}(L)$ this does indeed yield a well-defined operator:

$$\ldots \longrightarrow C_j(K, L) \xrightarrow{\partial} C_{j-1}(K, L) \xrightarrow{\partial} \ldots.$$

The *jth relative integral homology group* of the pair (K, L) is then defined by

$$H_j(K, L; \mathbb{Z}) = \text{Ker } \partial/\text{Im } \partial. \tag{3.2}$$

The elements of Ker ∂ and Im ∂ are here called *relative cycles* and *relative boundaries* respectively.

The *relative homology groups* $H_j(K, L; G)$ *with coefficients from an arbitrary abelian group* G are defined, analogously to the absolute homology groups over G, using the complex $C(K, L) \otimes G$; and the *relative cohomology groups* $H^j(K, L; G)$ *over* G are defined via the complex $\text{Hom}(C(K, L), G)$ and the relative coboundary operator ∂^*. A relative G-cochain may be thought of geometrically as a function with values in G, defined on the set of simplexes of K and vanishing on those of L.

Much as in the situation of the homotopy groups, one obtains the *exact homology and cohomology sequences of the pair* (K, L):

$$\cdots \xrightarrow{i_*} H_j(K) \xrightarrow{j_*} H_j(K, L) \xrightarrow{\partial} H_{j-1}(L) \xrightarrow{i_*} H_{j-1}(K) \longrightarrow \cdots, \tag{3.3}$$

$$\cdots \xleftarrow{i^*} H^j(K) \xleftarrow{j^*} H^j(K, L) \xleftarrow{\partial^*} H^{j-1}(L) \xleftarrow{i^*} H^{j-1}(K) \longleftarrow \cdots. \tag{3.4}$$

The constructions of the relative homology and cohomology groups and of the respective exact sequences (3.3), (3.4) are all functorial on the category of pairs (K, L) and simplicial maps of pairs $(K_1, L_1) \xrightarrow{f} (K_2, L_2)$. These objects are all homotopy invariants; the induced homomorphisms f_*, f^* are unaffected by homotopies of maps of pairs.

In a similar fashion one obtains, yet more generally, exact sequences of triples (K, L, M), $K \supset L \supset M$:

$$\cdots \xrightarrow{i_*} H_j(K,M) \xrightarrow{j_*} H_j(K,L) \xrightarrow{\partial} H_{j-1}(L,M) \xrightarrow{i_*} H_{j-1}(K,M) \longrightarrow \cdots, \tag{3.5}$$

$$\cdots \xleftarrow{i^*} H^j(K,M) \xleftarrow{j^*} H^j(K,L) \xleftarrow{\partial^*} H^{j-1}(L,M) \xleftarrow{i^*} H^{j-1}(K,M) \longleftarrow \cdots. \tag{3.6}$$

Noting that the quotient space $\widetilde{K} = K/L$ of a complex K by a subcomplex L admits a natural triangulation, one has the following result (the *Excision Theorem*):

$$\begin{aligned} H_j(K,L) &\cong H_j(K/L), \quad && j > 0, \\ H_0(K,L) &\cong 0, && \text{(if } K \text{ is connected and } L \neq \emptyset). \end{aligned} \tag{3.7}$$

The Excision Theorem follows from the observation that the quotient space K/L is clearly homotopically equivalent to the complex obtained by attaching the cone on L to K along L:

$$K/L \sim K \cup_L CL.$$

For the suspension $\Sigma K = (CK)_1 \cup_K (CK)_2$, one infers from the exact homology sequence of the pair $(\Sigma K, (CK)_2)$ in conjunction with the contractibility of the complex CK, the following important *suspension isomorphism*:

$$H_j(K, x_0) \cong H_{j+1}(\Sigma K, x_0), \quad j \geq 0. \tag{3.8}$$

Starting from the homology of the discrete two-point space $K = S^0 = \{x_0, x_1\}$, and iterating the suspension isomorphism, one obtains by induction the homology groups of the spheres:

$$\begin{gathered} H_j(S^n, s_0) = 0, j \neq n, \\ H_n(S^n, s_0; \mathbb{Z}) \cong H^n(S^n, s_0; \mathbb{Z}) \cong \mathbb{Z}, \\ H_n(S^n, s_0; G) \cong H^n(S^n, s_0; G) \cong G. \end{gathered} \tag{3.9}$$

According to a theorem of Eilenberg–Steenrod, homology (and cohomology) theory is uniquely determined (i.e. up to isomorphism) by the conditions (*Eilenberg–Steenrod axioms*) of functoriality, homotopy invariance, existence of the exact sequence of a pair (3.3), and the *normalization axiom* (essentially the property (3.9)), provided that the *excision axiom* (3.7) also holds; it is the latter axiom that distinguishes homology theory from homotopy theory.

In the 1960s, what are known as *generalized*, or *extraordinary, homology (and cohomology) theories* began to be intensively exploited as part of the apparatus of topology. These theories are the same as the ordinary theory except for the normalization axiom: the *generalized* homology groups of the sphere (S^n, s_0) may be non-trivial in dimensions other than n, or, equivalently, the (absolute) generalized homology groups of a contractible space (e.g. a disc or a point) may be non-trivial in non-zero (including negative) dimensions. We shall be considering questions related to this below.

Poincaré duality can be extended to the relative context (without changing the underlying geometric assumptions). Let M^m be a closed, orientable PL-manifold, and $K \subset M^m$ a subcomplex. There always exists a finite subcomplex L of M, which is a deformation retract of the open complementary set $U = M^m \setminus K$, $L \subset U$. We shall understand the homology of the pair (M^m, U) to be that of (M^m, L). We then have the following generalized version of Poincaré duality:

$$H_j(K) \cong H^{m-j}(M^m, U). \tag{3.10}$$

Example 1. Take $M^m = S^m$, the m-sphere. Provided $S^m \setminus U$ is non-empty, the exact cohomology sequence of the triple $(S^m, U, *)$ (see (3.5)) yields the isomorphism

$$H^{m-j}(S^m, U) \cong H^{m-j-1}(U, *), \quad j \neq 0.$$

From this we infer the isomorphism of *Alexander duality* — see (2.24) (valid also for $j = 0$):

$$H_j(K, x_0) \cong H^{m-j-1}(U, u_0), \quad x_0 \in K, \quad u_0 \in U. \ \square \tag{3.11}$$

Example 2. Let M^m be any orientable manifold and let $K = W^m$ be the closure of a non-empty open set in M^m, so that K is a submanifold W^m (with boundary ∂W^m) of the same dimension m. From (3.11) and the Excision Theorem (3.7) one obtains readily the following isomorphisms:

$$H_j(W^m, \partial W^m) \cong H^{m-j}(W^m),$$

$$H^j(W^m, \partial W^m) \cong H_{m-j}(W^m).$$

These isomorphisms constitute what is known as *Lefschetz duality.* $\square$

Homotopy-invariant homology theories are most conveniently constructed and investigated for the category of countable *cell complexes* (or *CW-complexes*). We have already encountered — for instance in the process of constructing the Poincaré dual of a simplicial complex — the necessity, or at least convenience, of considering consolidations of simplicial complexes, where the structural units are convex polyhedra or polytopes, for instance, rather than simplexes. CW-complexes are the most general and convenient objects of this type. To begin with, consider a simplicial complex K. We shall say that K is *decomposed*, or *subdivided, into cells* if we are given a family of cells (discs) D^n_α of various dimensions n and a collection of simplicial maps

$$\Psi^n_\alpha : D^n_\alpha \longrightarrow K,$$

with the following properties:

(i) Each map Ψ^n_α is one-to-one on the interior Int D^n_α of the cell D^n_α;

(ii) The union of the cells (i.e. of the images $\Psi^n_\alpha(D^n_\alpha)$) is all of the complex K, and their interiors $\Psi^n_\alpha(\text{Int } D^n_\alpha)$ are pairwise disjoint;

(iii) For each n, the image $\Psi^n_\alpha(\partial D^n_\alpha)$ (the boundary of the cell $\Psi^n_\alpha(D^n_\alpha)$) lies in the union of the cells of dimension $< n$.

Under these conditions we say that K has been endowed with the structure of a *CW-complex* $\widehat{K}$. It is clear from this definition that, given an orientation of K (assuming orientability) each n-dimensional cell $\Psi^n_\alpha(D^n_\alpha)$ of such a cellular subdivision is a linear combination of the (oriented) n-simplexes of K contained in the cell:

$$\Psi^n_\alpha(D^n_\alpha) = \sum_\gamma \lambda_{\alpha\gamma}\sigma^n_\gamma, \quad \lambda_{\alpha\gamma} = 0, \pm 1,$$

where the σ^n_γ are n-dimensional simplexes of K. (Note that since distinct cells intersect only along their boundaries if at all, the interior of each n-simplex σ^n_γ is contained in the interior of exactly one cell.) Denote the cell $\Psi^n_\alpha(D^n_\alpha)$ by κ^n_α. Since the boundary ∂D^n_α of each disc D^n_α is mapped simplicially to the union of the simplexes of K of dimension $< n$ (the $(n-1)$-skeleton of K), the boundary operator ∂ defined already (in §2) for simplicial chains, yields an expression for the boundary $\partial\kappa^n_\alpha$ of each n-cell of $\widehat{K}$ as an integral linear combination of cells of dimension $n-1$:

$$\partial\kappa^n_\alpha = \sum_\delta \left[\kappa^n_\alpha : \kappa^{n-1}_\delta\right]\kappa^{n-1}_\delta, \quad \partial \circ \partial = 0. \tag{3.12}$$

The integers $\left[\kappa^n_\alpha : \kappa^{n-1}_\delta\right]$ are called the *incidence numbers* of the CW-complex $\widehat{K}$.

The *cellular homology* and *cohomology groups* of the CW-complex $\widehat{K}$ are now defined, analogously to the simplicial versions, in terms of *integral cell-chains* (comprising $C_*(\widehat{K})$) and *cellular cochains*, and, over an arbitrary abelian group G of coefficients, in terms of $C_*(\widehat{K}) \otimes G$ and $\text{Hom}(C_*(\widehat{K}), G)$. The cellular homology and cohomology groups of a CW-complex coincide with (i.e. are naturally isomorphic to) their simplicial analogues as defined above (in §2).

The *n-skeleton* K^n of a CW-complex K is the union of the cells of dimension $\leq n$. Clearly each quotient space K^n/K^{n-1} is a bouquet of spheres

$$K^n/K^{n-1} \cong S_1^n \vee \cdots \vee S_m^n \vee \cdots,$$

representing the respective n-cells κ_α^n, $\alpha = 1, 2, \ldots, m, \ldots$. Hence the group $H_n(K^n, K^{n-1}; \mathbb{Z})$ is free abelian with free generators represented by the cells κ_α^n, and the boundary operator ∂ may be defined alternatively as the composite of the the maps ∂_*, j_* figuring in the exact homology sequence of the pairs (K^n, K^{n-1}), (K^{n-1}, K^{n-2}):

$$H_n(K^n, K^{n-1}; \mathbb{Z}) \xrightarrow{\partial_*} H_{n-1}(K^{n-1}; \mathbb{Z}) \xrightarrow{j_*} H_{n-1}(K^{n-1}, K^{n-2}; \mathbb{Z}).$$

The definition of a CW-complex can be somewhat broadened by dispensing with the dependence on an initially given simplicial decomposition. Thus a topological space X is called a *CW-complex* if there is given a collection of maps $\Psi_\alpha^n : D_\alpha^n \longrightarrow X$ of discs (balls) D_α^n of various dimensions (whose images, or, strictly speaking the maps themselves, are called *cells*) with the same properties (i), (ii), (iii) as before: each Ψ_α^n should be one-to-one on $\text{Int}D_\alpha^n$; the union of the cells should be all of X; the interiors $\Psi_\alpha^n(\text{Int}D_\alpha^n)$ of the cells should be pairwise disjoint; and for each n the set $\Psi_\alpha^n(\partial D_\alpha^n) = \Psi_\alpha^n(S^{n-1})$ should be contained in the union of the cells of dimension $\leq n-1$. It is also required that the topology on X be the appropriate identification topology, in the sense that for a mapping $f : X \longrightarrow Z$ to be continuous (Z any topological space) it suffices that the maps $f \circ \Psi_\alpha^n$ all be continuous (or in other words f should be continuous if it is the union of maps of the cells, each pair of which agrees on the shared portion of the cells' boundaries).

In this general case the *n-skeleton* of a CW-complex X is the subcomplex X^n made up of the cells of dimension $\leq n$. Clearly each quotient space X^j/X^{j-1} is again a bouquet of j-dimensional spheres:

$$X^j/X^{j-1} \cong S_1^j \vee \cdots \vee S_m^j \vee \cdots,$$

each sphere S_α^j arising from exactly one j-cell κ_α^j $(= \Psi_\alpha^j(D_\alpha^j))$, $\alpha = 1, \ldots, m, \ldots$. The boundary operator ∂ has the same form as before (see (3.12)), where now the *incidence number* $\left[\kappa_\alpha^{j+1} : \kappa_\beta^j\right]$ of a pair of (oriented) cells κ_α^{j+1}, κ_β^j is defined (equivalently) as the degree of the composite map:

$$\partial(D_\alpha^{j+1}) \longrightarrow \partial(\kappa_\alpha^{j+1}) \longrightarrow S_\beta^j$$

determined by Ψ_α^{j+1} and the above identification, i.e. essentially as the (signed) number of times the composite of $\Psi_\alpha^{j+1}|_{\partial(D_\alpha^{j+1})}$ with the identification $X^j/X^{j-1} \longrightarrow \bigvee S_\alpha^j$, "wraps" the j-sphere $\partial(D_\alpha^{j+1})$ around the corresponding

j-sphere of the bouquet. It follows that $\partial \circ \partial = 0$, and one then defines the homology and cohomology groups of the CW-complex X in the usual way, obtaining groups naturally isomorphic to those defined above.

Any CW-complex may be constructed by iterating the following operation: Given a space X and countable collection of maps $f_\alpha : S^{j-1} \longrightarrow X$, one constructs a new space Y by attaching discs D^j_α to X by identifying the boundary of each D^j_α with the corresponding $f_\alpha\left(S^{j-1}\right)$ via f_α:

$$Y = X \bigcup_{(f_1,\dots,f_\alpha,\dots)} \left(D^j_1 \cup \cdots D^j_\alpha \cup \cdots\right). \tag{3.13}$$

The topology on Y is defined in the natural way. Thus starting with a countable discrete set K^0 of points (to be eventually the 0-skeleton), one first attaches the 1-cells ($j = 1$), then to the resulting 1-skeleton the 2-cells are attached, and so on.

We note in conclusion the fact that every CW-complex is homotopically equivalent to some simplicial complex.

Example 1. The simplest cellular decomposition of the sphere S^n is as follows: to a single 0-dimesional cell (vertex) κ^0 one attaches a single n-dimensional cell κ^n by means of the obvious map $\Psi : D^n \longrightarrow S^n$ sending the boundary of D^n to the point κ^0: $\Psi(\partial D^n) = \left\{\kappa^0\right\}$. Clearly $\partial\kappa^0 = 0 = \partial\kappa^n$, so that the homology groups are as follows:

$$H_0(S^n) \cong \mathbb{Z} \cong H_n(S^n),$$
$$H_j(S^n) = 0 \quad \text{for } j \neq 0, n. \qquad \square$$

Example 2. a) The real projective spaces have the following cell decompositions, with boundary operators as indicated:

$$\mathbb{R}P^n = \kappa^0 \cup \kappa^1 \cup \cdots \cup \kappa^n,$$
$$\kappa^j = \mathbb{R}P^j \setminus \mathbb{R}P^{j-1} = \left\{(x^0 : \ldots : x^n) \mid x^j \neq 0, x^k = 0 \text{ for } k > j\right\}, \tag{3.14}$$
$$\partial\kappa^0 = \partial\kappa^1 = \partial\kappa^3 = \partial\kappa^5 = \cdots = \partial\kappa^{2k+1} = \cdots = 0, \quad \partial\kappa^{2k} = 2\kappa^{2k-1}, \quad k > 0.$$

b) For the complex projective spaces we have:

$$\mathbb{C}P^n = \kappa^0 \cup \kappa^2 \cup \cdots \cup \kappa^{2n},$$
$$\kappa^{2j} = \mathbb{C}P^j \setminus \mathbb{C}P^{j-1} = \left\{(z^0 : \ldots : z^n) \mid z^j \neq 0, z^k = 0 \text{ for } k > j\right\}, \tag{3.15}$$
$$\partial\kappa^{2j} = 0, \quad \text{for } j > 0. \qquad \square$$

Example 3. The construction of the closed orientable and non-orientable surfaces by identification of appropriate pairs of edges of convex polygons (see Figures 2.22, 2.23) yields the following minimal cellular decompositions of those surfaces:

(i) For the orientable surfaces M_g^2:

$$\kappa^0 \cup \kappa_1^1 \cup \cdots \cup \kappa_{2g}^1 \cup \kappa^2, \quad \partial\kappa_j^1 = 0, \quad \text{for } j = 1, \ldots, 2g, \quad \partial\kappa^2 = 0.$$

(ii) For the first family of non-orientable surfaces $N_{1,g}^2$, $g \geq 1$:

$$\kappa^0 \cup \kappa_1^1 \cup \cdots \cup \kappa_{2g}^1 \cup \kappa^2, \quad \partial\kappa_j^1 = 0, \quad \text{for } j = 1, \ldots, 2g, \quad \partial\kappa^2 = 2\kappa_0^1.$$

(iii) For the second family of non-orientable surfaces $N_{2,g}^2$, $g \geq 0$:

$$\kappa^0 \cup \kappa_0^1 \cup \cdots \cup \kappa_{2g}^1 \cup \kappa^2, \quad \partial\kappa_j^1 = 0, \quad \text{for } j = 0, \ldots, 2g, \quad \partial\kappa^2 = 2\kappa_{2g}^1.$$

The homology groups of the closed surfaces may be easily computed from these formulae.

The intersection indices of one-dimensional cycles on these surfaces (taken modulo 2 for the non-orientable ones) are determined by the intersection indices of pairs of the above basic 1-cycles, and these are as follows: Writing $a_i = \kappa_{2i}^1$, $b_i = \kappa_{2i-1}^1$, one has, for all surfaces M_g^2, $N_{1,g}^2$, $N_{2,g}^2$,

$$\begin{gathered} a_i \circ a_j = b_i \circ b_j = 0, \\ a_i \circ b_j = \delta_{ij}, \end{gathered} \tag{3.16}$$

where δ_{ij} is interpreted mod 2 for the non-orientable surfaces. For the surfaces $N_{2,g}^2$ there is the additional basic 1-cycle $\kappa_0^1 = c$ say; the additional basic intersection indices arising from this 1-cycle are as follows:

$$c \circ c = 1, \quad c \circ a_i = c \circ b_i = 0 \pmod 2. \tag{3.17}$$

Any collection of cycles $a_i, b_i, i = 1, \ldots, g$ (and c in the case of $N_{2,g}^2$), satisfying (3.16) (and (3.17) if appropriate) is called a *canonical basis* for the cycles on the surface.

Via the isomorphism of Poincaré duality, which in each case sends the basic element of the group H_0 to the fundamental cohomology class of H^2, we can infer from (3.16) (and (3.17 in the case of $N_{2,g}^2$)) the structure of the cohomology rings

$$H^*(M_g^2; \mathbb{Z}), \quad H^*(N_{1,g}^2; \mathbb{Z}/2), \quad H^*(N_{2,g}^2; \mathbb{Z}/2). \qquad \square$$

We have already mentioned *Hopf's Theorem*:

$$H_1(K; \mathbb{Z}) \cong \pi_1(K)/\left[\pi_1(K), \pi_1(K)\right], \tag{3.18}$$

valid for any CW-complex K. (Here $[\pi_1, \pi_1]$ denotes the commutator subgroup of the group π_1, i.e. the subgroup consisting of those elements of π_1 which become trivial if commutativity is imposed on π_1.) As an extension of this we have *Hurewicz' Theorem*:
If $\pi_i(K) = 0$ for $0 < i < n$, then $H_i(K;\mathbb{Z}) = 0$ for $0 < i < n$, and if $n > 1$ there is a natural isomorphism

$$H_n(K;\mathbb{Z}) \cong \pi_n(K). \tag{3.19}$$

(Note that since for a path-connected space the homotopy groups defined with reference to different base points are canonically isomorphic, we shall in future not indicate base points explicitly, and likewise for the relative homotopy groups $\pi_i(K, L)$ where L is path-connected.)

The relative version of Hurewicz' theorem is as follows:
If $\pi_i(K, L) = 0$ for $0 < i < n$, then $H_i(K, L;\mathbb{Z}) = 0$ for $0 < i < n$, and if $n > 1$ there is a natural isomorphism

$$H_n(K, L;\mathbb{Z}) \cong \pi_n(K, L)/\{a = t(a)\}, \tag{3.20}$$

where a ranges over $\pi_n(K, L)$, and t ranges over $\pi_1(L)$ considered as a group of operators acting as defined earlier on $\pi_n(K, L)$.

This relativized version of Hurewicz' theorem is actually more complete in that it applies in the non-simply-connected case also, and so embraces Hopf's theorem.

Example. If L is a path-connected subcomplex of the CW-complex K, such that K/L is a bouquet of k n-spheres, $n > 1$:

$$K/L = S_1^n \vee \cdots \vee S_k^n,$$

then $\pi_n(K, L)$ is free as a $\mathbb{Z}[\pi]$-module ($\pi = \pi_1(L)$) with k natural free generators corresponding to the spheres of the bouquet. □

§4. Simplicial complexes and other homology theories. Singular homology. Coverings and sheaves. The exact sequence of sheaves and cohomology.

In this section we shall use simplexes as a means for constructing a homology theory for arbitrary topological spaces X. We shall also define the cohomology groups with coefficients from a "sheaf", important in various questions touching on complex-manifold theory and complex analysis.

We begin with the "singular" homology theory of a general space X. We define an n-dimensional *singular simplex* of X to be a pair (σ^n, f) where σ^n is a standard n-simplex and f is a map of the simplex to X:

$$f : \sigma^n \longrightarrow X.$$

The singular n-simplexes of X thus form a continuum, as it were, of maps $f : \sigma^n \to X$. A *singular n-chain* in X is then a formal integral linear combination of finitely many singular simplexes. We denote the corresponding *singular chain complex* by $C_*(X)$, with boundary operator $\partial : C_n(X) \longrightarrow C_{n-1}$ defined on a singular simplex (σ^n, f) by the formula:

$$\partial(\sigma^n, f) = \sum_{j=0}^{n} (-1)^j \left(\sigma_j^{n-1}, f\right), \tag{4.1}$$

where σ_j^{n-1} is the j-th (oriented) $(n-1)$-dimensional face of σ^n. In terms of the singular chain complex $C_*(X)$, one defines in the usual way the *singular homology* and *cohomology groups* of X. These are easily verified as being homotopy invariants of X, and functorial. It is also easy to see that the singular (co)homology groups of a one-point space are zero in nonzero dimensions. The extension to arbitrary abelian coefficient group G is essentially as before. The *singular relative homology* and *cohomology groups* $H_j(X, A; G)$ and $H^j(X, A; G)$ are defined analogously to the earlier versions. If G is a ring then an analogous multiplication of singular cohomology classes can be defined making the direct sum $H^*(X; G)$ of the singular cohomology groups into a graded ring, the *singular cohomology ring* over G. Finally, it can be shown that if X is a simplicial or CW-complex then the singular homology and cohomology groups coincide (i.e. are canonically isomorphic to) those defined earlier.

Singular homology is particularly convenient for investigating spaces of maps from one complex to another, or, more precisely, their homotopy invariants, which are crucial for building up the technical apparatus used in computing the homotopy groups of spheres and other important complexes.

As mentioned above, for certain kinds of topological spaces, in particular those arising in complex geometry and complex analysis, cohomology theories with coefficients in "sheaves" are useful.

A *presheaf* $\mathcal{F}$ (of groups, rings, etc.) over a topological space X is a correspondence associating with each open set U of X a group (or ring, etc.) $\mathcal{F}_U$, and to each inclusion $V \subset U$ of open sets a homomorphism

$$\phi_{VU} : \mathcal{F}_U \longrightarrow \mathcal{F}_V, \tag{4.2}$$

called the *restriction homomorphism* corresponding to the pair $V \subset U$. It is further required that if $W \subset V \subset U$ are open sets then

$$\phi_{WV} \circ \phi_{VU} = \phi_{WU}. \tag{4.3}$$

A presheaf is called a *sheaf* if it has the following further properties:

(i) For any open cover of U by open sets $V_\alpha \subset U$, $\bigcup_\alpha V_\alpha = U$, the vanishing of all the restriction homomorphisms $\phi_{V_\alpha U}$ on an element $f \in \mathcal{F}_U$, $\phi_{V_\alpha U}(f) = 0$, should entail $f = 0$.

(ii) Given open sets V_α as in (i), and elements $f_\alpha \in \mathcal{F}_{V_\alpha}$, one from each $\mathcal{F}_{V_\alpha}$, at which for every α, β the two restriction homomorphisms corresponding to the pairs $W_{\alpha\beta} = V_\alpha \cap V_\beta \subset V_\alpha$ and $W_{\alpha\beta} \subset V_\beta$, agree in the sense that

$$\phi_{W_{\alpha\beta}V_\alpha}(f_\alpha) = \phi_{W_{\alpha\beta}V_\beta}(f_\beta), \tag{4.4}$$

there should exist an element f of $\mathcal{F}_U$ such that for all α

$$\phi_{V_\alpha U}(f) = f_\alpha.$$

The elements of the group (or ring etc.) $\mathcal{F}_U$ over a region $U \subset X$ are called *sections* of the sheaf $\mathcal{F}$ above U.

Example 1. The *constant sheaf* has $\mathcal{F}_U = G$ for every open set U, where G is some fixed group (or ring, etc.). □

Example 2. The sheaf of germs of continuous functions. Here each $\mathcal{F}_U$ is the ring of real-valued functions defined on U. One has, analogously, the sheaf of germs of smooth functions, and, allowing complex values, of holomorphic, meromorphic or algebraic functions, over respectively a smooth, complex manifold or algebraic variety X. One may consider more generally the sheaf of germs of vector fields, or general tensor fields on X of some smoothness class, or even the sheaf of germs of sections of any fiber bundle over X (see below). □

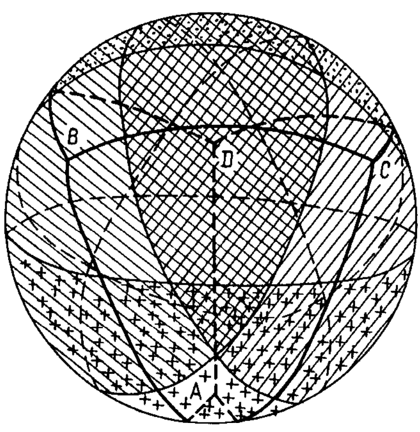


Fig. 3.7. Here the nerve is the tetrahedron $ABCD$

In the 1920s Lefschetz initiated a program of applying the algebraic methods of combinatorial topology to the study of general spaces, early advances in which were made by Vietoris, Alexandrov and Čech. Alexandrov introduced the concept of the *nerve* of a covering of a space X by countably many open subsets V_α, $\bigcup_\alpha V_\alpha = X$; this is a simplicial complex $K\{V_\alpha\}$ defined as follows: The indices $\alpha_0, \alpha_1, \dots$ of the $\{V_\alpha\}$ are taken as the vertices of $K\{V_\alpha\}$, and a finite subset $\{\alpha_{i_0}, \dots, \alpha_{i_n}\}$ of $n+1$ indices is regarded as an n-simplex if the corresponding open sets have non-empty intersection (see Figure 3.7):

$$V_{\alpha_{i_0}} \cap \dots \cap V_{\alpha_{i_n}} \neq \emptyset.$$

The cohomology groups $H^j(K;G)$ of the nerve $K = K\{V_\alpha\}$ are then called the *cohomology groups of the covering* $\{V_\alpha\}$. With each cover $\{W_\kappa\}$ "inscribed" in the covering $\{V_\alpha\}$, i.e. such that each W_κ is contained in some V_α, there is a natural simplicial map $K\{W_\kappa\} \longrightarrow K\{V_\alpha\}$ of nerves, namely that sending each vertex κ of $K\{W_\kappa\}$ to any vertex α of $K\{V_\alpha\}$ satisfying $W_\kappa \subset V_\alpha$. (If there is more then one such α, choose any one.) This map of vertices does indeed determine a simplicial map, since if $W_{\kappa_0} \cap \dots \cap W_{\kappa_n} \neq \emptyset$, then the intersection $V_{\alpha_0} \cap \dots \cap V_{\alpha_n}$ is also non-empty if $W_{\kappa_j} \subset V_{\alpha_j}$. Thus there is a simplicial map of nerves

$$\Psi_{WV} : K\{W_\kappa\} \longrightarrow K\{V_\alpha\}.$$

Note also that for any two coverings $\{V_\alpha\}$ and $\{U_\beta\}$ of the space X there is a covering $\{W_\kappa\}$ inscribed in these coverings (for example with $W_\kappa = U_\alpha \cap V_\beta$, $\kappa = \alpha\beta$) with corresponding maps of nerves:

$$\begin{array}{ccc} & & K\{V_\alpha\} \\ & \overset{\Psi_{WV}}{\nearrow} & \\ K\{W_\kappa\} & & \\ & \underset{\Psi_{WU}}{\searrow} & \\ & & K\{U_\beta\} \end{array},$$

whence there arises an *inverse system* (or "spectrum") $\{K\{V_\alpha\}, \Psi_{WV}\}$ of complexes $K\{V_\alpha\}$ and simplicial maps between them. The inverse system $\{K\{V_\alpha\}, \Psi_{WV}\}$ may be considered as providing a simplicial approximation of the space X; for instance if X is a compact metric space, then the inverse limit of $\{K\{V_\alpha\}, \Psi_{WV}\}$ is homeomorphic to X.

Each simplicial map $\Psi_{WV} : K\{W_\kappa\} \longrightarrow K\{V_\alpha\}$ induces a homomorphism of the cohomology groups of the corresponding coverings:

$$\Psi^*_{WV} : H^j(K\{V_\alpha\}; G) \longrightarrow H^j(K\{W_\kappa\}; G),$$

and one can take the *direct limit* of the system $\{H^j(K\{V_\alpha\}; G), \Psi^*_{WV}\}$ of groups and homomorphisms, called the *jth spectral* (or *Čech*) *cohomology group of the space* X. The direct limit is defined essentially as follows:

Starting with the elements of the $H^j(K\{V_\alpha\};G)$ for all coverings $\{V_\alpha\}$ as generators, take elements $x \in H^j(K\{V_\alpha\};G)$ and $y \in H^j(K\{U_\beta\};G)$, possibly corresponding to different coverings, to be equal (i.e. we impose this as an abelian-group relation among the elements) if there exists a finer covering $\{W_\kappa\}$ such that the images $\Psi^*_{WV}(x)$ and $\Psi^*_{WU}(y)$ of the two elements coincide in the group $H^j(K\{W_\kappa\};G)$:

$$\begin{array}{ccc} & & H^j(K\{V_\alpha\};G) \\ & \swarrow \Psi^*_{WV} & \\ H^j(K\{W_\kappa\};G) & & \\ & \nwarrow \Psi^*_{WU} & \\ & & H^j(K\{U_\beta\};G) \end{array}$$

The jth *spectral* (or *Čech*) *homology group* of X is defined similarly as the inverse limit of the jth homology groups of all coverings of X, where of course the homomorphisms all go the other way. The modern formulation of spectral cohomology and homology theory is due to Čech.

Given any triangulated manifold, it is not difficult to construct a covering whose homology and cohomology groups coincide with those of the original triangulation. (This may be done by taking a sufficiently fine triangulation K of the manifold, and observing that the Poincaré-dual complex DK yields a covering by the open convex polytopes $U_\alpha = \text{Int } D\sigma^0_\alpha$, where the σ^0_α are the vertices of the triangulation K. One can now see that the nerve $K\{U_\alpha\}$ of this covering coincides with the barycentric subdivision of the complex K.) Hence one infers, via the invariance of simplicial homology under barycentric subdivision, the equivalence of Čech homology with the simplicial version. Since the Čech homology groups are clearly invariant under homeomorphisms, one obtains in this manner the invariance under homeomorphisms of the simplicial homology groups (but not their homotopy invariance).

Using nerves of coverings of a topological space X one can give a natural definition of the *cohomology groups of X with coefficients from a sheaf $\mathcal{F}$ over X*, introduced originally, it would appear, by Leray. Let $K\{V_\alpha\}$ be the nerve of a covering $\{V_\alpha\}$ of X, as defined above. To each simplex $(\alpha_0, \ldots, \alpha_n)$ of the nerve one associates the obvious group (or ring, etc.):

$$(\alpha_0, \ldots, \alpha_n) \longrightarrow \mathcal{F}_{V_{\alpha_0} \cap \cdots \cap V_{\alpha_n}}.$$

The *cochains f with values in the sheaf $\mathcal{F}$* (corresponding to the covering $\{V_\alpha\}$) are then defined as those functions on the simplexes of $K\{V_\alpha\}$ which at an arbitrary simplex $(\alpha_0, \ldots, \alpha_n)$ have value in $\mathcal{F}_{V_{\alpha_0} \cap \cdots \cap V_{\alpha_n}}$. The coboundary operator ∂^* is naturally defined by

$$(\partial^* f, (\alpha_0, \ldots, \alpha_{n+1})) = \sum_{j=0}^{n+1} (-1)^j \phi_j f(\alpha_0, \ldots, \widehat{\alpha}_j, \ldots, \alpha_{n+1}),$$

where ϕ_j is the restriction homomorphism

$$\phi_j : \mathcal{F}_{\left(\bigcap_{i \neq j} V_{\alpha_i}\right)} \longrightarrow \mathcal{F}_{\left(\bigcap_i V_{\alpha_i}\right)}.$$

From this one obtains the sheaf cohomology group of dimension n corresponding to the particular covering $\{V_\alpha\}$, and the cohomology group $H^n(X; \mathcal{F})$ (nth cohomology group of X with coefficients in the sheaf $\mathcal{F}$) is then the limit of the resulting spectrum of such groups arising from all coverings of X.

Example 1. For the sheaf $\mathcal{F}$ of germs of continuous functions or functions of smoothness class C^k $(1 \leq k \leq \infty)$ on a smooth manifold M^n, it is not too difficult to show that

$$H^j(M^n; \mathcal{F}) = 0, \quad j > 0.$$

The zero-th cohomology group $H^0(M^n; \mathcal{F})$ is here just the infinite-dimensional space of (admissible) functions defined on the whole manifold M^n. □

Example 2. For the sheaf $\mathcal{F}$ of germs of holomorphic functions on a closed complex manifold X, one has:

(i) $H^0(X; \mathcal{F}) \cong \mathbb{C}$ (since the only globally holomorphic functions are the constant functions);

(ii) for $j > 0$, that $H^j(X; \mathcal{F})$ is a finite-dimensional vector space. □

Subsheaves $\mathcal{F}_1 \subset \mathcal{F}_2$, and *quotient sheaves* $\mathcal{F}_3 = \mathcal{F}_2 / \mathcal{F}_1$ can be defined in a natural fashion, although difficulties may arise in defining the quotient, so that in fact it is generally defined only as a presheaf. Thus we have a short exact sequence of sheaves:

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{i} \mathcal{F}_2 \xrightarrow{j} \mathcal{F}_3 \longrightarrow 0.$$

This yields the following exact sheaf-cohomology sequence having as particular cases all of the various versions of exact cohomology sequences:

$$\cdots \longrightarrow H^q(X; \mathcal{F}_1) \xrightarrow{i_*} H^q(X; \mathcal{F}_2) \xrightarrow{j_*} H^q(X; \mathcal{F}_3) \xrightarrow{\partial^*} H^{q+1}(X; \mathcal{F}_1) \longrightarrow \cdots. \tag{4.5}$$

Example 1. Let $\mathcal{F}_2$ be a constant sheaf over a CW-complex K, i.e. a fixed group G is associated with each open set of K. As the subsheaf $\mathcal{F}_1$ of $\mathcal{F}_2$ we take the sheaf over K which is constant $(= G)$ over a given subcomplex L of K, and zero outside L. The q-th cohomology group of K with values in $\mathcal{F}_2$ is then $H^q(K; G)$, and with values in $\mathcal{F}_1$ is $H^q(L; G)$. The exact sequence

(4.5) reduces to the exact cohomology sequence of the pair (K, L) (see §3, (3.4)). □

Example 2. Let $\mathcal{F}_1 \subset \mathcal{F}_2$ be constant sheaves of abelian groups over K, $\mathcal{F}_3 = \mathcal{F}_1/\mathcal{F}_2$. In this situation (4.5) yields a natural relationship between the cohomology groups of K with different groups of coefficients. Thus for instance if the constant group for $\mathcal{F}_1$ is $\mathbb{Z}/p$ and for $\mathcal{F}_2$ is $\mathbb{Z}/p^2$, then that for $\mathcal{F}_3$ is $\mathbb{Z}/p^2/\mathbb{Z}/p \cong \mathbb{Z}/p$, and the homomorphism ∂^* of the exact sequence (4.5) takes the form:

$$\partial^* : H^q(K; \mathbb{Z}/p) \longrightarrow H^{q+1}(K; \mathbb{Z}/p). \tag{4.6}$$

This homomorphism is usually denoted by β, and is called the *Bockstein homomorphism.* Here is the direct definition of it: Let $\widetilde{z}$ be an integral q-cochain in $C^*(K; \mathbb{Z})$ which on being reduced modulo p yields a cocycle $z \in C^*(K; \mathbb{Z}/p)$, $z = \widetilde{z} \bmod p$. Taking coboundaries gives

$$\partial^* z = 0, \quad \partial^* \widetilde{z} = pu, \quad \partial^* u = 0,$$

so that the expression

$$\frac{1}{p}\partial^* \widetilde{z} = \beta(z)$$

defines an element $\beta(z)$ of $H^{q+1}(K; \mathbb{Z}/p)$ for $z \in H^q(K; \mathbb{Z}/p)$. □

Remark. It is a worth noting that quite often there arise sheaves of non-abelian groups. For these one can naturally, and usefully, define objects $H^0(X; \mathcal{F})$ which come with a group structure, and $H^1(X; \mathcal{F})$ which are merely sets. For constant sheaves these objects were, in effect, introduced above (see §2). Their definition in the general case is completely analogous. We shall give specific examples below in the context of principal G-bundles. The group $H^0(X; \mathcal{F})$ is called the *group of sections* of the sheaf $\mathcal{F}$, and the $\mathcal{F}_U$ *sections over the regions* U (or *germs of sections*).

§5. Homology theory of non-simply-connected spaces. Complexes of modules. Reidemeister torsion. Simple homotopy type

The homology and cohomology theory of non-simply-connected complexes presents curious and isolated algebraic features leading in certain situations to important consequences. As has already been mentioned, for non-simply-connected complexes K a relative analogue of Hurewicz' theorem for a pair (K, L) can be formulated, concerning the structure of the module $\pi_n(K, L)$ over $\mathbb{Z}[\pi]$, where $\pi = \pi_1(L)$. If we regard $\mathbb{Z}$ as a trivial $\mathbb{Z}[\pi]$-module, i.e. consider the elements of π as acting identically on $\mathbb{Z}$, then this result may be considered as asserting the isomorphism (cf. (3.18))

$$H_n(K, L; \mathbb{Z}) \cong \pi_n(K, L) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}, \tag{5.1}$$

under the conditions $\pi_j(K, L) = 0$ for $0 < j < n$, $n > 1$.

Given two modules M, N over a ring R, we define their *tensor product* $M \otimes_R N$ over R to be the module generated by all symbols $m \otimes n$, $m \in M$, $n \in N$, subject to the usual bilinearity relations (cf. (2.3)):

$$\lambda(m \otimes n) = (\lambda m) \otimes n = m \otimes (\lambda n), \quad \lambda \in R. \tag{5.2}$$

In the case $R = \mathbb{Z}$ this reduces to the tensor product of abelian groups (considered in the natural way as $\mathbb{Z}$-modules) introduced in §2.

With any non-simply-connected simplicial (or CW-) complex K there is naturally associated an algebraic complex $C_*(\widetilde{K})$ (where $\widetilde{K}$ is the universal cover of K) of free $\mathbb{Z}[\pi]$-modules with basis consisting of chains in natural one-to-one correspondence with the simplexes (or cells) of K. This complex $C_*(\widetilde{K})$ is defined as follows: Let

$$p : \widetilde{K} \longrightarrow K, \quad (\pi_1(\widetilde{K}) = 0),$$

be the universal covering projection. As we saw earlier the group $\pi = \pi_1(K)$ acts freely on $\widetilde{K}$, and in fact sends simplexes to simplexes (or cells to cells); this follows from the fact that over a contractible subspace the universal cover is trivial, i.e. is the product of that subspace with the discrete fiber F, which is the group $\pi = \pi_1(K)$ in this case. In our present context we have as contractible subspaces the simplexes (or cells) of K, whence

$$p^{-1}(\sigma_\alpha^n) = \bigcup_{\gamma \in \pi} \sigma_{\alpha\gamma}^n.$$

Choose an "initial" n-simplex (or cell) $\sigma_{\alpha 1}^n \subset p^{-1}(\sigma_\alpha^n)$. The remaining n-cells of $p^{-1}(\sigma_\alpha^n) \subset \widetilde{K}$ are then obtained by applying the elements of π to this one:

$$\gamma(\sigma_{\alpha 1}^n) = \sigma_{\alpha\gamma}^n, \quad \gamma \in \pi.$$

Thus $C_n(\widetilde{K})$ is just the usual n-chain complex of $\widetilde{K}$, endowed with this $\mathbb{Z}[\pi]$-action. It is clearly a free module. Note that the action of $\mathbb{Z}[\pi]$ commutes with the boundary operator ∂:

$$\partial\lambda(c) = \lambda\partial(c), \quad \lambda \in \mathbb{Z}[\pi].$$

Thus we obtain $C_*(\widetilde{K})$ as a chain complex of free $\mathbb{Z}[\pi]$-modules.

If the covering $\widetilde{K}$ is not necessarily universal, but still regular, then the quotient $\Gamma = \pi_1(K)/N$, $N \cong \pi_1(\widetilde{K})$, acts freely on $\widetilde{K}$ and sends simplexes (cells) to simplexes (cells) in such a way that the orbit space $\widetilde{K}/\Gamma \cong K$. Then as before the chain complex $C_*(\widetilde{K})$ comes with the natural Γ-action, turning it into a free $\mathbb{Z}[\Gamma]$-module.

Any representation ρ of a group Γ, i.e. any homomorphism $\rho : \Gamma \longrightarrow \operatorname{Aut} V$, from Γ to the group of automorphisms of some vector space or abelian group V, can be used to endow V with the structure of a $\mathbb{Z}[\Gamma]$-module (V, ρ). Tensoring such a module (V, ρ) with the $\mathbb{Z}[\Gamma]$-module $C_*(\widetilde{K})$ just defined, we obtain the *chain complex of* K *with respect to* ρ

$$C_*(K; \rho) = C_*(\widetilde{K}) \otimes_{\mathbb{Z}[\Gamma]} (V, \rho). \tag{5.3}$$

From this one constructs, as appropriate factor modules, the *homology and cohomology groups of* K *with respect to the representation* ρ.

Example 1. If ρ is taken to be the trivial representation sending all of Γ to the identity automorphism of V, then the usual homology and cohomology groups $H_q(K; V)$, $H^q(K; V)$ are obtained. □

Example 2. Let K be a non-orientable manifold, $\dim K = n$. For each path class $[\gamma] \in \pi_1(K)$, define $\rho[\gamma] = \operatorname{sgn} \gamma$, where $\operatorname{sgn} \gamma = 1$ if transport of a co-ordinate frame around γ preserves orientation, and -1 otherwise. We take $V = \mathbb{Z}$, so that $\operatorname{Aut} \mathbb{Z} \cong \mathbb{Z}/2 \cong \{\pm 1\}$. For the resulting homology and cohomology groups $H_j(K; \rho)$ and $H^j(K; \rho)$ there are Poincaré-duality isomorphisms determined as earlier by the Poincaré dual complex DK (see §2):

$$\begin{aligned} D: \quad H_j(K; \rho) &\cong H^{n-j}(K; \rho), \\ D: \quad H^j(K; \rho) &\cong H_{n-j}(K; \rho). \quad \square \end{aligned} \tag{5.4}$$

Example 3. Take $V = \mathbb{Z}[\pi]$, and associate with each element g of π its action on $\mathbb{Z}[\pi]$ via multiplication on the right. This gives what is essentially the identity representation ρ, yielding

$$H_j(K; \rho) \cong H_j(\widetilde{K}; \mathbb{Z}), \quad H^j(K; \rho) \cong H^j(\widetilde{K}; \mathbb{Z}). \qquad \square$$

Of particular interest are those representations $\rho : \pi \longrightarrow \operatorname{Aut} V$, for which the corresponding homology groups $H_q(K; \rho)$ are trivial; in this situation we say that K is a ρ-*acyclic* complex. If K is ρ-acyclic of dimension n, the chain complex $C_*(K; \rho)$ yields an exact sequence:

$$0 \longrightarrow C_n(K; \rho) \xrightarrow{\partial} C_{n-1}(K; \rho) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(K; \rho) \xrightarrow{\partial} C_0(K; \rho) \longrightarrow 0. \tag{5.5}$$

It is important for what follows to note that if V is a free abelian group with a given free basis (or a vector space with given basis), then the free abelian groups $C_j(K; \rho)$ (vector spaces if V is a vector space) will likewise each have a distinguished basis, whose members correspond to the cells of K.

Remark. Assume V free abelian (or a vector space) with prescribed basis. The "distinguished basis" of $C_j(K; \rho)$ just described is uniquely determined only up to application of one or more of the following elementary geometric changes of basis:

1) that arising from a change of orientation of some or all cells: $\sigma_\alpha^n \longrightarrow -\sigma_\alpha^n$;
2) that arising from a different choice of "initial cell" $\sigma_{\alpha_1}^n$ from the complete inverse image $p^{-1}(\sigma_\alpha^n) = \bigcup \sigma_{\alpha\gamma}^n$, $\gamma \in \pi$; here the change of basis has the form $\sigma_\alpha^n \longrightarrow \pm\gamma\sigma_\alpha^n$, or, with respect to a representation ρ, $\sigma_\alpha^n \longrightarrow \pm\rho(\gamma)\sigma_\alpha^n$.

In what follows we assume for each $C_j(K;\rho)$ a fixed "distinguished basis". □

Suppose now that V is a finite-dimensional vector space over some field k, with prescribed basis, and that K is ρ-acyclic, so that we have the exact sequence (5.5). If that sequence has but two terms

$$0 \longrightarrow C_n(K;\rho) \xrightarrow{\partial} C_{n-1}(K;\rho) \longrightarrow 0,$$

then ∂ is simply an isomorphism, and relative to the distinguished bases for $C_n(K;\rho)$ and $C_{n-1}(K;\rho)$ is represented by a nonsingular matrix over k, with determinant $\det\partial = \det C_*(K;\rho)$. It turns out that with every finite ρ-acyclic complex K one may associate a "determinant" relative to distinguished bases for the $C_j(K;\rho)$ in the corresponding exact sequence (5.5). In the case where the exact sequence (5.5) has three terms:

$$0 \longrightarrow C_n(K;\rho) \xrightarrow{\partial} C_{n-1}(K;\rho) \xrightarrow{\partial} C_{n-2}(K;\rho) \longrightarrow 0,$$

this determinant is defined as follows: Denote the distinguished bases of $C_n(K;\rho)$, $C_{n-1}(K;\rho)$, $C_{n-2}(K;\rho)$ by c_n, c_{n-1}, c_{n-2}. The middle complex $C_{n-1}(K;\rho)$ also has $\left(\partial c_n, \partial^{-1}(c_{n-2})\right)$ as basis, where $\partial^{-1}(c_{n-2})$ denotes a set of arbitrary chosen inverse images of the members of c_{n-2}, ordered correspondingly. Since the latitude in the choice of $\partial^{-1}(c_{n-2})$ is "triangular" in character, the determinant of the change of basis from c_{n-1} to $\left(\partial c_n, \partial^{-1}(c_{n-2})\right)$ is uniquely defined. The determinant $\det C_*(K;\rho)$ is defined analogously in the general case: Denote the distinguished basis of each $C_j = C_j(K;\rho)$ by c_j, write $A_j \subset C_{j-1}$ for the image of C_j under ∂, (j = 1, ..., n-1) and choose bases a_j for these A_j arbitrarily. Set $A_0 = 0$, $A_n = C_n$ and $a_n = c_n$:

$$A_n = C_n, \quad A_j = \partial C_j \subset C_{j-1}, \quad j = 1, \dots, n-1, \quad A_0 = 0,$$

$$0 \longrightarrow A_n \longrightarrow C_{n-1} \xrightarrow{\partial} A_{n-1} \subset C_{n-2},$$

$$0 \longrightarrow A_{n-1} \longrightarrow C_{n-2} \xrightarrow{\partial} A_{n-2} \subset C_{n-3},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$0 \longrightarrow A_1 \longrightarrow C_0 \xrightarrow{\partial} A_0 = 0.$$

We then define

$$\det C_*(K;\rho) = \frac{\det\left[c_{n-1}:(a_n,\partial^{-1}a_{n-1})\right]\cdot\det\left[c_{n-3}:(a_{n-2},\partial^{-1}a_{n-3})\right]\cdots}{\det[c_{n-2}:(a_{n-1},\partial^{-1}a_{n-2})]\cdot\det[c_{n-4}:(a_{n-3},\partial^{-1}a_{n-4})]\cdots},$$

for the arbitrary ρ-acyclic complex $C_*(K;\rho)$.

Definition 5.1 The above quantity $\det C_*(K;\rho)$, defined for ρ-acyclic complexes K (i.e. with $H_j(K;\rho) = 0$ for $j \geq 0$), is called the *Reidemeister torsion* of K, and is denoted by $R(K,\rho)$.

It was established in the late 1930s (by Reidemeister, de Rham, Franz) that up to sign and multiplication by any number of the form $\det\rho(\gamma)$, $\gamma \in \pi$, the quantity $\det C_*(K;\rho) = R(K,\rho)$ is a combinatorial invariant of the simplicial complex K, i.e. is unchanged by subdivisions. Only much later, in the 1970s, was it shown that $R(K,\rho)$ is in fact a topological invariant. (The variation in $R(K,\rho)$ results from the above-mentioned latitude in choosing the distinguished bases of the C_j.)

Note that if L is any simply-connected complex, $\pi_1 L = 0$, then one has

$$R(K \times L, \rho) = R(K,\rho)\chi(L),$$

where $\chi(L)$ is the Euler-Poincaré characteristic of L.

The quantity $R(K,\rho)$ was introduced in the first place (by Reidemeister) for the purpose of investigating the "lens spaces", defined as follows: Let A be an $n \times n$ complex diagonal matrix with primitive mth roots of unity as its diagonal entries:

$$A = (a_{jk}), \quad a_{jk} = e^{2\pi i q_j/m}\delta_{jk}, \quad A^m = I_{n\times n}, \tag{5.6}$$

where $(\delta_{jk}) = I_{n\times n}$ is the identity $n\times n$ matrix. Since the a_{jj} are primitive mth roots of 1, each q_j must be relatively prime to m. The matrix A generates a group isomorphic to $\mathbb{Z}/m$ acting on the sphere S^{2n-1} according to the following rule:

$$z^j \xrightarrow{A} e^{2\pi i q_j/m} z^j, \quad \sum_{j=1}^{n} |z^j|^2 = 1. \tag{5.7}$$

Taking the orbit space yields the manifold known as the $(2n-1)$-dimensional *lens space*:

$$L^{2n-1}_{(q_1,\ldots,q_n)} = S^{2n-1}/A. \tag{5.8}$$

It may always be assumed that $q_1 = 1$ by choosing in place of A the appropriate generator of the cyclic group A generates.

When $n = 2$ we obtain the 3-dimensional manifolds $L^3_q = L^3_{(1,q)}$, where q is relatively prime to m. Two such manifolds $L^3_{q'}$ and $L^3_{q''}$ (defined for the same m) are homotopy equivalent if and only if

$$q'/q'' \equiv \pm\lambda^2 \bmod m. \tag{5.9}$$

This follows in a straightforward manner using the structure of the cohomology ring $H^*(L_q^3; \mathbb{Z}/m)$ and the Bockstein homomorphism $\beta : H^1(L_q^3; \mathbb{Z}/m) \longrightarrow H^2(L_q^3; \mathbb{Z}/m)$ (see (4.6)): For the fundamental homology class $\left[L_q^3\right]$ of the lens space L_q^3, the following congruence holds:

$$\left(a\beta(a), \pm \left[L_q^3\right]\right) \equiv \pm q \mod m,$$

for any generator a of $H^1(L_q^3; \mathbb{Z}/m) \cong \mathbb{Z}/m$. Replacing a in this congruence by any other generator λa then causes q to change to $\lambda^2 q$.

The lens spaces $L^{2n-1}_{(q_1,\dots,q_n)}$ may be decomposed as CW-complexes in a manner independent of $(q_1, \dots, q_n)$, much as this was done for the projective space $\mathbb{R}P^{2n-1}$. One obtains just one cell of each dimension: $\sigma^0, \sigma^1, \dots, \sigma^{2n-1}$, with boundary operator given by

$$\partial\sigma^{2j-1} = 0, \quad \partial\sigma^{2j} = m\sigma^{2j-1}, \quad j > 0. \tag{5.10}$$

However up on the sphere S^{2n-1}, the induced covering cell-decomposition, on which the cyclic group of order m generated by A acts freely and so as to respect cells, does depend in an essential way on $(q_1, \dots, q_n)$. In the case $n = 2$ we have the following cells induced on $S^{2n-1} = S^3$ from those of L_q^3:

$$\sigma_j^s = A^j\sigma^s, \quad s = 0, 1, 2, 3, \quad j = 0, 1, \dots, m-1. \tag{5.11}$$

The action of the boundary operator on these cells is as follows (writing $\sigma_0^s = \sigma^s$, $s = 0, 1, 2, 3$):

$$\begin{aligned} \partial\sigma^3 &= \left(1 - A^q\right)\sigma^2, \quad \sigma_1^3 = \sigma^3 \\ \partial\sigma^2 &= \left(1 + A + \dots + A^{m-1}\right)\sigma^1, \\ \partial\sigma^1 &= (1 - A)\,\sigma^0. \end{aligned} \tag{5.12}$$

For the representation $\rho : \mathbb{Z} \longrightarrow \left(e^{2\pi i/m}\right)$ in the space $V = \mathbb{C}^1$, the lens space L_q^3 is ρ-acyclic, and the exact sequence (5.5) arising from the complex $C_*(L_q^3; \rho)$ is

$$0 \longrightarrow C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \longrightarrow 0,$$

where each C_j is one-dimensional, with distinguished generator e_j say. In view of (5.12) the boundary operator is given by

$$\begin{aligned} \partial e_3 &= \left(1 - e^{2\pi i q/m}\right)e_2, \quad \partial e_2 = 0, \\ \partial e_1 &= \left(1 - e^{2\pi i q/m}\right)e_0, \end{aligned} \tag{5.13}$$

whence one computes the Reidemeister torsion, obtaining

$$R(L_q^3, \rho) = (1 - \zeta^q)(1 - \zeta) \mod (\pm\zeta^\alpha), \tag{5.14}$$

where $\zeta = e^{2\pi i/m}$. It follows from (5.14) that there are lens spaces that are homotopically equivalent but not combinatorially equivalent. Later results (of Moise, Chapman and Edwards) show that in fact homotopically equivalent lens spaces need not even be homeomorphic. By considering various representations ρ for $n > 2$, de Rham and Franz have shown that in fact only in trivial cases does one have combinatorial equivalence of lens spaces.

A complete analysis of this type of question from the point of view of homotopy theory, was carried out by Whitehead. Suppose we know that two finite complexes K and L are homotopically equivalent. Can we then realize a homotopy equivalence between them by means of elementary combinatorial operations? Rather than embark on a detailed, exact description of these operations in geometric terms, we shall confine ourselves to sketching the essential algebraic picture.

Suppose L is contained in K as a simplicial or CW-subcomplex. Since $K \sim L$, the complex of relative chains $C(\widetilde{K}, \widetilde{L}) = \bigoplus_t C_t(\widetilde{K}, \widetilde{L})$ regarded as a $\mathbb{Z}[\pi]$-module, is acyclic. Consider the following two types of elementary operation on an arbitrary algebraic complex $C = \bigoplus_t C_t$ over any associative ring R, where each C_t has a distinguished basis $\{\sigma_i^t\}$:

(i) For each fixed k, j, and $\gamma \in R$,

$$\sigma_j^k \longrightarrow \gamma\sigma_j^k + \sigma_j^k, \quad \sigma_i^t \longrightarrow \sigma_i^t \quad \text{if either } t \neq k \text{ or } i \neq j. \tag{5.15}$$

(ii) Stabilization: for each fixed k, the addition of a direct summand $\widetilde{C}^{(k)}$ to C of the following form:

$$0 \longrightarrow \widetilde{C}_k^{(k)} \xrightarrow{\partial} \widetilde{C}_{k-1}^{(k)} \longrightarrow 0, \tag{5.16}$$

where $\widetilde{C}_{k-1}^{(k)}$ and $\widetilde{C}_k^{(k)}$ are cyclic, generated respectively by $\widetilde{e}_{k-1}$, $\widetilde{e}_k$ with $\partial\widetilde{e}_k = \widetilde{e}_{k-1}$; thus the operation is defined as follows:

$$C \longrightarrow C \oplus \widetilde{C}^{(k)},$$

$$\text{i.e.} \quad C_k \longrightarrow C_k \oplus \widetilde{C}_k^{(k)}, \quad C_{k-1} \longrightarrow C_{k-1} \oplus \widetilde{C}_{k-1}^{(k)}; \tag{5.17}$$

$$C_j \longrightarrow C_j \quad \text{for } j \neq k-1, k.$$

If we define two acyclic algebraic complexes over R to be *equivalent* if one can be obtained from the other by means of a finite sucession of the operations (i), (ii) and their inverses, then under the operation of taking the direct sum the resulting equivalence classes form an abelian group $K_1(R)$. This is the "group of determinants"; it may be alternatively defined as a certain quotient group of the group of infinite matrices over R with only finitely-many non-zero off-diagonal entries and all but finitely many diagonal entries equal to the multiplicative identity element of R. To obtain $K_1(R)$ one factors this group by the normal subgroup generated by all commutators and all elementary matrices with diagonal entries 1 and the remaining non-zero entry above the diagonal. This group arises naturally in algebra as a result of the axiomatization of the fundamental properties of the determinant.

In the present topological context the ring R has the form $\mathbb{Z}[\pi]$, and in view of the aforementioned latitude in the choice of distinguished bases — namely that resulting from changes of orientation of cells of the covering space and of the initial cell — it is not so much $K_1(\mathbb{Z}[\pi])$ but rather its factor group

$$\mathrm{Wh}(\pi) = K_1(\mathbb{Z}[\pi])/(\pm\pi),$$

(the *Whitehead group*), that is significant.

Thus for a pair $L \subset K$ of homotopy equivalent complexes, we obtain as an invariant the "determinant" of the complex $C(\widetilde{K}, \widetilde{L})$ of $\mathbb{Z}[\pi]$-modules in the Whitehead group $\mathrm{Wh}(\pi)$. The terminology used in this connexion is "simple homotopy type": if one such complex can be changed to another by means of the above-defined elementary combinatorial operations then the complexes are said to be of the same *simple homotopy type.* Any representation $\rho : \pi \longrightarrow GL(n, \mathbb{R})$ gives rise to a homomorphism from $K_1(\mathbb{Z}[\pi])$ to the multiplicative group of $\mathbb{R}$

$$K_1(\mathbb{Z}[\pi]) \longrightarrow \mathbb{R}^*$$

yielding the Reidemeister torsion discussed above. For simply-connected complexes the group K_1 is trivial: $\mathbb{Z}[\pi] = \mathbb{Z}$, $K_1(\mathbb{Z}) = 0$, and simple homotopy equivalence reduces to the ordinary kind.

Already by the late 1930s, beginning with the work of Smith and Richardson, investigations of the homological properties of finite groups of transformations were being carried on. Suppose that a finite group G acts on a CW-complex K, sending cells to cells. From this action one obtains, much as above, a complex of $\mathbb{Z}[G]$-modules — even if G does not act freely. The subcomplex of K consisting of the points of K fixed by every element of G will be denoted by K^G, and the corresponding orbit complex by K/G. We have natural maps $i : K^G \longrightarrow K$, $\pi : K \longrightarrow K/G$, and the induced homomorphisms

$$H_*(K^G) \xrightarrow{i_*} H_*(K), \qquad H_*(K) \xrightarrow{\pi_*} H_*(K/G).$$

The finiteness of G allows one to define the *transfer homomorphism*

$$\mu_* : H_*(K/G) \longrightarrow H_*(K),$$

(defined over any group of coefficients) as follows: Every j-chain c_G in $C_j(K/G)$ corresponds to an orbit $\{g(c) \mid g \in G\}$ of j-chains of $C_j(K)$, $c \in C_j(K)$, under the action of the finite group G; the homomorphism μ_j is then induced by that associating each $c_G \in C_j(K/G)$ with $\sum_{g \in G} g(c) \in C_j(K)$, the (finite) sum of the chains in the corresponding orbit. It follows that the composite map

$$\pi_*\mu_* : H_*(K/G) \longrightarrow H_*(K/G) \tag{5.18}$$

is just multiplication by $|G|$, and

$$\mu_*\pi_* : H_*(K) \longrightarrow H_*(K) \tag{5.19}$$

is determined by the action of $\sum_{g\in G} g$ ($\in \mathbb{Z}[G]$ if the coefficient group is $\mathbb{Z}$). From (5.18) and (5.19) it follows that over a field of characteristic zero or finite characteristic relatively prime to $|G|$, we have

$$\begin{aligned} H_*(K;k) &\cong \mu_* H_*(K/G;k) \oplus \operatorname{Ker} \pi_*, \\ H_*(K/G;k) &\cong H_*(K;k)^G, \end{aligned} \tag{5.20}$$

where the superscript G denotes the subgroup pointwise stabilized by G.

Via the G-action each element ρ of $\mathbb{Z}[G]$ has associated with it the chain complex $\rho C(K)$ and the resulting homology groups $H_*^\rho(K) = H_*(\rho C(K))$. The case $G = \mathbb{Z}/p$, $k = \mathbb{Z}/p$ is of particular interest. Note that here $H_*^\rho(K;\mathbb{Z}/p)$ is to be understood as the homology of the complex $\rho C(K;\mathbb{Z}/p)$ and not of $\rho C(K;\mathbb{Z}) \otimes \mathbb{Z}/p$. Consider the following elements of the group ring $k[G]$, $k = \mathbb{Z}/p$:

$$\begin{gathered} \sigma = \sum_{g\in G} g = 1 + T + T^2 + \cdots + T^{p-1}, \\ \tau = 1 - T, \quad \sigma\tau = \tau\sigma = 0, \end{gathered} \tag{5.21}$$

where T is a generator of the cyclic group G of order p (written multiplicatively). The exact sequences due to Smith are as follows:

$$\begin{gathered} \cdots \to H_j(K;\mathbb{Z}/p) \xrightarrow{\sigma_*} H_j^\sigma(K;\mathbb{Z}/p) \xrightarrow{\partial} H_{j-1}^\tau(K;\mathbb{Z}/p) \oplus H_{j-1}(K^G;\mathbb{Z}/p) \xrightarrow{i_*} \\ H_{j-1}(K;\mathbb{Z}/p) \xrightarrow{\sigma_*} \cdots; \\ \cdots \to H_j(K;\mathbb{Z}/p) \xrightarrow{\tau_*} H_j^\tau(K;\mathbb{Z}/p) \xrightarrow{\partial} H_{j-1}^\sigma(K;\mathbb{Z}/p) \oplus H_{j-1}(K^G;\mathbb{Z}/p) \xrightarrow{i_*} \\ H_{j-1}(K;\mathbb{Z}/p) \xrightarrow{\tau_*} \cdots. \end{gathered} \tag{5.22}$$

There are analogous exact cohomology sequences (with the arrows going the other way).

From these exact sequences one infers the following result: If the group $\mathbb{Z}/p$ acts on a $\mathbb{Z}/p$-homology n-sphere, then the set of fixed points of the action is likewise a $\mathbb{Z}/p$-homology sphere of some dimension r. Moreover if $p > 2$, then $n-r$ is even. Analogous results are valid for homology n-discs in the category of pairs (K, L) of complexes. (A *$\mathbb{Z}/p$-homology n-sphere* is a complex K with $H_j(K;\mathbb{Z}/p) \cong H_j(S^n;\mathbb{Z}/p)$ for all j. This definition may be taken as applying also in the case $n = -1$, where S^n is the empty set.)

Within the bounds of the present exposition it is not appropriate to dwell at greater length on the purely homological theory of finite or compact transformation groups with its associated algebro-topological techniques, although several authors have investigated this topic. We shall touch on the subject of finite and compact groups of smooth transformations below in connexion with "K-theory", "elliptic operators", and "bordism theory".

§6. Simplicial and cell bundles with a structure group. Obstructions. Universal objects: universal fiber bundles and the universal property of Eilenberg-MacLane complexes. Cohomology operations. The Steenrod algebra. The Adams spectral sequence

We now turn to the elaboration of a concept already examined above from the point of view of homotopy, namely that of a *fiber bundle*. Our present definition will include an important new element — the "structure group" of the fiber bundle, occuring naturally as a subgroup of the group of self-homeomorphisms (transformations) of a fiber. Let

$$p : E \longrightarrow B \quad \text{(with fiber } F\text{)}$$

be a fiber bundle (as defined in Chapter 2, §3), where E, F and B are CW- or simplicial complexes and the projection p respects cells (simplexes). In this context the prescribed structural maps are assumed to be defined for some open neighbourhood $U(\sigma_\alpha^n)$ of each cell (or simplex) σ_α^n of the base B; hence on $p^{-1}(\sigma_\alpha^n)$ we have

$$\phi_\alpha : p^{-1}(\sigma_\alpha^n) \longrightarrow \sigma_\alpha^n \times F,$$

where the restrictions to fibers are homeomorphisms of F. It is further required that the transition function on some small open neighbourhood of the boundary (or faces) common to two cells (simplexes):

$$\lambda_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : U(\sigma_\alpha^n \cap \sigma_\beta^m) \times F \longrightarrow U(\sigma_\alpha^n \cap \sigma_\beta^m) \times F,$$

have the form

$$\lambda_{\alpha\beta}(x, y) = (x, \widehat{\lambda}_{\alpha\beta}(x)(y))$$

where $\widehat{\lambda}_{\alpha\beta}(x)$ is a transformation of the fiber F for each point $x \in U_{\alpha\beta} = U(\sigma_\alpha^n \cap \sigma_\beta^m)$. We assume that all maps $\widehat{\lambda}_{\alpha\beta}(x)$ belong to a prescribed group G of transformations of F; the group G is the *structure group* of the fiber bundle (E, B, F, p, G). It is also required that the composite of transformations $\widehat{\lambda}_{\alpha\beta}(x)$, $\widehat{\lambda}_{\beta\gamma}(x)$ and $\widehat{\lambda}_{\gamma\alpha}(x)$ is the identity element of the group G:

$$\widehat{\lambda}_{\alpha\beta}(x) \circ \widehat{\lambda}_{\beta\gamma}(x) \circ \widehat{\lambda}_{\gamma\alpha}(x) \equiv 1, \tag{6.1}$$

for each point x of the intersection $U_{\alpha\beta} \cap U_{\beta\gamma} \cap U_{\gamma\alpha}$, the common domain of the transition functions $\widehat{\lambda}_{\alpha\beta}$, $\widehat{\lambda}_{\beta\gamma}$ and $\widehat{\lambda}_{\gamma\alpha}$. Thus the fiber bundle is determined essentially by the maps

$$\widehat{\lambda}_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G, \tag{6.2}$$

with the condition (6.1) imposed.

A *cross-section* of the bundle is any map $f : B \longrightarrow E$ such that $p \circ f = 1_B$. A *map of fiber bundles* (or *fiber bundle map*) between fiber bundles with the same fiber F and group G:

$$\Phi : (E, B, F, p, G) \longrightarrow (E', B', F, p', G),$$

is defined in the natural way as given by maps $\Phi : E \longrightarrow E'$ and $\phi : B \longrightarrow B'$ commuting with the projections in the sense that

$$\phi \circ p = p' \circ \Phi.$$

It is further required that Φ respect fibers, i.e. that its restrictions to fibers be homeomorphisms. Moreover these homeomorphisms have to come from the structure group G in the following precise sense: over each point of the base B the map

$$\phi'_{\alpha'} \circ \Phi \circ \phi_{\alpha}^{-1} : F \longrightarrow F$$

(where $\phi'_{\alpha'}$, ϕ_{α} are any appropriate structural maps of the bundles E', E respectively) should be a homeomorphism from G.

Given a fiber bundle (E', B', F, p', G) and an arbitrary map $\phi : B \longrightarrow B'$, one obtains a fiber bundle over B, the *induced fiber bundle* (E, B, F, p, G) and a fiber bundle map

$$\Phi : E \longrightarrow E', \quad \phi : B \longrightarrow B',$$

by means of the following construction: The distinguished open sets of B on which the structural maps ϕ_{α} are to be defined, are taken to be the complete inverse images $U_{\alpha} = \phi^{-1}(U'_{\alpha})$ of the distinguished open sets U'_{α} of B' (and then if necessary these may be refined to open neighbourhoods of cells). Above each such U_{α} we set $p^{-1}(U_{\alpha}) = U_{\alpha} \times F$. The product spaces $p^{-1}(U_{\alpha})$ are then joined together to form E by means of the transition functions $\lambda_{\alpha\beta}$:

$$\lambda_{\alpha\beta}(x, y) = (x, \widehat{\lambda}_{\alpha\beta}(x)(y)), \quad x \in U_{\alpha} \cap U_{\beta}, \quad y \in F,$$

where the transformation $\widehat{\lambda}_{\alpha\beta}$ is defined in terms of the transformation $\widehat{\lambda}'_{\alpha\beta}$ from the fiber bundle (E', B', F, p', G) as follows:

$$\widehat{\lambda}_{\alpha\beta}(x) = \widehat{\lambda}'_{\alpha\beta}(\phi(x)) \in G.$$

The total space E may be considered formally as the subset of $B \times E'$ consisting of those pairs (x, y) such that $\phi(x) = p'(y)$. Thus the induced fiber bundle over B is obtained by "pulling back" via ϕ the fiber bundle over B'.

Remark. It is easily inferred from the condition (6.1) that the maps $\widehat{\lambda}_{\alpha\beta}$ represent a one-dimensional cocycle of the base B with coefficients in the sheaf $\mathcal{F}_G$ of germs of functions with values in the group G (see §4). Hence the fiber bundle is associated in this way with an element of $H^1(B; \mathcal{F}_G)$, and this correspondence affords a classification of bundles over B with structure group G in terms of one-dimensional sheaf cohomology. This classification is, however, of an essentially linguistic or tautological character, although it is occasionally useful, for instance if the group G is abelian. □

As observed above, the collection of maps $\widehat{\lambda}_{\alpha\beta}$ in (6.2) uniquely determines the structure of the bundle (E, B, F, p, G). Of course a given topological group G is in general realizable in many different ways as a group of homeomorphisms of a topological space. Fiber bundles over the same base B, with the same structure group G, and with the same collection of maps $\widehat{\lambda}_{\alpha\beta}$, but with possibly different fibers, are said to be *associated* with one another. Clearly any fiber bundle determines the structure of all of its associated bundles.

There is a canonical universal realization of a topological group G as a group of homeomorphisms of a fiber F, namely that where $F = G$ and G acts on itself by means of (right) translations (see Figure 3.8):

$$g : G \longrightarrow G, \quad g(h) = hg, \quad h, g \in G. \tag{6.3}$$

A fiber bundle (E, B, G, G, p) with this G-action is called *principal.* Such bundles arise most naturally in the situation of a topological group acting freely (i.e. $g(x) = x$ implies $g = 1$) on a space E; here the orbit space E/G plays the role of the base B, and the projection is the natural one $p : E \longrightarrow E/G$. In fact it is not difficult to show that every principal fiber bundle arises in this way, i.e. that there is a free action of G on the total space E such that the base B is naturally identifiable with the orbit space E/G, and the fibers $p^{-1}(x)$ with G.

Of particular importance is the case where G is a compact Lie group acting freely on a smooth manifold E by means of smooth transformations, and the orbit space $B = E/G$ is likewise a smooth manifold. This describes, in essence, the concept of a *smooth fiber bundle*, to be considered at greater length in the sequel. Another particular case of interest is that where G is a discrete group; here a principal G-bundle is simply a regular covering space, as considered earlier.

Definition 6.1 A fiber bundle (E, B, F, G, p) is called *universal* if the associated principal bundle (E', B, G, G, p) has the property that its total space E' is contractible.

The significance of this definition will appear below.

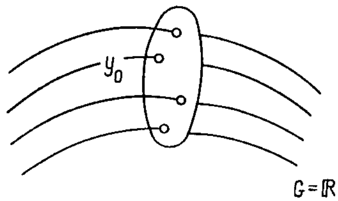


Fig. 3.8

The concept of an "obstruction" is of fundamental importance in algebraic topology; originally it arose in attempting to answer the following questions.

Question 1. Let L be a subcomplex of a complex K, and let $f : L \longrightarrow X$ be a map from L to an arbitrary topological space X. *How may f be extended, if it can be extended, to all of K?*

Let $f_\alpha : K \longrightarrow X$, $\alpha = 1, 2$, be two maps, and let $F : L \times I \longrightarrow X$ be a homotopy between the restrictions of these two maps to the subcomplex L. *Can the homotopy F be extended to all of K, i.e. to one between f_1 and f_2?*

Question 2. Let $p : E \longrightarrow B$ be a fiber bundle (with base B a simplicial or CW-complex), and let $\psi : L \longrightarrow E$ be a cross-section of the bundle over the subcomplex $L \subset B$: $p\psi = 1_L$. *How may one extend the cross-section over all of B?* The analogous problem for homotopies of two cross-sections is as follows: Given two cross-sections over B and a homotopy between them defined over $L \subset B$, *how can one extend this homotopy to one between the original cross-sections, i.e over all of B?*

Question 3. Let (E, B, F, G, p) and (E', B', F, G, p') be fiber bundles with the same fiber and structure group, $L \subset B$ be a subcomplex as above, and let $f : L \longrightarrow B'$ be a map inducing a map of fiber bundles $\widehat{f} : p^{-1}(L) \longrightarrow E'$. *How can one extend the map $\widehat{f}$ to a map of fiber bundles $E \longrightarrow E'$?* There is also the analogous question for homotopies between two maps of fiber bundles.

We begin with the first of these questions. Thus we seek to extend a map $f : L \longrightarrow X$ from the subcomplex $L \subset K$ to the whole CW-complex K. Suppose inductively that the map f has already been defined on all cells of dimension $\leq n-1$. Thus if σ_α^n is any cell of K not contained in L then f is assumed already defined on its boundary $\partial\sigma_\alpha^n$:

$$f|_{\partial\sigma_\alpha^n} : \partial\sigma_\alpha^n \longrightarrow X.$$

Clearly this map may be extended to the interior of σ_α^n if and only if the map of the sphere $S_\alpha^{n-1} \xrightarrow{\cong} \partial\sigma_\alpha^n \xrightarrow{f|_{\partial\sigma_\alpha^n}} X$ is null-homotopic. In any case the map $S_\alpha^{n-1} \longrightarrow X$ defines an element

$$\lambda\left(\sigma_\alpha^n\right) \in \pi_{n-1}(X).$$

Thus provided $n \geq 2$ we obtain (by considering all n-cells σ_α^n not contained in L) a relative cochain

$$\lambda(f) \in C^n\left(K, L; \pi_{n-1}(X)\right),$$

called an *obstruction to the extension* of f. Clearly the map f may be extended from the $(n-1)$-skeleton of K to the n-skeleton if and only if $\lambda(f) \equiv 0$.

It may be possible, however, to change the map f on the $(n-1)$-skeleton K^{n-1} without changing it on the $(n-2)$-skeleton K^{n-2} or on L. Observe that the restriction $f|_{\sigma_\beta^{n-1}}$ of the map f to each $(n-1)$-cell σ_β^{n-1} may be replaced by a map g agreeing with f on $K^{n-2} \cup L$ but differing on the interior of the cells σ_β^{n-1} not contained in L. Since f and g agree on the boundary $\partial\sigma_\beta^{n-1}$ of each such cell, they determine a map $S^{n-1} \longrightarrow X$, representing a homotopy class ("distinguishing element") $\mu(\sigma_\beta^{n-1}) \in \pi_{n-1}(X)$. Taking into account all such cells σ_β^{n-1} we obtain a relative $(n-1)$-cochain

$$\mu(f) \in C^{n-1}\left(K, L; \pi_{n-1}(X)\right).$$

Via the induced homomorphism $f_* : \pi_1(L) \longrightarrow \pi_1(X)$, the homotopy groups $\pi_j(X)$ acquire the structure of $\mathbb{Z}[\pi_1(L)]$-modules, whence the cochain complex $C^*\left(K, L; \pi_{n-1}(X)\right)$ acquires a $\mathbb{Z}[\pi_1(L)]$-module structure. It will be assumed in what follows that $\pi_1(L) \cong \pi_1(K) = \pi$ (via the homomorphism $f_* : \pi_1(L) \longrightarrow \pi_1(K)$) and that the map f is already defined on the $(n-2)$-skeleton of K. The following assertions are valid:

(i) *The obstruction cochain $\lambda(f)$ is a cocycle in the complex $C^*(K,L;\pi_{n-1}(X))$ considered as a complex of $\mathbb{Z}[\pi]$-modules: $\partial^*(\lambda(f)) = 0$.*

(ii) *Alteration of the map f (to the map g say) on that portion of the $(n-1)$-skeleton of K not contained in L, without changing it on the $(n-2)$-skeleton (so that $f = g$ on $K^{n-2} \cup L$) causes the obstruction cocycle $\lambda(f)$ to change by the addition of an arbitrary coboundary:*

$$\lambda(f) \mapsto \lambda(f) + \partial^*(\mu),$$

where $\mu \in C^{n-1}\left(K, L; \pi_{n-1}(X)\right)$, and ∂^* is the coboundary homomorphism: $\partial^* : C^{n-1}\left(K, L; \pi_{n-1}(X)\right) \longrightarrow C^n\left(K, L; \pi_{n-1}(X)\right)$.

We conclude that

For it to be possible to extend the given map f to the n-skeleton of K while keeping it fixed on the $(n-2)$-skeleton (but allowing f to be changed on that *portion of the $(n-1)$-skeleton not contained in L) it is nessesary and sufficient that the obstruction cohomology class $[\lambda(f)]$ in the complex of $\mathbb{Z}[\pi]$-modules $C^*(K, L; \pi_{n-1}(X))$ vanish:*

$$0 = [\lambda(f)] \in H^{n-1}(K, L; \pi_{n-1}(X)).$$

Allowing f to be changed on cells of dimension $\leq n-2$ (not contained in L) leads to more complicated rules for changing the obstruction cochain $\lambda(f)$. In particular, changes in $\lambda(f)$ arising from changes of f on the $(n-2)$-skeleton may be fully described in terms of the "cohomology operations" (introduced in the 1940s and early 1950s by Pontryagin, Steenrod, Postnikov, Boltyanskiĭ and Liao); we shall consider these cohomology operations below.

Observe that if the complex K is simply-connected (or the homomorphism $f_* : \pi_1(L) \longrightarrow \pi_1(X)$ is trivial) then the obstruction class $[\lambda(f)]$ lies in the ordinary relative cohomology group over the coefficient group $\pi_{n-1}(X)$.

The associated problem of extending a homotopy reduces to the above problem of extending a map, by taking $K_1 = K \times [a, b]$, and $L_1 \subset K_1$ to consist of the lid and the base:

$$K_1 = K \times [a, b], \quad L_1 = (K \times \{a\}) \cup (K \times \{b\}).$$

The prescribed map $f_1 : L_1 \longrightarrow X$ is defined by the two given maps $f_a : K \longrightarrow X$, $f_b : K \longrightarrow X$. Suppose we have already extended the homotopy $L \times I \longrightarrow X$ to the $(n-1)$-skeleton of K, $(n \geq 3)$, so that we have $F : \left(K^{n-1} \cup L\right) \times I \longrightarrow X$. Arguing as above one distils out the obstruction to extending the homotopy F to K^n as an element

$$\lambda \in C^{n+1}(K_1, L_1; \pi_n(X)), \quad n \geq 3.$$

It follows that the *obstruction to the extension of the homotopy F* to the n-skeleton of K is represented by the cohomology class of λ, or, equivalently, by the corresponding element

$$[\lambda] \in H^{n+1}(K_1, L_1; \pi_n(X)) \cong H^n(K; \pi_n(X)),$$

where the cohomology groups are defined with respect to the complexes of $\mathbb{Z}[\pi]$-modules

$$C^*(K; \pi_n(X)), \quad C^*(K_1, L_1; \pi_n(X)), \quad \pi = \pi_1(K).$$

The second of the above extension problems is treated analogously: Assuming that the cross-section ψ over L has been defined on the $(n-1)$-skeleton B^{n-1} of the base B, one seeks to extend it from the boundary $\partial\sigma_\alpha^n$ of each n-cell σ_α^n to the cell's interior. The known restriction map

$$\psi|_{\partial\sigma_\alpha^n} : \partial\sigma_\alpha^n \longrightarrow E, \quad p\psi = 1,$$

determines a map of the $(n-1)$-sphere

$$S^{n-1} \longrightarrow \partial\sigma_\alpha^n \xrightarrow{\psi} p^{-1}(\sigma_\alpha^n) \cong \sigma_\alpha^n \times F,$$

which determines in turn an element of the group $\pi_{n-1}(F)$.

Note that the action of $\pi_1(B)$ on the fibers (as determined by the "homotopy connexion" on the fiber bundle — see Chapter 2, §3), induces an action of $\pi_1(B)$ on the $\pi_j(F)$, turning them into $\mathbb{Z}[\pi_1(B)]$-modules. Hence there arises, much as before, an *obstruction cohomology class to the extension of the cross-section* ψ:

$$[\lambda] \in H^n(B; \pi_{n-1}(F)),$$

where the cohomology groups $H^j(B; \pi_{n-1}(F))$ are defined with respect to the complex of $\mathbb{Z}[\pi_1(B)]$-modules $C^*(B; \pi_{n-1}(F))$. By allowing ψ to change on the $(n-1)$-cells of B not contained in L, but keeping it fixed on $B^{n-2} \cup L$, the obstruction cocycle λ can, as before, be changed to any other cocycle from the same cohomology class $[\lambda]$.

The obstruction theory for homotopies between two cross-sections is similar: the obstruction class to extending the homotopy from B^{n-1} to B^n is an element of the group $H^n(B; \pi_{n-1}F)$.

The third problem is treated similarly to the first two, with one difference. Suppose we already have a bundle map $\Phi_{n-1} : p^{-1}(B^{n-1} \cup L) \longrightarrow E'$ defined on that portion of E above $B^{n-1} \cup L$, $L \subset B$, determining a map $\phi : B^{n-1} \longrightarrow B'^{n-1}$ between the $(n-1)$-skeletons of the bases. We might then define an obstruction to the extension of Φ_{n-1} to a bundle map Φ_n above B^n. However it is appropriate to consider this problem for *principal* bundles (i.e. where $F = G$, the structure group of the two fiber bundles), since the theory of such obstructions reduces to the case of the associated principal bundle.

Above each cell σ_α^n of the base B of the principal G-bundle we have, as usual,

$$p^{-1}(\sigma_\alpha^n) \cong \sigma_\alpha^n \times G.$$

The restriction of Φ_{n-1} to $\partial(\sigma_\alpha^n) \times \{1\}$ (covering $\phi|_{\partial(\sigma_\alpha^n)}$), where 1 is the identity element of G, determines a map from the $(n-1)$-sphere to E':

$$S^{n-1} \xrightarrow{\cong} \partial(\sigma_\alpha^n) \times \{1\} \xrightarrow{\Phi_{n-1}} E',$$

$$p' \circ \Phi_{n-1}|_{\partial(\sigma_\alpha^n)\times\{1\}} = \phi|_{\partial(\sigma_\alpha^n)}.$$

The map $S^{n-1} \longrightarrow E'$ so determined, defines an element of $\pi_{n-1}(E')$, and taking into consideration all cells σ_α^n on which ϕ is not yet defined, one obtains in this way an obstruction cochain λ with values in $\pi_{n-1}(E')$. The corresponding cochain complex may be given the structure of a $\mathbb{Z}[\pi_1(B)]$-module by pulling back the action of $\pi_1(B')$ via the induced homomorphism

$$\phi_* : \pi_1(B) \longrightarrow \pi_1(B').$$

If B is simply-connected then as earlier the obstruction class belongs to the ordinary cohomology group with coefficients from the group $\pi_{n-1}(E')$.

We infer the following result concerning our three questions:

If the homotopy groups of the space X of Question 1 (or of the fibre F in Question 2, or the total space E' in Question 3) are trivial in dimensions $\leq n$, then for CW-complexes K of dimension $\leq n$ (the base B in Questions 2 and 3) every map (cross-section in Question 2, bundle map in Question 3) defined on any skeleton of dimension $\leq n$ of K (or B) can be extended to the skeleton of next higher dimension, and likewise for homotopies (in the senses appropriate to the three questions) defined for any skeleton of dimension $\leq n-1$.

We obtain immediately as a particular case the consequence that for any fibre bundle (E, B, F, G, p) there is up to a homotopy just one bundle map

$$\Phi : (E, B, F, G, p) \longrightarrow (E', B', F, G, p')$$

to the corresponding universal bundle, since the total space E' is contractible. Writing $B' = B_G$ in standard notation for the base of the universal bundle, we deduce that every fibre bundle (E, B, F, G, p) determines a map $f : B \longrightarrow B_G$ uniquely to within a homotopy (which, conversely, induces the given fibre bundle from the universal bundle). Such a homotopy is of course a map $f : B \times I \longrightarrow B_G$, and so induces a fibre bundle over the cylinder $B \times I$. Since the cylinder $B \times I$, $I = [a, b]$, may be contracted onto its base $B \times \{a\}$ in such a way that the lid $B \times \{b\}$ goes "identically" onto $B \times \{a\}$ (in the sense that $(x, b) \longrightarrow (x, a)$ for all $x \in B$), we have that $H^j(B \times I, B \times \{a\}) = 0$ over any coefficient group. It follows that any fibre bundle over $B \times I$ may be mapped to the restricted bundle over the base $B \times \{a\}$ in such a way that the induced map of bases sends the lid $B \times \{b\}$ of $B \times I$ "identically" to $B \times \{a\}$. For the conclusion we wish to draw we require a preliminary definition.

Definition 6.2 Two fibre bundles over the same base B, with the same fibre F and the same structure group G, are said to be *equivalent* if there exists a bundle map between them:

$$\Phi : (E, B, F, G, p) \longrightarrow (E', B, F, G, p'),$$

inducing the identity map on the common base.

The foregoing discussion has as upshot the following important result.

Theorem 6.1 *The set of the equivalence classes of fibre bundles with the same base B, fibre F and group G, is in natural one-to-one correspondence with the set of homotopy classes of maps of the base B to the base B_G of the corresponding universal bundle.*

Example 1. From the exact homotopy sequence of the universal principal G-bundle $(F = G)$ (see Chapter 2, (4.7)) one obtains, on taking into account the contractibility of the total space, the following isomorphism:

$$\pi_n(B_G) \cong \pi_{n-1}(G) \quad (= \pi_{n-1}(F)), \quad n \geq 1. \tag{6.4}$$

This isomorphism and the preceding theorem together imply that the equivalence classes of G-bundles with base S^n are classified by the group $\pi_{n-1}(G)$. Recalling that for a topological group G the connected component G_0 containing the identity is a normal subgroup, so that $\pi_0(G) = G/G_0$ has a natural group structure, we infer that the action of $\pi_1(B_G)$ ($\cong \pi_0(G)$) on the groups $\pi_n(B_G)$ ($\cong \pi_{n-1}(G)$) is identifiable with the action of $\pi_0(G)$ on $\pi_{n-1}(G)$:

$$q(a) = qaq^{-1}, \quad q \in \pi_0(G), \quad a \in \pi_{n-1}(G). \tag{6.5}$$

Hence the free homotopy classes of maps $S^n \longrightarrow B_G$ (i.e. without distinguished base point) are in natural one-to-one correspondence with the orbits of $\pi_{n-1}(G)$ under the action of $\pi_0(G)$ given by (6.5), so that these orbits may be taken as determining the inequivalent G-bundles over S^n.

Geometrically speaking this reduction of the classification problem for G-bundles over S^n to the consideration of the group $\pi_{n-1}(G)$ is very natural: Consider the sphere S^n as the union $D^n_+ \cup D^n_-$ of the hemispheres with their boundaries (the equator) identified. Any G-bundle over S^n is trivial over each of the hemispheres, in view of their contractibility. Hence there are structural homeomorphisms:

$$\phi_1 : p^{-1}\left(D^n_+\right) \longrightarrow D^n_+ \times G,$$

$$\phi_2 : p^{-1}\left(D^n_-\right) \longrightarrow D^n_- \times G,$$

(where we are assuming the G-bundle to be principal), and the transition function

$$\lambda_{12} = \phi_1\phi_2^{-1} : S^{n-1} \times G \longrightarrow S^{n-1} \times G$$

on the equator has the form

$$\lambda_{12}(x, g) = (x, \widehat{\lambda}_{12}(x)(g)),$$

where $\widehat{\lambda}_{12} : S^{n-1} \longrightarrow G$ represents the appropriate homotopy class of $\pi_{n-1}(G)$. □

The above classification theorem was given its final form by Steenrod, it would seem, although investigations in this direction involving particular universal bundles of importance had been carried out earlier by Pontryagin, Ehresmann, Chern, and others. As far as obstructions are concerned, a version first appeared in the early 1930s in connexion with Van Kampen's construction of n-dimensional complexes K^n, $n \geq 2$, that are higher-dimensional analogues of the 1-complex figuring in Euler's problem of the three houses and three wells, in the sense that they are not embeddable without self-intersections in $\mathbb{R}^{2n}$. This idea was subsequently elaborated on by Whitney, in particular in relation to cross-sections of fiber bundles, and was actively exploited in concrete problems of homotopy theory by Pontryagin and Whitehead. Evidently the final formulation of the concept of an obstruction is due to Eilenberg in the case of maps, and Steenrod in the context of fiber bundles.

We note in passing that fiber bundles with contractible fiber possess cross-sections, which are, moreover, all homotopic to one another. This situation will arise frequently in the theory of smooth manifolds, to which Chapter 4 is devoted.

In concrete topological problems it is most often the so-called *first obstruction* that is used. This is the obstruction arising in attempting to extend maps (Question 1 above) or bundle cross-sections (Question 2) from the $(n-1)$-skeleton to the n-skeleton of the appropriate space, where n is the first dimension for which the homotopy group of the codomain X (in Question 1) is non-trivial:

$$\pi_n(X) \neq 0, \quad \pi_j(X) = 0 \ \text{ for } j < n,$$

or of the fiber F (in Question 2):

$$\pi_n(F) \neq 0, \quad \pi_j(F) = 0 \ \text{ for } j < n.$$

Example 1. Consider Hopf's problem of the classification up to homotopy, of the maps from an n-dimensional CW-complex K^n to the n-sphere S^n. Here there are no obstructions to map-extensions. There is just one obstruction to extending a homotopy, and this lies in the group $H^n(K^n;\mathbb{Z})$ since $\pi_n(S^n) \cong \mathbb{Z}$. It follows that the homotopy classes of maps $K^n \longrightarrow S^n$ are in natural one-to-one correspondence with the elements of the group $H^n(K^n;\mathbb{Z})$. Equivalently, the homotopy classes correspond one-to-one to the homomorphisms between the n-th integral homology groups of K^n and S^n, since

$$H^n(K^n;\mathbb{Z}) = \text{Hom}\ (H_n(K^n;\mathbb{Z}), \mathbb{Z}), \quad \mathbb{Z} \cong H_n(S^n;\mathbb{Z}). \qquad \square$$

Example 2. The previous example may be (mildly) generalized by replacing S^n by any $(n-1)$-connected space X (i.e. one for which $\pi_j(X) = 0$ for $j < n$). Thus one seeks to classify the homotopy classes of maps of an n-dimensional CW-complex K^n to X. As before, in view of the $(n-1)$-connectedness, only the first obstruction arises, and the analogous result is obtained: each map $K^n \longrightarrow X$ is to within a homotopy determined by a unique cohomology class in $H^n(K^n;\pi_n(X))$, or equivalently, a unique homomorphism between the nth integral homology groups of K^n and X, since

$$H^n(K^n;\pi_n(X)) = \text{Hom}\ (H_n(K^n;\mathbb{Z}), H_n(X;\mathbb{Z})), \quad H_n(X;\mathbb{Z}) \cong \pi_n(X). \quad \square$$

Example 3. Suppose X is an Eilenberg-MacLane space of type $K(\pi, n)$, i.e.

$$\pi_n(X) \cong \pi, \quad \pi_j(X) = 0 \quad \text{for } \ j \neq n.$$

The argument of the preceding two examples leads to the same conclusion, though here for any CW-complex K: The set of homotopy classes of maps $K \longrightarrow X$ is in natural one-to-one correspondence with $H^n(K;\pi)$, and so, provided $n > 1$, acquires the structure of an abelian group. Since $H_n(K(\pi,n);\mathbb{Z}) \cong \pi_n(K(\pi,n)) \cong \pi$ and

$$H^n(K(\pi,n);\pi) \cong \mathrm{Hom}\,(H_n(K(\pi,n);\mathbb{Z}),\pi) \cong \mathrm{Hom}\,(\pi,\pi),$$

there is a distinguished element u of $H^n(K(\pi,n);\pi)$, corresponding to the identity isomorphism $1:\pi \longrightarrow \pi$, called the *fundamental class* of $H^n(K(\pi,n);\pi)$. In terms of the fundamental class u, the above correspondence between homotopy classes $[f]$ of maps $f: K \longrightarrow K(\pi,n)$ and elements of $H^n(K;\pi)$ is given by

$$[f] \leftrightarrow f^*(u) \in H^n(K;\pi). \tag{6.6}$$

Note that in the case $n = 1$ the above is valid also for non-abelian groups π since, as observed earlier, $H^1(K;\pi)$ is defined (as a set) over such π. We conclude from (6.6) that:

The Eilenberg-MacLane space of type $K(\pi,n)$ is universal for the classification of n-dimensional cohomology classes over π (much as principal universal G-bundles are universal for the classification of equivalence classes of G-bundles).

As noted earlier, we know a series of examples of Eilenberg-MacLane complexes of type $K(\pi,1)$, and $K(\mathbb{Z},2) = \mathbb{C}P^\infty$ $(= \lim\limits_{n\to\infty} \mathbb{C}P^n)$, but of the remainder we know only that they exist. By virtue of (6.6) these complexes play an important technical role in topology. □

Definition 6.3 A *cohomology operation* is a "natural" function $\theta_X(z_1,\dots,z_k)$ of k variables $z_j \in H^{n_j}(X;G_j)$, taking its values in $H^*(X;G)$. Here "natural" means that the operation should be functorial in the sense that it commutes with homomorphisms induced by maps $f: Y \longrightarrow X$:

$$\text{if } \; f^*(z_j) = z_j', \quad \text{then} \quad \theta_Y(z_1',\dots,z_k') = f^*\theta_X(z_1,\dots,z_k). \tag{6.7}$$

The simplest example is afforded by the product $\theta = z_1 \cdots z_k$ of elements of any number k of factors, with all $G_j = G$. It is however important that there exist cohomology operations $\theta(z)$ of even one variable that are not definable merely in terms of the ring structure of $H^*(X;G)$, especially in the case $G = \mathbb{Z}/p$. The simplest such "non-trivial" example is afforded by the Bockstein homomorphism (see §4)

$$\beta : H^q(X;\mathbb{Z}/p) \longrightarrow H^{q+1}(X;\mathbb{Z}/p).$$

In view of the above-noted universal property of the Eilenberg-MacLane spaces in connexion with the classification of cohomology classes, the totality of cohomology operations of the above form is in natural one-to-one correspondence with the elements

$$\theta \in H^*\left(K(G_1, n_1) \times \cdots \times K(G_k, n_k); G\right).$$

This correspondence is easily defined in terms of the fundamental classes (see (6.6)):

$$u_j \leftrightarrow 1_{G_j}, \quad H^{n_j}(K(G_j, n_j); G_j) \cong \text{Hom}\,(G_j, G_j),$$

so that

$$\theta = \theta(u_1, \ldots, u_k). \tag{6.8}$$

For a given space X and elements $z_j \in H^{n_j}(X; G_j)$, $j = 1, \ldots, k$, there exist maps $f_j : X \longrightarrow K(G_j, n_j)$ such that $z_j = f^*(u_j)$, whence we obtain the map

$$f = f_1 \times \cdots \times f_k : X \longrightarrow K(G_1, n_1) \times \cdots \times K(G_k, n_k),$$

and the value $\theta_X(z_1, \ldots, z_k) \in H^*(X; G)$ of the operation θ is then given by

$$\theta_X(z_1, \ldots, z_k) = f^*\theta(u_1, \ldots, u_k).$$

(This was observed by Serre in the early 1950s.)

In addition to the operation β other particular non-trivial cohomology operations had been discovered earlier:

1) *Pontryagin squares and powers:*

$$\mathcal{P}_p : H^{2q}(X; \mathbb{Z}/p) \longrightarrow H^{2pq}(X; \mathbb{Z}/p^2),$$

where $\mathcal{P}_p(z) \equiv z^p \bmod p$, and p is prime.

2) *Steenrod squares and powers:*

a) Steenrod squares:

$$Sq^i : H^q(X; \mathbb{Z}/2) \longrightarrow H^{q+i}(X; \mathbb{Z}/2),$$

where, in particular, $Sq^0 = 1$ and Sq^1 is the Bockstein homomorphism for $p = 2$;

b) Steenrod powers:

$$St_p^i : H^q(X; \mathbb{Z}/p) \longrightarrow H^{q+i}(X; \mathbb{Z}/p),$$

where, in particular, $St_p^0 = 1$ and St_p^1 coincides with the Bockstein operator β.

The Steenrod squares are homomorphisms and have the following further basic properties:

(i) $Sq^0 = 1$, $Sq^1 = \beta$, $Sq^n(z) = z^2$, $z \in H^n(X; \mathbb{Z}/2)$.

(ii) The Steenrod squares are *stable*, i.e. they commute with the suspension isomorphisms

$$E : H^q(X;\mathbb{Z}/2) \longrightarrow H^{q+1}(\Sigma X;\mathbb{Z}/2).$$

(iii) $Sq^j(z_1 \cdot z_2) = \sum_{k+l=j} Sq^k(z_1) \cdot Sq^l(z_2)$.

(iv) $Sq^j = 0$ for $j < 0$; $Sq^n(z) = 0$ if z has dimension $< n$.

(v) The Steenrod squares are additive (in contrast with the Pontryagin powers):

$$Sq^j(z_1 + z_2) = Sq^j(z_1) + Sq^j(z_2).$$

The Steenrod powers have analogous properties:

(i) $St_p^0 = 1$, $St_p^1 = \beta$, $St_p^j = 0$ if $j < 0$ or $j \not\equiv 0, 1 \bmod (2p-2)$.

(ii) The Steenrod powers are stable, i.e. they commute with the suspension isomorphisms.

(iii) Writing $\mathcal{P}^j$ for $St_p^{2j(p-1)}$, one has

$$St_p^{2j(p-1)+1} = \beta\mathcal{P}^j, \tag{6.9}$$

and also the following formulae:

$$\begin{aligned} \mathcal{P}^j(z_1 \cdot z_2) &= \sum_{i+k=j} \mathcal{P}^i(z_1) \cdot \mathcal{P}^k(z_2), \\ \beta(z_1 \cdot z_2) &= \beta(z_1) \cdot z_2 + (-1)^m z_1 \cdot \beta(z_2), \end{aligned} \tag{6.10}$$

where $m = \dim z_1$.

(iv) $\mathcal{P}^n(z) = z^p$ for $z \in H^{2n}(X;\mathbb{Z}/p)$, and for $j > n$, $\mathcal{P}^j(z) = 0$.

(v) The Steenrod powers are additive:

$$St_p^j(z_1 + z_2) = St_p^j(z_1) + St_p^j(z_2).$$

By results of Serre (in the case $p = 2$) and Cartan ($p > 2$) from the early 1950s, the $\mathbb{Z}/p$-algebra of all stable operations (the *Steenrod algebra* $\mathcal{A}_p$) is generated multiplicatively by the Steenrod operations (the Sq^i when $p = 2$, and the St_p^i when p is an odd prime); it follows that all such cohomology operations are additive.

Let $\mathcal{A}_p^n$ denote the $\mathbb{Z}/p$-subspace of the Steenrod algebra $\mathcal{A}_p$ consisting of the operations of degree n, i.e. $a \in \mathcal{A}_p^n$ if $a : H^q \longrightarrow H^{q+n}$. In the case $p = 2$, a cohomology operation $a \in \mathcal{A}_2^n$ is the null operation precisely if the element $a(u)$ vanishes for the appropriate fundamental class u:

$$u = u_1 \cdots u_N \in H^N\left(\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_N^\infty;\mathbb{Z}/2\right), \quad u_j \in H^1\left(\mathbb{R}P_j^\infty;\mathbb{Z}/2\right),$$

for some $N > n$. (Recall that $\mathbb{R}P^\infty = K(\mathbb{Z}/2;1)$, and that the ring

$$H^*\left(\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_N^\infty; \mathbb{Z}/2\right)$$

is polynomial.) From the properties of the Steenrod squares listed above, it follows that the image $\mathcal{A}_2(u) \subset H^*\left(\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_N^\infty; \mathbb{Z}/2\right)$ is a $\mathbb{Z}/2$-vector space spanned by those symmetric polynomials u_ω in the variables $u_1, \ldots u_N$, of the form:

$$u_\omega = \left(\sum_{i_1 \neq i_2 \neq \cdots \neq i_s} u_{i_1}^{\omega_1} \cdots u_{i_s}^{\omega_s} \right) u, \qquad \sum_j \omega_j < n,$$

$$\omega_j = 2^{h_j} - 1, \qquad \omega = (\omega_1, \ldots, \omega_k), \qquad \omega_j \geq 0.$$

We denote by Sq^ω the operation defined by

$$Sq^\omega(u_1 \cdots u_s) = u_\omega, \qquad \deg Sq^\omega = \sum_j \omega_j. \tag{6.11}$$

Note that

$$Sq^i = Sq^\omega, \quad \text{with} \quad \omega = (\underbrace{1, \ldots, 1}_{i \text{ times}}).$$

One has the following product formula:

$$Sq^\omega(z_1 z_2) = \sum_{(\omega_1, \omega_2) = \omega} Sq^{\omega_1}(z_1) Sq^{\omega_2}(z_2).$$

For $p > 2$ we consider copies X_j of the infinite-dimensional lens space $K(\mathbb{Z}/p;1) = S^\infty/\mathbb{Z}/p$. Here the cohomology ring $H^*(X_j;\mathbb{Z}/p)$ has generators

$$v_j \in H^1(X_j;\mathbb{Z}/p), \quad u_j \in H^2(X_j;\mathbb{Z}/p), \quad u_j = \beta v_j.$$

In this case an operation $a \in \mathcal{A}_p^n$, $a : H^q \longrightarrow H^{q+n}$, is null precisely if $a(v) = 0$, where

$$v = v_1 \cdots v_{N_1} u_{N_1+1} \cdots u_{N_1+N_2} \in H^{N_1+N_2}\left(X_1 \times \cdots \times X_{N_1+N_2}; \mathbb{Z}/p\right),$$

where $N_1,\ N_2 > n$. The image $\mathcal{A}_p(v)$ is spanned by the polynomials of the form:

$$u_{\omega',\omega''} = u \cdot \sum_{\substack{i_1 < \cdots < i_s \leq N_1 \\ N_2 \geq j_1 > \cdots > j_q > N_1}} \left(u_{i_1}^{\omega_1'+1} \cdots u_{i_s}^{\omega_s'+1} u_{j_1}^{\omega_1''+1} \cdots u_{j_q}^{\omega_q''+1} \right) / v_{i_1} \cdots v_{i_s},$$

$$\omega_1' < \cdots < \omega_s',\ 1 + \omega_i' = p^{h_i},\ \omega_1'' < \cdots < \omega_q'',\ 1 + \omega_j'' = p^{l_j},\ h_i \geq 0,\ l_j \geq 0,$$

$$\deg\ a = \sum_i (2\omega_i' + 1) + \sum_j (2\omega_j'') = 2\left(\sum_i \omega_i' + \sum_j \omega_j'' \right) + s,$$

$$a : H^q(X;\mathbb{Z}/p) \longrightarrow H^{q+\deg a}(X;\mathbb{Z}/p), \quad a = S_{(\omega',\omega'')}.$$

The number $s \geq 0$ appearing in this formula is called the *τ-Cartan type* of the corresponding operation a, denoted more usually by $s(a)$ or $\tau(a)$. The type s and the degree deg a ($s \leq \deg a$) determine a bigrading of the algebra $\mathcal{A}_p$ (for $p \geq 2$):

$$a \mapsto [s(a), \deg\ a - s(a)]\,.$$

Of special interest are the operations $a = S_{k,\omega}$, where of course we have

$$k + 2\sum \omega_j < N_1 + 2N_2.$$

The operations $S_{k,0} = \theta_k$ say, are called *Milnor operations*. They define an exterior algebra:

$$\theta_k \theta_j = -\theta_j \theta_k, \quad s(\theta_k) = 1,$$

and have the further property

$$\theta_k(z_1 \cdot z_2) = \theta_k(z_1)z_2 \pm z_1\theta_k(z_2),$$

a special case of the general formula

$$S_{(\omega',\omega'')}(z_1 \cdot z_2) = \sum_{\substack{(\omega_1',\omega_2')=\omega' \\ (\omega_1'',\omega_2'')=\omega''}} (\pm 1) S_{(\omega_1',\omega_1'')}(z_1) S_{(\omega_2',\omega_2'')}(z_2).$$

It turns out that the algebra of all (not necessarily stable) $\mathbb{Z}/p$-cohomology operations is generated via composition by the Steenrod operations and the Bockstein operator β (together with the product operation in the cohomology ring), and that all relations among these operations follows from the properties listed above. Over other coefficient groups one has for instance the Pontryagin powers; a classification of all cohomology operations was given by Cartan in the mid-1950s. In the early years of that decade Serre had shown that cohomology operations over a field k of characteristic 0 (for instance $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) are all trivial, i.e. reduce essentially to the cohomological product operation. There are in such cases no stable operations aside from multiplication by a scalar: $z \mapsto \lambda z$, $\lambda \in k$.

As a generalization of single-valued, everywhere-defined cohomology operations one has multi-valued, partially defined operations, as in the following two examples.

Example 1. Consider the Bockstein homomorphism

$$\beta : H^k(X;\mathbb{Z}/p) \longrightarrow H^{k+1}(X;\mathbb{Z}/p).$$

As in §4 above, β is defined initially on integral cochains $\widetilde{z} \in C^k(X;\mathbb{Z})$ which reduce modulo p to a cocycle z over $\mathbb{Z}/p$, and then $\beta(z)$ is defined as $\frac{1}{p}\partial^*\widetilde{z}$ reduced mod p. If z is in Ker β, i.e. if the cocycle $\frac{1}{p}\partial^*\widetilde{z}$ (mod p) is cohomologous to 0, then $\frac{1}{p}\partial^*\widetilde{z} = pu + \partial^* y$, and we can define

$$\beta_2(z) = \frac{1}{p^2}\partial^*\widetilde{z} \pmod p,$$

provided we work modulo Im β. Proceeding inductively we similarly define on Ker β_{k-1}, $k \geq 2$, $\beta_1 = \beta$, an operation β_k with values determined modulo Im β_{k-1}, by

$$\begin{aligned}
&\beta_k(z) = \frac{1}{p^k}\partial^*\widetilde{z} \pmod p, \quad z \equiv \widetilde{z} \pmod p, \quad \widetilde{z} \in C^q(X;\mathbb{Z}),\\
&\beta_k : \text{Ker } \beta_{k-1} \longrightarrow H^{q+1}(X;\mathbb{Z}/p)/\text{Im } \beta_{k-1}, \qquad (6.12)\\
&\text{Ker } \beta_{k-1} \subset H^q(X;\mathbb{Z}/p).
\end{aligned}$$

If $\beta_k(z) = 0$ for all $k \geq 1$, then the class $z \in H^q(X;\mathbb{Z}/p)$ is obtained by reducing an integral class. Classes $\beta_k(z)$ always arise from integral classes of order p^k. □

Example 2. Let k be a ring or field and let $z_j \in H^{n_j}(X;k)$, $i = 1,2,3$, satisfy $z_1z_2 = 0$, $z_2z_3 = 0$. For such triples there is an operation called the *Massey bracket*

$$\langle z_1, z_2, z_3\rangle = \theta(z_1, z_2, z_3) \subset H^{n_1+n_2+n_3-1}(X;k), \qquad (6.13)$$

defined only up to the addition of an element of the form

$$w_1z_3 \pm z_1w_2, \quad w_1 \in H^{n_1+n_2-1}(X;k), \quad w_2 \in H^{n_2+n_3-1}(X;k).$$

The definition of the Massey bracket is very simple: on cochains in the $C^{n_j}(X;k)$ it is given by

$$\langle z_1, z_2, z_3\rangle = \partial^{-1}(z_1z_2)z_3 \pm z_1\partial^{-1}(z_2z_3).$$

In the case that k is a field of characteristic zero, the Massey brackets and their analogues constitute generators, with respect to composition, for all multivalued, partial, natural cohomology operations. □

We now return to our discussion of Steenrod squares. From the classification given above (see, in particular, (6.11)) we see that the Steenrod algebra $\mathcal{A}_2$ is generated multiplicatively by the Steenrod squares of the form Sq^{2^i}, $i \geq 0$. This fact has the following nice application.

Example. Let K be a CW-complex such that

$$\begin{aligned}
&H^j(K;\mathbb{Z}/2) \cong \mathbb{Z}/2 \qquad \text{for } j = 0, n, 2n,\\
&H^j(K;\mathbb{Z}/2) = 0 \qquad \text{for } j \neq 0, n, 2n.
\end{aligned}$$

Denoting by u_j the generator of $H^j(K;\mathbb{Z}/2)$ for $j = n, 2n$, we also assume that $u_n^2 = u_{2n}$. Examples of such K are the real, complex, quaternionic and

Cayley projective planes. It follows that $Sq^n u_n = u_{2n}$ (by property (i) of the Steenrod squares). If n is not a power of 2, this is not possible, since then Sq^n would be decomposable as a composite of the Sq^j with $j < n$, and for such j (with the exception of $j = 0$) we have $Sq^j u_n = 0$ in view of the assumption that $H^{n+j}(K; \mathbb{Z}/2) = 0$. Hence we must have $n = 2^s$. This was shown by Hopf, without using cohomology operations, in the 1940s, and by Steenrod in the late 1940s.

It was further (shown by Adams in the late 1950s) that the Steenrod squares Sq^n with $n = 2^i$, $i > 3$, decompose non-trivially as composites of partial operations. In particular, it follows that $Sq^n u_n = u_{2n}$ only if $n = 1, 2, 4, 8$. This result has a series of fundamental consequences, in particular the following ones:

1. *The only real division algebras are* $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^4$, $\mathbb{R}^8$.
2. *The stable tangent bundle of the sphere* S^{n-1} *is trivial only if* $n = 1, 2, 4, 8$.
3. *There exist elements with Hopf invariant 1 in the groups* $\pi_{2n-1}(S^n)$, *only if* $n = 2, 4, 8$.[6]
4. *There is a fibration* $S^{2n-1} \longrightarrow S^n$ *of Hopf type (i.e. where the base and total spaces are spheres) only if* $n = 2, 4, 8$. □

The essential idea behind the construction of the Steenrod squares for CW-complexes K is as follows. Consider the diagonal

$$\Delta(K) \subset K \times K, \quad \Delta(K) = \{(x, x)\},$$

and denote by $\overline{\Delta}$ the subcomplex of those cells of $K \times K$ containing Δ. (Note that Δ itself is not a subcomplex in general.) Let $\Delta_1 : K \longrightarrow \overline{\Delta} \subset K \times K$ be a cellular approximation of the diagonal, i.e. a cellular map close to the diagonal map $K \longrightarrow \Delta$, with the image of each cell σ^k of K approximated in $\overline{\Delta}(\sigma^k \times \sigma^k)$. This yields a product operation on the cell cochains in

$$C^*(K \times K) \cong C^*(K) \otimes C^*(K),$$

defined by

$$z_i z_j = \Delta_1^*(z_i \otimes z_j).$$

This operation turns $H^*(K)$ into a ring; it is the cohomological product defined earlier for simplicial complexes — see §2.

Consider the involution $\sigma : K \times K \longrightarrow K \times K$ given by

$$\sigma(x, y) = (y, x), \quad \sigma^2 = 1.$$

Although $\sigma\Delta = \Delta$, the map σ does not (except in trivial cases) preserve Δ_1: $\sigma\Delta_1 \neq \Delta_1$. This asymmetry is the source of the Steenrod squares. Let w be

[6] For the definition of the "Hopf invariant", see Chapter 4, §3.

any cocycle in $C^*(\Delta_1(K))$, and define cocycles $w_0, w_1, \ldots$, in terms of w as follows (over $\mathbb{Z}/2$ for simplicity):

$$w_0 = w, \qquad \partial^* w_1 = w_0 + \sigma^* w_0, \qquad \partial^*(w_1 + \sigma^* w_1) = 0,$$

$$\partial^* w_2 = w_1 + \sigma^* w_1, \qquad \partial^*(w_2 + \sigma^* w_2) = 0, \quad \ldots,$$

$$\partial^* w_i = w_{i-1} + \sigma^* w_{i-1}, \qquad \partial^*(w_i + \sigma^* w_i) = 0, \quad \ldots,$$

with the support of every w_i in $\Delta_1(K)$. For cocycles $z \in C^n(K; \mathbb{Z}/2)$ set

$$(z^2)_i = \Delta_1^*(w_i) \in C^{2n-i}(K; \mathbb{Z}/2),$$

with w_0 taken to be $\Delta_1^*(z \otimes z)$, and then define the $(n-i)$th Steenrod square of z as the cohomology class of the cocycle $(z^2)_i \in H^{2n-i}(K; \mathbb{Z}/2)$:

$$Sq^{n-i}(z) = (z^2)_i.$$

It is immediate from this definition that $Sq^{n+i}(z) = 0$ for $i > 0$, and $Sq^n(z) = z^2$. The remaining properties are more difficult to establish; in particular it is not clear from the above construction how the axiomatically simple property $Sq^0(z) = z$, i.e. $Sq^0 = 1$, follows. On the other hand this property follows easily from Serre's classification of the stable $\mathbb{Z}/2$-cohomology operations mentioned above (whereby the group of stable operations of degree j is isomorphic to $H^{j+n}(K(\mathbb{Z}/2, n); \mathbb{Z}/2)$). For, the cohomology of the Eilenberg-MacLane space $K(\mathbb{Z}/2, n)$ is trivial in dimensions $< n$, and $H^n(K(\mathbb{Z}/2, n); \mathbb{Z}/2) \cong \mathbb{Z}/2$, so that there is only one cohomology operation of degree zero, namely multiplication by the scalar $1 \in \mathbb{Z}/2$.

The above construction of the Steenrod squares has consequences in the purely algebraic theory of graded *Hopf algebras*. Let A be a graded algebra over $\mathbb{Z}/p$, with an identity element:

$$A = \bigoplus_{n \geq 0} A^n, \quad A^0 = \mathbb{Z}/p,$$

with multiplication

$$\phi : A \otimes A \longrightarrow A, \quad \phi(a \otimes b) = a \cdot b, \quad A^n \cdot A^m \subset A^{n+m}.$$

We have the *augmentation homomorphism* $\varepsilon : A \longrightarrow \mathbb{Z}/p = A^0$, where $\operatorname{Ker} \varepsilon = \bigoplus_{n \geq 1} A^n$. The tensor product (over $\mathbb{Z}/p$)

$$A \otimes_{\mathbb{Z}/p} A, \quad (A \otimes A)^j = \bigoplus_{q+m=j} A^q \otimes A^m,$$

is an algebra, where the product is defined by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

The algebra $A \otimes A$ has a "twisting" homomorphism

$$\sigma : A \otimes A \longrightarrow A \otimes A, \quad \sigma(a \otimes b) = (-1)^{\deg a \deg b} b \otimes a.$$

Such an algebra A is called a *Hopf algebra* if there is a *diagonal homomorphism*, i.e. a degree-zero algebra homomorphism of the form

$$\Delta^* : A \longrightarrow A \otimes A,$$

$$\Delta^*(a) = a \otimes 1 + 1 \otimes a + \sum a'_j \otimes a''_j, \quad \deg a'_j > 0, \quad \deg a''_j > 0.$$

Here $a \otimes b = (a \otimes 1) \cdot (1 \otimes b)$ and $A \otimes 1$ and $1 \otimes A$ commute elementwise to within multiplication by ± 1. A Hopf algebra A is said to be *symmetric* if the diagonal homomorphism Δ^* commutes (up to a sign $\pm$ depending on deg a) with the involution operator $\sigma : A \otimes A \longrightarrow A \otimes A$ defined by $\sigma(a \otimes b) = (b \otimes a)$:

$$\Delta^* \sigma = \pm \sigma \Delta^*.$$

It was observed by Milnor in the late 1950s that the Steenrod algebra $\mathcal{A}_p$ can be naturally endowed with the structure of a Hopf algebra by means of the formula for the action on a product of cocycles (see (6.10)). For $p = 2$, for instance, one sets

$$\Delta^*(Sq^i) = Sq^i \otimes 1 + 1 \otimes Sq^i + \sum_{j+k=i} Sq^j \otimes Sq^k.$$

For any symmetric Hopf algebra A over $\mathbb{Z}/p$, analogues of the Steenrod operations can be defined on the corresponding cohomology algebra $\mathrm{Ext}_A^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ (defined below):

$$St_p^i : \mathrm{Ext}_A^{m,n}(\mathbb{Z}/p, \mathbb{Z}/p) \longrightarrow \mathrm{Ext}_A^{m+i,pn}(\mathbb{Z}/p, \mathbb{Z}/p). \tag{6.14}$$

For example, the operation St_p^0 coincides with a nontrivial operator α^* determined by the Frobenius-Adams operator[7] $\alpha : x \mapsto x^p$ on the algebra A. The operations St_p^i have more-or-less the same properties as the original Steenrod operations, while $St_p^i \equiv 0$ only for $i \not\equiv 0, 1 \bmod (p-1)$, $p > 2$. Thus here there are more nonzero operations St_p^i, than was the case for the original Steenrod operations.

These operations were discovered in the process of developing the algebraic techniques needed for calculating stable homotopy groups using the Adams spectral sequence (Novikov in the late 1950s, Liulevicius in the early

[7] Well-defined when A is a Hopf algebra over $\mathbb{Z}/p$.

1960s).[8,9] In the Adams spectral sequence for spheres, the structure of the Steenrod agebra $\mathcal{A}_p$ and of its cohomology algebra

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p) \tag{6.15}$$

play a crucial role. In accordance with Adams' method, the computation of the algebra (6.15) uses an "approximation" of the Steenrod algebra $\mathcal{A}_p$ by finitely-generated symmetric Hopf algebras; here the above analogues of the Steenrod operations turn out to be very useful. In the case of the Steenrod algebra $\mathcal{A}_p$ with $p > 2$, one must take into account the bigrading (Cartan type) of the this algebra as well as its approximating finite Hopf algebras.

In applying the Adams spectral sequence to the computation of the stable homotopy groups of any CW-complex K, one considers the cohomology algebra $H^*(K; \mathbb{Z}/p) = M$ as a graded left module over the Steenrod algebra. Consider more generally any left module M over a graded Hopf algebra A (over $\mathbb{Z}/p$). The *cohomology algebra*

$$\mathrm{Ext}^{*,*}_{A}(M, \mathbb{Z}/p) \tag{6.16}$$

of the module M may be defined as follows (after Cartan-Eilenberg in the mid-1950s). We denote by A^+ the kernel $\bigoplus_{n\geq 1} A^n$, of the augmentation homomorphism ε, and consider the complex

$$M \leftarrow A^+ \otimes M \leftarrow A^+ \otimes A^+ \otimes M \leftarrow \cdots \leftarrow A^+ \otimes A^+ \otimes \cdots \otimes A^+ \otimes M \leftarrow \cdots, \tag{6.17}$$

where an "n-chain" has the form

$$\sigma^n = (a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes x) \in A^+ \otimes A^+ \otimes \cdots \otimes A^+ \otimes M,$$

$$a_j \in A^{n_j}, \quad n_j > 0, \quad x \in M.$$

The n-chain (or "simplex") σ^n is taken to have dimension n, and is graded also by its degree, defined by

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[8] These operations were introduced by Novikov in order to compute the cohomology of the Steenrod algebra $\mathcal{A}_p$ and modules over $\mathcal{A}_p$. It emerged in particular that for any prime p there exists a nontrivial torsion element x of the ring of stable homotopy of spheres such that $x^p \neq 0$ (1959). This result was proved also by Toda (1960) using a different approach. Nishida has proved the following theorem: *For any nontrivial torsion element x of the homotopy ring π_*^s of spheres there exists n such that $x^n = 0$.* Hence the ring π_*^s is locally nilpotent although not nilpotent in the strong sense. In the late 1960s P. May published an article establishing (in slightly more general categorical language) all foundational results needed for these operations.

[9] *Translator's Note:* A general Nilpotence Theorem has been proved for any spectra (Hopkins, Devinatz, and Smith, in the mid-1980s). The simplest version of the Nilpotence Theorem states that if a "ring spectrum" X (or "stable space" with a multiplicative structure, such as the above-mentioned stable Thom spaces) has torsion-free cohomology $H^*(X; Z)$, then every element of Tor $\pi_*^s(X)$ is nilpotent. (For details and further developments, see *D. Ravenel, Nilpotence and Periodicity in Stable Homotopy Theory, Annals of Mathematics Studies no. 128, Princeton, 1992.*)

$$\deg \sigma^n = \sum_{j=1}^{n} n_j + n.$$

The boundary homomorphism is defined by

$$\partial \sigma^n = \sum_{j=1}^{n-1} (a_1 \otimes \ldots \otimes a_j \cdot a_{j+1} \otimes \ldots \otimes a_n \otimes x) + (-1)^n (a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n \cdot x),$$

and preserves the degree of chains. The homology of the complex (6.17) with respect to the above differential furnishes the desired cohomology algebra (6.16). The particular case $M = \mathbb{Z}/p = H^*(S^0)$ yields the cohomology algebra

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathbb{Z}/p, \mathbb{Z}/p)$$

mentioned above. If the module M possesses a diagonal homomorphism

$$\Delta^* : M \longrightarrow M \otimes M$$

compatible with the diagonal of $\mathcal{A}_p$, then the cohomology algebra $\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(M, \mathbb{Z}/p)$ acquires the structure of a bigraded $\mathbb{Z}/p$-algebra.

One can now construct the *Adams spectral sequence* of groups (actually $\mathbb{Z}/p$-vector spaces) $E_m = \sum_{q,l} E_m^{q,l}$, and differentials (homomorphisms)

$$d_m : E_m^{q,l} \longrightarrow E_m^{q+m,l+m-1}, \quad d_m^2 = 0,$$

such that

(i) $E_2^{q,l} \cong Ext^{q,l}_{\mathcal{A}_p}(M, \mathbb{Z}/p)$, $E_{m+1} = H(E_m, d_m)$;

(ii) $\sum_{q-l=t} E_\infty^{q,l} \cong G\pi_t^s(K)$, where $G\pi$ is the "adjoined group".

Here $\pi_t^s(K) = \pi_{n+t}(\Sigma^n K)$, where $\pi_j(\Sigma^n K) = 0$ for $j < n$, $0 \le t \le n-1$. The *adjoined group* $G\pi$ (relative to a filtration $\pi = \pi^{(0)} \supset \pi^{(1)} \supset \ldots$) is, as for rings etc., the direct sum of the successive factors:

$$G\pi = \bigoplus_{j \ge 0} \pi^{(j)}/\pi^{(j+1)} = \bigoplus_{j \ge 0} G_j \pi.$$

The intersection of all members of the filtration arising from the Adams spectral sequence (for a given prime p) is precisely the set of elements of the stable homotopy groups of K of finite order relatively prime to p.

In the case $K = S^n$ (and in several other important cases) the Adams spectral sequence has as its terms bigraded $\mathbb{Z}/p$-algebras E_m with differentials d_m satisfying $d_m(uv) = d_m(u)v \pm u d_m(v)$ for bigraded elements $u, v \in E_m$, and the algebra E_∞ is adjoined to the ring of stable homotopy groups of the sphere S^n:

$$\pi_*^s = \bigoplus_{k \geq 0} \pi_k^s, \quad \pi_k^s = \pi_{n+k}(S^n), \quad n > k+1,$$

with the multiplication induced by the composition $S^{n+i+j} \xrightarrow{a} S^{n+i} \xrightarrow{b} S^n$, which, via the realization of the stable homotopy groups of spheres in terms of the "cobordism ring of framed manifolds" (see Chapter 4, §3), corresponds to the multiplication induced by the direct product of manifolds.

In the case of the sphere $K = S^n$, the second term of the Adams spectral sequence is the algebra

$$E_2^{*,*} = \operatorname{Ext}_{\mathcal{A}_p}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p),$$

where $\mathbb{Z}/p$ has the $\mathcal{A}_p$-module structure determined by the identical action of $1 \in \mathcal{A}_p$, and the annihilating action of the elements of $\mathcal{A}_p$ of positive degree. Here there is an element

$$h_0 \in E_2^{1,1} = \operatorname{Ext}_{\mathcal{A}_p}^{1,1}(\mathbb{Z}/p, \mathbb{Z}/p),$$

with the property that multiplication by h_0 is adjoint to multiplication by p.

Example 1. From the definition of the cohomology of an $\mathcal{A}_p$-module M (see (6.17)) it follows that

$$E_2^{0,k} = \operatorname{Ext}_{\mathcal{A}_p}^{0,k}(M, \mathbb{Z}/p) = \operatorname{Hom}_{\mathcal{A}_p}^k(M, \mathbb{Z}/p), \tag{6.18}$$

where only those homomorphisms are admitted that commute with the action of $\mathcal{A}_p$. The action of $\mathcal{A}_p$ on $\mathbb{Z}/p$ is given by

$$a(u) = 0 \quad \text{if} \quad \deg a > 0,$$

$$\lambda(u) = \lambda u \quad \text{if} \quad a = \lambda \in \mathbb{Z}/p,$$

where $u \in \mathbb{Z}/p$. The superscript k in $\operatorname{Hom}_{\mathcal{A}_p}^k(M, \mathbb{Z}/p)$ signifies that we only admit $\mathbb{Z}/p$-linear forms that are zero off M^k (and of course commute with the action of $\mathcal{A}_p$). When $M = H^*(K; \mathbb{Z}/p)$ the zero-line $E_2^{0,*}$ has a clear topological meaning: any map $S^k \xrightarrow{f} K$ induces a homomorphism of $\mathcal{A}_p$-modules $M = H^*(K; \mathbb{Z}/p) \xrightarrow{f^*} \mathbb{Z}/p$, and, consequently determines an element $f^* \in \operatorname{Hom}_{\mathcal{A}_p}^*(M, \mathbb{Z}/p)$; the non-vanishing of this element guarantees that the map f is essential (i.e. not homotopic to zero). However, an element $x \in \operatorname{Hom}_{\mathcal{A}_p}^*(M, \mathbb{Z}/p)$ is induced by some map $S^n \longrightarrow K$ only if all differentials in the Adams spectral sequence act trivially on x:

$$d_m(x) \equiv 0, \quad m \geq 2.$$

Note that there are many essential maps $S^k \xrightarrow{f} K$ inducing the trivial homomorphism $H^*(K;\mathbb{Z}/p) \xrightarrow{f^*} \mathbb{Z}/p$; for instance none of the nontrivial elements of positive degree of the stable homotopy groups of spheres can be "detected" by the corresponding homomorphisms of the cohomology ring. □

Example 2. As noted above, for the sphere S^n we have $M = \mathbb{Z}/p$. In this case $\mathrm{Ext}^{0,0}_{\mathcal{A}_p} = \mathbb{Z}/p$ and $\mathrm{Ext}^{0,k}_{\mathcal{A}_p} = 0$ for $k \neq 0$. It is not difficult to identify the groups $\mathrm{Ext}^{1,k}_{\mathcal{A}_p}(\mathbb{Z}/p,\mathbb{Z}/p)$. We sketch the argument, assuming for simplicity that $p = 2$. Since the algebra $\mathcal{A}_2$ is generated by the elements Sq^{2^i}, there are non-zero elements

$$h_i \in \mathrm{Ext}^{1,2^i}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$$

dual to the Sq^{2^i}. These in fact account for all non-zero elements of the groups $\mathrm{Ext}^{1,k}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$. Note that $d_q h_i = 0$ for $q \geq 2$ if and only if there exists an element with Hopf invariant 1 in the appropriate stable homotopy group of spheres (see Chapter 4, §3). Adams proved in the late 1950s that

$$d_2(h_i) = h_0 h_{i-1}^2,$$

where $h_0 h_{i-1}^2 \neq 0$ for $i \geq 4$. It is not difficult to establish Adams' formula in the case $i = 4$: $d_2(h_4) = h_0 h_3^2$. Since $d_q(h_4) = 0$ for $q \geq 3$ for "dimensional reasons" (there is no nontrivial element to be a target), the element h_3 represents a non-zero element $\overline{h}_3 = \sigma$ of Hopf invariant 1 in the group $\pi_7^s = \pi_{n+7}(S^n)$, $n \geq 8$. By skew-symmetry we have $\overline{h}_3\overline{h}_3 = -\overline{h}_3\overline{h}_3$, whence $2\overline{h}_3^2 = 0$. Since multiplication by h_0 is adjoint to multiplication by 2, the element $h_0 h_3^2$ must be trivial in the infinite term E_∞ of the Adams spectral sequence, whence

$$h_0 h_3^2 = dz \quad \text{for some} \quad z \in \mathrm{Ext}^{1,16}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2),$$

and since h_4 is the only nonzero element, we must have $z = h_4$, which establishes the formula. We note that the 2-line $\mathrm{Ext}^{2,*}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$ is generated by the products

$$h_i h_j \in \mathrm{Ext}^{1,2^i+2^j}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2), \tag{6.19}$$

where $h_i h_{i+1} = 0$ and all other products are nontrivial. The products h_j^2 are of particular importance in manifold theory, to be considered in the next chapter.[10] □

[10] *Translator's Note:* The elements h_j^2 (which are related to the "Arf invariant problem") are the only double products $h_i h_j$ about which it is not known whether they support nontrivial differentials. It is known that the elements h_j^2 are cycles of all differentials for $j = 1, 2, 3, 4, 5$ (M. Barrat, J. D. S. Jones and M. Mahowald, mid-1970s). The elements $h_1 h_j$ for $j \geq 4$ are nontrivial, and of order at least 4 for $j \geq 5$ (M. Mahowald, 1978). For a general account of the results of computations in the Adams spectral sequence for spheres see *D. Ravenel, Complex Cobordism and Stable Homotopy of Spheres, Academic Press, 1986.*

Applications of the Adams spectral sequence to the computation of bordism and cobordism rings of smooth manifolds were made around 1960 by Milnor and Novikov. The Adams spectral sequence admits a natural generalization in terms of extraordinary (generalized) cohomology theory (Novikov, in the late 1960s) revealing a deep connexion with stable homotopy theory, while the Cartan-Serre method of computing stable (and unstable) homotopy groups does not admit such a generalization.

Over the rationals the Adams spectral sequence converges to the stable rational homotopy groups $\pi_j^s(K) \otimes \mathbb{Q} = \pi_{n+j}(\Sigma^n K)$ (for large enough n), reducing to a trivial procedure yielding the earlier result of Serre (from the early 1950s) that the Hurewicz homomorphism is in fact an isomorphism:

$$\pi_j^s(K) \otimes \mathbb{Q} \xrightarrow{\cong} H_j(K; \mathbb{Q}). \tag{6.20}$$

As mentioned before, the algebra $\mathcal{A}_{\mathbb{Q}}$ of operations of rational cohomology theory consists only of scalar multiplications.

As already noted, in the case of a prime p application of the Adams spectral sequence to particular problems of homotopy theory entails investigation of various Hopf algebras over $\mathbb{Z}/p$:

$$B = \bigoplus_{n \geq 0} B^n, \quad B^0 = \mathbb{Z}/p,$$

with a symmetric diagonal $\Delta : B \longrightarrow B \otimes B$ (see above). The category of such algebras (where the morphisms preserve grading and the Hopf structure), together with the category of modules over Hopf algebras, constitutes an interesting algebraic model[11] which imitates certain properties of the homotopy category of CW-complexes. For instance a fiber bundle with simply-connected base is modelled algebraically by an appropriate epimorphism $f : A \longrightarrow B$ of Hopf algebras (where A is the "total space", B the "base") with the following special properties: there should exist a Hopf subalgebra $C \subset A$ (the "fiber"):

$$C = \bigoplus_{n \geq 0} C^n \subset A = \bigoplus_{n \geq 0} A^n, \quad C^0 = A^0 = \mathbb{Z}/p,$$

with augmentation ideal

$$C^+ = \bigoplus_{n \geq 1} C^n \subset A^+ = \bigoplus_{n \geq 1} A^n,$$

such that

(i) C is a *central subalgebra* of A ($ac = ca$ for all $c \in C$, $a \in A$), and Ker f has the form Ker $f = AC^+ = C^+A$;

[11] In particular, in these categories cohomology algebras $H^{*,*}(A; R)$ are defined for a Hopf algebra A and an A-module R.

(ii) the algebra A is a free left C-module, i.e. there is a basis of elements $a_i \in A^{n_i}$, $i = 0, 1, 2, \ldots$, $0 = n_0 < n_1 \leq n_2 \leq n_3 \leq \ldots$, with $a_0 = 1$, such that each element a of A can be uniquely expressed in the form

$$a = \sum_{i \geq 0} c_i a_i, \quad c_i \in C.$$

Under the above conditions there is a spectral sequence (a special case of the Serre-Hochschild spectral sequence discovered in the mid 1950s) of a "fiber bundle" $f : A \longrightarrow B$ with "fiber" C which imitates certain properties of the Leray spectral sequence for a fiber bundle of CW-complexes (see §7). This takes the form of a tri-graded spectral sequence of algebras

$$E_m^{q,l,r}, \quad d_m : E_m^{q,l,r} \longrightarrow E_m^{q+m,l-m+1,r}, \quad d_m^2 = 0,$$

where:

1. $E_{m+1}^{*,*,*} = H^*(E_m^{*,*,*}, d_m), \quad d_m(uv) = d_m(u)v \pm u d_m(v)$;
2. $E_2^{q,l,r} = \sum_{r_1+r_2=r} H^{q,r_1}(B; H^{l,r_2}(C)) = \sum_{r_1+r_2=r} H^{q,r_1}(B) \otimes_{\mathbb{Z}/p} H^{l,r_2}(C)$;
3. $\bigoplus_{q+l=n} E_\infty^{q,l,r} \cong GH^{n,r}(A)$,

where $GH^{n,r}(A)$ denotes the adjoint ring with respect to a certain filtration generating the above spectral sequence. The gradings indicated by the indices r_1, r_2, r_3 correspond to the gradings of A, B, C respectively.

The analogues of the Steenrod operations St_p^i (the Steenrod squares Sq^i for $p = 2$) mentioned above (see (6.14)) are related to the above spectral sequence in much the same way that the ordinary Steenrod operations are related to the Leray spectral sequence for fiber bundles, which we shall consider in the next section.

§7. The classical apparatus of homotopy theory. The Leray spectral sequence. The homology theory of fiber bundles. The Cartan-Serre method. The Postnikov tower. The Adams spectral sequence

The method of "spectral sequences", first discovered by Leray in the mid-1940s in connexion with maps of spaces, and so applying in particular to fiber bundles, is of fundamental importance as an effective tool in homological algebra, making possible (among other things) the computation of the homology groups of an extensive class of spaces by means which avoid direct detailed examination of their geometrical structure.

Let $p : Y \longrightarrow X$ be a map of topological spaces. For each $j \geq 0$ the map p determines a presheaf $\mathcal{F}^j$ on Y given by

$$\mathcal{F}^j_U = H^j(p^{-1}(U); D), \quad j \geq 0, \tag{7.1}$$

on each open set U of X (and for any fixed abelian group D). Hence there arise in turn the cohomology groups

$$E_2^{q,l} = H^q(X; \mathcal{F}^l)$$

(defined to be zero for $q < 0$ or $l < 0$). *Leray's theorem* asserts the existence of a spectral sequence converging to $H^m(Y; D)$:

$$E_m^{q,l}, \quad d_m : E_m^{q,l} \longrightarrow E_m^{q+m,l-m+1}, \quad d_m^2 = 0, \tag{7.2}$$

where

$$E_{m+1}^{q,l} = H^*(E_m^{q,l}, d_m) \cong \text{Ker } d_m / \text{Im } d_m,$$

$$E_\infty^{q,l} = E_m^{q,l} \quad \text{for} \quad m > q + l,$$

such that

$$\sum_{q+l=m} E_\infty^{q,l} \cong GH^m(Y, D) = \sum_{l=0}^{m} B^{m-l}/B^{m-l-1}. \tag{7.3}$$

Here $GH^m(Y, D)$ is the group adjoined to $H^m(Y, D)$ by means of a certain filtration

$$B^m = H^m(Y; D) \supset B^{m-1} \supset B^{m-2} \supset \ldots \supset B^0 \supset 0.$$

For certain indices q, l the group $E_\infty^{q,l}$ is easily identified; for instance $E_\infty^{0,m} = B^0 \subset H^m(Y; D)$ is just the image under the projection homomorphism p^*:

$$E_\infty^{0,m} = B^0 = \text{Im } p^*, \quad p^* : H^m(X; D) \longrightarrow H^m(Y; D). \tag{7.4}$$

For a locally trivial fibration $Y \xrightarrow{p} X$ with fiber F, the above *Leray spectral sequence* may be described as follows. Let us assume the base X is a CW-complex with n-skeleton X^n, $n = 0, 1, 2, \ldots$. The filtration of $H^m(Y; D)$ is determined by the following natural filtration of the space Y:

$$Y_n = p^{-1}(X^n), \quad Y_n \xrightarrow{\phi_n} Y,$$

$$Y_0 \subset Y_1 \subset Y_2 \subset \ldots \subset Y_n \subset \ldots, \quad Y = Y_\infty.$$

From the inclusion map $\phi_k : Y_k \longrightarrow Y$, we obtain for each m the induced homomorphism $\phi_k^* : H^m(Y) \longrightarrow H^m(Y_k)$; clearly ϕ_k^* is an isomorphism for $k = m$. The desired filtration

$$H^m(Y; D) = B^m \supset B^{m-1} \supset B^{m-2} \supset \ldots \supset B^0 \supset 0$$

of $H^m(Y; D)$, is obtained by setting

$$B^k = \text{Ker } \left\{\phi^*_{m-k} : H^m(Y; D) \longrightarrow H^m(Y_{m-k}; D)\right\}, \quad k = m, m-1, \ldots, 0.$$

If in addition the base X is connected and has just one vertex (0-cell) — which may without loss of generality always be assumed — then the quotient B^m/B^{m-1} in (7.3) is isomorphic to the image of $H^m(Y; D)$ in $H^m(F; D)$ under the homomorphism induced from the inclusion of the fiber F in the total space Y:

$$\begin{gathered} i : F \longrightarrow Y, \quad F = p^{-1}(x_0), \\ \text{Im}\,\{i^* : H^m(Y; D) \longrightarrow H^m(F; D)\} \cong B^m/B^{m-1} \subset H^m(F; D). \end{gathered} \tag{7.5}$$

Furthermore for locally trivial fiber bundles (and also for Serre fibrations — in fact for any fibration with the covering homotopy property) Leray's theorem gives the explicit form of the term $E_2^{q,l}$:

$$E_2^{q,l} = H^q(X; (H^l(F; D), \rho)). \tag{7.6}$$

Here a representation ρ of the fundamental group $\pi_1(X)$ on $H^l(F; D)$ is determined by the action of $\pi_1(X)$ on the fiber F. If $\pi_1(X)$ acts trivially (in particular if $\pi_1(X) = 0$), then it follows that $E_2^{q,l} = H^q(X; H^l(F))$, i.e. the term $E_2^{q,l}$ coincides with the cohomology of the direct product $X \times F$:

$$\sum_{q,l} E_2^{q,l} \cong H^*(X \times F; D). \tag{7.7}$$

For the trivial bundle $Y = X \times F$ (the direct product) we have that $E_2^{q,l} = E_\infty^{q,l}$, i.e. the differentials d_m are all zero for $m \geq 2$.

However for a non-trivial fiber bundle (or "skew product") with simply-connected base, the cohomology of the total space becomes different from $H^*(X \times F; D)$ because of the existence of non-zero differentials d_m, $m \geq 2$. Thus for a skew product there is "less" cohomology than for the direct product. This is not surprising, for the following reasons. Assuming that the fiber F, as well as the base X, is a CW-complex, the cell decomposition of the total space is the same as that of the direct product $X \times F$ since above each cell σ^m of K the bundle is trivial:

$$p^{-1}(\sigma^m) \cong \sigma^m \times F.$$

The "skewing" in the cohomology groups $H^*(Y; D)$ of the total space Y appears when we apply the boundary operator ∂:

$$\partial \sigma_Y^{q+l} = \partial(\sigma_X^q \times \sigma_F^l) = \sigma^q \times (\partial \sigma_F^l) + (\partial_1 + \partial_2 + \ldots)\sigma_Y^{q+l},$$

where $\partial_j \sigma_Y^{q+l}$ lies above the $(q-j)$-skeleton X^{q-j} of the base X. The operator ∂_1 can be explicitly calculated (as shown by Leray) in terms of the cohomology of the base with coefficients from the $\pi_1(X)$-module $H^*(F)$.

If D is a ring then each $E_m^{*,*}$ becomes a bigraded ring and the differential d_m satisfies $d_m(uv) = d_m(u)v \pm ud_m(v)$ for bigraded elements u, v (and the resulting ring structure is incorporated into Leray's theorem). There is a completely analogous homological version of the theorem, and over a field the cohomology and homology spectral sequences are mutually dual.

The *transgression homomorphism* τ in the Leray spectral sequence is defined as follows: consider the pair (Y, F) and the homomorphisms

$$H^q(F) \xrightarrow{\delta} H^{q+1}(Y, F) \qquad \text{(the coboundary homomorphism),}$$

$$H^{q+1}(X, x_0) \xrightarrow{p^*} H^{q+1}(Y, F) \qquad \text{(the projection homomorphism).}$$

The composition $(p^*)^{-1} \circ \delta = \tau$ is the desired transgression. Note that τ need not be everywhere defined on $H^q(F)$, and may be many-valued. An element z of $H^q(F)$ is called *transgressive* if $\tau(z)$ is defined. In terms of the Leray spectral sequence of the fiber bundle the transgression τ may be equivalently defined as $\tau = d_q$, where

$$H^q(F) \supset E_q^{0,q} \xrightarrow{d_q} E_q^{q+1,0} = H^{q+1}(X)/\text{Ker } p^*. \tag{7.8}$$

It follows from (7.8) that for a transgressive element $z \in H^q(F)$, we have $d_j(z) = 0$ for $j < q$.

In the case $D = \mathbb{Z}/p$, all cohomology rings are modules over the algebra $\mathcal{A}_p$, and it turns out that the differentials in the Leray spectral sequence commute with the Steenrod squares and powers; indeed, since δ and p^* commute with the Steenrod squares and powers, it follows that the differential d_q does also. This is of crucial importance in certain difficult computations. In particular, if $z \in H^q(F)$ is transgressive then so is $St_p^i(z)$, and

$$\tau St_p^i(z) = St_p^i \tau(z).$$

Example 1. Borel's theorem (early 1950s): If $H^*(F; \mathbb{Q})$ is an exterior algebra over $\mathbb{Q}$ and $H^*(Y; \mathbb{Q}) = 0$, then $H^*(X; \mathbb{Q})$ is the polynomial algebra in the images under transgression of certain exterior generators. Over the field $\mathbb{Z}/2$, the hypothesis that $H^*(F; \mathbb{Z}/2)$ be an exterior algebra is replaced by the requirement that there exist a finite collection of transgressive generators $v_j \in H^{n_j}(F; \mathbb{Z}/2)$ forming an additive basis for $H^*(F; \mathbb{Z}/2)$:

$$v_{j_1}, v_{j_2}, \dots, v_{j_k}, \quad j_1 < j_2 < \dots < j_k.$$

The same conclusion follows: the ring $H^*(X; \mathbb{Z}/2)$ is polynomial. By means of this theorem the cohomology over $\mathbb{Z}/2$ of a space of type $K(\mathbb{Z}/2, n)$, and the Steenrod algebra $\mathcal{A}_2$ were computed (Serre, early 1950s). □

Example 2. Using the Leray spectral sequence the following cohomology rings can be calculated without difficulty:

$$H^*(\mathbb{R}P^n;\mathbb{Z}/2);\quad H^*(\mathbb{R}P^n;\mathbb{Z});\quad H^*(\mathbb{C}P^n;\mathbb{Z});$$

$$H^*(S^\infty/\mathbb{Z}/m;\mathbb{Z});\quad H^*(S^\infty/\mathbb{Z}/m;\mathbb{Z}/m);$$

$$H^*(SO_n;\mathbb{Q});\quad H^*(U_n;\mathbb{Z});\quad H^*(Sp_n;\mathbb{Z});$$

$$H^*(BSO_n;Q);\quad H^*(BU_n;\mathbb{Z});\quad H^*(BSO_n;\mathbb{Z}/2);$$

$$H^*(BO_n;\mathbb{Z}/2).$$

The calculations are carried out using the following fibrations:

1. $S^{2n+1} \longrightarrow \mathbb{C}P^n$ (fiber S^1);
2. $\mathbb{R}P^{2n+1} \longrightarrow \mathbb{C}P^n$ (fiber S^1);
3. $S^{2n+1}/\mathbb{Z}/m \longrightarrow \mathbb{C}P^n$ (fiber S^1);
4. $SO_n \longrightarrow S^{n-1}$ (fiber SO_{n-1});
5. $U_n \longrightarrow S^{2n-1}$ (fiber U_{n-1});
6. $E \longrightarrow BO_n$ (fiber O_n), $E \sim *$;
7. $E \longrightarrow BU_n$ (fiber U_n), $E \sim *$. □

Example 3. As a trivial case of Borel's theorem above, one has for the space $X = K(\mathbb{Z}, n)$:

$$H^*(X;\mathbb{Q}) = \begin{cases} \Lambda(u)\ , & u \in H^n(X)\ , \quad \text{if } n \text{ is odd,} \\ \mathbb{Q}[u]\ , & u \in H^n(X)\ , \quad \text{if } n \text{ is even.} \end{cases}$$

For finite groups π the cohomology ring $H^*(K(\pi,n);\mathbb{Q})$ is zero. For finitely generated abelian groups π the cohomology ring $H^*(K(\pi,n);R)$ over any coefficient ring R is finitely generated. □

In Chapter 1 we described how to convert, by means of a construction preserving homotopy types, any map $f : X \longrightarrow Y$ into a Serre fibration

$$p : \tilde{X} \longrightarrow \tilde{Y},\quad \tilde{X} \sim X,\quad \tilde{Y} \sim Y,\quad p \sim f,$$

where $\tilde{Y}$ is the mapping cylinder of f, contractible to Y, $\tilde{X}$ is the space of paths in $\tilde{Y}$ beginning at points of X (identified with $X \times \{0\} \subset \tilde{Y}$) and ending in points of Y, and p sends each such path to its end-point in Y. The space $\tilde{X}$ is contractible to X.

We now describe the Cartan–Serre method of computing the homotopy groups $\pi_j(X)$ of a simply-connected space X. Suppose $\pi_j(X) = 0$ for $j < n$

and $Y = K(\pi_n(X), n)$. There is then a map $f : X \longrightarrow Y$, for which the induced homomorphism $f_* : \pi_n(X) \longrightarrow \pi_n(Y)$ is an isomorphism. Applying the above construction to the map f, we obtain a fiber bundle (Serre fibration):

$$p : \tilde{X} \longrightarrow \tilde{Y} \sim K(\pi_n(X), n).$$

It follows that the fiber F of the fibration $p : \tilde{X} \longrightarrow \tilde{Y}$ has the same homotopy groups as X in dimensions above n:

$$\begin{aligned} &\pi_j(F) \cong \pi_{j+1}(X) \quad \text{for} \quad j \geq n, \\ &\pi_j(F) = 0 \qquad\qquad \text{for} \quad j < n. \end{aligned} \tag{7.9}$$

Hence computing $\pi_*(F)$ is as good as computing $\pi_*(X)$. In its simplest form the Cartan–Serre method operates as follows:

Step 1. The homology groups $H_*(F)$ of the fiber F are calculated by means of the homological Leray spectral sequence applied to the fibration $p : \tilde{X} \longrightarrow \tilde{Y}$.

Step 2. Hurewicz' theorem is invoked to compute the group $\pi_{n+1}(X)$:

$$H_n(F) \cong \pi_n(F) \cong \pi_{n+1}(X).$$

Step 3. The entire process is repeated.

However there is a more convenient procedure for calculating the homotopy groups of a simply-connected space X. By attaching cells of dimensions $\geq q$ to the space X so as to "kill" the homotopy groups of dimensions $\geq q$, one obtains a space $X_q \supset X$ for which

$$\begin{aligned} &\pi_j(X) \cong \pi_j(X_q) \quad \text{for} \quad j < q, \\ &\pi_j(X_q) = 0 \qquad\quad \text{for} \quad j \geq q. \end{aligned} \tag{7.10}$$

Thus if X is such that $\pi_j(X) = 0$ for $j < n$, then $\pi_j(X_n) = 0$ for all j, so that X_n is contractible, while X_{n+1} is of type $K(\pi, n)$, $\pi = \pi_n(X)$. Consider the following sequence of fibrations (obtained by conversion of the corresponding inclusions $X_{l+1} \subset X_l$ into fibrations):

$$\begin{array}{lll} (n+1) & \tilde{X}_{n+2} \longrightarrow \tilde{X}_{n+1} & (\text{with fiber } F_{n+1} \sim K(\pi_{n+1}(X), n+1)); \\ (n+2) & \tilde{X}_{n+3} \longrightarrow \tilde{X}_{n+2} & (\text{with fiber } F_{n+2} \sim K(\pi_{n+2}(X), n+2)); \\ & \cdots\cdots & \cdots\cdots \\ (n+k) & \tilde{X}_{n+k+1} \longrightarrow \tilde{X}_{n+k} & (\text{with fiber } F_{n+k} \sim K(\pi_{n+k}(X), n+k)). \end{array}$$

Now a fiber bundle with fiber of type $K(\pi, m)$ and with simply-connected base B, is homotopically determined by a single cohomology class

$$u_m \in H^{m+1}(B, \pi).$$

Thus to each of the above fibrations $\tilde{X}_{n+k+1} \longrightarrow \tilde{X}_{n+k}$ with fiber $F_{n+k} \sim K(\pi_{n+k}, n+k)$, there corresponds a particular cohomology class

$$u_{n+k} \in H^{n+k+1}(X_{n+k}, \pi_{n+k}(X)).$$

The cohomology classes u_{n+k} are known as the *Postnikov invariants* of the space X. The sequence of fibrations with fibers $F_{n+k} \sim K(\pi_{n+k}, n+k)$:

$$\cdots \longrightarrow \tilde{X}_{n+3} \longrightarrow \tilde{X}_{n+2} \longrightarrow \tilde{X}_{n+1} \longrightarrow * \tag{7.11}$$

(where $*$ here denotes a contractible space) is called the *Postnikov tower* for X. It follows that the collection of groups $\pi_j(X)$ and Postnikov invariants u_j determine the homotopy type of the the simply-connected space X.

The Cartan–Serre method enables one to compute the Postnikov invariants in the course of recursively constructing the spaces X_{n+j}, invoking knowledge of the cohomology of the Eilenberg-MacLane spaces $K(\pi, m)$. For the *rational* homotopy groups $\pi_i(X)\otimes\mathbb{Q}$ this computational procedure becomes very much simpler by virtue of the simplicity of the structure of the cohomology rings $H^*(K(\pi, m); \mathbb{Q})$.

Example 1. For the sphere S^n the groups $\pi_i(S^n) \otimes \mathbb{Q}$ are given by:

$$\pi_n(S^n) \cong \mathbb{Z}, \quad \pi_{4n-1} \otimes \mathbb{Q} = \mathbb{Q},$$

with all other rational homotopy groups trivial (whence it follows that in all other cases $\pi_i(S^n)$ is finite).

Example 2. For simply-connected, finite CW-complexes X the homotopy groups $\pi_j(X)$ are all finitely generated. (These two results were obtained by Serre in the early 1950s.)

Example 3. The *Cartan–Serre theorem* (early 1950s). If a simply-connected space X is such that the algebra $H^*(X; \mathbb{Q})$ is a free skew-symmetric algebra, then $\pi_j(X) \otimes \mathbb{Q}$ is isomorphic to the vector space of those linear $\mathbb{Q}$-forms on $H^j(X; \mathbb{Q})$ that vanish on the nontrivially decomposable elements of $H_j(X; \mathbb{Q})$. This theorem applies to H-spaces X since by Hopf's theorem $H^*(X; \mathbb{Q})$ is free skew-symmetric for such spaces. Hence in particular the result applies to loop spaces ΩX, so that knowledge of $H^*(\Omega X; \mathbb{Q})$ for any finite simply-connected CW-complex X yields complete information about the groups $\pi_j(X) \otimes \mathbb{Q}$. However such cohomology algebras are not always easily computed in detail. (For the definition of H-spaces and for further details, see Chapter 4, §2.)

Much later, in the early 1970s, Sullivan gave precise form to a general theory of rational homotopy type ("$\mathbb{Q}$-type") applying to finite, simply-connected CW-complexes, where all computations are carried out in the rational category, i.e. when all invariants are tensored by $\mathbb{Q}$. He showed that the rational homotopy type of such a CW-complex is determined by an equivalence class of certain differential skew-symmetric algebras. Via Kählerian geometry this led, in particular, to the result that the $\mathbb{Q}$-type of a simply-connected "Kähler manifold" is completely determined by its rational cohomology algebra (Deligne, Sullivan, Griffiths, Morgan, in the mid-1970s). □

In certain special cases the Cartan-Serre method allows the homotopy groups to be computed exactly, rather than just up to tensoring by $\mathbb{Q}$. Adams reorganized the Cartan-Serre method into the form of a spectral sequence known as the "Adams spectral sequence". He observed that for the computation of stable homotopy it is actually more convenient to use the so-called "Adams resolution", rather than Postnikov towers.

Let X be a finite CW-complex (or homotopic equivalent thereof) of dimension $< 2n - 1$ and with $\pi_j(X) = 0$ for $j \leq n - 1$. We now regard the cohomology ring $H^*(X; \mathbb{Z}/p)$ as a module over the Steenrod algebra $\mathcal{A}_p$. Let $\{m_1, \dots, m_k\}$, $m_j \in H^{n_j}(X; \mathbb{Z}/p)$, be a basis for this module, and consider maps

$$f_j : X \longrightarrow K(\mathbb{Z}/p, n_j), \quad j = 1, 2, \dots, k,$$

such that $f_j^*(u_j) = m_j$, where $u_j \in H^{n_j}(K(\mathbb{Z}/p, n_j); \mathbb{Z}/p)$ is the fundamental class of the complex $K(\mathbb{Z}/p, n_j)$. We obtain a map

$$f = \prod_{j=1}^{k} f_j : X \longrightarrow \prod_{j=1}^{k} K(\mathbb{Z}/p, n_j).$$

We convert the map f into a fibration with total space $X_{-1} \sim X$ and fiber $X_0 = F \subset X_{-1}$:

$$\tilde{f} : X_{-1} \longrightarrow \prod_{j=1}^{k} Y_j, \quad Y_j \sim K(\mathbb{Z}/p, n_j).$$

We then repeat the entire procedure with X_0 in place of X_{-1}, and so on, to obtain a filtration of the space X:

$$X \sim X_{-1} \supset X_0 \supset X_1 \supset X_2 \supset \cdots , \tag{7.12}$$

with the property that the $\mathcal{A}_p$-modules $M_{j+1} = H^*(X_j, X_{j+1}; \mathbb{Z}/p)$ are free for $j \geq 0$. The boundary homomorphism

$$\delta : H^q(X_j, X_{j+1}; \mathbb{Z}/p) \longrightarrow H^{q+1}(X_{j-1}, X_j; \mathbb{Z}/p)$$

in the cohomology exact sequence of the triple (X_{j-1}, X_j, X_{j+1}) yields the differential $d_j : M_{j+1} \longrightarrow M_j$, determining the following complex of free $\mathcal{A}_p$-modules:

$$\cdots \longrightarrow M_{j+1} \xrightarrow{d_j} M_j \xrightarrow{d_{j-1}} M_{j-1} \longrightarrow \cdots \xrightarrow{d_0} M_0 \xrightarrow{\varepsilon} M_{-1}. \tag{7.13}$$

Here $M_{-1} \cong H^*(X_{-1}; \mathbb{Z}/p)$ and ε is defined to be the homomorphism

$$H^*(X_{-1}, X_0; \mathbb{Z}/p) \longrightarrow H^*(X_{-1}; \mathbb{Z}/p)$$

induced by the inclusion of pairs $(X_{-1}, \emptyset) \longrightarrow (X_{-1}, X_0)$. In particular, in "stable dimensions" we have

$$M_0 = H^*(X_{-1}, X_0; \mathbb{Z}/p) \cong H^*(\prod_{j=1}^{k} K(\mathbb{Z}/p, n_j); \mathbb{Z}/p),$$

so that in the algebraic resolution (7.13) all M_j with $j \geq 0$ are free (and $\operatorname{Ker} d_j = \operatorname{Im} d_{j+1}$); this therefore represents an acyclic resolution of the $\mathcal{A}_p$-module $M_{-1} \cong H^*(X; \mathbb{Z}/p)$. The complex (7.13) is known as the *Adams resolution of the $\mathcal{A}_p$-module $H^*(X; \mathbb{Z}/p)$*, and the diagram of "stable spaces" or *spectra*

$$\begin{array}{ccccccccccccc}
X \sim X_{-1} & \longleftarrow & & X_0 & \longleftarrow & & X_1 & \leftarrow \cdots \leftarrow & X_j & \longleftarrow & & X_{j+1} & \leftarrow \cdots \\
& \searrow & & \nearrow & \searrow & & \nearrow & \searrow \quad \nearrow & & \searrow & & \nearrow & \searrow \\
& & X_{-1}/X_0 & & & X_0/X_1 & & \cdots & & & X_{j+1}/X_j & & \cdots
\end{array} \tag{7.14}$$

is called the *Adams resolution of the space X* or the *geometric realization* of the Adams resolution (7.13). By taking stable homotopy groups one obtains from the diagram (7.14) the *Adams spectral sequence.* We note that there is the following isomorphism for a product of Eilenberg-MacLane spaces $K(\mathbb{Z}/p, n_j)$:

$$\operatorname{Hom}_{\mathcal{A}_p}(H^*(\prod_j K(\mathbb{Z}/p, n_j); \mathbb{Z}/p),\ \mathbb{Z}/p) \cong \pi_*^s\Big(\prod_j K(\mathbb{Z}/p, n_j)\Big),$$

and this isomorphism allows the Adams spectral sequence to be handled very efficiently by means of homological algebra.

By using the above methods of Cartan, Serre and Adams, often in combination with other more special tools and ideas, several topologists have managed to carry out far-reaching computations of the stable and non-stable homotopy groups of spheres (Serre, Toda and others, from the early 1950s to the mid-1960s). In the late 1960s and into the 1970s these techniques were supplemented by those of extraordinary homology theory (see §8, 9 below).

The stable homotopy groups of spheres $\pi_i^s = \pi_{N+i}(S^N)$, $N > i+1$, and the associated skew-symmetric ring π_*^s:

$$\pi_*^s = \sum_{i \geq 0} \pi_i^s, \quad \pi_i^s \pi_j^s = (-1)^{ij} \pi_j^s \pi_i^s,$$

are important in the theory of smooth manifolds. The first few stable homotopy groups of spheres are given in the following table:

$i=0$	1	2	3	4	5	6	7	8	9	10	11	12	13
$\pi_i^s=\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/240$	$\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$	$\mathbb{Z}/6$	$\mathbb{Z}/504$	0	$\mathbb{Z}/4$
1	η	η^2	ν			ν^2	σ	$\eta\sigma,\ u$	$\eta^2\sigma,\ \eta u,\ v$	w	x		y

The multiplicative and additive relations among the indicated generators are as follows:[12]

$$\begin{aligned} &2\eta=0, && \eta^3=12\nu, && \eta\nu=0, && 240\sigma=0,\\ &\eta^2\sigma=\nu^3, && \eta^3\sigma=0, && \nu^2u=0, && 3w=\eta v,\\ &\eta^2v=256x, && y=\nu w, && 6w=0, && 504x=0. \end{aligned} \tag{7.15}$$

Note that the elements η, ν, σ, $\eta\sigma$, $\eta^2\sigma$, x and their multiples are realizable by means of framed normal bundles on the sphere S^n, i.e. are images under the homomorphism

$$J:\pi_j(SO)\longrightarrow\pi_j^s,$$

whose definition is given in Chapter 4, §3. (Some of the stable groups $\pi_j(SO)$ are tabulated in Chapter 4, §2.) Nontrivial p-components $\pi_j^{(p)}\subset\pi_j^s$ appear first for $j=2p-1$; for $p>2$, $j\le 2p(p-1)-1$, they are as follows:

$$\begin{aligned} &\pi_j^{(p)}\cong\mathbb{Z}/p, && j=2k(p-1), \quad k=1,\dots,p-1,\\ &\pi_j^{(p)}\cong\mathbb{Z}/p, && j=2p(p-1)-2,\\ &\pi_j^{(p)}\cong\mathbb{Z}/p^2, && j=2p(p-1)-1. \end{aligned} \tag{7.16}$$

For $j\ne 2p(p-1)-2$, these $\pi_j^{(p)}$ are contained in Im J, i.e. $\pi_j^{(p)}\subset J\pi_j(SO)$. It turns out that the image under J contains only a small part of π_*^s; in particular, the group $\pi_{2p(p-1)-2}^{(p)}$ contains nontrivial elements that do not belong to that image.

[12]We recall that in the algebra $\mathrm{Ext}_{\mathcal{A}_2}(\mathbb{Z}/2,\mathbb{Z}/2)$ the elements 2, η, ν, σ are represented by the elements h_0, h_1, h_2, h_3, respectively.

§8. Definition and properties of K-theory. The Atiyah-Hirzebruch spectral sequence. Adams operations. Analogues of the Thom isomorphism and the Riemann-Roch theorem. Elliptic operators and K-theory. Transformation groups. Four-dimensional manifolds

The first and the most important generalizations of cohomology theory — K-theory and cobordism theory — were introduced (or more accurately first considered from the appropriate point of view) in the late 1950s and 1960s by Atiyah. Subsequent development of the associated methodology very significantly augmented the algebraic apparatus of topology. Moreover for the investigation of many topological problems either K-theory or cobordism (bordism) theory has turned out to provide the most appropriate context. The general axiomatics of extraordinary homology (and cohomology) theory were worked out by G. W. Whitehead in the early 1960s.

We begin with the basic concepts of K-theory. For a pair (K, L) of finite CW-complexes the groups $K^0_{\mathbb{R}}(K, L)$ and $K^0_{\mathbb{C}}(K, L)$ are defined as the *Grothendieck groups* of classes of *stably equivalent vector bundles* (real and complex respectively) with base $B = K/L$, where "stable equivalence" of two vector bundles ν_1 and ν_2 means that

$$\nu_1 \oplus \varepsilon^{N_1} = \nu_2 \oplus \varepsilon^{N_2} \tag{8.1}$$

where ε^{N_i} $(i = 1, 2)$ is the trivial vector bundle with fiber $\mathbb{R}^{N_i}$ or $\mathbb{C}^{N_i}$. Every vector bundle ν over K/L has a stable inverse. This is explained in Chapter 4, §1: one first realizes K/L as a deformation retract of a manifold $U \supset K/L$ over which ν can be extended to a bundle stably equivalent to the tangent bundle of U; the inverse $-\nu$ is then realized as the normal bundle over U with respect to an embedding $U \subset \mathbb{R}^q$, q sufficiently large:

$$\nu \oplus (-\nu) = \varepsilon^N \sim 0. \tag{8.2}$$

It is clear that under the direct sum operation on vector bundles, $K^0_\Lambda(K, L)$ $(\Lambda = \mathbb{R}, \mathbb{C})$ becomes a group, and, via the tensor product, in fact a commutative ring.

For a finite CW-complex K we set

$$K^0_\Lambda(K) = K^0_\Lambda(K \cup *, *), \quad \Lambda = \mathbb{R}, \mathbb{C}, \tag{8.3}$$

where $*$ denotes the one-point space, i.e. in effect we consider vector bundles over K.

Motivated by the suspension isomorphism (see (3.6)) we define

$$K^{-j}_\Lambda(K, L) = K^0_\Lambda(\Sigma^j K, \Sigma^j L), \quad j > 0, \tag{8.4}$$

where ΣK, ΣL denote the suspensions of K, L. From properties of the universal G-bundles with $G = \lim O_n$, $\lim U_n$, one obtains natural isomorphisms

$$\begin{aligned} K_{\mathbb{R}}^{-j}(K,L) &\cong [\Sigma^j(K/L),\ BO], \\ K_{\mathbb{C}}^{-j}(K,L) &\cong [\Sigma^j(K/L),\ BU], \end{aligned} \tag{8.5}$$

where $[X, Y]$ denotes the set of homotopy classes of maps $X \longrightarrow Y$. Bott periodicity (see Chapter 4, §2) then yields

$$\begin{aligned} K_{\mathbb{R}}^{-j-8}(K,L) &\cong K_{\mathbb{R}}^{-j}(K,L), \\ K_{\mathbb{C}}^{-j-2}(K,L) &\cong K_{\mathbb{C}}^{-j}(K,L), \end{aligned} \tag{8.6}$$

which allows the groups $K_{\Lambda}^{j}(K,L)$ ($\Lambda = \mathbb{R}, \mathbb{C}$) to be defined, via periodicity, for all j, $-\infty < j < \infty$.

The functor $K_{\Lambda}^{j}(\cdot,\cdot)$ satisfies the axioms of generalized cohomology theory (see §3): homotopy invariance, functoriality, the excision axiom (since by definition $K_{\Lambda}^{j}(K/L,*) = K_{\Lambda}^{j}(K,L)$), and existence of exact sequences of pairs:

$$\cdots \longrightarrow K^j(K,L) \longrightarrow K^j(K) \longrightarrow K^j(L) \xrightarrow{\delta} K^{j+1}(K,L) \longrightarrow \cdots . \tag{8.7}$$

The dual homology theory $K_j^{\Lambda}(\cdot,\cdot)$ is defined for the category of finite complexes by

$$\begin{aligned} K_j^{\Lambda}(K) = K_j^{\Lambda}(K,*) &= K_{\Lambda}^{n-j-1}(S^n \backslash K, *'), \quad n \longrightarrow \infty, \\ K_j^{\Lambda}(K,L) &= K_j^{\Lambda}(K/L,*), \end{aligned} \tag{8.8}$$

i.e. as an appropriate limit, with the finite CW-complex K embedded in S^n (for sufficiently large n). There is no simple geometric description of the groups $K_j^{\Lambda}(K)$, and they are not generally used in homotopy theory.

Since each vector bundle over the suspension ΣK is determined by a homotopy class of maps

$$K \longrightarrow O_n \quad \text{or} \quad K \longrightarrow U_n,$$

for some n, we infer natural isomorphisms

$$K_{\mathbb{R}}^{-1}(K) \cong [K, O], \quad K_{\mathbb{C}}^{-1}(K) \cong [K, U].$$

Hence for a contractible CW-complex (in particular a one-point space) we obtain via Bott periodicity the isomorphisms:

$$\begin{aligned} K_{\mathbb{R}}^{-j}(*) &\cong \pi_j(BO) \cong \pi_{j-1}(O), \quad (j > 0), \\ K_{\mathbb{C}}^{-j}(*) &\cong \pi_j(BU) \cong \pi_{j-1}(U), \quad (j > 0). \end{aligned}$$

Note that in general $K_{\Lambda}^{*}(K,L)$ is a graded skew-commutative ring. The above observations lead to the following description of the coefficient rings $K_j^{\mathbb{R}}(*)$, $K_j^{\mathbb{C}}(*)$:

1. $K^*_{\mathbb{R}}(*)$ has as generators $1 \in K^0_{\mathbb{R}}(*)$, $h \in K^{-1}_{\mathbb{R}}(*)$, $u \in K^{-4}_{\mathbb{R}}(*)$, $v \in K^{-8}_{\mathbb{R}}(*)$, $v^{-1} \in K^8_{\mathbb{R}}(*)$, with relations

$$2h = 0, \quad h^3 = 0, \quad u^2 = 4v, \quad vv^{-1} = 1; \tag{8.9}$$

2. $K^*_{\mathbb{C}}(*)$ has as generators

$$1 \in K^0_{\mathbb{C}}(*), \quad w \in K^{-2}_{\mathbb{C}}(*), \quad w^{-1} \in K^2_{\mathbb{C}}(*), \tag{8.10}$$

with the single relation $ww^{-1} = 1$, so that

$$K^*_{\mathbb{C}}(*) \cong \mathbb{Z}[w, w^{-1}]. \tag{8.11}$$

Note that multiplication by v in the case of $K^*_{\mathbb{R}}(*)$ (and multiplication by w in the case of $K^*_{\mathbb{C}}(*)$) corresponds to application of the Bott periodicity operator.

The ring $K^*_{\Lambda}(K)$ may be computed by means of the *Atiyah-Hirzebruch spectral sequence*, which exists for any generalized cohomology theory. In the present case the Atiyah-Hirzebruch spectral sequence has as its terms rings $E^{*,*}_m$ with differentials d_m satisfying

$$\begin{gathered} E^{*,*}_{m+1} = H(E^{*,*}_m, d_m), \\ d_m : E^{p,q}_m \longrightarrow E^{p+m,q-m+1}_m, \quad d^2_m = 0, \\ E^{p,q}_2 = H^p(K; K^q_{\Lambda}(*)), \\ \sum_{p+q=s} E^{p,q}_{\infty} = GK^s_{\Lambda}(K), \end{gathered} \tag{8.12}$$

i.e. the ring $E^{*,*}_{\infty} = \sum_{p,q} E^{p,q}_{\infty}$ is adjoined to $K^*_{\Lambda}(K)$. The images of the differentials d_m are finite groups, and for a finite CW-complex K one has $d_m = 0$ for m sufficiently large. The orders of the groups Im d_m, and the relationship of the differentials d_m to cohomology operations, were studied by Buchstaber in the late 1960s.

For CW-complexes K with torsion-free integral cohomology groups, the Atiyah-Hirzebruch spectral sequence collapses in the case $\Lambda = \mathbb{C}$, i.e. $d_m = 0$ for $m \geq 3$, yielding the isomorphism:

$$H^*(K; K^*_{\mathbb{C}}(*)) \cong K^*_{\mathbb{C}}(K).$$

With $\Lambda = \mathbb{R}$ the same condition on K implies that all groups Im d_m have order at most 2.

Example 1. Let $K = \mathbb{R}P^k$. The ring $K^0_{\mathbb{R}}(\mathbb{R}P^k)$ is generated by the element $u = \eta - 1$, where η is the canonical line bundle over $\mathbb{R}P^k$ (with nontrivial

Stiefel-Whitney class $w_1(\eta)$; see Chapter 4, §1). It follows that $d_m = 0$ for $m \geq 3$, in the Atiyah-Hirzebruch spectral sequence, whence

$$GK^0_{\mathbb{R}}(\mathbb{R}P^k, *) \cong \sum_{j\geq 0} H^j(\mathbb{R}P^k; K^{-j}_{\mathbb{R}}(*)),$$

$$K^0_{\mathbb{R}}(\mathbb{R}P^k, *) \cong \mathbb{Z}_{2^{h(k)}}, \tag{8.13}$$

$$\text{where} \quad k = 8q + s, \quad h(k) = 4q + \lambda_s,$$

$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = 2, \quad \lambda_4 = 3, \quad \lambda_5 = 3, \quad \lambda_6 = 3, \quad \lambda_7 = 3.$$

The following relations hold:

$$-u^2 = 2u, \quad 2^{h(k)}u = 0, \quad \psi^l(u) = (1+u)^l - 1,$$

where ψ^l is the "Adams operation", shortly to be defined.

For $\Lambda = \mathbb{C}$, the generator of $K^0_{\mathbb{C}}(\mathbb{R}P^k)$ is $v = cu$, the complexification of u, so that again $d_m = 0$ for $m \geq 3$ in the Atiyah-Hirzebruch spectral sequence, yielding

$$K^0_{\mathbb{C}}(\mathbb{R}P^k) \cong \mathbb{Z}_{2^q}, \quad -v^2 = 2v, \quad \psi^l(v) = (1+v)^l - 1. \quad \square \tag{8.14}$$

By means of real and complex representations, one can define on the Grothendieck group K^0_Λ, $\Lambda = \mathbb{R}, \mathbb{C}$, various operations commuting with continuous mappings. These are the K-theoretic analogues of the ordinary cohomology operations. We now define the *Adams operations* of complex K-theory.

Given a complex bundle η over K/L, we may form the exterior power bundle $\Lambda^j(\eta)$, $j \geq 0$. The formal power series in t:

$$\Lambda_t(\eta) = \sum_{j\geq 0} \Lambda^j(\eta)t^j = 1 + \Lambda^1(\eta)t + \Lambda^2(\eta)t^2 + \cdots,$$

defines an operator

$$\Lambda_t : K^0(K, L) \longrightarrow K^0(K, L)\,[[t]]\,,$$

whose coefficients Λ^j, however, are not additive since (see Chapter 4, §1)

$$\Lambda_t(\eta_1 \oplus \eta_2) = \Lambda_t(\eta_1)\Lambda_t(\eta_2). \tag{8.15}$$

To obtain additive operations consider the coefficients of the series

$$-\log(1 - t\Lambda^1 + \cdots) = -\log \Lambda_{-t} = \sum_{k\geq 1} \frac{\psi^k}{k} t^k.$$

Here $\psi^k(\Lambda^1, \ldots, \Lambda^k)$ is a polynomial over $\mathbb{Z}$ in the Λ^j. It is convenient to express ψ^k in terms of symmetric polynomials. Let $s_1, s_2, \ldots$ be the elementary symmetric polynomials in $u_1, u_2, \ldots$; then the symmetric polynomial $u_1^k + \cdots + u_k^k$ may be expressed as a polynomial in $s_1, \ldots, s_k$:

$$u_1^k + \cdots u_k^k = P_k(s_1, s_2, \ldots, s_k), \quad k = 1, 2, \ldots.$$

The following expression (the k-th *Adams operation*) is very convenient for particular calculations:

$$\psi^k(\eta) = P_k(\eta, \Lambda^1\eta, \ldots \Lambda^k\eta).$$

It follows that

$$\psi^k(\eta_1 \oplus \eta_2) = \psi^k(\eta_1) \oplus \psi^k(\eta_2), \quad \psi^1 = \Lambda^1 = 1. \tag{8.16}$$

If η is a line bundle, then

$$\psi^k(\eta) = \eta^k. \tag{8.17}$$

For line bundles (considered as representing elements of $K_{\mathbb{C}}^0$) and consequently their direct sums, it is easy to establish the following properties:

$$\begin{gathered} \psi^k(\eta_1\eta_2) = \psi^k(\eta_1)\psi^k(\eta_2), \\ ch^n(\psi^k(\eta)) = k^n(ch^n\eta), \\ \psi^k\psi^l = \psi^{kl}, \end{gathered} \tag{8.18}$$

where ch is the Chern character. These properties then follow without difficulty for all vector bundles (or rather elements of $K_{\mathbb{C}}^0$).

Denoting by η the generator of $K_{\mathbb{C}}^0(S^{2n}, *) \cong \mathbb{Z}$, for which $ch\eta = \mu \in H^{2n}(S^{2n}; \mathbb{Z})$, the dual of the fundamental class, we have

$$\psi^k\eta = k^n\eta. \tag{8.19}$$

The formula (8.19) implies the following rule for commuting the operator ψ^k and the Bott periodicity operator w (see above):

$$\psi^k(wz) = kw\psi^k(z).$$

This motivates the extension of ψ^k to all $K_{\mathbb{C}}^{-j}(K, L)$ with $j > 0$ by setting $\psi^k w = kw$, and exploiting multiplicativeness:

$$\psi^k(ab) = \psi^k(a)\psi^k(b).$$

Extension of the operators ψ^k to the $K_{\mathbb{C}}^j$ with $j > 0$ is possible only after tensoring with $\mathbb{Z}\left[\frac{1}{k}\right]$, i.e. going over to the groups $K_{\mathbb{C}}^j \otimes \mathbb{Z}\left[\frac{1}{k}\right]$.

In real K-theory the Adams operations ψ^k are defined in a similar fashion. These operations commute with the complexification homomorphism:

$$c : K_{\mathbb{R}}^0(K, L) \longrightarrow K_{\mathbb{C}}^0(K, L), \quad \psi^k \cdot c = c \cdot \psi^k.$$

The operations ψ^k, introduced by Adams in the early 1960s, have turned out to be extremely useful in the solution of several problems. In order to formulate these problems, we first need some further definitions.

Let $S(\eta)$ denote the spherical fibration associated with a vector bundle η, i.e. $S(\eta)$ has the same base as η, but each fiber $\mathbb{R}^n$ of the vector bundle η is replaced by the sphere $S^{n-1} \subset \mathbb{R}^n$. Two vector bundles η_1, η_2 over the same base K with fiber $\mathbb{R}^n$ or $\mathbb{C}^n$ are said to be *fiber-wise homotopy equivalent* if there is a map

$$S(\eta_1) \longrightarrow S(\eta_2)$$

between the total spaces of the associated spherical fibrations, inducing the identity map on the base, and of degree $+1$ on the fibers. This leads to the concept of *stable fiber-wise homotopy equivalence* of real or complex vector bundles (namely $\eta_1 \sim \eta_2$ if there exist trivial bundles ε^{N_1}, ε^{N_2}, such that $\eta_1 \oplus \varepsilon^{N_1}$ and $\eta_2 \oplus \varepsilon^{N_2}$ are fiber-wise homotopy equivalent as above). The set of classes determined by this stable equivalence (first considered by Dold in the mid-1950s) are denoted by $J_{\mathbb{R}}(K)$ (or $J_{\mathbb{C}}(K)$). There are obvious epimorphisms

$$K^0_{\mathbb{R}}(K) \longrightarrow J_{\mathbb{R}}(K), \quad K^0_{\mathbb{C}}(K) \longrightarrow J_{\mathbb{C}}(K).$$

Let G_q be the group of homotopy equivalences of the sphere S^q (i.e. of homotopy classes of maps $S^q \longrightarrow S^q$ of degree 1), and let $G = \lim\limits_{q\to\infty} G_q$, the direct limit of the groups G_q under the natural inclusions $G_q \subset G_{q+1}$. The natural embedding $O \subset G$ (see Chapter 4, §4) induces a map $j : BO \longrightarrow BG$, whence

$$K^0_{\mathbb{R}}(K) \cong [K, BO],$$

$$J_{\mathbb{R}}(K) = j_*(K^0_{\mathbb{R}}(K)) \subset [K, BG],$$

where (as before) $[X, Y]$ denotes the set of homotopy classes of maps $X \longrightarrow Y$.

For smooth manifolds M^n the group $J_{\mathbb{R}}(M^n)$ has several important applications:

1. To obtaining lower bounds for the orders of homotopy groups of spheres ($M^n = S^n$).
2. To the study of smooth structures on spheres ($M^n = S^n$).
3. To estimating the number of independent vector fields on spheres S^{2n-1} ($M^n = \mathbb{R}P^n$).
4. To classifying the stable normal bundles of smooth closed manifolds (see Chapter 4, §3,4).

It is appropriate to mention here also the following fact, a corollary of a result of Atiyah:

Let η^N be a vector bundle over M^n. The class $[\eta] \in J_{\mathbb{R}}(M^n)$ contains the normal bundle over M (i.e. η is stably homotopy equivalent to the normal bundle) if and only if the cycle $\phi([M]) \in H_{N+n}(T\eta; \mathbb{Z})$ is spherical.

(Here $T\eta$ is the Thom space, and ϕ is the Thom isomorphism $\phi : H_n(M^n) \xrightarrow{\cong} H_{N+n}(T\eta)$; a cycle $x \in H_*(X;\mathbb{Z})$ is *spherical*, if it belongs to the image of the Hurewicz homomorphism $\pi_*(X) \longrightarrow H_*(X;\mathbb{Z})$.) By the Browder-Novikov theorem (see Chapter 4, §4), for a simply-connected manifold M^n of dimension $n \neq 4k+2$, $n \geq 5$, the above condition that $\phi([M^n])$ be spherical, together with the Hirzebruch formula for the signature, suffice for the vector bundle η to be realizable as a normal bundle of a possibly different manifold L homotopy equivalent to M^n.

Consider the subgroup H of $K^0_{\mathbb{R}}(K)$ generated by all elements of the form

$$p^{N_p}(\psi^k - 1)\eta, \tag{8.20}$$

where η is any element of $K^0_{\mathbb{R}}(K)$, p ranges over all primes, and N_p over all sufficiently large integers. According to a result of Adams the quotient group

$$\overline{J}_{\mathbb{R}}(K) \cong K^0_{\mathbb{R}}(K)/\langle p^{N_p}(\psi^k - 1)\eta\rangle = K^0_{\mathbb{R}}(K)/H$$

provides a lower estimate for $J_{\mathbb{R}}(K)$ since the kernel of the epimorphism $K^0_{\mathbb{R}}(K) \longrightarrow J_{\mathbb{R}}(K)$ is contained in Adams' subgroup H. In the case $K = S^n$, Adams was able to establish the reverse estimate (to within the factor 2 in certain dimensions), thereby obtaining a measure (to within the indicated accuracy) of the orders of the groups $J_{\mathbb{R}}(S^{4n}) \subset \pi^s_{4n-1} = \pi_{4n-1+N}(S^N)$. This estimate of Adams coincides (up to the factor 2 for even n) with a lower estimate obtained by Kervaire-Milnor (see Chapter 4, §3). For $K = \mathbb{R}P^{2n}$ Adams' lower estimate yields

$$K^0_{\mathbb{R}}(\mathbb{R}P^{2n}) \cong J_{\mathbb{R}}(\mathbb{R}P^{2n}),$$

from which there follows, by means of certain reductions due to Toda, an exact value for the largest number of linearly independent vector fields on an odd-dimensional sphere S^{2q-1}. If $2q = 2^m(2l+1)$, $m = c + 4d$, where $0 \leq c \leq 3$, this number is $2^c + 8d - 1$ (Adams, early 1960s); and this number of linearly independent vector fields may be constructed on the sphere by using infinitesimal rotations.

Adams's conjecture that all elements of $K^0_{\mathbb{R}}(K)$ of the form $k^N(\psi^k - 1)\eta$ are, for sufficiently large N, J-trivial (i.e. lie in the kernel of the epimorphism $K^0_{\mathbb{R}}(K) \longrightarrow J_{\mathbb{R}}(K)$), was proved in the late 1960s by Sullivan and Quillen by means of elegant general categorical ideas involving augmentation of the homotopy category in such a way that multiplication by k becomes invertible.

Since there is no simple geometric interpretation of homological K-theory, the appropriate K-theoretic analogue of Poincaré duality for closed manifolds requires special treatment. In particular the following questions arise: What is a "fundamental class" in $K_n(M^n)$? When does it exist? An essentially equivalent, but more accessible, question is the following one: For which vector bundles η over K, with fiber $\mathbb{R}^n$, is there a "Thom isomorphism"

$$\phi_K : K^j_\Lambda(K) \longrightarrow K^{n+j}_\Lambda(T\eta), \quad \phi_K(z) = z\phi_K(1), \quad \Lambda = \mathbb{R}, \mathbb{C}\ ? \tag{8.21}$$

For a complex vector bundle η with fiber $\mathbb{C}^m$, $n = 2m$, the value of $\phi_K(1)$ was determined (by Grothendieck-Atiyah-Hirzebruch in the late 1950s) in the context of complex K-theory (i.e. when $\Lambda = \mathbb{C}$):

$$\phi_K(1) = \Lambda^{\text{even}}(\eta) - \Lambda^{\text{odd}}(\eta) \in K^0_{\mathbb{C}}(T\eta),$$

where

$$\Lambda^{\text{even}}(\eta) = \sum_{j\geq 0} \Lambda^{2j}(\eta), \quad \Lambda^{\text{odd}}(\eta) = \sum_{j\geq 0} \Lambda^{2j+1}(\eta).$$

The following formula is valid:

$$\phi_H^{-1}(ch(\phi_K(1))) = T(\eta) \in H^*(K;\mathbb{Q}), \quad i : K \longrightarrow T\eta,$$

where $T(\eta)$ is the "Todd genus" of η, and ϕ_H is the Thom isomorphism in cohomology (see Chapter 4, §3). This yields a formula for the Chern character:

$$ch(\phi_K(\eta)) = T(\eta)ch(\eta).$$

For maps $f : M^n \longrightarrow N^k$ of quasicomplex manifolds (i.e. having their stable normal bundles endowed with a complex structure) there is an analogue of the Gysin homomorphism, of the form

$$Df_*D = f_! : K^0(M^n) \longrightarrow K^0(N^k),$$

satisfying

$$ch(f_!(\eta))T(N^k) = f_!(ch(\eta)T(M^n)). \tag{8.22}$$

(This is a generalization of the Riemann-Roch theorem.) For embeddings $M^n \subset N^k$ the operator $f_!$ coincides with the Thom isomorphism for the normal bundle $\nu_{M^n \subset N^k}$ of the embedding.

In real K-theory ($\Lambda = \mathbb{R}$) the construction of the Thom isomorphism requires a spinor structure to exist on the bundle η ($\dim \eta = n$), which is the case if the Stiefel-Whitney classes $w_1(\eta), w_2(\eta) = 0$. One then defines

$$\phi_K(1) = \Delta^+(\eta) - \Delta^-(\eta) \in K^0_{\mathbb{R}}(T\eta), \tag{8.23}$$

where $\Delta^\pm$ are the "semi-spinor" representations of the group $Spin(n)$ on the vector space Λ ($\mathbb{R}^n$), and $\Delta^\pm(\eta)$ are the vector bundles associated with these representations. The Thom isomorphism ϕ_K in real K-theory is used to define the $\hat{A}$-genus:

$$\hat{A}(\eta) = \phi_H^{-1}(ch(\phi_K(1))). \tag{8.24}$$

As we shall see (in Chapter 4, §3) $\hat{A}(M^{4k}) = \hat{A}(\nu_M)$ is an integer for a $Spin$-manifold M^{4k}. (This may be inferred from the real analogue of the generalized Riemann-Roch theorem. Indeed, if in 8.22 we take $N^k = *$, we conclude that the T-genus is integer-valued. If the Chern class $c_1(M) = 0$ for a quasicomplex manifold M, then the $\hat{A}$-genus $\hat{A}(M)$ will be integral as well.)

As a result of investigations by several people throughout the 1960s and early 1970s, the ring $K^*_\Lambda(X)$ has been calculated for most of the important homogeneous spaces X. Of particular interest is the result that for a compact, connected and simply-connected Lie group G, the ring $K^*_{\mathbb{C}}(G)$ is generated by the "fundamental representations" $\rho_1, \ldots, \rho_m \in K^1(G)$ ($m = \text{rank } G$) as the exterior algebra in these generators over the ring $\mathbb{Z}[w, w^{-1}]$:

$$K^*_{\mathbb{C}}(G) \cong \Lambda(\rho_1, \ldots, \rho_m), \quad \Lambda = \mathbb{Z}[w, w^{-1}],$$

where w is the Bott periodicity operator (Hodgkin's theorem, proved in the early 1960s). For $G = U(n)$ the representations $\rho_1, \ldots, \rho_n$ are just the exterior powers of the canonical bundle η over $BU(n)$: $\rho_j = \Lambda^j(\eta)$. For the classifying space BG of a compact (or finite) Lie group, we have:

$$K^{-1}_{\mathbb{C}}(BG) = 0, \quad K^0_{\mathbb{C}}(BG) \cong \widehat{R}_G = \mathbb{Z} + I/I^2 + I^2/I^3 + \cdots,$$

the completion of the complex representation ring R_G with respect to powers of a prime ideal I with $R_G/I \cong \mathbb{Z}$ (Atiyah, in the mid - 1960s).

In computing the K-theory of infinite-dimensional CW-complexes X additional difficulties arise; in particular, elements of infinite filtration may occur, i.e. nontrivial elements of $K^*_\Lambda(X)$, whose restriction to any n-skeleton X^n is zero (Mishchenko, Buchstaber, Hodgkin, in the late 1960s). The appearance of elements of infinite filtration is a common feature of generalized cohomology theory, not occuring in the ordinary theory.

One of the most elegant applications of K-theory occurs in connexion with the index formula for elliptic pseudo-differential operators on closed manifolds. First we give the requisite definitions. Let η_1, η_2 be vector bundles over a closed manifold M^n, and $\Gamma(\eta_i)$ the linear space of smooth cross-sections[13] of the bundle η_i, $i = 1, 2$. A general differential operator L of order m is defined on the space $\Gamma(\eta_1)$ taking its values in the space $\Gamma(\eta_2)$:

$$L : \Gamma(\eta_1) \longrightarrow \Gamma(\eta_2).$$

In local coordinates L has the form

$$L = \sum a_{i_1 \ldots i_m}(x) \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_m}} + L_1,$$

where $a_{i_1 \ldots i_m}(x)$ is a matrix-valued function and L_1 is an operator of order $< m$.

When L is elliptic, both kernel and cokernel are finite-dimensional, and the difference of these dimensions is by definition the *index* $\text{Ind } L = \dim \text{Ker } L - \dim \text{Coker } L$, of the operator L. The "index problem" is to give a description of the integer $\text{Ind } L$ in terms of topological data implicit in the

[13] Here the vector spaces are assumed to be endowed with a proper norm determining the structure of a Sobolev space on $\Gamma(\eta_i)$.

elliptic operator L. The following interpretation of the highest-order terms in the operator L, provides the key to the solution of the "index problem". Let $\pi : \tau^*(M) \longrightarrow M$ be the projection of the cotangent bundle over M^n, and let $\tau_0^*(M) = \tau^*(M) \setminus M$ be the subset of nonzero vectors in $\tau^*(M)$. The highest terms of L then determine a homomorphism $\sigma_L : \pi^*\eta_1 \longrightarrow \pi^*\eta_2$, which in local co-ordinates is given by:

$$\sigma_L(x,p) = \sum a_{i_1 \dots i_m}(x) p_{i_1} \cdot \dots \cdot p_{i_m} : \mathbb{C}^N \longrightarrow \mathbb{C}^N, \tag{8.25}$$

where $p = (p_1, \dots, p_n) \in \tau^*(M)$ is a covector. The homomorphism σ_L is called the *symbol* of the operator L. The operator L is elliptic if and only if its symbol σ_L is an isomorphism on $\tau_0^*(M)$, or, in terms of local co-ordinates, if and only if each map $\sigma_L(x,p)$ is invertible, i.e. det $\sigma_L(x,p) \neq 0$ if $p \neq 0$. The isomorphism σ_L — or rather its stable homotopy class $[\sigma_L]$ — is to be thought of as a topological "twist" of the elliptic operator L, and determines the operator up to the addition of a compact operator.

For singular integral operators we have $m = 0$. In the case $M = S^1$ and any elliptic operator L of order $m = 0$, the symbol σ_L reduces to a pair of matrix-valued functions $\sigma_L^+(z) = \sigma_L(z, 1)$, $\sigma_L^-(z) = \sigma_L(z, -1)$ on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The Noether-Muskhelishvili formula (1920-1930) for the index of L in this case is as follows:

$$2\pi \text{Ind } L = \oint \log \det \sigma_L^+(z) dz - \oint \log \det \sigma_L^-(z) dz.$$

The general construction of Atiyah-Singer (from the early 1960s) is as follows: given our two bundles $\pi^*(\eta_1)$, $\pi^*(\eta_2)$ over the total space $\tau^*(M)$ and an isomorphism $\sigma_L : \pi^*(\eta_1) \longrightarrow \pi^*(\eta_2)$ between their restrictions to the submanifold $\tau_0^*(M)$, there is defined a *difference element*

$$d(\pi^*(\eta_1), \pi^*(\eta_2), \sigma_L) \in K^0(\tau^*(M), \tau_0^*(M)).$$

Let $D(M)$ and $S(M)$ stand for the unit disc and unit sphere bundles obtained from $\tau^*(M)$ endowed with some Riemannian structure. Going over to the Thom complex $T\tau^* = D(M)/S(M)$, which is homotopy equivalent to the complex $\tau^*(M)/\tau_0^*(M)$, we obtain a well-defined element

$$d(\pi^*(\eta_1), \pi^*(\eta_2), \sigma_L) \in K^0(T\tau^*).$$

The *Atiyah-Singer index theorem* consists in the formula:

$$\text{Ind } L = \langle \phi^{-1}(ch\ d) \cdot T(M^n), [M^n] \rangle,$$

where $T(M)$ is the Todd genus of the complexification of the tangent bundle τ over M.

One of the most beautiful applications of this formula yields the interpretation of the $\widehat{A}$-genus of a *Spin*-manifold M^{4k} (i.e. for which $w_1(M) = w_2(M) = 0$) as the index of the Dirac operator, yielding the integrality of the $\widehat{A}$-genus.

At the present time several different proofs of this theorem are known, as well as various generalizations of it. Atiyah-Singer's first proof was, like that of the Riemann-Roch-Hirzebruch theorem, based on cobordism theory. Both of these results extend to manifolds–with–boundary, provided that the restriction of the operator L to the boundary is Noetherian.

Thus we see that the link with the theory of linear operators has been of great benefit for topology, leading as it has to the discovery of several deep algebro-topological connexions.

By way of an example we shall now give the Atiyah-Bott formula for the number of isolated fixed points of a holomorphic self-map

$$f : M^n \longrightarrow M^n$$

of a compact Kähler manifold M^n. Let $p_1, \dots, p_k$ be the fixed points of such a map f, all assumed to be isolated and non-degenerate. Let $T(f) : \tau \longrightarrow \tau$ be the derivative of the map f, inducing bundle maps

$$T(f) : \eta_1 \longrightarrow \eta_1, \qquad \eta_1 = \Lambda^{\text{even}}\tau,$$

$$T(f) : \eta_2 \longrightarrow \eta_2, \qquad \eta_2 = \Lambda^{\text{odd}}\tau.$$

Let $\lambda_{1q}, \dots, \lambda_{nq}$, $\lambda_{jq} \neq 1$, denote the eigenvalues of the induced self-map $T(f)$ of the tangent space $\tau(p_q)$ to M^n at each fixed point p_q. The *Atiyah-Bott formula* (an analogue of the Lefschetz formula) for the number of fixed points, is then as follows:

$$\chi(f) = \sum_{q=1}^{k} \left(\prod_{j=1}^{n} \frac{1}{1 - \lambda_{jq}} \right) = \sum_{q=1}^{k} \left(\operatorname{Tr} T(f)|_{\eta_1(p_q)} - \operatorname{Tr} T(f)|_{\eta_2(p_q)} \right),$$

where Tr denotes the trace. (The *Lefschetz formula* for the number of fixed points is:

$$\chi(f) = \sum_{m} (-1)^m \operatorname{Tr} T(f)^*|_{Hol^m(M)},$$

where $Hol^m(M)$ denotes the vector space of holomorphic m-forms on M^n.)

The above formula has several analogues, including a real analogue where M^n is endowed with a Riemannian metric and the map f is an isometry, or, more generally, one has a f-invariant elliptic complex (or f-invariant elliptic) operator $L : \Gamma(\eta_1) \longrightarrow \Gamma(\eta_2)$ over M^n. In the latter case the spaces Ker L, Coker L and the fibers of η_1, η_2 above the fixed points p_q, become representation spaces for the operator $T(f)$, and the following formula (of Atiyah-Bott) holds:

$$\operatorname{Tr} T(f)|_{\operatorname{Ker} L} - \operatorname{Tr} T(f)|_{\operatorname{Coker} L} = \sum_{q} \left(\operatorname{Tr} T(f)|_{\eta_1(p_q)} - \operatorname{Tr} T(f)|_{\eta_2(p_q)} \right).$$

In the early 1970s, in the course of investigating the "Euclidean" 4-dimensional theory of Yang-Mills fields (connexions on vector bundles), "instantons" were discovered. These are special solutions in $\mathbb{R}^4$ of the Yang-Mills equations, decreasing to zero as $|x| \to \infty$, so that, in particular, they may be extended to the sphere $S^4 = \mathbb{R}^4 \cup \infty$ (Belavin, Polyakov, Schwarz, Tyupkin).

Of particular interest is the self-duality equation of Belavin-Polyakov:

$$\begin{gathered} F = F_{ab} dx^a \wedge dx^b, \qquad *F = \pm F, \\ F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b], \end{gathered} \tag{8.26}$$

for the 1-form of a connexion on a vector bundle over S^4, which is in fact satisfied by instantons. (Of course instantons satisfy the full second-order Maxwell-Yang-Mills system of equations, but the equation (8.26) is first-order.) It turns out that all functions globally minimizing the functional

$$-S = \int_{S^4} \mathrm{Tr}\ \left(F_{ab} F^{ab}\right) \sqrt{g}\ d^4x = \int_{S^4} \mathrm{Tr}\ \left(F \wedge *F\right),$$

satisfy the self-duality equation, for a given first Pontryagin class

$$p_1 = \text{const.} \int_{S^4} \mathrm{Tr}\ \left(F \wedge *F\right), \quad G = SU(2).$$

By exploiting properties of the manifold of self-dual connexions on a given manifold M^4, Donaldson discovered, in the early 1980s, deep topological properties of 4-dimensional closed manifolds. He was able to show, for instance, that for a simply-connected manifold M^4 the quadratic form defined by the intersection index on $H_2(M^4; \mathbb{Q})$ can be positive definite only if it is reducible over $\mathbb{Z}$ to a sum of squares. The remarkable equation (8.26) seems to be specific, in a certain sense, to the geometry of 4-dimensional manifolds. Thus the theory of non-linear elliptic equations also turns out to be very useful for topology.

It is appropriate to mention also the analogues of K-theory for the category of G-spaces — the K_G-theory developed by Atiyah and Segal in the mid-1960s. Of particular interest is the special case of the category of spaces with an involution ($G = \mathbb{Z}/2$), where the group $\mathbb{Z}/2$ acts on the fibers of vector bundles via complex "anticonjugation". ("$K\mathbb{R}$-theory" was developed by Atiyah, and the important special case "$KS\mathbb{C}$-theory", by Anderson.) These K-theories allow one to obtain various relations between invariants associated with fixed points and global invariants of a manifold, and to clarify the algebraic nature of the 8-fold periodicity of real K-theory.

§9. Bordism and cobordism theory as generalized homology and cohomology. Cohomology operations in cobordism. The Adams-Novikov spectral sequence. Formal groups. Actions of cyclic groups and the circle on manifolds

From a geometrical point of view, *bordism theory* is *the* most natural homology theory. We start with the classical situation where the cycles are taken to be any smooth manifolds. In *unoriented bordism theory* $\Omega_*^O(\cdot) = \mathfrak{N}_*(\cdot)$, a *singular n-cycle* of a space X is a pair (M^n, f), where M^n is a closed manifold and f is a map of the manifold to X:

$$(M^n, f); \quad f : M^n \longrightarrow X.$$

Two singular n-cycles (M_1^n, f_1), (M_2^n, f_2) are *equivalent* (or *cobordant*): $(M_1^n, f_1) \sim (M_2^n, f_2)$, if there exists a manifold L^{n+1} with boundary $\partial L^{n+1} = M_1^n \cup M_2^n$ (the disjoint union) and a map $g : L^{n+1} \longrightarrow X$ such that

$$g|_{M_1^n} = f_1, \quad g|_{M_2^n} = f_2.$$

Here all manifolds involved are assumed to be embedded into $\mathbb{R}^N$ for $N >> n$. We now define the *n-dimensional bordism group* by

$$\Omega_n^O(X) = \{n\text{-cycles}\} / \sim ,$$

where the (abelian) group structure is defined in terms of the disjoint union of n-cycles:

$$[(M_1^n, f_1)] + [(M_2^n, f_2)] = [(M_1^n \cup M_2^n, f_1 \cup f_2)] .$$

For a pair of spaces $Y \subset X$ the *relative bordism groups* $\Omega_n^O(X, Y)$ are defined analogously, using maps of manifolds (M^n, f) now possibly with boundary $\partial M^n \neq \emptyset$, where the boundary ∂M is mapped into Y, and defining appropriately the equivalence $\sim$.

In the general case, there is a bordism theory $\Omega_*^G(X, Y)$ associated with any stable sequence of Lie groups $G = (G_n)$, $G_n \subset O_n$. The basic examples are

$$G = O, \ SO, \ U, \ SU, \ Sp, \ e,$$

where $G_n = O_n$, $G_n = SO_n$, $G_{2n} = U_n$, $G_{2n} = SU_n$, $G_{4n} = Sp_n$, $G_n = 1$. Besides these, more examples may be produced by considering representations

$$\rho : G \longrightarrow O, \quad \rho = (\rho_n), \quad \rho_n : G_n \longrightarrow O_n,$$

for instance $G = Spin = (G_n) = (Spin_n)$, $\rho_n : Spin_n \longrightarrow SO_n$ (a double cover).

A manifold M is called a *G-manifold* when its stable normal bundle is endowed with a G-structure. The bordism theory $\Omega_*^G(\cdot)$ is defined as above, assuming that all manifolds involved are embedded into $\mathbb{R}^N$, $N >> n$, and that their normal bundles are furnished with a G-structure. The bordism groups $\Omega_n^G(X,Y)$ have all the basic properties of a generalized homology theory: homotopy invariance, functoriality, fulfilment of the Excision Axiom for pairs $Y \subset X$ of CW-complexes: $\Omega_n^G(X,Y) = \Omega_n^G(X/Y,*)$, and existence of an exact sequence of a pair (X,Y):

$$\cdots \longrightarrow \Omega_n^G(Y) \longrightarrow \Omega_n^G(X) \longrightarrow \Omega_n^G(X,Y) \longrightarrow \Omega_{n-1}^G(Y) \longrightarrow \cdots .$$

The bordism groups $\Omega_n^G(*)$ of the one-point space $*$ coincide with those of spheres; this follows from the suspension isomorphism

$$\Sigma : \Omega_n^G(X,Y) \xrightarrow{\cong} \Omega_{n+1}^G(\Sigma X, \Sigma Y),$$

in view of which

$$\Omega_n^G(*) \cong \Omega_n^G(* \cup *', *') \xrightarrow{\Sigma} \Omega_{n+k}^G(S^k, *'').$$

The bordism groups of a point are called the *cobordism groups.* Their direct sum $\Omega_*^G = \bigoplus_n \Omega_n^G(*)$, furnished with the product operation induced by the direct product of manifolds, is the *cobordism ring.* In Chapter 4, §3, the relationship of the cobordism groups $\Omega_m^G(*)$ to the "Thom spaces" $MG = (MG_n)$ of universal vector G_n-bundles with base BG_n is indicated. "Thom's theorem" yields

$$\Omega_m^G(*) \cong \pi_m^s(MG) = \pi_{m+k}(MG_k), \quad k > m+1.$$

(Note that the Thom space MG_n is $(n-1)$-connected.) In particular for $G = e$, the groups Ω_m^G coincide with the stable homotopy groups of spheres, which were considered at the end of §7 above. Results concerning the rings Ω_*^G for other groups G will be given in Chapter 4, §3.

Dual to the bordism theory $\Omega_*^G(\cdot)$ is the *cobordism theory* $\Omega_G^*(\cdot)$, the corresponding *cobordism groups* being defined by

$$\Omega_G^m(X,*) = \Omega_{N-m-1}^G(S^N \setminus X, *'), \quad N \longrightarrow \infty,$$

where the finite CW-complex X is embedded in a sphere of sufficiently large dimension.

By proceeding according to the scheme laid out by Thom (see Chapter 4, §3) one obtains the following natural isomorphism:

$$\Omega_G^{-m}(X,*) \cong \lim_{n\to\infty} \left[\Sigma^{n+m}X, MG_n\right], \tag{9.1}$$

where [,] denotes the set of homotopy classes of maps. The direct sum $\Omega_G^*(X,Y) = \bigoplus_m \Omega_G^m(X,Y)$ can be given a multiplicative structure turning it into a skew-symmetric ring. This multiplication is defined as follows. For each of the above G, the groups G_n are "closed" with respect to the operation of taking the direct sum of vector bundles in the sense that this operation gives rise to maps of the corresponding classifying spaces and of vector bundles over them:

$$\begin{gathered} BG_{n_1} \times BG_{n_2} \longrightarrow BG_{n_1+n_2}, \\ BG \times BG \longrightarrow BG. \end{gathered} \tag{9.2}$$

The resulting map of universal vector bundles

$$\Phi_{n_1,n_2} : \eta^G_{n_1} \times \eta^G_{n_2} \longrightarrow \eta^G_{n_1+n_2},$$

induces a "multiplication" between the corresponding Thom spaces:

$$M\Phi_{n_1,n_2} : M(G_{n_1}) \wedge M(G_{n_2}) \longrightarrow M(G_{n_1+n_2}),$$

where $V \wedge W = V \times W/V \vee W$ is the "tensor" or *smash product* of the pointed spaces V, W. Applying the map $M\Phi_{n_1,n_2}$ to the product of maps f, g:

$$f : \Sigma^{m_1+n_1}X \longrightarrow M(G_{n_1}), \quad g : \Sigma^{m_2+n_2}X \longrightarrow M(G_{n_2}),$$

representing, via (9.1), elements of the appropriate cobordism groups, we obtain the map

$$M\Phi_{n_1,n_2}(f \times g) : \Sigma^{m_1+m_2+n_1+n_2}X \xrightarrow{f \wedge g} M(G_{n_1}) \wedge M(G_{n_2}) \xrightarrow{M\Phi_{n_1,n_2}} M(G_{n_1+n_2}),$$

yielding the desired biliniar, associative multiplication on the direct sum $\Omega^*_G(X)$ of the corresponding cobordism groups:

$$\begin{gathered} \Omega^{m_1}_G(X) \cdot \Omega^{m_1}_G(X) \subset \Omega^{m_1+m_2}_G(X), \\ z_1 \cdot z_2 = (-1)^{m_1 m_2} z_2 \cdot z_1. \end{gathered} \tag{9.3}$$

A closed manifold M^n with a normal G-structure possesses a natural *fundamental class* $[M^n] \in \Omega^G_n(M^n)$ defined by the identity map

$$1 : M^n \longrightarrow M^n; \quad [M^n] = [(M^n, 1)].$$

There is a *cap product operation* which can be defined between cobordism and bordism classes, analogous to that defined in ordinary homology and cohomology (see Chapter 2, §3):

$$\Omega^m_G(X) \cap \Omega^G_k(X) \longrightarrow \Omega^{k-m}_G(X),$$

with the property

$$f^*(a) \cap b = a \cap f_*(b)$$

for maps f of CW-complexes. In the case $X = M^n$ the map

$$a \mapsto a \cap [M^n]$$

defines the *Poincaré-Atiyah duality isomorphism*

$$D : \Omega_G^k(M^n) \xrightarrow{\cong} \Omega_{n-k}^G(M^n).$$

This isomorphism induces the operation of *intersection of bordism classes* dual to the product of cobordism classes defined above:

$$a \circ b = D^{-1}(Da \cdot Db).$$

As for any generalized cohomology theory, there is a spectral sequence (of rings) in cobordism theory (the *Atiyah-Hirzebruch spectral sequence*):

$$\begin{gathered}(E_m^{p,q}, d_m), \quad d_m : E_m^{p,q} \longrightarrow E_m^{p+m,q-m+1}, \quad d_m^2 = 0, \\ E_2^{p,q} = H^p(X; \Omega_G^q(*)),\end{gathered} \tag{9.4}$$

with adjoined ring $\sum_{p,q} E_\infty^{p,q} = G\Omega_G^*(X)$.

Examples. 1. In the case $G = O$, by a theorem of Thom from the mid-1950s, the groups $\Omega_G^n(X)$ are isomorphic to direct sums of ordinary $\mathbb{Z}/2$-cohomology groups of X of various dimensions. The ring structure of $\Omega_O^*(X)$ is given by a canonical isomorphism

$$\Omega_O^*(X) \cong H^*(X; \Omega_O^*(*)),$$

where $\Omega_O^*(*)$ is the polynomial algebra

$$\Omega_O^*(*) = \mathbb{Z}/2[v_2, v_4, v_5, v_6, v_8, \dots]$$

with generators v_j of dimensions $j = 2, 4, 5, \dots$, $j \neq 2^k - 1$. The Atiyah-Hirzebruch sequence collapses ($d_m = 0$ for $m \geq 2$, $E_\infty^{p,q} = E_2^{p,q}$), and the adjoined ring $G\Omega_O^*(X) = \sum_{p,q} E_\infty^{p,q}$ is isomorphic to $\Omega_O^*(X)$.

2. In the case $G = SO$, the structure of the ring $\Omega_{SO}^*(X) \otimes \mathbb{Z}_{(2)}$, where $\mathbb{Z}_{(2)}$ is the ring of rationals with odd denominators (i.e. $\mathbb{Z}$ localized at the prime 2) also reduces to that of the ordinary cohomology ring with coefficients from $\mathbb{Z}_{(2)}$ (by results of Novikov, Rohlin and Wall from around 1960). On the other hand, the theory $\Omega_{SO}^*(\cdot) \otimes \mathbb{Z}_{(p)}$ does not reduce to ordinary cohomology for any prime $p \neq 2$. In this case the ring $\Omega_{SO}^* \otimes \mathbb{Z}_{(p)}$ is a polynomial ring $\mathbb{Z}_{(p)}[x_1, x_2, \dots]$ with generators in each negative dimension of the form $4k$ (Milnor, Novikov; around 1960).

3. The case $G = U$ is the most interesting one, and provides a basis for applications of the algebraic techniques of cobordism theory. The ring Ω_U^* is a polynomial ring with one generator of each even negative dimension (Milnor, Novikov; see Chapter 4, §3):

$$\Omega_U^* \cong \mathbb{Z}[u_1, u_2, , \dots], \quad u_j \in \Omega_U^{-2j} = \Omega_{2j}^U.$$

The Atiyah-Hirzebruch sequence collapses only for complexes X with torsion-free integral cohomology groups; however even for such X the adjoined ring $G\Omega_U^*(X)$ need not be isomorphic to $\Omega_U^*(X)$.

4. The structure of the ring $\Omega_{SU}^* \otimes \mathbb{Z}_{(p)}$, for $p > 2$, as well as some of 2-primary torsion of Ω_{SU}^*, were described by Novikov in the early 1960s. The structure of $\Omega_{SU}^* \otimes \mathbb{Z}_{(2)}$ was subsequently completely elucidated by Conner and Floyd (in the mid-1960s).

5. The case of the symplectic cobordism ring Ω_*^{Sp} turns out to be much more complicated than those of the above cobordism rings. It is known that all torsion of the ring Ω_*^{Sp} is 2-primary (Novikov, 1960), and that the Ray elements $\phi_j \in \Omega_{8j-5}^{Sp}$ (indecomposables of order 2) play an exclusively important role in all computations of the ring Ω_*^{Sp}. The additive structure of Ω_*^{Sp} has been computed up to dimension 120 by means of the Adams spectral sequence (Kochman, mid 1980), and the ring structure in the same dimensions has now essentially been determined by Vershinin via the Adams-Novikov spectral sequence (1990). Vershinin also shows that the cobordism ring of symplectic manifolds with singularities (Ray elements) reduces, in fact, to the cobordism ring Ω_*^U. Using this Botvinnik has described the Adams-Novikov spectral sequence for symplectic cobordism in terms of cobordism with singularities; further computations have led to the detection of nontrivial elements of Ω_*^{Sp} of order 2^k for any k (Botvinnik, Kochman, 1992). □

The work of Conner and Floyd focussed attention on the fact that there exist *Chern characteristic classes* in complex cobordism theory $\Omega_U^*(\cdot)$ (and, analogously, there exist Stiefel-Whitney classes in unoriented cobordism theory $\Omega_O^*(\cdot)$). These characteristic classes with values in cobordism groups are direct analogues of the Chern and Stiefel-Whitney classes in ordinary cohomology.

In the case of a manifold M^n and a vector bundle η over M^n, this implies that the cycles in M dual to the characteristic classes $c_j(\eta)$, $w_i(\eta)$ may always be realized as images of manifolds. The first indication of this property of the characteristic cycles was given by Gamkrelidze (in the early 1950s) when he proved that the characteristic cycles of algebraic varieties in $\mathbb{C}P^n$ may be realized by algebraic subvarieties.

As will be indicated in Chapter 4, §3 (see also §6 above) there exists a close link between the Stiefel-Whitney characteristic classes in $\mathbb{Z}/2$-cohomology and the Steenrod squares, given by the formula of Thom; in fact one has a representation of all Steenrod operations $a \in \mathcal{A}_2$ in the cohomology of the product $\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty$:

$$a \mapsto a(u), \quad u = u_1 \cdots u_n \in H^n(\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty).$$

By means of this representation one can define appropiate analogues of the Steenrod operations for the cobordism theories $\Omega_O^*(\cdot)$ and $\Omega_U^*(\cdot)$.

In what follows we confine ourselves to complex cobordism theory $\Omega_U^*(\cdot)$. (The theory $\Omega_O^*(\cdot)$ is analogous, with considerable trivialization occuring.) In the cobordism group $\Omega_U^2(X)$ of any CW-complex X, there is a subset $W(X)$ of "geometric cobordism" classes, defined as follows. Since the Thom space $M(U_2)$ may be identified with $\mathbb{C}P^\infty = K(\mathbb{Z}, 2)$ (see Chapter 4, §3), there corresponds to each cohomology class $z \in H^2(X;\mathbb{Z})$ a unique homotopy class of maps

$$f_z : X \longrightarrow M(U_2) = \mathbb{C}P^\infty,$$

such that $f_z^*(u) = z$, where $u \in H^2(\mathbb{C}P^\infty;\mathbb{Z})$ is a generator; the map f_z thus determines a cobordism class (*geometric cobordism class*) $[z] \in \Omega_U^2(X)$. The set $W(X)$ of all geometric cobordism classes corresponds one-to-one with $H^2(X;\mathbb{Z})$; however this correspondence is not additive.

For U-manifolds M^{2n} one obtains a dual (to $W(M^{2n})$) set of *divisors*

$$\{D[z]\} \subset \Omega^U_{2n-2}(M^{2n}),$$

also not additive in $\Omega^U_{2n-2}(M^{2n})$.

For each complex vector bundle η over X we define its *first Chern characteristic class* $\sigma_1(\eta) \in \Omega_U^2(X)$ to be the geometric cobordism class $[c_1(\eta)]$ corresponding, as above, to the first Chern class $c_1(\eta)$ in ordinary cohomology. The further *Chern characteristic classes* $\sigma_j(\eta) \in \Omega_U^{2j}(X)$, $1 < j \leq dim_{\mathbb{C}}\eta$, are then determined by general functorial properties and the Whitney formula:

$$\sigma(\eta_1 \oplus \eta_2) = \sigma(\eta_1)\sigma(\eta_2), \quad \sigma(\eta) = 1 + \sigma_1(\eta) + \sigma_2(\eta) + \cdots. \tag{9.5}$$

The images of the $\sigma_j(\eta)$ in the ordinary cohomology groups are again the ordinary Chern characteristic classes. The Chern characteristic classes are closely related to cohomology operations in complex cobordism theory $\Omega_U^*(\cdot)$; the latter are determined by the following conditions:

1. For each finite sequence $\omega = (\omega_1, \ldots, \omega_k)$ of integers there is a stable, additive operation

$$s_\omega : \Omega_U^j(X) \longrightarrow \Omega_U^{j+2|\omega|}(X), \quad \deg s_\omega = 2|\omega|, \quad |\omega| = \sum_{i=1}^k \omega_i. \tag{9.6}$$

2. The following product formula holds:

$$s_\omega(ab) = \sum_{(\omega',\omega'')=\omega} s_{\omega'}(a)s_{\omega''}(b).$$

3. The operation s_0 is the identity operator: $s_0 = 1$; if $a \in \Omega_U^2(X)$ is a geometric cobordism class, then

$$s_\omega a = \begin{cases} a^{\omega_1+1} & \text{for} \quad \omega = (\omega_1), \\ 0 & \text{for} \quad \omega = (\omega_1, \ldots, \omega_k), \quad k > 1. \end{cases}$$

4. If a product $s_{\omega'} \circ s_{\omega''}$ of total degree $< 2n$, vanishes on the element

$$u_1 \cdots u_n \in \Omega_U^{2n}(\mathbb{C}P_1^\infty \times \cdots \times \mathbb{C}P_n^\infty),$$

where the $u_j \in \Omega_U^2(\mathbb{C}P_j^\infty)$ are the standard generators, then the product is the null operation: $s_{\omega'} \circ s_{\omega''} = 0$. From this condition the composition formula follows:

$$s_{\omega'} \circ s_{\omega''}(u) = \sum_\omega \lambda_\omega(\omega', \omega'') s_\omega,$$

where the $\lambda_\omega(\omega', \omega'')$ are integers. It follows also that the element $s_\omega(u)$, $u = u_1 \cdots u_n$, has the form of a monomial "symmetrized with respect to $\omega_1, \dots, \omega_n$":

$$s_\omega(u) = \left(\sum u_{i_1}^{\omega_1} \cdot \ldots \cdot u_{i_n}^{\omega_n}\right) u_1 \cdot \ldots \cdot u_n.$$

The operations s_ω are called *Landweber-Novikov operations.* These afford a basis for the Landweber-Novikov (LN) algebra S over $\mathbb{Z}$, which is a Hopf algebra with the obvious (diagonal) comultiplication, defined according to the same scheme as Milnor uses for defining the classical Steenrod algebra: the diagonal is determined by the "Leibniz formula" for the action of the operations on the product of two elements (see above). Such actions of Hopf algebras on modules equipped with a multiplication, are therefore called *Milnor modules.*

It is not difficult to show that the algebra A^U of all stable operations in complex cobordism theory consists precisely of the linear combinations

$$\sum_j \lambda_j s_{\omega(j)},$$

where the $\lambda_j \in \Lambda = \Omega_U^*$ have negative dimensions (degrees):

$$\lambda_j \in \Omega_U^{-2q_j} = \Lambda^{-2q_j}, \quad \deg \lambda_j = -2q_j,$$

and the elements s_ω have positive degrees:

$$\deg s_\omega = 2|\omega| = \sum_{j=1}^k \omega_j.$$

The algebra A^U has the grading

$$A^U = \sum_{-\infty}^{\infty} A_{2n}^U,$$

where A_{2n}^U consists of the operations of degree $2n$, i.e. of the linear combinations

$$\sum_j \lambda_j s_{\omega(j)}, \quad \deg \lambda_j + \deg s_{\omega(j)} = 2n.$$

Here infinite formal series are allowed, with $\deg s_{\omega(j)} \longrightarrow \infty$ as $j \longrightarrow \infty$.

The action of the operations s_ω on the cobordism ring $\Omega_U^* = \Lambda$ can be described explicitly using particular manifolds. Each element of Λ may be represented as a polynomial with rational coefficients in the cobordism classes of the manifolds $\mathbb{C}P^1, \dots, \mathbb{C}P^n, \dots$. The tangent bundle $\tau(\mathbb{C}P^n)$ decomposes as follows (see Chapter 4, §1):

$$\tau(\mathbb{C}P^n) + \varepsilon_{\mathbb{C}}^1 = (n+1)\eta,$$

where η is the canonical line bundle over $\mathbb{C}P^n$, for which we have

$$\sigma_1(\eta) = u \in \Omega_U^2(\mathbb{C}P^n),$$

$$D(u) = \left[\mathbb{C}P^{n-1}\right] \in \Omega_{2n-2}^U(\mathbb{C}P^n).$$

It follows that the stable normal bundle $\nu(\mathbb{C}P^n)$ may be identified with $-(n+1)\eta$. For the element $-\eta \in K^0(\mathbb{C}P^n)$ the Chern characteristic classes in cobordism are as follows:

$$\sigma_j(-\eta) = (-1)^j u^j, \quad Du^j = \left[\mathbb{C}P^{n-j}\right].$$

We now define the characteristic class σ_ω, $\omega = (\omega_1, \dots, \omega_k)$, as follows. Consider the symmetric polynomial

$$Q^\omega(u_1, \dots, u_k) = \sum_\alpha u_1^{\omega_{\alpha(1)}} \cdot \ldots \cdot u_k^{\omega_{\alpha(k)}},$$

where summation is over all elements α of the symmetric group on k letters. The polynomial $Q(u_1, \dots, u_k)$ is uniquely expressible as a polynomial $P^\omega(\sigma_1, \sigma_2, \dots)$ in the generators σ_j, where σ_j is the jth elementary symmetric polynomial in $u_1, \dots, u_k$. Let σ_ω, $\omega = (\omega_1, \dots, \omega_k)$, denote the characteristic class

$$\sigma_\omega = P^\omega(\sigma_1, \sigma_2, \dots).$$

We now apply the class σ_ω to the bundle $-(n+1)\eta$ (stably equivalent to the normal bundle $\nu(\mathbb{C}P^n)$), whose Chern class $\bar{\sigma}(-(n+1)\eta)$ is the following:

$$\bar{\sigma}(-(n+1)\eta) = 1 + \bar{\sigma}_1 + \bar{\sigma}_2 + \cdots = \frac{1}{(1+u)^{n+1}}.$$

A simple computation shows that the dual of the class $\sigma_\omega(-(n+1)\eta) \in \Omega_U^{2|\omega|}(\mathbb{C}P^n)$ is the cobordism class $\left[\mathbb{C}P^{n-|\omega|}\right]$ times an integer $\lambda(\omega)$. This determines the cobordism operation s_ω as follows:

$$s_\omega(\mathbb{C}P^n) = \lambda(\omega)\left[\mathbb{C}P^{n-|\omega|}\right], \quad |\omega| = \sum_{j=1}^k \omega_j. \tag{9.7}$$

A complete description of the action of the cobordism operations on the cobordism ring $\Omega_U^* = \Lambda$ is now obtained from (9.7) using the properties:

$$\begin{aligned} s_\omega(\mu) &= \sum_{(\omega',\omega'')=\omega} s_{\omega'}(\mu)\cdot s_{\omega''}(\mu), \quad \mu\in\Lambda, \\ s_\omega\circ\mu &= \sum_{(\omega',\omega'')=\omega} s_{\omega'}(\mu)\circ s_{\omega''}, \qquad \mu\in\Lambda, \end{aligned} \tag{9.8}$$

together with $s_0 = 1$ (for the empty ω). (This was established by Novikov in the mid-1960s.) This representation of the algebra A^U of operations of complex cobordism theory $\Omega_U^*(\cdot)$ by means of its action on the coefficient ring $\Omega_U^* = \Lambda$, is faithful, i.e. if $a \in A^U$ acts trivially on Λ then $a = 0$. This property represents a cardinal distinction between complex cobordism theory and ordinary cohomology.

It is of interest to note that the algebra A^U may be given as the completion of the product of two algebras with respect to the topology of formal series:

$$A^U = (\Lambda\otimes S)^T.$$

This product is isomorphic as vector space to the tensor product of these algebras, but the multiplication is not the usual one. They form non-commuting subalgebras; the Novikov formula for their commutator given above (see 9.8) depends on the action of the Landweber-Novikov algebra S on the complex cobordism ring described above. What is significant here is that this action determines a "Milnor module" over the Hopf algebra S, i.e. that it satisfies the Leibniz product formula with respect to the comultiplication. The algebra A^U is realized as the algebra of operators on the (left) Milnor module Λ over the Hopf algebra S, realized in turn as the algebra of differential operators with constant coefficients, acting on the functions "belonging to the ring Λ". This is a special case of a rather general construction of "operator algebras" by Novikov (in the early 1990s) in the context of the theory of Hopf algebras and Milnor modules over them.

In 1978 Buchstaber and Shokurov observed that the natural dual Hopf algebra to the Landweber-Novikov algebra S, contains the ring Λ of "complex cobordisms for the one-point space":

$$\Lambda\subset X^*,\quad \Lambda\otimes\mathbb{Q} = S^*\otimes\mathbb{Q}.$$

(Via the Chern numbers (giving the integral structure of Λ as known from earlier results of Milnor and Novikov) the ring Λ was analysed by Buchstaber from this new point of view in the late 1970s; its structure turned out to be highly non-trivial.) We may therefore regard this construction as a special case of the very general construction of the "operator double" of the Hopf algebra S and its dual S^*, in the sense of Novikov (1992). There are natural actions R_a^*, L_b^* of an arbitrary algebra S on its dual space S^*, which are adjoint to right and left multiplication respectively:

$$R_b(a) = L_a(b) = ab,\quad (R_y^*(z), x) = (z, R_y(x)) = (z, xy).$$

In the situation of Hopf algebras one has the Leibniz property, i.e. S^* is a left (or right) Milnor module over S in the natural way; in fact S^* can be made into a left Milnor module over the larger Hopf algebra $S \otimes S^t$ using the representation $\rho_\pm$ satisfying

$$\rho_{\pm,a\otimes b}(x) = R_a^*(L_{s^{\pm 1}b}^*(x)),$$

where s indicates the "antipodes" of a Hopf algebra, and S^t denotes the algebra S with "transposed" comultiplication and consequently with the opposite antipodes. Restriction of the action of $\rho_{\pm 1}$ to the diagonal $\Delta(S) \subset S \otimes S^t$ yields an "ad-module". The diagonal restriction of the operator algebra acting on S^* (as Milnor module) leads to Drinfeld's "quantum double" of the Hopf algebra S, again a Hopf algebra, with diagonal as in S^* and S^t.

Buchstaber and Shokurov have identified the algebra A^U with the algebra of differential operators on the group of formal diffeomorphisms of the real line fixing 0 and with first derivative at 0 equal to 1. It follows (as was pointed out by Novikov in 1992) that the algebra A^U is the operator double of the Landweber-Novikov algebra S corresponding to the Milnor module S^* equipped with the action R_a^*; this affords a basis for the general notion of the "quantum analogue of rings of differential operators" according to Novikov's scheme (1992), where such an analogue was considered in the case of Fourier transforms. Various "almost-Hopf" properties of such operator doubles were investigated by Novikov (in 1992) and Buchstaber (1994). A special case of this construction of operator doubles was investigated independently in 1992 by Semenov-Tyanshanskiĭ, Faddeev and Alexeev, who used instead the term "Heisenberg double".

It turns out that complex cobordism theory $\Omega_U^*(\cdot)$ may be used very effectively for computing stable homotopy classes of maps of finite complexes by means of the *Adams-Novikov spectral sequence* (an analogue of the Adams spectral sequence described in §6,7 above). The group of stable homotopy classes of maps $K \longrightarrow L$ is defined by

$$[K, L]^s = \lim_{j\to\infty} [\Sigma^j K, \Sigma^j L].$$

Set

$$[K, L]_q^s = [\Sigma^q K, L]^s, \quad [K, L]_*^s = \bigoplus_q [K, L]_q^s.$$

Thus from the stable homotopy classes of maps $K \longrightarrow L$, we obtain the graded abelian group $[K, L]_*^s$. For $K = L = S^0$, in particular, we obtain the direct sum of stable homotopy groups of spheres:

$$[S^0, S^0]_*^s = \bigoplus_{q\geq 0} \pi_q^s = \bigoplus_{q\geq 0} \pi_{N+q}(S^N), \quad N > q + 1.$$

How does one compute $[K, L]_*^s$? Let M, N denote the graded A^U-modules $\Omega_U^*(K)$, $\Omega_U^*(L)$ respectively. There exists a spectral sequence (*the Adams-Novikov spectral sequence*)

$$E_m^{p,q}, \quad d_m : E_m^{p,q} \longrightarrow E_m^{p-m,q+m-1}, \quad d_m^2 = 0, \quad E_{m+1}^{*,*} = H(E_m^{*,*}, d_m), \quad (9.9)$$

with second term

$$E_2^{p,q} = \mathrm{Ext}_{A^U}^{p,q}(N, M),$$

and adjoined group

$$G\,[K, L]_n^s = \sum_{p-q=n} E_\infty^{p,q}.$$

Note that $\mathrm{Ext}_{A^U}^{0,q}(N, M)$ coincides with the set of A^U-module homomorphisms

$$N = \Omega_U^*(K) \longrightarrow \Omega_U^*(\Sigma^q L) = \Sigma^q M,$$

of degree q:

$$\mathrm{Ext}_{A^U}^{0,q}(N, M) \cong \mathrm{Hom}_{A^U}^q(N, M).$$

Here $\Sigma^q M$ coincides with M, but the grading is shifted by q.

The definition of the functor $\mathrm{Ext}_A^{*,*}(\ ,\ )$ (basic to homological algebra), where A is a ring, is as follows (in the case when $A = A^U$). Given any A^U-module N, we can form its *free acyclic resolution* (respecting the grading on N); this is a complex C of A^U-modules of the form

$$\cdots \longrightarrow C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} C_{n-2} \xrightarrow{d} \cdots \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{d} N \longrightarrow 0,$$

where all of the C_j are free A^U-modules (whose free generators can be chosen compatibly with the grading), the operators d preserve the grading, and, finally, $\mathrm{Ker}\ d = \mathrm{Im}\ d$ everywhere. We therefore have $N \cong C_0/\mathrm{Im}\ d$. We now form the dual complex $C^* = \mathrm{Hom}_{A^U}(C, M)$:

$$\cdots \xleftarrow{\partial^*} C_n^* \xleftarrow{\partial^*} C_{n-1}^* \xleftarrow{\partial^*} \cdots \xleftarrow{\partial^*} C_1^* \xleftarrow{\partial^*} C_0^* \longleftarrow 0,$$

where

$$C_n^* = \bigoplus_q C_n^{*,q} = \bigoplus_q \mathrm{Hom}_{A^U}^q(C_n, M) = \bigoplus_q \mathrm{Hom}_{A^U}^0(C_n, \Sigma^q M).$$

The homology groups of the complex C^* are then the objects $\mathrm{Ext}_{A^U}^{*,*}$:

$$\bigoplus_q \mathrm{Ext}_{A^U}^{p,q}(N, M) = \bigoplus_q \mathrm{Ker}\ \partial^*/\mathrm{Im}\ \partial^*.$$

It can be shown that these groups do not depend on the choice of the free A^U-module resolution C with the prescribed properties.

Remark. The objects $\mathrm{Ext}_A^{*,*}$ for an arbitrary ring A, were first introduced into homological algebra by Eilenberg-MacLane in the late 1940s. The following theorem is due to them:

If $A = \mathbb{Z}[\pi]$, the integral group ring of a group π, and $\mathbb{Z}$ is given the trivial A-module structure, then there is an isomorphism for any A-module M:

$$\mathrm{Ext}^q_A(\mathbb{Z}, M) \cong H^q(K(\pi,1); M). \qquad \square$$

In the case $K = S^0$, the Adams-Novikov spectral sequence converges to the stable homotopy groups of the complex L:

$$M = \Lambda = \Omega^*_U, \quad N = \Omega^*_U(L),$$

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{A^U}(N, \Lambda).$$

The existence of the Adams-Novikov spectral sequence (9.9) with second term $\mathrm{Ext}^{*,*}_{A^U}$ represents a quite general categorical property of a generalized cohomology theory $h^*(\cdot)$ defined on the *stable category* of stable homotopy types and stable homotopy classes of maps. (Note that the stable category is not abelian in the sense of Grothendieck, although stable homotopy classes of maps always come with the natural structure of a graded abelian group. Note also that in K-theory there is no such spectral sequence since that theory is defined in terms of non-stable objects.) The usefulness of a generalized cohomology theory $h^*(\cdot)$ for solving particular problems depends of course on the specific properties of that theory: how efficiently the algebra A^h of stable operations can be computed; to what extent the structure of the A^h-module $h^*(X)$ can be worked out for particular spaces X, and, finally, how many non-trivial differentials d_m there are in the Adams-Novikov spectral sequence corresponding to the theory $h^*(\cdot)$ (the fewer, the better). In the latter respect complex cobordism theory $\Omega^*_U(\cdot)$ has a significient advantage over ordinary $\mathbb{Z}/p$-cohomology theory (especially for primes $p > 2$), and is essentially no worse as far as computations of the appropriate algebraic objects are concerned. For example for the stable homotopy groups of spheres one has

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{A^U}(\Lambda, \Lambda),$$

and the groups $\mathrm{Ext}^{1,*}_{A^U}$ contain the lower estimate of Kervaire-Milnor for the groups $J\pi_{q-1}(SO)$, which is not the case for the ordinary Adams spectral sequence. (For the definition of the J-homomorphism, see §8 above, and Chapter 4, §3.) The groups $\mathrm{Ext}^{2,*}_{A^U}$ yield much new information, as do the higher terms. In the 1970s several authors carried out far-reaching calculations of the Adams-Novikov spectral sequence for spheres (Miller, Ravenel, Wilson, Buchstaber and others).[14]

The first Chern class σ_1 in complex cobordism theory has certain remarkable properties. It was mentioned above that for a complex line bundle η, the class $\sigma(\eta)$ is a geometric cobordism class. How may one compute the class $\sigma_1(\eta_1 \otimes \eta_2)$ for line bundles η_1, η_2?

Write $\sigma_1(\eta_1) = u$, $\sigma_1(\eta_2) = v$. It can be shown that the class $\sigma_1(\eta_1 \otimes \eta_2)$ is representable as a power series in u, v (Novikov, in the mid-1960s):

[14] *Translator's note:* For a general account of the results on computations in the Adams-Novikov spectral sequence for spheres see the book *D. Ravenel, Complex Cobordism and Stable Homotopy of Spheres, Academic Press, 1986.*

$$\sigma_1(\eta_1 \otimes \eta_2) = f(u,v) = u + v + \sum_{i,j\geq 1} \lambda_{ij} u^i v^j, \quad \lambda_{ij} \in \Omega_U^* = \Lambda,$$

for which the defining conditions of a commutative *formal group* hold:

$$f(u, f(v,w)) = f(f(u,v),w), \qquad f(u,v) = f(v,u). \tag{9.10}$$

Let η^{-1} be a complex line bundle such that $\eta \otimes \eta^{-1}$ is the trivial line bundle. The Chern class $\sigma_1(\eta^{-1}) = \bar{u}$ say, is a power series in $u = \sigma_1(\eta)$ of the following form:

$$\bar{u} = -u + \sum_{i\geq 2} \mu_i u^i, \quad \mu_i \in \Lambda,$$

with

$$f(u, \bar{u}) = 0. \tag{9.11}$$

The resulting formal group $f(u,v)$ of geometric cobordism classes can be explicitly calculated in term of the series

$$g(u) = \sum_{n\geq 0} \mathbb{C}P^n \frac{u^{n+1}}{n+1},$$

namely as (Mishchenko, in the late 1960s):

$$f(u,v) = g^{-1}(g(u) + g(v)). \tag{9.12}$$

By means of the formal group $f(u,v)$ one may define (and calculate) analogues in $\Omega_U^*(\cdot)$-theory of the Adams operations constructed in K-theory using representations of the groups U_n. We define

$$\Psi^k(u) = \frac{1}{k} g^{-1}(kg(u)), \quad u \in \Omega_U^2(\mathbb{C}P^\infty),$$

where u is as before the geometric cobordism class $\sigma_1(\eta)$. We require that the operations Ψ^k have the following properties, analogous to those of the Adams operations in K-theory:

$$\Psi^{kl} = \Psi^k \Psi^l, \qquad \Psi^k(x+y) = \Psi^k(x) + \Psi^k(y), \qquad \Psi^k(x\cdot y) = \Psi^k(x)\Psi^k(y). \tag{9.13}$$

The operation Ψ^k is then well-defined in the theory $\Omega_U^*(\cdot) \otimes \mathbb{Z}\left[\frac{1}{k}\right]$ by (9.13) (Novikov, in the late 1960s).

Let k be any algebra over the rationals. In many situations it is useful to have a classification of the ring homomorphisms $Q : \Omega_U^* \longrightarrow k$. It was shown by Hirzebruch that such a homomorphism Q is determined by a series $Q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ (see Chapter 4, §1–3). Writing $Q(\mathbb{C}P^N)$ for the value of Q at $\mathbb{C}P^n$, one has the formula (Novikov, late 60s)

$$g_Q(u) = \sum_{n\geq 0} Q(\mathbb{C}P^n)\frac{u^{n+1}}{n+1} = Q(g(u)),$$

whence

$$Q(z) = \frac{z}{g_Q^{-1}(z)}.$$

Thus we see that the series $g(u)$ determining the above formal group, makes an appearance also in this context.

A natural analogue of the Chern character (the *Chern-Dold character*, whose existence was first established by Dold) can be effectively constructed by means of the series $g(u)$. This character is given by a Λ-algebra homomorphism

$$ch_U : \Omega_U^*(K) \longrightarrow H^*(K; \Lambda \otimes \mathbb{Q}), \quad \Lambda = \Omega_U^*, \tag{9.14}$$

determined by the requirement that for $K = \mathbb{C}P^\infty$ and u the same generator as above,

$$ch_U(g(u)) = t \in H^2(\mathbb{C}P^2; \mathbb{Z}), \tag{9.15}$$

where t is the canonical generator. (The theory of the character ch_U and various applications of it were introduced by Buchstaber in the late 1960s.)

At the end of the 1960s Quillen showed that the above formal group of geometric cobordism classes is in fact a geometric realization of the universal Lazard formal group of the theory of one-dimensional formal groups. He was able to calculate effectively important projective operators in the algebra $A^U \otimes \mathbb{Z}_{(p)}$ (where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at the prime p), thereby completing the computation of the "reduced" complex cobordism theories (now known as "Brown-Peterson" theories $BP^*(\cdot)$) into which the theory $\Omega_U^*(\cdot)\otimes\mathbb{Q}_p$ decomposes as their direct sum. (The existence of the theories $BP^*(\cdot)$ was established (non-effectively) by Brown and Peterson in the mid 1960s. Such projectors had earlier, though not sufficiently effectively, been constructed by Novikov.) This approach has proved effective for computing the groups $\mathrm{Ext}_{A^U}^{*,*} \otimes \mathbb{Z}_{(p)}$ figuring in stable homotopy theory (see above).[15]

Later (in the early 1970s) two-valued analogues of the formal groups were introduced into cobordism theory by Buchstaber and Novikov. Through the 1970s the associated algebraic theory was developed by Buchstaber, who also

[15] *Translator's note:* Formal group theory has been applied to great effect in computions of the Adams-Novikov spectral sequence for spheres (Miller, Morava, Ravenel, Wilson, and others). There are other cohomology theories generalizing both K-theory and complex cobordism theory, namely the "Morava K-theories" $K(n)^*(\cdot)$, that have been effectively formulated in terms of formal group theory, and have proved very useful in computations with the Adams-Novikov spectral sequence as well as for investigations of the general properties of the stable homotopy category. (For details, see D. Ravenel, *Complex Cobordism and Stable Homotopy of Spheres*, Academic Press, 1986.)

gave several applications of this theory to symplectic and self-dual cobordism theories. Algebraic topology from the point of view of cobordism theory has been elaborated by several authors (Dieck, Gottlieb, Becker, and others); however we shall not pursue these further developments here.[16]

In the early 1960s it was observed by Conner and Floyd that bordism theory can be used to obtain various results concerning smooth actions of finite groups and compact Lie groups on closed manifolds. We shall now consider some of these results. Suppose for instance that we are given a diffeomorphism of order p (prime) $T : M^{2n} \longrightarrow M^{2n}$, $T^p = 1$, where M^{2n} is a closed U-manifold, with isolated fixed points $p_1, \dots, p_m$. The restriction of the diffeomorphism T to a sufficiently small sphere in M^{2n} centered at p_j has no fixed points, and determines a singular complex bordism, namely a map of the appropriate lens space (as orbit space) to the Eilenberg-MacLane space $K(\mathbb{Z}/p, 1)$:

$$\begin{gathered} f_j : S_j^{2n-1}/\mathbb{Z}/p \longrightarrow S_j^{\infty}/\mathbb{Z}/p = K(\mathbb{Z}/p, 1), \\ w_j^T = \left(S_j^{2n-1}/\mathbb{Z}/p, f_j\right), \quad w_j^T \in \Omega_{2n-1}^U\left(S_j^{\infty}/\mathbb{Z}/p\right). \end{gathered} \tag{9.16}$$

The bordism class of w_j^T depends only on the differential $dT : \tau(M)_{p_j} \longrightarrow \tau(M)_{p_j}$ at the point p_j, i.e. on the eigenvalues $\lambda_1, \dots, \lambda_n$ of the complex matrix $A_j = dT_{p_j}$:

$$w_j^T = w_j^T(\lambda_1, \dots, \lambda_n), \quad \lambda_q(A_j) = \exp(2\pi x_{jq}/p), \quad x_{jq} \in (\mathbb{Z}/p)^*,$$

where the x_{jq} are non-zero residues modulo p. By removing open neighbourhoods bounded by the spheres S_j^{2n-1} from the manifold M^{2n}, one obtains a manifold on which $\mathbb{Z}/p$ acts without fixed points. In particular, this yields the relation

$$\sum_j w_j^T = 0.$$

in the group $\Omega_{2n-1}^U\left(S_j^{\infty}/\mathbb{Z}/p\right)$.

In the case $p = 2$ we have $x_{jq} = 1$, and from knowledge of the structure of the group

$$\Omega_{2n-1}^U\left(S_j^{\infty}/\mathbb{Z}/2\right) = \Omega_{2n-1}^U\left(\mathbb{R}P^{\infty}\right)$$

we infer (see below)

$$w_j^T = a_n, \quad 2^{n-1}a_n \neq 0, \quad 2^n a_n = 0,$$

so that the number of fixed points must be divisible by 2^n (Conner and Floyd, in the mid-1960s).

[16] The theory of two-valued formal groups (developed in the late 1970s — mid 1980s by Buchstaber) provides a powerful tool for computations in the Adams-Novikov spectral sequence for the simplectic cobordism ring (Buchstaber, Ivanovskiĭ, Nadiradze, Vershinin).

The case $p > 2$ is more complicated since the bordism classes w_j^T (our invariants of the fixed points) depend on the residues $x_{jq} \in (\mathbb{Z}/p)^*$ for which of course there are more choices than before. It turns out that the classes w_j^T can be effectively calculated as functions of the residues (Novikov, Mishchenko, Kasparov, in the late 1960s): writing $w_j^T = \alpha(x_{j1}, \dots, x_{jn})$, we have

$$\alpha(x_{j1}, \dots, x_{jn}) = \prod_{q=1}^{n} \frac{u}{g^{-1}(x_{jq} g(u))} \cap \alpha(1, \dots, 1). \tag{9.17}$$

The groups $\Omega_U^*\left(S^{2n-1}/\mathbb{Z}/p\right)$ and $\Omega_*^U\left(S^{2n-1}/\mathbb{Z}/p\right)$ for the lens spaces $S^{2n-1}/\mathbb{Z}/p$ may be described explicitly using formal group theory as follows. (We note that the first partial results about these cobordism and bordism groups were obtained by Conner and Floyd in the early 1960s.) Consider the spaces $S_j^{2n+1}/\mathbb{Z}/p \subset S_j^\infty/\mathbb{Z}/p$ as representatives of the bordism classes $\alpha(x_{j1}, \dots x_{jn}) \in \Omega_{2n+1}^U\left(S^\infty/\mathbb{Z}/p\right)$. The geometric cobordism class $u \in \Omega_U^2\left(S^{2n-1}/\mathbb{Z}/p\right)$ generates the ring $\Omega_U^*\left(S^{2n+1}/\mathbb{Z}/p\right)$ (as Ω_U^*-module) with relations:

$$u^{n+2} = 0, \quad g^{-1}(pg(u)) = 0,$$

$$Du^k = \alpha(\underbrace{1, \dots, 1}_{n-k \text{ times}}) = \alpha_{n-k} \in \Omega_U^{2(n-k)+1}\left(S^{2n+1}/\mathbb{Z}/p\right), \tag{9.18}$$

$$\alpha_{n-k} \circ \alpha_{n-l} = \alpha_{n-k-l}.$$

The sets $\{x_{j1}, \dots, x_{jn}\}$ of residues $(j = 1, \dots, m)$ actually realizable as corresponding to the set of fixed points $p_1, \dots, p_m \in M^{2n}$ of some $\mathbb{Z}/p$-action on M^{2n}, must satisfy

$$\sum_{j=1}^{m} \alpha(x_{j1}, \dots, x_{jn}) = 0. \tag{9.19}$$

Bringing in the formula (9.17), we obtain from (9.19) a system of equations in the mn variables x_{jq} ($\not\equiv 0 \bmod p$) determining the set of admissible residues x_{jq}.

We remark that the bordism class $\left[M^{2n}\right]$ in the group $\Omega_U^{2n} \otimes \mathbb{Z}/p$ can also be computed via the residues x_{jq}.

The above discussion carries over to oriented cobordism theory, with some simplification, by means of the homomorphism $\Omega_U^* \longrightarrow \Omega_{SO}^*$. Here, however, the case $p = 2$ is special and requires different techniques. The formula (9.17) generalizes to the situation where the fixed point set of the transformation is a submanifold of M with a nontrivial normal bundle. The situation where a cyclic group of composite order acts on a manifold has also been investigated (by Mishchenko, Gusein-Zade, Krichever, in the early 1970s).

Especially interesting results have been obtained in the situation where the circle S^1 acts smoothly on an oriented or U-manifold. For instance, Gusein-

Zade (ca. 1970s) has completely described the bordism groups arising in connexion with actions of S^1 on manifolds such that there are no points fixed by the whole group S^1. It turns out that these bordism groups contain in a natural way invariants of (and relations between) the fixed points for *any* smooth action of the circle on a manifold. Several elegant results of this type for smooth actions of S^1 on U-manifolds, using complex cobordism theory and formal group theory, were obtained by Krichever in the first half of the 1970s. We shall now describe two such results.

1. Let $A_{(k)}$ be the characteristic class determined by the series

$$A_{(k)} = \frac{kue^u}{e^{ku} - 1},$$

(see Chapter 4, §3); then for each U-manifold M^{2n} this characteristic class determines the rational number $A_{(k)}(M^{2n})$. If the circle S^1 acts smoothly on a U-manifold M^{2n} for which k divides $c_1(M^{2n})$, then $A_{(k)}(M^{2n}) = 0$. In the case $k = 2$, the number $A_{(2)}(M^{2n})$ coincides with the signature of M^{2n}. (The latter result was proved by Atiyah-Hirzebruch via the Atiyah-Bott formulae for *Spin*-manifolds.)

2. If S^1 acts on a U-manifold M^{2n}, then the linearization of this action on the normal bundle to the submanifold $N_j^k \subset M^{2n}$ $(j = 1, \dots, m)$ of fixed points, has eigenvalues

$$\lambda_{1j}, \dots, \lambda_{n-k,j}, \quad \lambda_{qj} = \exp(2\pi i n_{qj}),$$

where the n_{qj} (the *weights*) are integers. Let l_j denote the number of negative weights $n_{qj} < 0$, and let T_y be the characteristic class with values polynomials in y, determined by the Hirzebruch series

$$T_y(u) = \frac{(y+1)u}{1 - \exp(-u(y+1))} - yu.$$

Then the following formula holds:

$$\sum_{j=1}^{m} y^{l_j} T_y(N_j^k) = T_y(M^{2n}). \tag{9.20}$$

For $y = 1$, the value of T_1 on manifolds coincides with their signature, and this case of the above result was obtained earlier by Atiyah-Hirzebruch using the theory of elliptic operators. Putting $y = -1$ yields the Euler-Poincaré characteristic: $T_{-1}(M) = \chi(M)$, and (9.20) reduces to the classical formula for $\chi(M)$.[17]

[17] Around 1990 further important results were obtained concerning actions of the circle on manifolds. Certain of the ideas of "global analysis on loop spaces" first appeared here (Witten, Taubes), the elegant notion of "elliptic genera" (Oshanine, Landweber), and certain generalizations of these such as the Baker-Akhieser functions (Krichever).

Chapter 4
Smooth Manifolds

§1. Basic concepts. Smooth fiber bundles. Connexions. Characteristic classes

The topology and geometry of *smooth (differentiable) manifolds* is the most important area of study in topology, the source of and most fruitful field for applications of the whole complex of topological methods, closely linked with analysis and, particularly in recent times, with contemporary mathematical physics. The elementary theory of smooth manifolds was created by Whitney in the mid-1930s.

The internal definition of a differentiable manifold is as follows: it is in the first place an (n-dimensional) *topological* manifold (see the conclusion of §1 of Chapter 2), i.e. a Hausdorff topological space X for which there is a covering collection of open sets V_α, $\bigcup_\alpha V_\alpha = X$, each homeomorphic to (an open region of) $\mathbb{R}^n$. Such homeomorphisms

$$\phi_\alpha : V_\alpha \longrightarrow \mathbb{R}^n$$

then determine *local co-ordinates* $(x_\alpha^1, \dots, x_\alpha^n)$ on each open set (or *chart* or *local co-ordinate neighbourhood*) V_α. The manifold X is said to be *smooth of class* C^k if on each region of intersection $V_\alpha \cap V_\beta$ the *transition functions* expressing one set of co-ordinates in terms of the other are smooth of class C^k, i.e. the maps $\phi_\alpha \phi_\beta^{-1}$ and $\phi_\beta \phi_\alpha^{-1}$ between the appropriate regions of Euclidean n-space are functions of class $C^k, k > 0$. (If $k = \infty$ the manifold is said to be *infinitely differentiable*, or just *smooth*. A *real-analytic* manifold is one for which the transition functions are real-analytic.)

Manifolds will be denoted by M^n, N^k, etc. Note that the Jacobian

$$\det(\partial x_\alpha^j / \partial x_\beta^k)$$

of the transition function $x_\alpha(x_\beta)$ is non-zero everywhere on $V_\alpha \cap V_\beta$, since if it were zero at any point, then the inverse function $x_\beta(x_\alpha)$ would not be smooth.

The development, at the elementary level, of techniques for studying smooth manifolds, requires the existence of *partitions of unity*, i.e. the existence, for each open cover $\{V_\alpha\}$ of a manifold M^n, of open sets $W_\alpha \subset W'_\alpha \subset V_\alpha$ such that $\bigcup_\alpha W_\alpha = M^n$, and of real-valued functions $\psi_\alpha \geq 0$ of class C^∞ defined on M^n, with the following properties (see Figure 4.1):

$\psi_\alpha \equiv 1$ on W_α; $\psi_\alpha \equiv 0$ outside W'_α;

$\infty > \sum_\alpha \psi_\alpha(x) > 0$ for each $x \in M^n$; $\sum \phi_\alpha = 1$, where $\phi_\alpha = \psi_\alpha / \sum \psi_\alpha$.

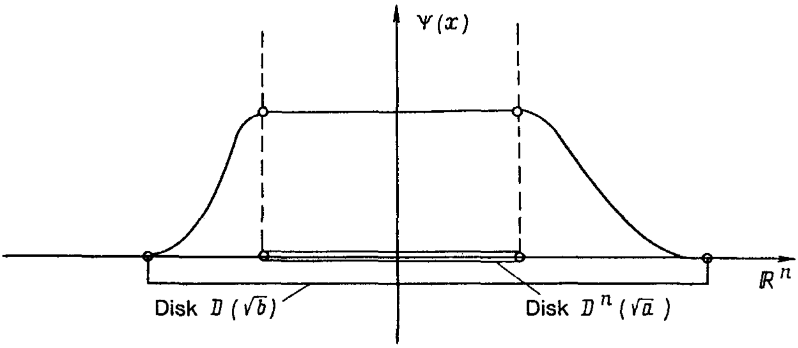

Fig. 4.1

Example. For compact manifolds M^n, partitions of unity exist for all finite covers $\{V_\alpha\}$ of M^n by local co-ordinate neighbourhoods. If we multiply the local co-ordinates x^j_α on each region V_α by the function ψ_α (see above) of a corresponding partition of unity, then the resulting functions $\psi_\alpha x^j_\alpha = y^j_\alpha$ say, are defined and smooth on the whole manifold M^n. If the cover $\{V_\alpha\} = \{V_1, \dots, V_N\}$, then the collection of functions $\{y^j_\alpha\}, j = 1, \dots, n$, $\alpha = 1, \dots, N$, defines a C^∞-embedding

$$\{y^j_\alpha\} : M^n \longrightarrow \mathbb{R}^{nN},$$

since $\psi_\alpha \equiv 1$ on W_α and any two points of W_α are distinguished by their local co-ordinates x^j_α. (For non-compact manifolds the analogous argument yields an embedding into Hilbert space.) Using the idea of "general position" (see below) and projections of the above embedded manifold onto k-dimensional hyperplanes, it can be shown that for $k \geq 2n+1$ the set of smooth embeddings $M^n \longrightarrow \mathbb{R}^k$ is everywhere dense in the space of smooth (even just continuous) maps $M^n \longrightarrow \mathbb{R}^k$, and the set of smooth immersions likewise for $k \geq 2n$. In the case $n = 1$ this is visually obvious — see Figure 4.2. □

A *smooth map* $f : M^n \longrightarrow N^k$ between two smooth manifolds is a map each of whose expressions in terms of local co-ordinate systems $\{x^j_\alpha \mid j = 1, \dots, n\}$ on the charts V_α of M^n and $\{y^s_\gamma \mid s = 1, \dots, k\}$ on the charts U_γ of N^k is made up of smooth functions $y^s_\gamma(x^1_\alpha, \dots, x^n_\alpha)$ (in the usual sense). (Here each domain of definition of these functions has the form

$$f^{-1}(U_\gamma) \cap V_\alpha.)$$

At each point x of M^n the *rank* (rank $f(x)$) of the smooth map f is defined to be the rank of the matrix $(\partial y^s_\gamma / \partial x^j_\alpha)$ evaluated at x. The rank at each point is an invariant of f (in the sense that it is independent of the particular local co-ordinates in use).

(i) $S^1 \longrightarrow \mathbb{R}^3$.
In general position
this is an embedding

(ii) $S^1 \longrightarrow \mathbb{R}^2$.
The projection to $\mathbb{R}^2$
has self-intersections not
removable by means of small
perturbations: the tangent
vector field (in general
position) is non-degenerate

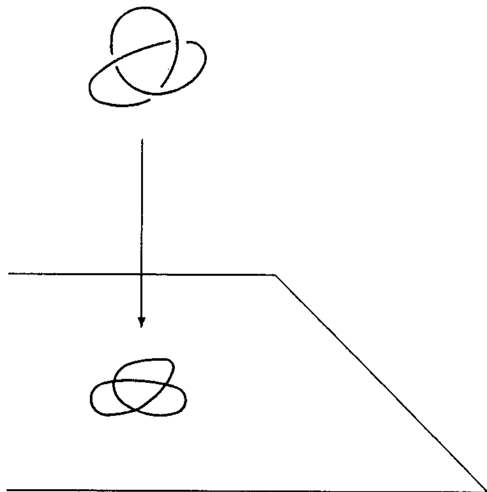

Fig. 4.2

An *immersion* $f : M^n \longrightarrow N^k$ is a smooth map with the property that at every point $x \in M^n$ the rank of f is n (so that we must have $n \leq k$). A *submersion* is defined by the requirement rank $f \equiv k$ (so that $n \geq k$). An *embedding* f is a one-to-one immersion: $f(x) \neq f(y)$ if $x \neq y$. A *diffeomorphism* $f : M_1^n \longrightarrow M_2^n$ between two smooth manifolds is a smooth homeomorphism with smooth inverse.

The smoothness class C^k of a manifold is, beyond the basic assumption that $k \geq 1$, not of great significance for topology, in view of a theorem of Whitney to the effect that any C^k-manifold is diffeomorphic to a unique C^∞-manifold, in fact to a (unique) real-analytic manifold. (It was further shown by Morrey (in the late 1950s) that any real-analytic manifold can be embedded real-analytically in some $\mathbb{R}^n$; this is a difficult result requiring more recent methods of the theory of complex manifolds.) In the sequel we shall therefore frequently assume, without explicit mention, that our smooth manifolds and maps have smoothness class C^∞, and shall not distinguish diffeomorphic manifolds.

Beginning with Whitney, lemmas (due also to Brown, Sard, and others in the 1930s) concerning "general position" have come to play an important role in connexion with the techniques employed in studying the topology of smooth manifolds. The simplest such result states that a real-valued map $f : M^n \longrightarrow \mathbb{R}$ of class C^k for k sufficiently large, always has the property that the image $f(Z)$ of the set Z of critical points (i.e. points $x \in M^n$ for which grad $f(x) = 0$), has measure zero in $\mathbb{R}$. (For compact M^n the set $f(Z)$ is closed, and its complement has positive measure.) More generally, for any map $f : M^n \longrightarrow N^k$ with $k \leq n$, of sufficiently high smoothness class, the set $f(Z)$ has measure zero in N^k, where Z consists of the points $x \in M^n$ for which rank $f(x) < k$. Note also that for any C^1-map $M^n \longrightarrow N^k$ with $k > n$, the image $f(M^n)$ has measure zero in N^k. In all of these situations the points in the codomain N^k outside the set $f(Z)$ ($f(M^n)$ in the latter situation where $k > n$) are said to be *in general position* with respect to f. We shall describe the various essential lemmas about bringing maps into general position, as the need for them arises.

Recall that a *tangent vector* to a (smooth) manifold M^n at a point x_0 of M^n is the velocity vector $(dx^j_\alpha/dt)_{x_0}$ at x_0 to a parametrized curve $x^j_\alpha(t)$ (in local co-ordinates on a chart V_α containing x_0) passing through x_0. The vector space $\mathbb{R}^n_{x_0}$ of all tangent vectors to M^n at x_0 is called the *tangent space* to M^n at x_0.

Let $(\eta^1_\alpha, \ldots, \eta^n_\alpha)$ be any tangent vector to a point $x \in M^n$, in terms of local co-ordinates $x^1_\alpha, \ldots, x^n_\alpha$ on a local co-ordinate neighbourhood containing x. Under a change to co-ordinates $x^1_\beta, \ldots, x^n_\beta$ (with Jacobian matrix $I_{\alpha\beta} = (\partial x^k_\beta/\partial x^j_\alpha)$), the components of the tangent vector transform according to the rule

$$\eta^k_\beta = \eta^j_\alpha \frac{\partial x^k_\beta}{\partial x^j_\alpha} \quad \text{(with summation over } j\text{)}. \tag{1.1}$$

On the other hand the gradient, and, more generally, differential 1-forms given in local co-ordinates by $\sum \eta_{i\alpha} dx^i_\alpha = \eta_{i\alpha} dx^i_\alpha$, transform according to the rule

$$\eta_{i\beta} = \eta_{k\alpha} \frac{\partial x^k_\alpha}{\partial x^i_\beta}. \tag{1.2}$$

More general *tensors of type* (m, k), with components

$$\overset{\alpha}{T}{}^{i_1 \ldots i_m}_{j_1 \ldots j_k}$$

relative to local co-ordinates $x^1_\alpha, \ldots, x^n_\alpha$, transform under co-ordinate changes $\alpha \longrightarrow \beta$ in the appropriate way (obtained by "extrapolating" from (1.1) and (1.2)). A little more detail is in order. In terms of local co-ordinates $x^1_\alpha, \ldots, x^n_\alpha$ in some neighbourhood of the point x of interest, there are the usual standard basis vectors $e_{1\alpha}, \ldots, e_{n\alpha}$ for the tangent space at x, and the corresponding dual basis of covectors $e^1_\alpha, \ldots, e^n_\alpha$ for the dual space, with scalar product $\langle e^i_\alpha, e_{j\alpha}\rangle = \delta^i_j$, given by evalulating each e^j_α at the $e_{j\alpha}$. Thus each vector η in $\mathbb{R}^n_x$ has the unique form $\eta^j_\alpha e_{j\alpha} \in \mathbb{R}^n_x$, and each covector η^* the unique form $\eta_{i\alpha} e^i_\alpha \in \mathbb{R}^{n*}_x$. At each point $x \in M^n$ we then have the space of tensors of type (m, k)

$$\underbrace{\mathbb{R}^n_x \otimes \cdots \otimes \mathbb{R}^n_x}_{m} \otimes \underbrace{\mathbb{R}^{n*}_x \otimes \cdots \otimes \mathbb{R}^{n*}_x}_{k},$$

with standard basis (relative to the prevailing local co-ordinates) consisting of the expressions

$$e_{i_1\alpha} \otimes \cdots \otimes e_{i_m\alpha} \otimes e^{j_1}_\alpha \otimes \cdots \otimes e^{j_k}_\alpha.$$

An arbitrary tensor (of type (m, k)) at the point x has the form (in the local co-ordinates $x_\alpha^1, \ldots, x_\alpha^n$)

$$\left(T_{j_1 \ldots j_k}^{i_1 \ldots i_m}\right) = T_{j_1 \ldots j_k}^{i_1 \ldots i_m} e_{i_1} \otimes \cdots \otimes e_{i_m} \otimes e^{j_1} \otimes \cdots \otimes e^{j_k},$$

where the summation convention is assumed (and the index α suppressed). The basic vectors $e_{j\alpha}$ correspond naturally to the operators $\partial/\partial x_\alpha^j$, and the dual-basis covectors e_α^j to the differentials dx_α^j:

$$e_{j\alpha} \leftrightarrow \partial/\partial x_\alpha^j; \quad e_\alpha^j \leftrightarrow dx_\alpha^j.$$

A *tensor field* of type (m, k) is then a smooth function assigning to each point x of M^n a tensor of type (m, k).

The usual tensor (field) operations (contraction, product, permutation of indices, etc.) and the Einstein summation convention (whereby any index appearing as both superscript and subscript in a single expression is automatically summed over) will be used without comment in what follows.

Skew-symmetric tensors (more precisely tensor *fields*) with lower indices only, i.e. of type $(0, k)$, are of particular importance; they have the alternative name *differential forms* and are usually written as follows (in local co-ordinates):

$$\Omega_k = \frac{1}{k!} T_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{1.3}$$

in view of their use in integration theory. The *exterior product* $\Omega_k \wedge \Omega_l$ of differential forms is defined algebraically by means of the tensor product and the "alternation" operation making the exterior product skew-symmetric:

$$\Omega_k \wedge \Omega_l = (-1)^{kl} \Omega_l \wedge \Omega_k.$$

In terms of the local notation (1.3) for differential forms, the exterior product is determined uniquely by the conditions of associativity, commutativity with scalars, and

$$dx^i \wedge dx^j = -dx^j \wedge dx^i. \tag{1.4}$$

The *differential operator* d on forms is defined by the following conditions:

(i) $df = \dfrac{\partial f}{\partial x_\alpha^j} dx_\alpha^j$ (locally) for real-valued functions f;

(ii) $d(dx^{i_1} \wedge \cdots \wedge dx^{i_k}) = 0$;

(iii) $d(f\Omega_k) = df \wedge \Omega_k + f d\Omega_k$, f any scalar function.

From these we infer

$$d \circ d = 0, \quad d(\Omega_k \wedge \Omega_l) = d\Omega_k \wedge \Omega_l + (-1)^k \Omega_k \wedge d\Omega_l.$$

Finally we mention the important concept of a *Riemannian* or *pseudo-Riemannian metric* on a manifold M^n; this is a non-degenerate symmetric tensor (field) of type $(0,2)$ depending smoothly on the point x of M^n. In local co-ordinates (x^j_α), it will have the form (g_{ij}), with $g_{ij} = g_{ji}$ and det $(g_{ij}) \neq 0$. Such a tensor determines a symmetric scalar product of pairs of tangent vectors (at each point x), pairs of covectors (and in general pairs of tensors of any type) in the natural way:

$$\begin{aligned} \langle \eta_1, \eta_2 \rangle &= \eta_1^i \eta_2^j g_{ij}, \\ \langle \eta_1^*, \eta_2^* \rangle &= \eta_{1i}^* \eta_{2j}^* g^{ij}, \quad g^{ij} g_{jk} = \delta^i_k. \end{aligned}$$

As a tool for investigating the topology of manifolds, usually Riemannian metrics are used, i.e. those for which det $(g_{ij}) > 0$. However pseudo-Riemannian metrics occur not infrequently; they are important not only in the theory of relativity, but also for instance in studying the geometry of semisimple Lie groups.

A *Lie group* is a smooth manifold M^n equipped with a smooth binary operation

$$\Psi : M^n \times M^n \longrightarrow M^n; \quad \Psi(x,y) = x \cdot y, \quad x, y \in M^n,$$

under which M^n is a group: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; there is an identity element 1; each element x of M^n has an inverse element x^{-1} $(xx^{-1} = x^{-1}x = 1)$, and this inverse should vary smoothly with x.

Let $x^1, \ldots, x^n$ be local co-ordinates of a Lie group M^n in a neighbourhood of $x_0 = 1$. In terms of such co-ordinates the group operation $z = \Psi(x,y)$ has the form

$$z^j = \Psi^j(x^1, \ldots, x^n, y^1, \ldots, y^n),$$

whence by Taylor's theorem

$$\begin{aligned} z^j &= x^j + y^j + b^j_{ks} x^k y^s + o(|x| \cdot |y|), \\ (x^{-1})^j &= -x^j + o(|x|). \end{aligned} \tag{1.5}$$

The *Lie algebra* of the Lie group M^n is the tangent space to M^n at $x_0 = 1$, equipped with the bracket operation defined (in terms of local coordinates in a neighbourhood of 1) by

$$[\xi, \eta]^k = (b^k_{js} - b^k_{sj}) \xi^j \eta^s, \tag{1.6}$$

where $\xi = (\xi^1, \ldots, \xi^n)$, $\eta = (\eta^1, \ldots, \eta^n)$. It is immediate that this operation is bilinear, skew-symmetric $([\xi, \eta] = -[\eta, \xi])$, and satisfies Jacobi's identity:

$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0. \tag{1.7}$$

The most important Lie groups are groups of matrices over various fields, in particular $\mathbb{R}$, $\mathbb{C}$ and also the skew-field $\mathbb{H}$ of quaternions. The group $O_{p,q}$ consists of the linear transformations of the vector space $\mathbb{R}^n$, $n = p + q$, preserving a symmetric scalar product of signature p, q. The *orthogonal group* O_n $(= O_{n,0})$ is of course of particular importance. Analogously, the group $U_{p,q}$ consists of the linear transformations of $\mathbb{C}^n$, $n = p + q$, preserving an Hermitian scalar product of signature (p, q), and $U_n = U_{n,0}$ is the *unitary group*. The respective subgroups $SO_{p,q}$, SO_n, $SU_{p,q}$, SU_n, and $SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$, $SL_n(\mathbb{C}) \subset GL_n(\mathbb{C})$ (where SG denotes the subgroup of matrices of determinant 1 of the matrix group G) are also important. Finally, Sp_n denotes the subgroup of $n \times n$ orthogonal quaternionic matrices. The Lie groups O_n, U_n, SO_n, SU_n, Sp_n are all compact, and by the theorem of Cartan-Killing there are only finitely many (namely 6) compact, simply-connected, simple Lie groups not locally isomorphic to a group in this list of matrix groups. In fact every compact Lie group G is of the form

$$(T^n \times G_1 \times G_2 \times \cdots \times G_k) / D,$$

where T^n is the n-torus, the G_i are compact, simply connected, simple Lie groups (and so figuring in the Cartan-Killing list), and D is a finite subgroup contained in the center.

A *smooth fiber bundle* is defined as in the general case (see Chapter 3, §6) except that now the spaces E, B, F are all required to be smooth manifolds, the projection p is a smooth map, the transition functions are likewise smooth, and the structure group $G \subset \text{Diff } F$, i.e. is a subgroup of the group of diffeomorphisms of the fiber F. (In the most important situations G is a Lie group.)

A *(smooth) vector bundle* is a smooth fiber bundle where the fiber F is a vector space on which the structure group G acts as a group of linear transformations. The two most important types are *real vector bundles*, where $F \cong \mathbb{R}^n$, $G \subset GL_n(\mathbb{R})$, and *complex vector bundles*, where $F \cong \mathbb{C}^n$, $G \subset GL_n(\mathbb{C})$.

An important example is the *tangent bundle* $\tau = \tau(M^n)$ of a smooth manifold M^n, with $F \cong \mathbb{R}^n$, $G = GL_n(\mathbb{R})$ and base $B = M^n$; the total space is denoted by $T(M^n)$. Note that a cross-section of the tangent bundle is just a (tangent) vector field on M^n. If a Riemannian metric is given on M, then the structure group reduces to O_n.

From the tangent bundle over M^n one can construct many important associated vector bundles over M^n, with the group $GL_n(\mathbb{R})$ acting on the various fiber–manifolds. For example if we take as fiber the dual space $\mathbb{R}_x^{n*}$ at each point $x \in M^n$ (i.e. the space of tensors of type $(0, 1)$) we obtain the *cotangent vector bundle* τ^* with total space denote by $T^*(M^n)$. More generally, the action of the group $GL_n(\mathbb{R})$ on the vector space of tensors of type (m, k) yields an associated vector bundle over M^n with fiber this space of tensors; this bundle is denoted by

$$\underbrace{\tau \otimes \cdots \otimes \tau}_{m \text{ times}} \otimes \underbrace{\tau^* \otimes \cdots \otimes \tau^*}_{k \text{ times}}.$$

Similarly, one can form the exterior powers $\Lambda^k(\tau^*)$ of the cotangent space τ^*. Differential forms on M^n are then cross-sections of such vector bundles. A Riemannian metric on M^n may analogously be regarded as a cross-section of the symmetric square $S^2\tau$ of the tangent bundle, obtained as the symmetric part of $\tau \otimes \tau$.

A further example of a vector bundle associated with the tangent space, is that where the fiber above each point consists of the ordered k-tuples of linearly independent tangent vectors (k-frames).

In the presence of a Riemannian metric it is appropriate to use representations of the group O_n rather than $GL_n(\mathbb{R})$ (by means of linear transformations of the fiber). Important homogeneous spaces for the orthogonal and unitary groups are *Stiefel manifolds* and *Grassmannian manifolds*. The points of the real (or complex) Stiefel manifold $V^{\mathbb{R}}_{n,k}$ (or $V^{\mathbb{C}}_{n,k}$) are the orthogonal k-frames $(v_1, \dots, v_k)$ in $\mathbb{R}^n$ (or in $\mathbb{C}^n$ endowed with the Hermitian metric). It is easy to see that

$$\begin{aligned} V^{\mathbb{R}}_{n,k} &\cong SO_n/SO_{n-k} \cong O_n/O_{n-k}, \\ V^{\mathbb{C}}_{n,k} &\cong SU_n/SU_{n-k} \cong U_n/U_{n-k}. \end{aligned} \tag{1.8}$$

The points of the real (or complex) Grassmannian manifold $G^{\mathbb{R}}_{n,k}$ (or $G^{\mathbb{C}}_{n,k}$) are the k-dimensional subspaces of $\mathbb{R}^n$ (or $\mathbb{C}^n$). Since the usual action of O_n on $\mathbb{R}^n$ induces a transitive action of that group on the set of all k-dimensional subspaces (and analogously for U_n on $\mathbb{C}^n$), it is not difficult to deduce that

$$\begin{aligned} G^{\mathbb{R}}_{n,k} &\cong SO_n/(SO_{n-k} \times SO_k) \cong O_n/(O_{n-k} \times O_k), \\ G^{\mathbb{C}}_{n,k} &\cong SU_n/(SU_{n-k} \times SU_k) \cong U_n/(U_{n-k} \times U_k). \end{aligned} \tag{1.9}$$

The *orientable Grassmannian manifold* $\widehat{G}^{\mathbb{R}}_{n,k}$ has as points the oriented k-dimensional subspaces of $\mathbb{R}^n$. This is a double cover (with fiber $\mathbb{Z}/2$) of the corresponding Grassmannian manifold $G^{\mathbb{R}}_{n,k}$:

$$\widehat{G}^{\mathbb{R}}_{n,k} \longrightarrow G^{\mathbb{R}}_{n,k}.$$

There are natural embeddings

$$G^{\mathbb{R}}_{n,k} \longrightarrow G^{\mathbb{R}}_{n+1,k}, \quad \widehat{G}^{\mathbb{R}}_{n,k} \longrightarrow \widehat{G}^{\mathbb{R}}_{n+1,k}, \quad G^{\mathbb{C}}_{n,k} \longrightarrow G^{\mathbb{C}}_{n+1,k},$$

induced by the standard embeddings $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$, $\mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$. We denote by $G^{\mathbb{R}}_k$, $\widehat{G}^{\mathbb{R}}_k$, $G^{\mathbb{C}}_k$ the direct limits of the respective sequences of spaces and embeddings:

$$G^{\mathbb{R}}_k = \lim_{n\to\infty} G^{\mathbb{R}}_{n,k}, \quad \widehat{G}^{\mathbb{R}}_k = \lim_{n\to\infty} \widehat{G}^{\mathbb{R}}_{n,k}, \quad G^{\mathbb{C}}_k = \lim_{n\to\infty} G^{\mathbb{C}}_{n,k}.$$

It is easy to see directly that $\widehat{G}^{\mathbb{R}}_1$ is contractible, $G^{\mathbb{R}}_1 = \mathbb{R}P^\infty$, and $G^{\mathbb{C}}_1 = \mathbb{C}P^\infty$.

From the isomorphisms (1.13) below, it follows that

$$\begin{aligned} \pi_j(V^{\mathbb{R}}_{n,k}) &= 0 \quad \text{for } j < n-k, \\ \pi_j(V^{\mathbb{C}}_{n,k}) &= 0 \quad \text{for } j < 2(n-k). \end{aligned} \tag{1.10}$$

The principal bundles

$$\begin{aligned} E^{\mathbb{R}}_n &= V^{\mathbb{R}}_{n,k} \longrightarrow G^{\mathbb{R}}_{n,k}, \quad F = O_k = G, \\ \widehat{E}^{\mathbb{R}}_n &= V^{\mathbb{R}}_{n,k} \longrightarrow \widehat{G}^{\mathbb{R}}_{n,k}, \quad F = SO_k = G, \\ E^{\mathbb{C}}_n &= V^{\mathbb{C}}_{n,k} \longrightarrow G^{\mathbb{C}}_{n,k}, \quad F = U_k = G, \end{aligned} \tag{1.11}$$

(where the projections are the obvious ones), taken together with the observation that the spaces

$$E^{\mathbb{R}}_\infty = \lim_{n\to\infty} E^{\mathbb{R}}_n, \quad \widehat{E}^{\mathbb{R}}_\infty = \lim_{n\to\infty} \widehat{E}^{\mathbb{R}}_n, \quad E^{\mathbb{C}}_\infty = \lim_{n\to\infty} E^{\mathbb{C}}_n$$

are contractible, lead to the conclusion that:

The principal bundles (1.11) go in the limit as $n \longrightarrow \infty$ to the universal G-bundles for the Lie Groups $G = O_k, SO_k, U_k$. Hence the classifying spaces for these groups are

$$BO_k = G^{\mathbb{R}}_k, \quad BSO_k = \widehat{G}^{\mathbb{R}}_k, \quad BU_k = G^{\mathbb{C}}_k. \tag{1.12}$$

Recall that any map $M^n \longrightarrow BG$ determines up to equivalence a (principal) G-bundle over M^n as the induced bundle (and this determines a one-to-one "classifying" correspondence between the equivalence classes of (principal) G-bundles over M^n and the set $[M^n, BG]$ of homotopy classes of maps $M^n \longrightarrow BG$). In the case of the tangent bundle of a manifold M^n the corresponding map $M^n \longrightarrow BO_k$ (or $M^n \longrightarrow BSO_k$) arises naturally as a generalization of the spherical Gauss map of a surface. One first embeds M^n in $\mathbb{R}^N$ for sufficiently large N (this is possible by Whitney's theorem), and then associates with each point x of M^n the tangent plane to M^n at x (with an orientation indicated if M^n is orientable) parallel-translated to the origin in $\mathbb{R}^n$. This defines an appropriate classifying map

$$f : M^n \longrightarrow G^{\mathbb{R}}_{N,n}, \quad (\text{or } f : M^n \longrightarrow \widehat{G}^{\mathbb{R}}_{N,n}, \text{ if } M^n \text{ is orientable}),$$

which is covered in the natural way by the induced map from the tangent bundle of M^n to the universal bundle.

The generalization of this construction to any smooth vector bundle with base B a manifold presents no difficulty. Note that the total space E of such a bundle is contractible to B (identified with the zero-th cross-section). Embed E in $\mathbb{R}^N$ for suitably large N, and consider at each point of $B \subset E$ the plane in $\mathbb{R}^N$ tangent at x to the fiber $F_x \subset \mathbb{R}^N$ through x. On parallel-translating these planes to the origin in $\mathbb{R}^N$, we obtain the desired map $B \longrightarrow G^{\mathbb{R}}_{N,n}$ (or $B \longrightarrow \widehat{G}^{\mathbb{R}}_{N,n}$ if B is orientable). Since for any finite CW-complex K there is a homotopy equivalent smooth manifold U (obtained by embedding K in $\mathbb{R}^N$ for some N, and taking as U a small neighbourhood of K having K as deformation retract), the above construction of the smooth *Gauss map* determined by a vector bundle is, in essence, no less general than the construction of a universal G-bundle over a finite CW-complex.

For general smooth fiber bundles with $G \subset \operatorname{Diff} F$, the universal G-bundle may be constructed as follows. Denote by $E_N(F)$ the space of embeddings $F \longrightarrow \mathbb{R}^N$. The group $\operatorname{Diff} F$ then acts in the obvious way on $E_N(F)$, and the direct limit $E_\infty(F) = \lim\limits_{N\to\infty} E_N(F)$ is contractible. The orbit space under the action of G on $E_\infty(F)$ then gives the desired base BG of the universal G-bundle. In the literature an alternative construction (due to Milnor in the late 1950s) is generally employed, based on a different idea, allowing the construction of BG for a wide class of groups G.

Returning to vector bundles, note that the homotopy exact sequences of the following principal bundles:

$$SO_{n-1} \longrightarrow SO_n \longrightarrow SO_n/SO_{n-1} \cong S^{n-1},$$

$$U_{n-1} \longrightarrow U_n \longrightarrow U_n/U_{n-1} \cong S^{2n-1}$$

(where $SO_{n-1} \longrightarrow SO_n$, $U_{n-1} \longrightarrow U_n$ are embeddings) taken together with the fact that $\pi_j(S^{k-1})$ is trivial for $j \le k-2$, yield isomorphisms:

$$\begin{aligned} \pi_k(SO_n) &\cong \pi_k(SO_{n-1}), \quad k < n-1, \\ \pi_k(U_n) &\cong \pi_k(U_{n-1}), \quad k < 2(n-1). \end{aligned} \tag{1.13}$$

Hence the natural embeddings $SO_{n-k} \longrightarrow SO_n$, $U_{n-k} \longrightarrow U_n$ (and $SU_{n-k} \longrightarrow SU_n$) induce isomorphisms between the corresponding homotopy groups of dimensions $< n-k-1$, $2(n-k)$ respectively. From this (1.10) above follows. (For Sp_n one has, analogously, $\pi_k(Sp_n) \cong \pi_k(Sp_{n-1})$ for $k < 4n-2$.) The homotopy groups $\pi_k(O_n) \cong \pi_k(SO_n)$ for $k < n-1$, $\pi_k(U_n)$ for $k < 2n-2$, and $\pi_k(Sp_n)$ for $k < 4n-2$, are called *stable*, since they are independent of n in these ranges. They were first computed by Bott (in the late 1950s) using the global calculus of variations (see §2 below).

The differential geometry of smooth G-bundles with structure group G a Lie group, depends crucially on the concept of a "connexion". A *differential-geometric G-connexion* on a principal G-bundle over a manifold B of dimension n, is a smooth G-invariant field associating with each point of the total space E an n-dimensional subspace of the tangent space to E, transverse to the fiber through the point. (Recall that the group G acts freely on E (on the left say) with the orbits of the action as the fibers $F \cong G$.) The n-dimensional subspaces attached to each point of E are said to be *horizontal*.

On the Lie group G there is a standard right-invariant 1-form w_0 with values (at tangent vectors to G) in the Lie algebra $\mathfrak{G}$ of G, where $\mathfrak{G}$ is identified with the algebra of right-invariant vector fields on G, namely the 1-form which at each tangent vector η to G at $x \in G$, takes as value the unique right-invariant vector field ξ satisfying $\xi(x) = \eta$. The 1-form then satisfies

$$-\frac{1}{2}[w_0, w_0] = dw_0, \quad g^* w_0 = (\mathrm{Ad}\ g)w_0, \tag{1.14}$$

where g acts by left multiplication $h \mapsto gh$, and $[\ ,\ ]$ denotes the exterior product operation on forms on G with values in $\mathfrak{G}$. (Recall that forms and cochains with values in any ring may be multiplied, although in general the multiplication will not be skew-commutative. Note also that for a matrix Lie group G the linear transformation Ad g of the (matrix) Lie algebra $\mathfrak{G}$ is effected by conjugation of the matrices $u \in \mathfrak{G}$ by g: Ad $g(u) = gug^{-1}$.) A connexion on a principal bundle (E, B, G, G, p) is then determined by any 1-form ω on E with values in the Lie algebra $\mathfrak{G}$, satisfying

$$g^*\omega = (\mathrm{Ad}\ g)\omega, \quad \omega|_F = \omega_0, \tag{1.15}$$

where F ($\cong G$) is the fiber over any point; the horizontal n-dimensional tangent subspaces at each point y of E are defined by the equation $\omega = 0$ at y. It can be shown that, conversely, any connexion on the G-bundle determines such a 1-form ω.

The *curvature form* Ω on E is the 2-form defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]. \tag{1.16}$$

It follows that $g^*\Omega = (\mathrm{Ad}\ g)\Omega$ for all $g \in G$. Furthermore the form Ω is, as they say, "purely horizontal": it vanishes on any pair of tangent vectors of which at least one is vertical, i.e. tangent to the fiber. Thus Ω has non-zero values essentially only on pairs of "horizontal" tangent vectors.

Let $x^1, \ldots, x^n$ be co-ordinates on any (distinguished) chart $U \subset B$ satisfying $p^{-1}(U) \cong U \times G$ (via ϕ_U say). These then furnish coordinates on $U \times \{1\}$, and in terms of such local co-ordinates the form ω of the connexion, restricted to $U \times \{1\}$, is given by

$$\omega_{\phi_U} = \omega|_{U\times\{1\}} = A_a dx^a,$$

(where the A_a are $\mathfrak{G}$-valued functions of the point (x^a)), and the curvature form Ω has locally the form

$$\Omega_{\phi_U} = \Omega|_{U\times\{1\}} = F_{ab} dx^a \wedge dx^b,$$

where the F_{ab} are $\mathfrak{G}$-valued functions of (x^a). From (1.16) it follows that

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b], \quad \partial_a = \partial/\partial x^a. \tag{1.17}$$

In view of (1.15), these local formulae for the restrictions of ω and Ω to $U \times \{1\}$ determine ω and Ω completely on the whole region $p^{-1}(U) \cong U \times G$.

How do the above local expressions for ω_{ϕ_U} and Ω_{ϕ_U} change under changes of bundle coordinates, i.e. changes of the diffeomorphism ϕ_U? Let ψ_U be another diffeomorphism $p^{-1}(U) \longrightarrow U \times G$, and consider the diffeomorphism

$$\phi_U \psi_U^{-1} : U \times G \longrightarrow U \times G, \quad (x, g) \mapsto (x, \lambda(x) g), \tag{1.18}$$

where λ defines the appropriate transition function, $\lambda(x) \in G$. In particular, the cross-section $U \times 1$, where $1 \in G$ is the identity element, goes to a different one:

$$\phi \psi^{-1} : (x, 1) \mapsto (x, \lambda(x)), \quad \lambda(x) \in G. \tag{1.19}$$

(A transformation of the type (1.19) is in this context called a *gauge transformation.*) Under this transformation the connexion and curvature forms transform as follows (assuming that G and $\mathfrak{G}$ are comprised of matrices):

$$\begin{aligned} &\text{(i)} \quad \omega_\phi \mapsto \omega_\psi, \quad \omega_\phi = A_a dx^a, \quad \omega_\psi = A'_a dx^a, \\ &\qquad A_a dx^a \mapsto A'_a dx^a, \quad A'_a = \lambda A_a \lambda^{-1} + (\partial_a \lambda(x)) \lambda^{-1}(x); \\ &\text{(ii)} \quad \Omega_\phi \mapsto \Omega_\psi, \\ &\qquad F_{ab} \mapsto \lambda(x) F_{ab} \lambda^{-1}(x) = F'_{ab}. \end{aligned} \tag{1.20}$$

Example 1. Let G be an abelian Lie group (for example $G = S^1 \cong SO_2 \cong U_1$). Then in view of (1.16) the curvature form is in essence the closed 2-form on the base defined in terms of the connexion form ω by

$$p^* \Omega = d\omega.$$

The cohomology class $[\Omega] \in H^2(B; \mathbb{R})$ turns out to be independent of the choice of connexion, and to be "integral" in the sense that the integrals of the form Ω over 2-cycles are integers.

Example 2. Let $G = U_n$. In this case the form $\operatorname{Tr} \Omega$ ($= \operatorname{Tr} F_{ab} dx^a \wedge dx^b$ in local co-ordinates) of any curvature form Ω, is again a closed 2-form on the base with cohomology class

$$[\operatorname{Tr} \Omega] = c_1 \in H^2(B; \mathbb{R})$$

independent of the connexion, and integral. This cohomology class is called the *first Chern class* of B. The forms $\operatorname{Tr}(\Omega^m)$ obtained by taking powers of the matrix (F_{ab}), represent integral cohomology classes $\widetilde{c}_m \in H^{2m}(B; \mathbb{R})$ (or over $\mathbb{C}$), and are also called *Chern classes* of B. However for the integral Chern classes (see below for their precise definition) there is a more appropriate basis

$c_1, c_2 \ldots$, in terms of which the classes $\widetilde{c}_i$ have integer polynomial expressions (but not vice versa)

$$\widetilde{c}_i = P_i(c_1, \ldots, c_i).$$

The polynomials $P_i(c_1, \ldots, c_i)$ are in fact the Newton polynomials expressing the symmetric polynomials $\sum_{k=1}^{n} u_k^i$ in terms of the elementary symmetric polynomials

$$c_j = \sum_{i_1 < \cdots < i_j} u_{i_1} \cdot \ldots \cdot u_{i_j} : \quad \widetilde{c}_1 = c_1, \quad \widetilde{c}_2 = c_1^2 - 2c_2, \quad \ldots.$$

This representation of the classes $\widetilde{c}_i$, c_i as symmetric polynomials has topological significance, and turns out to be very useful (see below). □

Example 3. In the case $G = SO_n$, the curvature form Ω (of any connexion) yields the *Pontryagin classes* $\widetilde{p}_i = \left[\mathrm{Tr}\ \Omega^{2i}\right] \in H^{4i}(B; \mathbb{R})$ of B, also integral. As in the case of the Chern classes, we shall choose below a different ("indivisible" over $\mathbb{Z}$) basis p_i for the Pontryagin classes, in terms of which the $\widetilde{p}_i$ are once again expressed as Newton polynomials. □

The field of horizontal n-dimensional tangent subspaces on the total space of a principal G-bundle, determining, in accordance with the definition, a connexion on the bundle, induces such a field, likewise transverse to the fibres, on the total space E' of any associated bundle (E', B, F, G, p'). Via this field of horizontal tangent subspaces every smooth or piecewise smooth path $\gamma(t)$, $a \leq t \leq b$, in the base B, is covered in the total space E' by a unique horizontal path $\widetilde{\gamma}(t)$, $p'\widetilde{\gamma}(t) = \gamma(t)$, once the initial point $\widetilde{\gamma}(a)$ is prescribed (see Figure 4.3).

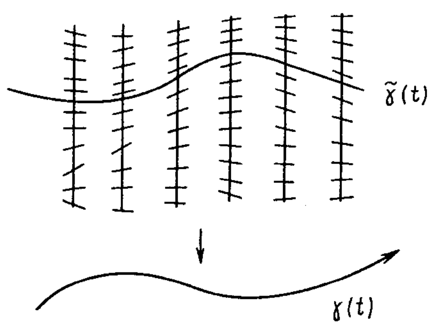

Fig. 4.3

The covering motion of the point $\widetilde{\gamma}(t)$ above $\gamma(t)$ is termed *parallel translation.* This construction represents the smooth version of the "homotopy connexion" introduced in Chapter 2, §3, with the concomitant "covering homotopy property", and the homotopic consequences flowing thence.

A connexion on a principal G-bundle determines an operation of *covariant differentiation* in the direction of any tangent vector to the base B, on the cross-sections of any associated G-bundle. As noted above, in terms of local co-ordinates on suitable regions U of the base B the connexion 1-form is given by

$$\omega_{\phi_U} = A_a(x)dx^a.$$

Since A_a is a matrix-valued function, we may write $A_a = (A_a)^i_j = (A^i_{aj})$, $i, j = 1, \dots, q$. A cross-section of an associated G-bundle can locally be represented as a vector-valued function $(\eta^i(x))$ with values in $\mathbb{R}^n$. The covariant derivatives are then the operators D_a defined by

$$D_a = \partial_a + A_a, \quad D_a\eta^i(x) = \partial_a\eta^i + A^i_{aj}\eta^j. \tag{1.21}$$

The commutators $[D_a, D_b]$ yield the curvature components (1.17):

$$[D_a, D_b] = \partial_a A_b - \partial_b A_a + [A_a, A_b] = F_{ab}. \tag{1.22}$$

In the case of the tangent bundle $\tau(M^n)$ the above discussion yields the classical concepts of Riemannian geometry. It appears that the idea of a connexion on a fibre bundle was first introduced in the 1930s by Weyl in a special case, and in general form by Cartan. Physicists came to the concept somewhat later (Yang and Mills in the mid-1950s), although doubtless independently — at least of Cartan. In physical terminology connexions are called "gauge fields"; they are of fundamental importance in the physics of elementary particles. The simplest example of such a field (due to Weyl) is that of a connexion on a G-bundle with $G = U_1 \cong SO_2 \cong S^1$, having the physical interpretation as an electromagnetic field with field strength the curvature tensor F_{ab}. As noted above (see (1.20) the gauge transformations are just those transformations of the local formula for the connexion, arising from changes from the identity cross-section to another: $(x, 1) \mapsto (x, \lambda(x))$. In Einstein's general theory of relativity one considers connexions on the tangent bundle arising from the gravitational field.

Returning to the main line of development of our exposition, we now introduce the concept of a "characteristic class" of a smooth G-bundle.

Definition 1.1 A *characteristic class* θ with respect to the category of G-bundles (E, B, F, G, p) is a function associating with each G-bundle an element θ of $H^*(B)$ (over some coefficient ring) in such a way that under bundle maps $\Phi : E \longrightarrow E'$, $\phi : B \longrightarrow B'$, the characteristic class behaves naturally (more precisely, as a contravariant functor):

$$\phi^*(\theta') = \theta, \quad \phi : B \longrightarrow B'.$$

It is easy to see that the characteristic classes of G-bundles correspond one-to-one to the elements of the cohomology ring of the base BG of the universal G-bundle: For, as noted earlier, every G-bundle with base B is induced by a map $\phi : B \longrightarrow BG$ unique up to homotopy, so that if $\theta' \in H^*(BG)$ is any given element, then we obtain a characteristic class θ of the G-bundle over B induced by ϕ, as the pullback of θ': $\theta = \phi^*(\theta')$.

For compact abelian groups G the spaces BG are known:

$$G = S^1, \qquad BG = \mathbb{C}P^\infty;$$

$$G = T^n, \qquad BG = \mathbb{C}P_1^\infty \times \cdots \times \mathbb{C}P_n^\infty;$$

$$G = (\mathbb{Z}/2)^n, \qquad BG = \mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty;$$

$$G = \mathbb{Z}/m, \qquad BG = \lim_{n\to\infty} S^{2n-1}/\mathbb{Z}/m = S^\infty/\mathbb{Z}/m,$$

(where, in the last case, $S^\infty/\mathbb{Z}/m$ is the infinite-dimensional lens space). The cohomology rings of these BG are also known; of particular importance are the following ones:

$$H^*\left(\mathbb{C}P_1^\infty \times \cdots \times \mathbb{C}P_n^\infty; \mathbb{Z}\right) \cong \mathbb{Z}\left[u_1, \ldots, u_n\right],$$

(the polynomial ring over $\mathbb{Z}$ in $u_j \in H^2\left(\mathbb{C}P_j^\infty; \mathbb{Z}\right)$);

$$H^*\left(\mathbb{R}P_1^\infty \times \cdots \times \mathbb{R}P_n^\infty; \mathbb{Z}/2\right) \cong \mathbb{Z}/2\left[v_1, \ldots, v_n\right], \quad v_j \in H^1\left(\mathbb{R}P_j^\infty; \mathbb{Z}/2\right);$$

$$\begin{gathered} H^*\left(B\mathbb{Z}/p; \mathbb{Z}/p\right) = \Lambda_p(v) \otimes \mathbb{Z}/p\left[u\right], \\ v \in H^1\left(B\mathbb{Z}/p; \mathbb{Z}/p\right), \quad u = \beta v \in H^2\left(B\mathbb{Z}/p; \mathbb{Z}/p\right), \end{gathered} \tag{1.23}$$

where β is the Bockstein homomorphism (see Chapter 3, §5), $\Lambda_p(v)$ is the exterior algebra over $\mathbb{Z}/p$ generated by v ($v^2 = 0$), and the tensor product has imposed on it skew-commutativity.

The most important facts about the cohomology rings $H^*(BG)$ for $G = O_n$, SO_n, U_n, SU_n with coefficients from $\mathbb{Z}$, $\mathbb{Z}/2$ and $\mathbb{R}$ are as follows:

In the group U_n we have the subgroup T^n, a *maximal torus*, comprised of the diagonal matrices g, and the *Weyl group* S_n of permutation matrices g normalizing T^n: $gT^ng^{-1} = T^n$. The restriction of the cohomology ring of BU_n to BT^n has zero kernel, i.e. the homomorphism

$$i^* : H^*(BU_n; \mathbb{Z}) \longrightarrow H^*(BT^n; \mathbb{Z}) \cong \mathbb{Z}\left[u_1, \ldots, u_n\right],$$

induced by the inclusion $i : BT^n \longrightarrow BU_n$, is a monomorphism invariant under the action of the Weyl group. It follows that the cohomology ring of the space BU_n is given via the homomorphism i^* by that of BT^n, and that the image under i^* consists of all symmetric polynomials in $u_1, \ldots, u_n$, thus yielding a complete description of the ring $H^*\left(BU_n; \mathbb{Z}\right)$. The *Chern class* $c_i \in H^{2i}\left(BU_n; \mathbb{Z}\right)$ is defined as the element mapped by i^* to the i-th elementary symmetric polynomial in the "Wu variables", i.e. the elements $u_j \in H^2\left(\mathbb{C}P_j^\infty; \mathbb{Z}\right)$:

$$c_i \mapsto \sum_{j_1<\dots<j_i} u_{j_1}\cdot\ldots\cdot u_{j_i}, \quad \widetilde{c}_i \mapsto \sum_{j=1}^{n} u_j^i. \tag{1.24}$$

Thus the elements c_i form a basis for $H^*(BU_n;\mathbb{Z})$. (The elements $\widetilde{c}_i$ were defined earlier in terms of the curvature form.)

For the groups SO_{2n}, SO_{2n+1} the maximal torus T^n is as for U_n, but the Weyl groups are larger. The kernel of the restriction homomorphism

$$H^*(BSO_k;\mathbb{Z}) \longrightarrow H^*(BT^n;\mathbb{Z}), \quad k = 2n, 2n+1,$$

analogous to i^* above, contains only the 2-torsion elements. The Weyl group of SO_{2n}, modelled via the action on $H^*(BT^n;\mathbb{Z})$, is generated by the symmetric group S_n of all permutations of $u_1,\dots,u_n$, together with the transformations σ_{ij} defined by

$$\sigma_{ij} : u_i \mapsto -u_i, \quad u_j \mapsto -u_j, \quad u_k \mapsto u_k, \quad k \neq i,j. \tag{1.25}$$

For SO_{2n+1} one needs instead of these (as generators) the transformations given by

$$\sigma_i : u_i \mapsto -u_i, \quad u_j \mapsto u_j, \quad j \neq i. \tag{1.26}$$

Thus the set of invariant polynomials for SO_{2n+1} is smaller than for SO_{2n}. The invariance of the homomorphism $H^*(BSO_{2n};\mathbb{Z}) \longrightarrow H^*(BT^n;\mathbb{Z})$ under the action of the Weyl group yields as image the linear span of the elementary symmetric polynomials in the squares u_j^2, representing the *Pontryagin classes* p_i, supplemented by the polynomial $\chi_{2n} = u_1\cdots u_n$, representing the *Euler characteristic class*. The image under the restriction homomorphism $H^*(BSO_{2n+1};\mathbb{Z}), \longrightarrow H^*(BT^n;\mathbb{Z})$ on the other hand, is spanned only by the images of the Pontryagin classes. In summary (for both SO_{2n}, SO_{2n+1}):

$$\begin{gathered} p_i \mapsto \sum_{j_1<\dots<j_i} u_{j_1}^2\cdot\ldots\cdot u_{j_i}^2 \in H^{4i}(BT^n;\mathbb{Z}), \\ \chi_{2n} \mapsto u_1\cdots u_n \in H^{2n}(BT^n;\mathbb{Z}), \\ p_i \in H^{4i}(BSO_k;\mathbb{Z}), \quad k \geq 2n, \quad \chi_{2n} \in H^{2n}(BSO_{2n};\mathbb{Z}). \end{gathered} \tag{1.27}$$

Over the field $\mathbb{R}$ of reals (or over $\mathbb{Q}$) the situation is the same for all compact Lie groups G (with maximal torus T^n): the restriction homomorphism $H^*(BG;\mathbb{R}) \longrightarrow H^*(BT^n;\mathbb{R})$ has trivial kernel and image consisting all polynomials in $u_1,\dots,u_n$, invariant under the induced action of the Weyl group.

In order to describe the mod-2 cohomology of BG, with $G = O_n$, SO_n, one uses the obvious discrete analogue of the torus, namely the subgroup of diagonal matrices $(\mathbb{Z}/2)^n \subset O_n$. Here the restriction homomorphism

$$H^*(BO_n;\mathbb{Z}/2) \longrightarrow H^*(B(\mathbb{Z}/2)^n;\mathbb{Z}/2) \cong \mathbb{Z}/2\,[v_1,\dots,v_n],$$

$$v_j \in H^1(\mathbb{R}P_j^\infty;\mathbb{Z}/2), \quad B\mathbb{Z}/2 = \mathbb{R}P^\infty,$$

has trivial kernel, and the image consists of the symmetric polynomials in $v_1, \dots, v_n$. The elements of $H^*(BO_n; \mathbb{Z}/2)$ sent under this homomorphism to the elementary symmetric polynomials in $v_1, \dots, v_n$, are called the *Stiefel-Whitney characteristic classes* w_j:

$$w_j \mapsto \sum_{i_1<\dots<i_j} v_{i_1} \cdot \dots \cdot v_{i_j} \in H^j(B(\mathbb{Z}/2)^n; \mathbb{Z}/2). \tag{1.28}$$

The natural homomorphism $H^*(BO_n; \mathbb{Z}/2) \longrightarrow H^*(BSO_n; \mathbb{Z}/2)$ is onto (i.e. an epimorphism) with kernel the principal ideal generated by w_1, i.e. the set of multiples of w_1.

Note that reduction modulo 2 of the Chern classes c_i yields the Stiefel-Whitney classes w_{2i}, and of χ_{2n} yields w_{2n}:

$$w_{2i} = c_i \pmod 2, \quad w_{2n} = \chi_{2n} \pmod 2.$$

Observe also that

$$p_i(\eta) \pmod 2 = w_{4i}(c\eta) = \sum_{2j+2k=2i} w_{2j}(\eta) w_{2k}(\eta),$$

(where c denotes "complexification"), and for U_n-bundles $\chi_{2n} = c_n$.

The *characteristic classes of a manifold* are defined as the characteristic classes of the tangent bundle of the manifold.

Such are the basic facts concerning characteristic classes. We now describe some of their further properties.

Note first that the condition $w_1(\eta^n) = 0$ is nessesary and sufficient for orientability of the G-bundle η^n, i.e. for reducibility of the bundle structure group G to SO_n.

Recall that with any real (or complex) vector bundle η^n with base B one may associate a fibration (over B) with fiber

$$F = V^{\mathbb{R}}_{n,n-k}, \quad (\text{or } F = V^{\mathbb{C}}_{n,n-k}),$$

the corresponding Stiefel manifold. The Stiefel-Whitney (or Chern, in the complex case) classes arise as the "first obstructions" to the existence of cross-sections (see Chapter 3, §6) of this fiber bundle:

$$V^{\mathbb{R}}_{n,n-k}(\eta^n) \xrightarrow{V^{\mathbb{R}}_{n,n-k}} B \quad (\text{or } V^{\mathbb{C}}_{n,n-k}(\eta^n) \xrightarrow{V^{\mathbb{C}}_{n,n-k}} B \text{ in the complex case}),$$

where

$$\pi_j(F) = 0, \quad j < k-1, \quad \pi_{k-1}(F) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2, \quad F = V^{\mathbb{R}}_{n,n-k},$$

or

$$\pi_j(F) = 0, \quad j < 2k-1, \quad \pi_{2k-1}(F) \cong \mathbb{Z}, \quad F = V^{\mathbb{C}}_{n,n-k}.$$

The fundamental group $\pi_1(B)$ acts trivially on the group $\pi_{k-1}(F)\otimes\mathbb{Z}/2$ in the first case $(F = V^{\mathbb{R}}_{n,n-k})$, and on $\pi_{2k-1}(F)$ $(F = V^{\mathbb{C}}_{n,n-k})$, where the structure groups of the original vector bundle η^n are O_n, U_n (or may even be $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$) respectively.

The construction of the Poincaré duals Dw_i, Dc_i of the characteristic cycles w_i, c_i as cycles in the appropriate homology groups, is more direct. For the classes w_i, take any O_n-bundle η^n with fiber $\mathbb{R}^n$ and base B a smooth closed manifold, and consider a field on B of k-tuples of vectors $v_1(x),\dots,v_k(x)$, $x \in B$, in general position; the set of all points $x \in B$ where the vectors $v_1(x),\dots,v_k(x)$ are linearly dependent then constitutes a cycle (mod 2) of dimension $n - k + 1$ representing the element Dw_i. In other words if one considers the associated fiber bundle $V^{\mathbb{R}}_{n,n-k}(\eta^n)$ over B and a cross-section of this bundle in general position, then all "zeros" of this section form the cycle representing Dw_i. In the case of an oriented (or SO_n-) bundle η^n, $n = 2m$ and $k = 1$, the above field is just a vector field, whose zeros form an integral cycle of dimension n representing the element $D\chi_{2m}$. The duals of the Chern classes may be obtained similarly.

For G-bundles a subgroup $G_1 \subset G$ will frequently serve as structure group; the operation of taking a subgroup as a structure group, where possible, is called *reduction* of the structure group G. A structure Lie group G may always be reduced to a maximal compact subgroup in a class of equivalent C^∞-bundles. (However for a class of equivalent complex-analytic G-bundles this is not always possible.)

Any complex-vector bundle η may be regarded as real, by "forgetting" the complex structure: $\eta \mapsto r\eta$; this operation corresponds to the embedding $U_n \xrightarrow{r} SO_{2n}$ (or $GL_n(\mathbb{C}) \xrightarrow{r} GL_{2n}(\mathbb{R})$). In the other direction, one can apply to any real vector bundle η the operation of *complexification* $\eta \mapsto c\eta$, using the embedding $GL_n(\mathbb{R}) \xrightarrow{c} GL_n(\mathbb{C})$. Given any representation (i.e. homomorphism) $\rho : G_1 \longrightarrow G_2$ of one group to another, one obtains, in the obvious way, from any G_1-bundle η a G_2-bundle $\rho\eta$. We now list these and other operations on vector bundles (some mentioned earlier):

1. The *direct sum* $\eta_1 \oplus \eta_2$.

2. The *tensor product* $\eta_1 \otimes \eta_2$ (over $\mathbb{R}$ or $\mathbb{C}$).

3. The *exterior powers* $\Lambda^j\eta$ (coinciding when $j = n = \dim\eta$ with the *determinant* of the bundle: $\det\eta = \Lambda^n\eta$).

4. The *symmetric powers* $S^j\eta$ (the symmetric part of the tensor power $\otimes^j\eta$). Thus
$$\Lambda^2\eta \oplus S^2\eta = \eta\otimes\eta. \tag{1.29}$$

5. The *dual bundle* η^*; the *complex conjugate bundle* $\overline{\eta}$ (when η is a complex bundle); the operations r and *complexification* c noted above, satisfying
$$rc(\eta) \cong \eta\oplus\eta^*, \quad \overline{c\eta} \cong c\eta, \quad cr(\eta) \cong \eta\oplus\overline{\eta}. \tag{1.30}$$

It follows from their definitions that the Pontryagin classes of a vector bundle η coincide (up to sign) with the Chern classes of the complexified bundle $c\eta$:

$$(-1)^i p_i(\eta) = c_{2i}(c\eta), \quad 2c_{2i+1}(c\eta) = 0. \tag{1.31}$$

For direct sums of vector bundles the following formulae are valid:

$$\begin{aligned}
&\text{(i)} & w_i(\eta_1 \oplus \eta_2) &= \sum_{j+k=i} w_j(\eta_1) w_k(\eta_2), \quad w_0 = 1; \\
&\text{(ii)} & c_i(\eta_1 \oplus \eta_2) &= \sum_{j+k=i} c_j(\eta_1) c_k(\eta_2), \quad c_0 = 1; \\
&\text{(iii)} & p_i(\eta_1 \oplus \eta_2) &= \sum_{j+k=i} p_j(\eta_1) p_k(\eta_2), \quad \text{to within 2-torsion}; \\
&\text{(iv)} & \chi_{2n}(\eta_1 \oplus \eta_2) &= \chi_{2k}(\eta_1)\chi_{2m}(\eta_2), \quad k+m=n,
\end{aligned} \tag{1.32}$$

where in the case (iv) it is assumed that $\dim \eta_1 = 2k$, $\dim \eta_2 = 2m$.

It follows from these formulae that except for χ_{2n} these classes are all stable, i.e. if $\varepsilon_{\mathbb{R}}^N$ (or $\varepsilon_{\mathbb{C}}^N$) is the trivial bundle with fiber $\mathbb{R}^N$ (or $\mathbb{C}^N$), then

$$w_i(\eta \oplus \varepsilon_{\mathbb{R}}^N) = w_i(\eta), \quad c_i(\eta \oplus \varepsilon_{\mathbb{C}}^N) = c_i(\eta), \quad p_i(\eta \oplus \varepsilon_{\mathbb{R}}^N) = p_i(\eta).$$

The behaviour of characteristic classes under the tensor product operation (and likewise the operation of forming exterior and symmetric powers) of vector bundles over B is somewhat more complicated. This behavior may be described in rational (or real) cohomology by means of "Wu generators" $u_1, \dots, u_n$ of the cohomology ring $H^*(T^n; \mathbb{Q})$ of the maximal torus. For U_n-bundles η the *Chern character* $ch(\eta)$ is defined as the formal sum

$$ch(\eta) = \sum_{j=1}^{n} \exp(u_j) = \sum_{j=1}^{n} \sum_{m\geq 0} \frac{u_j^m}{m!} = \sum_{m\geq 0} \frac{1}{m!}\widetilde{c}_m = \sum_{m\geq 0} ch_m(\eta), \tag{1.33}$$

$$ch_m(\eta) = \tfrac{1}{m!}\widetilde{c}_m = P_m(c_1, \dots, c_m),$$

and for SO_n-bundles η,

$$ch(\eta) = ch(c\eta),$$

where $ch(c\eta)$ is given by (1.33). Here we identify the elementary symmetric polynomials in $u_1, \dots, u_n$ with the Chern classes c_i (and in $u_1^2, \dots, u_n^2$ with the Pontryagin classes p_i). One then has the simple formula

$$ch(\eta_1 \otimes \eta_2) = ch\eta_1 ch\eta_2. \tag{1.34}$$

In what follows we shall often omit the symbol $\otimes$ for the tensor product, since there is no other product for bundles. Returning for the moment to characteristic classes of sums $\eta_1 \oplus \eta_2$ of vector bundles over B, we consider the formal sums (defined for each vector bundle η):

$$1 \oplus \eta \oplus \Lambda^2\eta \oplus \Lambda^3\eta \oplus \cdots = \Lambda(\eta),$$

$$1 \oplus \eta \oplus S^2\eta \oplus S^3\eta \oplus \cdots = S(\eta),$$

which are easily seen to satisfy

$$\begin{aligned} \Lambda(\eta_1 \oplus \eta_2) &= \Lambda(\eta_1)\Lambda(\eta_2), \\ S(\eta_1 \oplus \eta_2) &= S(\eta_1)S(\eta_2). \end{aligned} \tag{1.35}$$

Similarly, if we write (formally)

$$w = 1 + w_1 + w_2 + \ldots, \quad c = 1 + c_1 + c_2 + \ldots, \quad p = 1 + p_1 + p_2 + \ldots,$$

then the formulae (1.32) take the simple form:

$$\begin{aligned} &\text{(i)} \quad & w(\eta_1 \oplus \eta_2) &= w(\eta_1)w(\eta_2), \\ &\text{(ii)} \quad & c(\eta_1 \oplus \eta_2) &= c(\eta_1)c(\eta_2), \\ &\text{(iii)} \quad & p(\eta_1 \oplus \eta_2) &= p(\eta_1)p(\eta_2) \quad \text{(modulo 2-torsion)}. \end{aligned}$$

The symbolic generators $u_1, \ldots, u_n$ (representing appropriate elements of $H^*(BT^n; \mathbb{R})$) are also convenient for defining additive and multiplicative expressions $f(\eta)$ (like $w(\eta)$, $c(\eta)$ and $p(\eta)$) in the characteristic classes. It turns out that all such additive expressions (i.e. such that $f(\eta_1 \oplus \eta_2) = f(\eta_1) + f(\eta_2)$) can be obtained from those in a single variable u, expressed as a power series (for complex and real bundles respectively):

$$\text{(i)} \quad f(u) = \sum_{m \geq 0} a_m u^m, \quad f(\eta) = f(c_1, \ldots, c_k, \ldots) = \sum_{j=1}^{n} f(u_j);$$

$$\text{(ii)} \quad f(u^2) = \sum_{m \geq 0} b_m u^{2m}, \quad f(\eta) = f(p_1, \ldots, p_k, \ldots) = \sum_{j=1}^{n} f(u_j^2).$$

Hence all additive expressions $f(u)$ are of the form (again in the cases of complex and real bundles respectively):

$$\begin{aligned} &\text{(i)} \quad f(\eta) = f(c_1, \ldots, c_k, \ldots) = \sum_{m \geq 0} m! a_m \widetilde{c}_m = \sum_{m \geq 0} m! a_m ch_m(\eta); \\ &\text{(ii)} \quad f(\eta) = f(p_1, \ldots, p_k, \ldots) = \sum_{m \geq 0} m! b_m \widetilde{c}_{2m} = \sum_{m \geq 0} m! b_m ch_{2m}(\eta). \end{aligned} \tag{1.36}$$

Analogously, all multiplicative characteristics $f(\eta)$, i.e. satisfying $f(\eta_1 \otimes \eta_2) = f(\eta_1) \cdot f(\eta_2)$, are determined by power series in one variable, of the following form:

$$\text{(i)} \qquad f(u) = \sum_{m \geq 0} a_m u^m, \quad f(\eta) = f(c_1, \dots, c_k, \dots) = \prod_{j=1}^{n} f(u_j);$$

$$\text{(ii)} \qquad f(u^2) = \sum_{m \geq 0} b_m u^{2m}, \quad f(\eta) = f(p_1, \dots, p_k, \dots) = \prod_{j=1}^{n} f(u_j^2).$$

On closed manifolds (real or complex) the form of such a multiplicative characteristic f is determined by the requirement

$$f(M_1^n \times M_2^m) = f(M_1^n) \cdot f(M_2^m),$$

yielding:

$$\text{(i)} \qquad f(M^{2n}) = \langle f(c_1, \dots, c_n), [M^{2n}] \rangle \quad \text{(the complex case)};$$

$$\text{(ii)} \qquad f(M^{4n}) = \langle f(p_1, \dots, p_n), [M^{4n}] \rangle \quad \text{(the real orientable case)}.$$

In particular, in the case (i) above we have for the Euler characteristic $\chi\left[M^{2n}\right]$ that

$$f(u) = 1 + u, \quad f(c_1, \dots, c_k, \cdots) = 1 + c_1 + c_2 + \dots.$$

We shall meet with other important cases below.

Example 1. For the tangent bundle $\tau = \tau(\mathbb{R}P^n)$ of n-dimensional real projective space, we have the simple bundle isomorphism

$$\tau(\mathbb{R}P^n) \oplus \varepsilon_{\mathbb{R}} \cong \underbrace{\eta \oplus \cdots \oplus \eta}_{n+1 \text{ times}},$$

where $\varepsilon_{\mathbb{R}}$ denotes the trivial bundle over $\mathbb{R}P^n$ with fiber $\mathbb{R}^1$, and η is the unique nontrivial line bundle over $\mathbb{R}P^n$ with structure group $O_1 \cong \mathbb{Z}/2$. Over $\mathbb{R}P^1$ the total space of η is the usual Möbius band (see Figure 4.4) — though with the fiber taken to be an open interval. The Stiefel-Whitney classes of η are as follows:

$$0 \neq w_1(\eta) \quad = \quad v \in H^1(\mathbb{R}P^n; \mathbb{Z}/2),$$

$$w_i(\eta) \quad = \quad 0, \quad i > 1.$$

The cycle $Dv \in H_{n-1}(\mathbb{R}P^n; \mathbb{Z}/2)$ is represented by the hypersurface $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$; see Figure 4.5. Using the formula for the class of a direct sum of vector bundles, we obtain thence

$$w(\tau) = 1 + w_1(\tau) + \dots + w_n(\tau) = (1 + v)^{n+1} \in H^*(\mathbb{R}P^n; \mathbb{Z}/2).$$

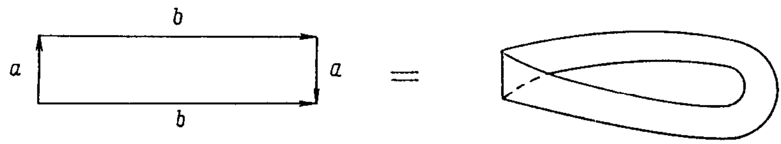

Fig. 4.4. The Möbius band: the simplest nontrivial bundle, with base S^1 and fiber an interval

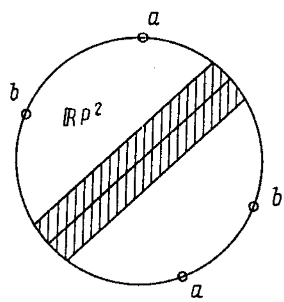

Fig. 4.5

Example 2. The tangent bundle $\tau = \tau(\mathbb{C}P^n)$ is a complex $GL_n(\mathbb{C})$-bundle. For reasons similar to those applying in the previous example we have

$$\tau(\mathbb{C}P^n) \oplus \varepsilon_{\mathbb{C}} \cong \underbrace{\eta \oplus \cdots \oplus \eta}_{n+1 \text{ times}},$$

$$c_1 = u \in H^2(\mathbb{C}P^n; \mathbb{Z}),$$

$$c(\tau) = 1 + c_1(\tau) + \ldots + c_n(\tau) = (1+u)^{n+1} \in H^*(\mathbb{C}P^n; \mathbb{Z}),$$

where u is a generator of $H^2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}$. The Poincaré dual cycle $Du \in H_{2n-2}(\mathbb{C}P^n; \mathbb{Z})$, is represented by the cycle $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Example 3. Let M^n be a smooth submanifold of a smooth manifold N^k $(n < k)$. We then have, in addition to the tangent bundle $\tau = \tau(M^n)$, the normal bundle $\nu = \nu(M^n)$ in N^k with respect to some Riemannian metric on N^k (although in fact the metric is not significant here). One has the simple formula

$$i^*\tau(N^k) = \tau(M^n) \oplus \nu(M^n)$$

(where i^* is induced by the inclusion $i : M^n \longrightarrow N^k$), whence

$$i^*w(N^k) = w(\tau)w(\nu),$$

$$i^*p(N^k) = p(\tau)p(\nu) \quad \text{(modulo the 2-torsion)},$$

$$i^*c(N^k) = c(\tau)c(\nu),$$

for complex U_n-manifolds $M^n \subset N^k$ in the last case. In the case that $N^k = \mathbb{R}^k$, the tangent bundle $\tau(\mathbb{R}^k)$ is trivial, so that

$$i^*w(\mathbb{R}^k) = 1, \quad i^*p(\mathbb{R}^k) = 1.$$

Consequently $w(\tau)w(\nu) = 1$ and $p(\tau)p(\nu) = 1$ (modulo the 2-torsion), yielding (see (1.32)) formulae expressing the corresponding characteristic classes of the tangent and normal bundles of M^n in terms of one another.

Example 4. For the tangent bundle τ of an even-dimensional orientable manifold M^{2n} the Euler characteristic class satisfies

$$\langle \chi_{2n}(\tau), [M^{2n}] \rangle = \chi(M^{2n}),$$

i.e. yields the Euler-Poincaré characteristic when evalulated at the fundamental homology class $[M^{2n}]$. (For U-manifolds one replaces the class χ_{2n} by the Chern class c_n.)

Example 5. For Lie groups G the tangent bundle is trivial and all of the characteristic classes are zero. Lie groups are parallelizable manifolds, i.e. admit fields of n-frames ($n = \dim G$) that are everywhere non-degenerate (obtained by means of left (or right) translation of any basis of the tangent space at the identity element).

Example 6. Suppose a submanifold M^n of $\mathbb{R}^N$ is defined by a set of simultaneous equations $f_1 = 0, \ldots, f_{N-n} = 0$, where the gradients of the f_i, $i = 1, \ldots, N-n$, are everywhere linearly independent on the manifold of common zeros:

$$M^n = \{x \in \mathbb{R}^N \mid f_1(x) = 0, \ldots, f_{N-n}(x) = 0\}.$$

The normal bundle $\nu(M^n)$ (with fiber $\mathbb{R}^{N-n}$) is then trivial and in view of the formula (1.32) all of the stable characteristic classes w_i, p_k of the manifold M^n will be zero. Only the Euler class χ_{2k} may be non-zero, provided $n = 2k$. (For instance $\chi(S^{2k}) = 2$.) The tangent bundle $\tau(M^n)$ is here stably trivial, i.e. $\tau \oplus \varepsilon_{\mathbb{R}}^{N-n} = \varepsilon_{\mathbb{R}}^{N}$. □

At various points above we have mentioned the concept of a *complex manifold*: this is an even-dimensional real smooth manifold with charts V_α co-ordinatized by complex co-ordinates $z_\alpha^1, \ldots z_\alpha^n$, and transition functions on the regions $V_\alpha \cap V_\beta$ of overlap complex-analytic. Here n is the *complex dimension* of the complex manifold M^n; as a real manifold its dimension is $2n$. The simplest examples of compact complex manifolds are the complex projective spaces $\mathbb{C}P^n$, and the even-dimensional tori $T^{2n} = \mathbb{C}^n/\Gamma$, where Γ is a lattice determined by any $2n$ linearly independent real vectors in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. Of particular importance are the compact complex submanifolds $M^k \subset \mathbb{C}P^n$, the *projective algebraic varieties.*

An *Hermitian metric* on a complex manifold M^n is given in terms of local co-ordinates on each chart V_α by

$$ds^2 = g_{ij} dz_\alpha^i d\bar{z}_\alpha^j, \quad g_{ij} = g_{ji}.$$

With such a metric one may associate the real 2-form given locally by

$$\Omega = \frac{i}{2} g_{ij} dz_\alpha^i \wedge d\bar{z}_\alpha^j.$$

The requirement that parallel translation of tangent vectors along any path determine a unitary transformation, leads to the condition that the 2-form Ω be closed:

$$d\Omega = 0.$$

Complex manifolds endowed with an Hermitian metric whose associated 2-form Ω is closed, are called *Kähler manifolds.* A Kähler manifold M^n for which the form Ω represents an integral cohomology class, i.e. is such that its integrals over 2-cycles in M^n are integer-valued, is called a *Hodge manifold.* The complex projective space $\mathbb{C}P^n$ and its complex submanifolds (varieties) are Hodge manifolds. In fact, every Hodge manifold is a projective algebraic variety; this was proved by Siegel for complex tori, and by Kodaira in the general case (in the late 1940s). A complex torus endowed with an Hermitian metric with constant components g_{ij} is Kähler, but is a complex projective algebraic variety if and only if the defining lattice $\Gamma \subset \mathbb{C}^n$ satisfies the conditions of Riemann for a corresponding θ-function to be definable.

All complex submanifolds of $\mathbb{C}^n$ without boundary and of dimension > 0, are non-compact. Further results concerning the topology of complex manifolds will be given in the following sections.

§2. The homology theory of smooth manifolds. Complex manifolds. The classical global calculus of variations. H-spaces. Multi-valued functions and functionals

Every differentiable manifold M^n of smoothness class C^k, $k \geq 1$, can be triangulated, i.e. turned into a simplicial complex. A *smooth simplex* is a pair consisting of a standard simplex $\sigma^k \subset \mathbb{R}^k$ and a regular smooth embedding of some k-dimensional open region U such that $\sigma^k \subset U \subset \mathbb{R}^k$, into the manifold:

$$f_k : U \longrightarrow M^n.$$

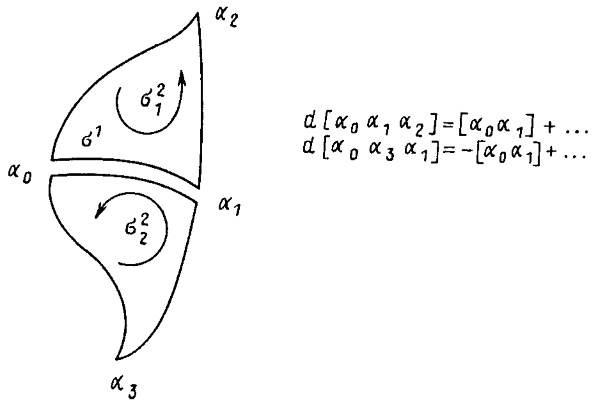

Fig. 4.6

A *smooth triangulation* of M^n is then a simplicial decomposition of M^n into smooth simplexes:

$$M^n = \bigcup_{\alpha,k} f_{k,\alpha}(\sigma^k),$$

making M^n a "PL-manifold," i.e. such that the simplicial complex made up of all simplexes containing any given simplex is combinatorially equivalent to the standard n-simplex σ^n (see Chapter 3, §1).

Up to combinatorial equivalence any smooth manifold M^n has just one smooth triangulation (Cairns, Whitehead, in the 1930s). Hence the standard apparatus of simplicial homology and cohomology ($H_*(M^n), H^*(M^n)$, the multiplication on H^* making it a ring, intersections of cycles in $H_*(M^n)$, Poincaré duality, etc. — see Chapter 3, §1,2) goes over in its entirety, via smooth triangulations, to smooth manifolds M^n (see Figure 4.6).

For closed submanifolds W^k, $V^j \subset M^n$, the intersection operation has a clear geometric interpretation. First one subjects W^k to an arbitrarily small perturbation to bring it into "general position" relative to V^j, in the sense that at each point $x \in W^k \cap V^j$, the vectors tangent to W^k or V^j should together span the full tangent space $\mathbb{R}^n_x$ of the ambient manifold M^n. Thus if $k + j < n$ the intersection $W^k \cap V^j$ will, in general position, be empty. If $k + j = n$, then (in general position) the intersection will consist of finitely many isolated points $x_1, \dots, x_m$ (see Figure 4.7); in this case the number m of these, reduced modulo 2, gives the intersection index modulo 2:

$$W^k \circ V^j \equiv m \pmod 2. \tag{2.1}$$

If the manifolds M^n, W^k, V^j come with orientations, then each point x_j of the intersection $W^k \cap V^j$ has associated with it a sign $sgn_{x_j}(W^k \cap V^j)$, namely the sign of the determinant of the change from a frame τ_{x_j} for the tangent space to M^n at x_j, agreeing with the given orientation of M^n, to the frame

$(\tau'_{x_j}, \tau''_{x_j})$ where τ'_{x_j} is a tangent frame for W^k at x_j according with the given orientation of W^k, and similarly for τ''_{x_j} with respect to V^j. We then have

$$\begin{aligned} W^k \circ V^j &= \sum_j sgn_{x_j} \left(W^k \cap V^j\right), \\ V^j \circ W^k &= (-1)^{kj} \left(W^k \circ V^j\right). \end{aligned} \tag{2.2}$$

For a smooth orientable manifold M^n, one may choose a pair of bases

$$\left\{z_j^{\alpha_j}\right\}, \quad \left\{\tilde{z}_j^{\alpha_j}\right\}, \quad \alpha_j = 1, \dots, b_j,$$

for the groups $H^j(M^n; \mathbb{Z})$, $j = 0, 1, \dots, n$, mutually Poincaré-dual in pairs:

$$\begin{aligned} & z_j^{\alpha_j} \circ \tilde{z}_{n-j}^{\alpha_{n-j}} = \delta_{\alpha_j \alpha_{n-j}}, \\ & z_0 = \tilde{z}_0 = 1, z_n = \tilde{z}_n = [M^n]. \end{aligned} \tag{2.3}$$

For $n = 2m$, one has in particular two bases $\{z_m^{\alpha_m}\}$, $\{\tilde{z}_m^{\alpha_m}\}$, $\alpha_m = 1, \dots, b_m$, for the group $H^m(M^{2m}; \mathbb{Z})$, satisfying

$$z_m^i \circ \tilde{z}_m^j = \delta_{ij},$$

and so Poincaré-dual. (For non-orientable manifolds there exist analogous mutually dual bases modulo 2.)

Example. Let M^n be a closed, orientable manifold, and $f : M^n \longrightarrow M^n$ any regular smooth self-map. Consider the following cycles (the diagonal and the graph of f) in $M^n \times M^n$:

$$\begin{aligned} \Delta &= \{(x, x)\} \in H_n(M^n \times M^n; \mathbb{Q}); \\ \Delta_f &= \{(x, f(x))\} \in H_n(M^n \times M^n; \mathbb{Q}). \end{aligned}$$

As noted above, in general position the intersection $\Delta \cap \Delta_f$ consists of isolated points x_j: $f(x_j) = x_j$, which, in view of the regularity of f and the assumption that the cycles Δ, Δ_f are in general position relative to one another, are non-degenerate, i.e. in terms of local co-ordinates x_α^q in a neighbourhood of the point x_j,

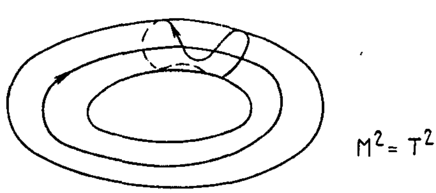

Fig. 4.7

$$\det\left(\delta_{pq} - \frac{\partial f^p}{\partial x^q_\alpha}\right)_{x=x_j} \neq 0.$$

The intersection index is then the sum of the attached signs of the fixed points x_j:

$$\Delta \circ \Delta_f = \sum_{x_j \in \Delta \cap \Delta_f} sgn \det\left(\delta_{pq} - \frac{\partial f^p}{\partial x^q}\right)_{x=x_j}. \tag{2.4}$$

From bases $\{z_j^{\alpha_j}\}$, $\{\tilde{z}_j^{\alpha_j}\}$ satisfying (2.3), one obtains the basis $z_j^{\alpha_j} \otimes \tilde{z}_{n-j}^{\alpha_{n-j}}$ for $H_n(M^n \times M^n; \mathbb{Q})$, using which one can establish (via (2.4)) the Lefschetz formula (ca. 1920)

$$\Delta \circ \Delta_f = \sum_{k \geq 0} (-1)^k \mathrm{Tr} f_*|_{H_k(M^n;\mathbb{Q})}. \tag{2.5}$$

It follows that if f is null-homotopic then $\Delta \circ \Delta_f = 1$. Applying this to $M^n = S^n$, and considering the disc D^n as a hemisphere of S^n, $D^n \subset S^n$, one can deduce Brouwer's theorem (ca. 1910) on the existence of a fixed point of any map $D^n \longrightarrow D^n$. □

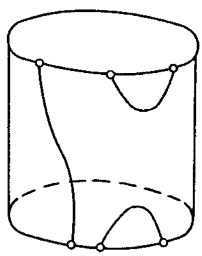

Fig. 4.8. The intersection index is not changed by a deformation of cycles

Returning to our closed submanifolds W^k, $V^j \subset M^n$, we now consider the case $k + j > n$. Here the intersection $W^k \cap V^j$ (if non-empty and assuming general position) is always a submanifold (with an orientation imposed if the manifolds are oriented) of dimension $j + k - n$:

$$V^j \circ W^k = N^{j+k-n} \tag{2.6}$$

Thus in this case the intersection of cycles has immediate geometric meaning. However it is far from being the case that all cycles in a manifold (representing elements of the integral or $\mathbb{Z}/2$-homology groups) are representable as submanifolds or linear combinations of submanifolds. In fact in integral homology not all cycles need even be representable as images of closed manifolds

$$f_*[N^k] = z \in H_k(M^n; \mathbb{Z}), \quad f : N^k \longrightarrow M^n.$$

The problem of determining when cycles are so representable is called "Steenrod's problem". (Of course the fundamental cycle $[M^n]$ is trivially so representable.) In $\mathbb{Z}/2$-homology Steenrod's problem has a positive solution (Thom, in the early 1950s). We shall discuss Thom's theory in connection with this sort of problem below (see §3).

As noted earlier, orientability of a manifold M^n is determined by the vanishing of the first Stiefel-Whitney class $w_1(M^n)$, whose Poincaré dual $Dw_1 \in H_{n-1}(M^n;\mathbb{Z}/2)$ is represented by the cycle consisting of the "zeroes of the determinant", i.e. the points where a field of tangential n-frames $\tau^n(x)$ in general position becomes degenerate (i.e. linearly dependent). It may also be obtained as the image under the Bockstein homomorphism

$$\beta_* : H_n(M^n;\mathbb{Z}) \longrightarrow H_{n-1}(M^n;\mathbb{Z}/2)$$

of the fundamental class $[M^n]$:

$$Dw_1 = \beta_*[M^n]. \tag{2.7}$$

(Recall from Chapter 3, end of §4, that the Bockstein homomorphism is defined (on integral chains) by

$$\beta_*(\tilde{z}) = \partial z/2, \quad z = \tilde{z} \mod 2.)$$

From (2.7) we infer the homotopy invariance of w_1; in the sequel we shall discuss Thom's theorem on the homotopy invariance of all of the Stiefel-Whitney classes w_i, and also the explicit formula of Wu for calculating them (dating from the early 1950s). (See also the preceding section.)

For a non-orientable manifold M^n there is an *orientation epimorphism* $\sigma : \pi_1(M^n) \to \mathbb{Z}/2$ (where $\mathbb{Z}/2 = \{-1, 1\}$), defined as follows: we set $\sigma(a) = -1$ if parallel transport of a tangent n-frame around a loop γ in the homotopy class a changes the orientation of the n-frame; otherwise set $\sigma(a) = 1$. (Thus in any surface a sufficiently small neighborhood of a non-self-intersecting loop γ parallel transport around which changes the orientation of a tangent n-frame, will be a Möbius band (see Figure 4.6).) There is therefore an action σ of $\pi_1(M^n)$ on $\mathbb{Z}$ which simply multiplies $\mathbb{Z}$ by ± 1; the homology groups $H_i(M^n;\sigma)$ with coefficients from the corresponding representation are the Poincaré duals of the usual integral cohomology groups $H^{n-j}(M^n;\mathbb{Z})$ (see Chapter 3, §5) (and analogously for $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$).

Cellular decompositions of manifolds, and consequently cellular homology, arise in connection with the following situation (considered by Morse in the late 1920s): Suppose we are given a real-valued smooth function f on a closed or open manifold M^n, with the following properties:

1. "Compactness": every closed region of M^n determined by an inequality of the form $f(x) \leq c$ should be compact.

2. All *critical* (or *stationary*) points x_j (points where grad f vanishes) should be non-degenerate, i.e. the quadratic form $d^2 f$ on the tangent space

$\mathbb{R}^n_{x_j}$ should have non-zero determinant; in terms of local coordinates this is the condition that at x_j

$$\det\left(\frac{\partial^2 f}{\partial x^i_\alpha \partial x^k_\alpha}\right) \neq 0.$$

Such functions are called *Morse functions.* The "height functions" on closed orientable surfaces suitably embedded in $\mathbb{R}^3$, furnish examples of Morse functions–see Figure 4.9. The number of negative squares in the diagonalized quadratic form $d^2 f$ at the point x_j, is called the *Morse index* of the critical point.

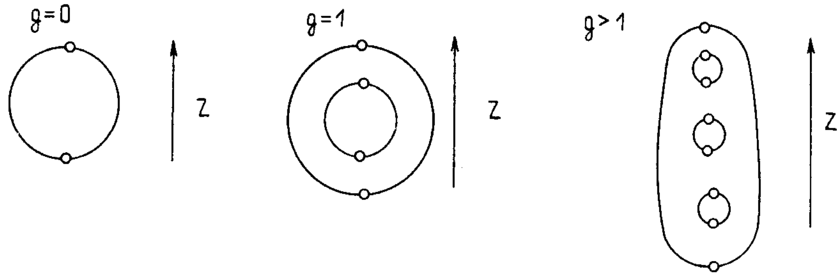

Fig. 4.9

Note that a Morse function f on a manifold M^n may always be chosen in such a way, that for any critical value c of f there is an $\varepsilon > 0$ such that $f^{-1}[c-\varepsilon, c+\varepsilon]$ contains only one critical point x_0, $f(x_0) = c$. If x_0 is such a critical point of index λ, then the manifolds

$$M^n_{c+\varepsilon} \quad (f(x) \leq c+\varepsilon) \quad \text{and} \quad M^n_{c-\varepsilon} \quad (f(x) \leq c-\varepsilon),$$

and their boundaries

$$N^{n-1}_{c+\varepsilon} \quad (f(x) = c+\varepsilon) \quad \text{and} \quad N^{n-1}_{c-\varepsilon} \quad (f(x) = c-\varepsilon),$$

are related in the following way: Consider the "n-dimensional handle of index λ"

$$H^n_\lambda = D^{n-\lambda} \times D^\lambda,$$

which has boundary

$$\partial H^n_\lambda = \left(S^{n-\lambda-1} \times D^\lambda\right) \cup \left(D^{n-\lambda} \times S^{\lambda-1}\right).$$

One identifies the portion $D^{n-\lambda} \times S^{\lambda-1}$ of the boundary ∂H^n_λ with a suitable small tubular neighborhood of an embedded $(\lambda-1)$-sphere in $N^{n-1}_{c-\varepsilon} = \partial M^n_{c-\varepsilon}$, via a diffeomorphism

$$\psi : D^{n-\lambda} \times S^{\lambda-1} \longrightarrow N^{n-1}_{c-\varepsilon}.$$

One then attaches the handle H^n_λ to $M^n_{c-\varepsilon}$ via the diffeomorphism ψ, smoothing out "corners" to obtain a smooth manifold-with-boundary. It can be shown that this can be effected in such a way as to obtain a manifold diffeomorphic to $M^n_{c+\varepsilon}$, i.e. one obtains

$$M^n_{c+\varepsilon} \cong M^n_{c-\varepsilon} \cup_\psi (D^{n-\lambda} \times D^\lambda).$$

This procedure induces a "Morse surgery"

$$N^{n-1}_{c-\varepsilon} \mapsto N^{n-1}_{c+\varepsilon},$$

where

$$N^{n-1}_{c+\varepsilon} \cong \left(N^{n-1}_{c-\varepsilon} \setminus \widetilde{D}^{n-\lambda} \times S^{\lambda-1}\right) \cup_\psi \left(S^{n-\lambda-1} \times D^\lambda\right).$$

(Here $\widetilde{D}^{n-\lambda}$ denotes the interior of the disc $D^{n-\lambda}$.)

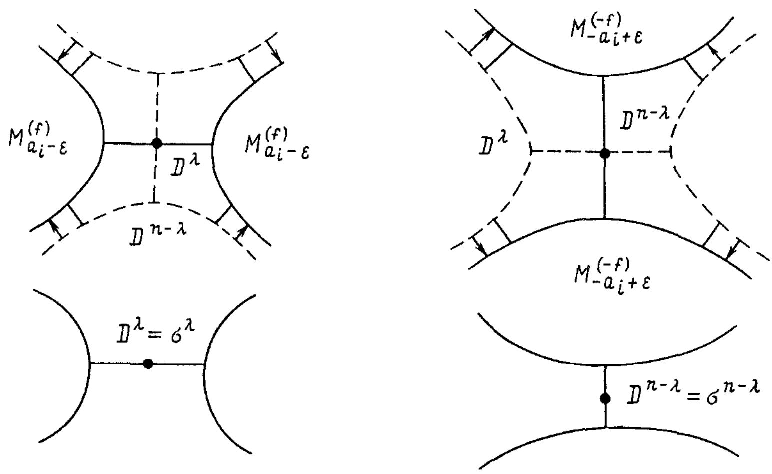

Fig. 4.10

Thus a Morse function on M^n provides a "handle-decomposition" of the manifold M^n, i.e. leads to a decomposition of the manifold into a "sum" of handles. (Figure 4.11 shows the two possibilities for attaching a handle H^2_1 to a disc H^2_0, and then in Figure 4.12 a "Smale function" (i.e. a Morse function with critical values ordered according to their index) is used to effect a Morse surgery on the torus by starting with H^2_0 and successively attaching handles as one passes through the critical points.)

Morse's theorem consists in the following lower estimate for the number m_λ of critical points of index λ of a Morse function f on a closed manifold M^n, relating that number to the integral homology of M^n:

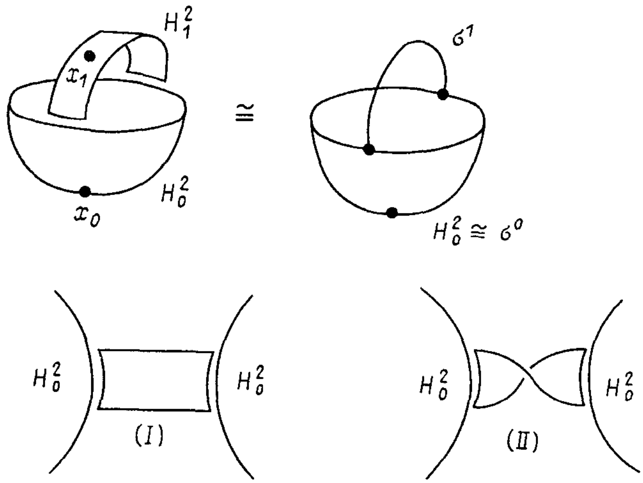

Fig. 4.11

$$m_\lambda(f) \geq b_\lambda, \quad m_\lambda(f) \geq b_\lambda^{(p)}, \tag{2.8}$$

where b_λ is the λth Betti number ($=$ rank $H_\lambda(M^n;\mathbb{Z})$), and $b_\lambda^{(p)}$ the λth mod-p Betti number (see Chapter 3, §2).

Writing q_λ for the least number of generators of the torsion subgroup Tor $H_\lambda(M^n;\mathbb{Z})$ of $H_\lambda(M^n;\mathbb{Z})$, we infer from the relationships between the homology groups over different coefficient groups (see Chapter 3, §2) that

$$m_\lambda(f) \geq b_\lambda + q_\lambda + q_{\lambda-1}. \tag{2.9}$$

These are the *Morse inequalities.*

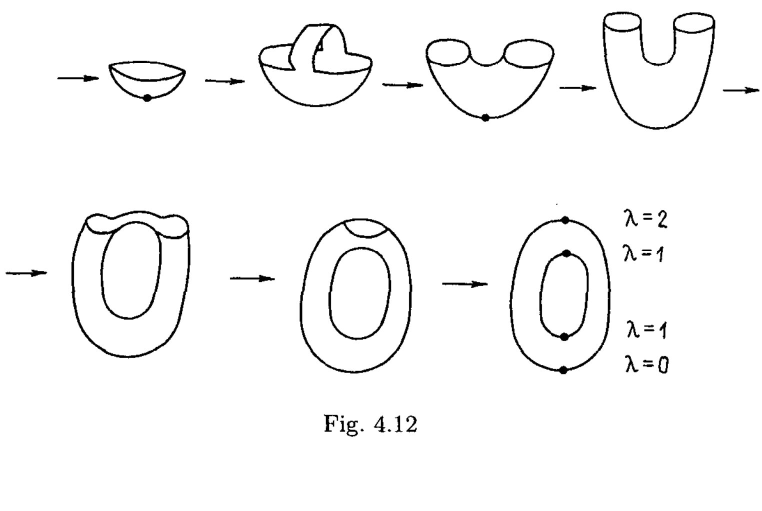

Fig. 4.12

For simply-connected manifolds of dimension ≥ 6 these inequalities are strict in the sense that there exists a Morse function f on M^n for which equality obtains in all of the equations (2.9) (Smale, in the early 1960s).

A Morse function f on a closed manifold M^n can be used to construct a CW-complex of the same homotopy type as the manifold, with the same number of cells of each dimension λ as the Morse function has critical points of index λ. The basic idea of the construction is as follows: Endow the manifold M^n with a Riemannian metric g_{ij} and consider the vector field $-\text{grad}\, f$, which in local coordinates $\{x_\alpha^j\}$ defines the dynamical system given locally by

$$\frac{dx_\alpha^i}{dt} = -g^{ij}\frac{\partial f}{\partial x_\alpha^j}, \tag{2.10}$$

where $g^{ij}g_{jq} = \delta_q^i$. Let x_0 be a critical point of index λ with $f(x_0) = c$ say, and, as before, denote by $M_{c-\varepsilon}^n$, $M_{c+\varepsilon}^n$ the manifolds-with-boundary determined respectively by the inequalities $f(x) \leq c - \varepsilon$, $f(x) \leq c + \varepsilon$, where ε is sufficiently small for there to be no critical points other than x_0 in $f^{-1}[c-\varepsilon, c+\varepsilon]$. Choose a coordinate system $\{x^j\}$ in a neighborhood U of x_0 so that the identity

$$f(x) = c - (x^1)^2 - \ldots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \ldots + (x^n)^2$$

holds throughout U for the Morse function f. Now consider an embedding $\phi : D^\lambda \longrightarrow U$ of the λ-disc, determined by the formula

$$(x^1)^2 + \ldots + (x^\lambda)^2 \leq \varepsilon \quad \text{and} \quad x^{\lambda+1} = 0, \ldots, x^n = 0.$$

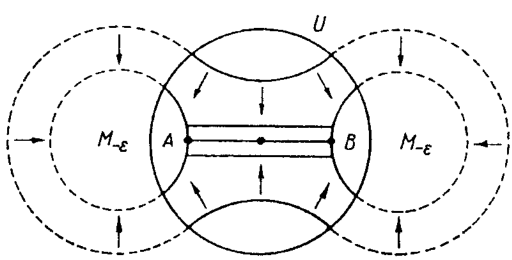

Fig. 4.13

Note that:

(i) $\phi(0) = x_0$;

(ii) the function $\phi^* f$ on the disc D^λ has the center of the disc as its only (non–degenerate) critical point (a maximum);

(iii) the image $\phi(\partial D^\lambda)$ of the boundary $\partial D^\lambda = S^{\lambda-1}$ is contained in the manifold $\partial M^n_{c-\varepsilon} = \{x \in M^n \mid f(x) = c - \varepsilon\}$.

The manifold $M^n_{c+\varepsilon}$ is homotopy equivalent to the space $M^n_{c-\varepsilon} \cup_\phi D^\lambda$. An appropriate homotopy $\varphi_\tau : M^n_{c+\varepsilon} \longrightarrow M^n_{c+\varepsilon}$, $0 \le \tau \le 1$, is defined as follows: On $M^n_{c-\varepsilon}$ it is the identity map for all τ; outside U and $M^n_{c-\varepsilon}$, φ_τ is the deformation of $M_{c+\varepsilon}$ which moves the points along the integral trajectories of the dynamical system (2.10), transverse to the level surfaces. It can be shown that on U the homotopy φ_τ can be defined so as to deform those parts of the level surfaces $f^{-1}[c-\varepsilon, c+\varepsilon]$ contained in U, onto the λ-dimensional disc D^λ — see Figure 4.13. The upshot is that φ_τ provides a homotopy between the identity map $\varphi_0 : M_{c+\varepsilon} \longrightarrow M_{c+\varepsilon}$ and the desired map $\varphi_1 : M_{c+\varepsilon} \longrightarrow M^n_{c-\varepsilon} \cup_\phi D^\lambda$.

By carrying out this construction for each critical point in turn one obtains the desired CW-complex homotopically equivalent to M^n. At each stage one can ensure, by means of a small perturbation of φ_1 bringing it into "general position", that the boundary ∂D^λ is attached only to cells of dimension $< \lambda$ already constructed in $M_{c-\varepsilon}$. (Alternatively one may use a *Smale function*, i.e. a Morse function where the critical values have the same ordering as the indices of the corresponding critical points.)

The Morse inequalities (see above) follow quickly from this construction of a CW-complex of the same homotopy type as M^n. They may in fact be supplemented, since one infers from that construction that

$$\sum_{j=0}^{k}(-1)^j m_j(f) \ge \sum_{j=0}^{k}(-1)^j b_j, \tag{2.11}$$

so that in particular taking $k = n$, one has

$$\sum_{j=0}^{n}(-1)^j m_j(f) = \chi(M^n). \tag{2.12}$$

The Morse inequalities do not apply if the critical points of a real-valued smooth function on a closed manifold M^n are isolated but are allowed to be degenerate. There is however another useful homotopy invariant — the *Lyusternik-Shnirelman category* of the manifold (introduced in the late 1920s) — which provides a lower bound for the number of critical points of smooth functions on M^n regardless of non-degeneracy. For an arbitrary topological space X, this invariant, denoted by $cat(X)$, is defined as the least number k of closed sets $X_1, \ldots, X_k$, such that $X = \bigcup_i X_i$, and each X_i is contractible in X (to a point). One then has the result that for a closed manifold M^n the number $m(f)$ of critical points of an arbitrary smooth, real-valued function f on M^n satisfies

$$m(f) \ge cat(M^n).$$

The invariant $cat(M^n)$ may in turn be estimated from below using the cohomology ring of M^n (with appropriate coefficients): Defining the *cohomological* *length* $l(M^n)$ of M^n to be the largest number l for which there exist non-zero products $\alpha_1 \cdots \alpha_l$ of elements $\alpha_1, \ldots, \alpha_l$ of positive degree in $H^*(M^n; \mathbb{Z})$ ($H^*(M^n; \mathbb{Z}/2)$ if M^n is non-orientable), we have

$$cat(M^n) > l(M^n).$$

One always has $cat(M^n) \leq n+1$. It is not difficult to see that $cat(S^n) = 2$, and that $cat(M^2) = 3$ for every closed surface M^2 other than S^2, and also that $cat(\mathbb{R}P^n) = n+1$, $cat(\mathbb{C}P^n) = n+1$. In the sequel we shall consider inequalities of Morse and Lyusternik-Shnirelman type in the context of certain path spaces arising in the calculus of variations.

Finally we consider smooth functions f on a manifold M^n whose critical points constitute a submanifold of M^n; such functions arise inevitably in the presence of a Lie group of symmetries of M^n, where it is required that the level sets of f be preserved by the Lie action (see Figure 4.14).

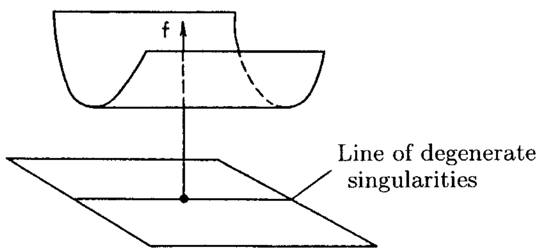

Fig. 4.14

Let f be a smooth function on M^n. A submanifold W^k is called *critical* if grad $f|_{W^k} \equiv 0$. Note that the Hessian d^2f determines a quadratic form on a subspace $V_x^{n-k} \subset TM_x^n$ orthogonal to W^k with respect to some Riemannian metric on M^n. A connected critical submanifold W^k is called *nondegenerate of index* λ if for any $x \in W^k$ the quadratic form $d^2f|_{V_x^{n-k}}$ is nongenerate and has index λ. (The index is independent of the point x since W^k is connected.) A particular case of a critical manifold was considered by Pontryagin in the late 1930s in connection with a calculation of the mod-2 Betti numbers of Lie groups; the general theory together with its most important applications are due to Bott in the 1950s.

Over a critical submanifold $W_\alpha^k \subset M^n$ there is a naturally defined vector bundle $\nu^{\lambda_\alpha} \longrightarrow W_\alpha^k$ with fibers $\mathbb{R}_x^{\lambda_\alpha}$ given by the directions of "steepest descent" (i.e. decrease in f) at each point $x \in W_\alpha^k$, i.e. those determined by the negative squares in the Hessian d^2f restricted to the normal vector bundle of W_α^k.

Suppose now that the solution set of the equation grad$f = 0$ constitutes a collection of smooth critical submanifolds W_α^k of M^n, with indices λ_α. If for every W_α^k the vector bundle $\nu^{\lambda_\alpha} \longrightarrow W_\alpha^k$, just defined, is orientable (i.e. one may reduce the stucture group of ν^{λ_α} to the group SO_{λ_α}), then one has the inequalities

$$\sum_\alpha b_{i-\lambda_\alpha}\left(W_\alpha^k\right) \geq b_i\left(M^n\right) \tag{2.13}$$

for the ranks b_j of the jth rational homology groups (and also over $\mathbb{Z}/p$). If the vector-bundles $\nu^{\lambda_\alpha} \longrightarrow W_\alpha^k$ are not all orientable then the inequalities (2.13) hold only over $\mathbb{Z}/2$. These inequalities may be explained in the following way: Each s-dimensional cycle in a critical submanifold W_α^k corresponds to a relative cycle of dimension $s+\lambda_\alpha$ in each of the groups $H_{s+\lambda_\alpha}(M_c, M_{c-\varepsilon})$ (where M_c denotes the closed region of M^n defined by $f \leq c$) formed, roughly speaking, out of the rays in $\mathbb{R}_x^{\lambda_\alpha}$ of "steepest descent" emanating from the points x of the s-cycle.

In general, analytical or geometrical information concerning real-valued, smooth functions definable on a manifold, with restrictions of various kinds imposed on the nature of their critical points, may yield significant conclusions concerning the topology of the manifold.

Example 1. If M^n is a closed manifold on which there exists a Morse function with just 2 critical points, then it is not difficult to see that M^n must be homeomorphic (even piecewise linearly) to S^n. By the Smale-Wallace theorem (of the early 1960s), for $n \geq 5$ there exists on every manifold M^n homotopically equivalent to the sphere S^n a Morse function with exactly 2 critical points. The idea for such a characterization of spheres is due to Reeb (in the early 1950s); the first deep application of it was made by Milnor in the late 1950s in connection with his discovery of exotic smooth structures on the sphere S^7, in the sense that the manifolds with these smooth structures are not diffeomorphic to the standard smooth 7-sphere, though they are PL-homeomorphic to it. In §4 below we shall discuss the more elaborate Milnor-Kervaire theory of manifolds M^n homotopically equivalent to S^n for $n \geq 5$. □

Example 2. Let W^{n+1} be a manifold-with-boundary, with boundary a disjoint union: $\partial W^{n+1} = V_1^n \cup V_2^n$, and suppose that W^{n+1} is contractible onto each of these portions of the boundary:

$$W^{n+1} \sim V_1^n \sim V_2^n, \quad \pi_j\left(W^{n+1}, V_i^n\right) = 0, \ j \geq 0.$$

In this situation we say that the manifold W^{n+1} is an *h-cobordism* between V_1^n and V_2^n, and that the latter manifolds are *h-cobordant.* The "h-cobordism theorem" of Smale states that if $n \geq 5$ and V_1^n, V_2^n are simply-connected then there exists a smooth, real-valued function f on W^{n+1}, constant on V_1^n and V_2^n, without any critical points (and with gradient nowhere tangential to V_1^n or V_2^n). It follows that under the same hypotheses, W^{n+1} is diffeomorphic to the cylinder $V_1^n \times I$, and V_1^n is diffeomorphic to V_2^n. □

Example 3. The closed surfaces M^2 may be readily classified by considering particular Morse functions on them. The generalization of this approach, on the basis of the h-cobordism theorem above, leads to the classification of certain classes of simply-connected manifolds, for instance those having non-zero homology groups only in the middle dimension $\left[\frac{n}{2}\right]$. The method of Novikov (of the early 1960s) for classifying arbitrary simply-connected manifolds uses a different approach although it does invoke the h-cobordism theorem (see §4 below). On any connected, closed 3-dimensional manifold M^3 there exists a Morse function f with one maximum point, one minimum point, a collection of critical points of index 1 all on one level surface $f = c_1$, and a collection of critical points of index 2 all on another level surface $f = c_2$. (The number of critical points of index 1 can then be shown to be equal to the number of index 2; we denote this number by g.) If the manifold is cut along an intermediate level surface $f = c$, $c_1 < c < c_2$, one obtains the two submanifolds

$$M^3_- = \{x \in M \mid f(x) \le c\}, \quad M^3_+ = \{x \in M \mid f(x) \ge c\}$$

with the surface M_g^2 as common boundary:

$$M_g^2 = \{x \in M \mid f(x) = c\}.$$

It is not difficult to see that if M^3 is orientable then the manifolds M_-, M_+ are each homeomorphic to the solid handlebody obtained as the region of $\mathbb{R}^3$ filling the sphere-with-g-handles M_g^2 embedded in standard fashion in $\mathbb{R}^3$. It follows that the structure of the 3-manifold M^3 is determined wholly by the way in which these two (identical) solid handlebodies M_- and M_+ are glued together along their boundaries, i.e. by a diffeomorphism $h: M_g^2 \to M_g^2$. (Such a decomposition of a 3-manifold is called a *Heegaard splitting*.) Although the homotopy and topological classes of diffeomorphisms $M_g^2 \longrightarrow M_g^2$ have been classified (by Nielsen in the 1930s), this does not yield a classification of the (closed, connected) 3-manifolds since different diffeomorphisms $M_g^2 \longrightarrow M_g^2$ may yield homeomorphic manifolds M^3. In the case $g = 1$ one obtains, as the only 3-manifolds with Heegaard splittings of genus one, S^3, $S^2 \times S^1$, and certain lens spaces. □

Example 4. If $g(z)$ is a holomorphic function in general position on a complex manifold M^n of complex dimension n, then the real part $f(z) = \mathrm{Re}g(z)$ of $g(z)$ is a Morse function all of whose critical points have index n. This is true in particular for the manifold obtained from a projective algebraic variety $M^n \subset \mathbb{C}P^n$ by deleting the hyperplane section $\mathbb{C}P^{N-1} \cap M^n = W^{n-1}$ say: on the remaining affine portion $M^n \setminus W^{n-1}$ there exist many everywhere-holomorphic functions g in general position, and their real parts will then be Morse functions with all critical points of index n. From this one can conclude that: Every map $K \longrightarrow M^n$, where K is a CW-complex of dimension $\le n-1$, is contractible onto the hyperplane section. This has the consequence for the homology groups that the inclusion homomorphism

$$H_j\left(W^{n-1}\right) \longrightarrow H_j\left(M^n\right)$$

is an isomorphism for $j < n-1$ and an epimorphism for $j = n-1$.

The homology theory of projective algebraic varieties, developed by Lefschetz in the 1920s, including intersections of cycles, the theory of algebraic cycles and the special role of the hyperplane section, will be examined in detail in a cognate volume surveying complex algebraic geometry. We shall confine ourselves here to the consideration of particular properties of meromorphic maps

$$f : M^n \longrightarrow \mathbb{C}P^1 \cong S^2$$

(i.e. complex analytic functions with poles). The critical points of such a function have associated with them a nice topological phenomenon. Let x_j be a critical point ($\operatorname{grad} f|_{x_j} = 0$, $d^2 f|_{x_j}$ non–degenerate) with $f(x_j) = c$ say. On each non-singular level set (or "fiber") F_ε defined by the equation $f(x) = c+\varepsilon$, for $|\varepsilon|$ sufficiently small, there is a "vanishing cycle" $z_0 \in H_{n-1}(F_\varepsilon)$, vanishing in the sense that the cycle goes to 0 as $\varepsilon \to 0$. Moving the fiber F_ε around the loop $|\varepsilon| =$ const. yields a "monodromy" map

$$A : F_\varepsilon \longrightarrow F_\varepsilon.$$

The induced self-map $A_j : H_j(F_\varepsilon) \longrightarrow H_j(F_\varepsilon)$ of the homology groups has the form

$$A_j = 1 \quad \text{for} \quad j \neq n-1,$$

$$A_{n-1}(z) = z \pm (z_0 \cdot z) z_0, \qquad z, z_0 \in H_{n-1}(F_\varepsilon).$$

(Note that the fiber F_ε is a complex manifold of complex dimension $n-1$.)

The monodromy transformations associated with degenerate critical points of algebraic functions have been the object of deep investigations in algebraic geometry over a long period (including the last decade), by Deligne and many others. One of the most interesting situations is that where there arise such "fiber bundles with singularities" as a result of sectioning a variety $M^n \subset \mathbb{C}P^N$ by a one-parameter family of hyperplanes, yielding fibers F which are all hyperplane sections. Homologically, a bundle with singular fibers over critical values $c_j = g(x_j)$ is described by means of homology with coefficients in a "Leray sheaf" $\mathcal{F}$ on the sphere $S^2 = \mathbb{C}P^1$, given by $\mathcal{F}_U = H_*(g^{-1}(U))$. (Note that here for small discs U, the region $p^{-1}(U)$ is contractible onto a single fiber.) In the case of a meromorphic function $f : M^n \longrightarrow \mathbb{C}P^1$ having only nondegenerate critical points, the only homological invariants of f are the homology groups of the fibers F, and the collection of critical points together with their associated vanishing cycles (see above). Actually this situation was already covered by the Picard-Lefschetz theory of the 1920s, although rigorous proofs were given only somewhat later. □

A completely different approach, via tensors, to the real or complex cohomology theory of a manifold M^n was initiated around 1930 by Cartan, using skew-symmetric tensor fields (with lower indices only) on the manifold, i.e. differential forms, and the associated differential operator d on such forms, introduced above. Particular important types of 1-forms and 2-forms, together with some of their deeper topological and algebro-topological properties, including applications, had been considered earlier by Poincaré in connection with the theory of automorphic forms and his investigations of Hamiltonian systems; however the full systematic theory is due to Cartan.

Consider the algebraic complex $\Lambda^*(M^n)$, defined to have as its elements the smooth globally-defined differential forms Ω_k on M^n, given in terms of local co-ordinates on each chart $V_\alpha \subset M^n$ by

$$\Omega_k = \sum_{j_1<\cdots<j_k} T_{j_1\cdots j_k} dx_\alpha^{j_1} \wedge \cdots \wedge dx_\alpha^{j_k}.$$

Thus here k is the degree of the form, and the k-forms Ω_k together comprise $\Lambda^k(M^n)$. The differential operator $d : \Lambda^k(M^n) \longrightarrow \Lambda^{k+1}(M^n)$ is defined locally by

$$d\Omega_k = \sum_{j_1<\cdots<j_k} \frac{\partial T_{j_1\cdots j_k}}{\partial x_\alpha^j} dx_\alpha^j \wedge dx_\alpha^{j_1} \wedge \cdots \wedge dx_\alpha^{j_k},$$
$$dx_\alpha^j \wedge dx_\alpha^s = -dx_\alpha^s \wedge dx_\alpha^j. \tag{2.14}$$

The real vector spaces $\Lambda^k(M^n)$, $k = 0, \ldots, n$, together with the differential d determine the algebraic complex $\Lambda^*(M^n)$:

$$0 \longrightarrow \Lambda^0(M^n) \xrightarrow{d} \Lambda^1(M^n) \to \cdots \xrightarrow{d} \Lambda^n(M^n) \to 0. \tag{2.15}$$

Note that $\Lambda^k(M^n)$ may be defined as the sheaf of germs of k-forms (of class C^∞), and the sequence (2.15) gives rise to an exact sequence of sheaves (see below). The cohomology groups of the manifold M^n are now defined via the homology of the algebraic complex (2.15):

$$H_\Lambda^k(M^n; \mathbb{R}) = \text{Ker } d/\text{Im } d, \quad \text{Im } d \subset \text{Ker } d \subset \Lambda^k(M^n),$$

and their direct sum forms a graded ring

$$H_\Lambda^*(M^n) = \sum_{k\geq 0} H_\Lambda^k(M^n; \mathbb{R}),$$

where the multiplicative operation is the exterior product of forms. "Poincaré's lemma" states that in some neighbourhood of every point of M^n every closed form $\Omega \in \Lambda^k(M^n)$, $k > 0$, (i.e. $d\Omega = 0$) is exact (i.e. $\Omega = d\Omega'$).

For compact, simply-connected Lie groups and certain homogeneous spaces, namely simply-connected symmetric spaces of covariantly constant Riemann curvature (i.e. where the covariant derivative of the Riemann curvature is identically zero), the cohomology ring $H_\Lambda^*(M^n)$ is readily calculated.

Each class $z \in H^*_\Lambda(M^n)$ contains a unique two-sided invariant k-form, (and conversely each such k-form belongs to some cohomology class) on $M^n = G$ (in the Lie group case), or a unique G'-invariant k-form on $M^n = G'/G'_0$ in the case of a symmetric space. Note that in the latter case the Lie algebra $\mathfrak{G}'$ of the Lie group G' always admits a $\mathbb{Z}/2$-grading:

$$\begin{gathered}\mathfrak{G}' = \mathfrak{G}'_0 + \mathfrak{G}'_1, \\ [\mathfrak{G}'_0, \mathfrak{G}'_0] \subset \mathfrak{G}'_0, \quad [\mathfrak{G}'_0, \mathfrak{G}'_1] \subset \mathfrak{G}'_1, \quad [\mathfrak{G}'_1, \mathfrak{G}'_1] \subset \mathfrak{G}'_0.\end{gathered} \tag{2.16}$$

A Lie group G endowed with a Riemannian metric may be regarded as a symmetric space by taking the group $G' = G \times G$ as acting on G, with the action given by

$$(g_1, g_2) : g \mapsto g_1 g g_2^{-1}.$$

This yields $M^n = G \cong G'/G$, and

$$\begin{gathered}\mathfrak{G}' = \mathfrak{G}'_0 + \mathfrak{G}'_1, \\ \text{where} \quad \mathfrak{G}'_0 = \{(x, x)\}, \quad \mathfrak{G}'_1 = \{(x, -x)\}.\end{gathered} \tag{2.17}$$

In fact for any homogeneous space of a compact Lie group G the computation of the cohomology ring reduces to the computation of the complex $\Lambda^*_{inv}(M^n)$ consisting of the G-invariant forms, although the restriction of the operator d to Λ^*_{inv} may act like the null operator.

In the 1930s de Rham showed that these cohomology groups (i.e. defined in terms of differential forms) are naturally isomorphic to the usual simplicial cohomology groups with coefficients in $\mathbb{R}$. We now sketch the modern proof of this result using sheaf theory. Consider the following sheaves on a manifold M^n:

$$\Lambda^k = \text{the sheaf of germs of } k\text{-forms (of class } C^\infty);$$

$$Z^k = \text{the sheaf of germs of closed } k\text{-forms.}$$

(Thus Z^0 is the sheaf of constants.) One has the following exact sequence of sheaves

$$0 \longrightarrow Z^k \xrightarrow{i} \Lambda^k \xrightarrow{d} Z^{k+1} \longrightarrow 0 \tag{2.18}$$

(where i is the inclusion), which , as noted in Chapter 3, §4, gives rise to the exact cohomology sequence

$$\cdots \to H^j(Z^k) \xrightarrow{i_*} H^j(\Lambda^k) \xrightarrow{d_*} H^j(Z^{k+1}) \xrightarrow{\partial^*} H^{j+1}(Z^k) \to \cdots, \tag{2.19}$$

(where $H^j(\mathcal{F})$ denotes the jth cohomology group with coefficients from the sheaf $\mathcal{F}$, i.e. the group $H^j(M^n; \mathcal{F})$). Using the fact that sections of $\Lambda^k (h \geq 0)$ are required only to be of class C^∞ and not analytic, it is not difficult to show that

$$H^j(\Lambda^k) = 0 \quad \text{for } j > 0. \tag{2.20}$$

Note also that $H^0(\Lambda^k) = \Lambda^k(M^n)$, since by definition $H^0(\Lambda^k)$ coincides with the group of all k-forms of class C^∞ defined on the whole of M^n. From (2.20) and the exactness of the sequence (2.19), it follows that for $j > 0$ the homomorphisms

$$\partial^* : H^j(Z^{k+1}) \longrightarrow H^{j+1}(Z^k)$$

are isomorphisms. Thus we have a sequence of isomorphisms

$$H^1(Z^k) \xrightarrow{\partial^*} H^2(Z^{k-1}) \xrightarrow{\partial^*} \cdots \xrightarrow{\partial^*} H^{k+1}(Z^0), \tag{2.21}$$

where

$$H^{k+1}(Z^0) \cong H^{k+1}(M^n; \mathbb{R}), \tag{2.22}$$

the ordinary real cohomology group, since Z^0 is the sheaf of constants. We have also from (2.19) and (2.20) the exact sequence

$$H^0(\Lambda^k) \xrightarrow{d_*} H^0(Z^{k+1}) \xrightarrow{\partial^*} H^1(Z^k) \to 0,$$

where $H^0(\Lambda^k)$ is the group of all k-forms (of class C^∞) globally defined on M^n (so that $d_* = d$), and $H^0(Z^{k+1})$ the group of closed $(k+1)$-forms on M^n. From this exact sequence, together with (2.20), (2.21) we have immediately

$$H^{k+1}(M^n; \mathbb{R}) \cong H^1(Z^k) \cong \operatorname{Ker} d/\operatorname{Im} d \cong H_\Lambda^{k+1}(M^n),$$

$$\operatorname{Ker} d \subset \Lambda^k(M^n),$$

which is de Rham's theorem. (Note that the Poincaré lemma is used implicitly in this argument to establish local exactness, i.e. to obtain the exactness of the sequence of sheaves (2.18).)

We next glance at a different kind of invariance of the cohomology groups $H_\Lambda^j(M^n)$, namely their invariance under smooth homotopies of manifolds

$$F : M_1^n \times I \to M_2^m, \quad I = [a, b].$$

Observe first that for any form $\omega \in \Lambda^k(M_2^m)$ the pullback has the form

$$F^* w = \Phi^{(0)}_{i_1 \cdots i_{k-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dt + \Phi^{(1)}_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

If we define an operator D on forms $\omega \in \Lambda^k(M_2^m)$ by

$$D\omega = \int_a^b F^*\omega, \quad D\omega \in \Lambda_{k-1}(M_1^n),$$

where the integral is with respect to t, then the contribution to $D\omega$ from the second term in the above expression for $F^*\omega$ is zero, since that term does not involve dt. Using this it is straightforward to verify that

$$Dd \pm dD = f_a^* - f_b^*,$$

where $f_a = F(\ ,a)$, $f_b = F(\ ,b)$. It follows (much as in Chapter 3, §2, the analogous conclusion followed from equation (2.17) there) that for any cocycle z $(dz = 0)$ the difference $f_a^*(z) - f_b^*(z) = dD(z)$, and so is a coboundary. Hence the induced homomorphisms

$$f_a^*, f_b^* : H_\Lambda^j(M_2^m) \longrightarrow H_\Lambda^j(M_1^n)$$

coincide (for all j).

As far as homotopy theory is concerned, the category of CW-complexes and cellular maps may be replaced by the category of manifolds and smooth maps, since any CW-complex K, after being embedded in $\mathbb{R}^N$ for some N, may be replaced by a neighborhood of it in $\mathbb{R}^N$ contractible to K, and then cellular maps may be approximated arbitrarily closely by smooth ones, or replaced by smooth maps deformable to them by means of arbitrarily small homotopies. From this consideration and the homotopy invariance of the cohomology groups H_Λ^j just established in outline (together with the fact that they satisfy the other cohomology axioms, given in Chapter 3, §3), it is clear that the coincidence of the H_Λ^j with the usual cohomology groups follows. In fact there is no difficulty in taking this route to establishing the equivalence of cohomology theory defined via forms on manifolds, with the usual cohomology theory; however a detailed and rigorous exposition of this line of argument would not seem to be worthwhile since it would not yield a proof shorter than other known proofs.

There is a natural operator on forms on an orientable manifold M^n, determined by a Riemannian metric g_{ij} on M^n, called the *duality operator* $* : \Lambda^k(M^n) \longrightarrow \Lambda^{n-k}(M^n)$. It is given locally by

$$*\omega = (*T)_{i_1 \cdots i_{n-k}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-k}},$$

$$(*T)_{i_1 \cdots i_{n-k}} = \sqrt{\det(g_{ij})}\; g_{i_1 j_1} \cdots g_{i_{n-k} j_{n-k}} \varepsilon^{j_1 \cdots j_{n-k} j_{n-k+1} \cdots j_n} T_{j_{n-k+1} \cdots j_n},$$

$$\omega = T_{j_1 \cdots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

(The operator $*$ is well-defined for oriented manifolds M^n since for these the local expression

$$d\sigma = \sqrt{\det(g_{ij})}\; dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n$$

determines a well-defined n-form.) The duality operator $*$ has the following properties:

1) $(*)^2 = (-1)^{k(n-k)}$ on k-forms;

2) the operator $\delta = *^{-1} d * : \Lambda^{k+1}(M^n) \longrightarrow \Lambda^k(M^n)$ is formally dual to d with respect to the scalar product given by

$$\langle \omega, \omega' \rangle = \int_{M^n} \omega \wedge *\omega', \tag{2.23}$$

that is,

$$\pm\langle d\omega, \omega'\rangle = \langle \omega, \delta\omega'\rangle.$$

3) The *Laplace operator* $\Delta = d\delta + \delta d = (d+\delta)^2$ commutes with both d and δ (and is self-dual).

For any closed manifold M^n one has the following orthogonal "Hodge decomposition" of $\Lambda^k(M^n)$ considered as a Hilbert space with respect to the above scalar product (2.23):

$$\Lambda^k(M^n) = \text{Im } d \oplus \text{Im } \delta \oplus \text{Ker } \Delta, \tag{2.24}$$

where Ker Δ consists of the *harmonic forms* on M^n. One has also

$$\begin{gathered} \text{Ker } d = \text{Im } d \oplus \text{Ker } \Delta, \\ \text{Ker } \delta = \text{Im } \delta \oplus \text{Ker } \Delta, \quad \text{Ker } \Delta = \text{Ker } d \cap \text{Ker } \delta, \end{gathered} \tag{2.25}$$

whence it follows that the dimension of the space of harmonic forms in $\Lambda^k(M^n)$ is equal to the kth Betti number $b_k(M^n)$. It can be shown that the Poincaré duality homomorphism is determined by the action of the duality operator $*$ on the harmonic forms.

The Laplace operator Δ can be defined also for simplicial complexes, in fact for CW-complexes more generally, by taking ∂^* and ∂ in place of d and δ respectively, so that the Laplace operator in this context is defined as $\Delta = \partial\partial^* + \partial^*\partial$. With this understanding, the formulae (2.24) and (2.25) for the Hodge decomposition hold for any CW-complex K, with respect to the scalar product determined by taking the cells to form an orthogonal basis.

If the CW-complex K is not simply connected, then as described in Chapter 3, §5, any (unitary) representation $\rho : \pi_1(K) \to U_n$ determines a chain complex $C_*(K;\rho)$ with the structure of a $\mathbb{Z}[\pi_1]$-module. In the case where all homology groups of positive dimension of this chain complex are zero, the following formula for the Reidemeister torsion holds (Singer, in the late 1960s):

$$2\log|R(K,\rho)| = \sum_q q(-1)^q \log\det \Delta|_{C_q(K;\rho)}. \tag{2.26}$$

By extending this formula to the complex of differential forms on a manifold, using a suitably defined relative determinant of two different representations, Ray and Singer (in the early 1970s) were able to give an analytic treatment of Reidemeister torsion, showing in particular that their analytic version of the concept was independent of the metric g_{ij}. A rigorous verification that the Ray-Singer torsion actually does coincide with the usual combinatorially defined Reidemeister torsion was given in general form somewhat later (by Müller and Cheeger in the late 1970s).

It is of interest to note in this connection A. S. Schwarz' curious quantum-theoretical interpretation of Ray-Singer torsion using a functional integral of a simple gauge-invariant expression in differential forms where the torsion appears in the process of constructing, in the standard way, a theory hypothetically violating gauge-invariance.[1]

Differential forms are especially important in the homology theory of Kähler manifolds M^n of n complex dimensions, with Kähler metric g_{kj}. Recall (from the end of the preceding section) that by definition this metric is Hermitian, given locally (on a chart V_α co-ordinatized by $z_\alpha^1, \dots, z_\alpha^n$) by

$$ds^2 = g_{kj} dz_\alpha^k d\bar{z}_\alpha^j,$$

with the property that the associated real 2-form

$$\Omega = \frac{i}{2} g_{kj} dz_\alpha^k \wedge d\bar{z}_\alpha^j$$

is closed: $d\Omega = 0$. The space $\Lambda^k(M^n)$ of k-forms (and the associated sheaf Λ^k on M^n) decomposes naturally as a direct sum of forms of type (p,q):

$$\Lambda^k(M^n) = \sum_{p+q=n} \Lambda^{p,q}, \tag{2.27}$$

where a *form of type* (p,q) is defined (locally) as

$$\sum f_{i_1 \dots i_p j_1 \dots j_q} dz_\alpha^{i_1} \wedge \dots \wedge dz_\alpha^{i_p} \wedge d\bar{z}_\alpha^{j_1} \wedge \dots \wedge d\bar{z}_\alpha^{j_q}. \tag{2.28}$$

Differentiation of forms on the Kähler manifold M^n is defined on 0-forms (functions) by

$$d = d_c + \bar{d}_c, \quad d_c f = \frac{\partial f}{\partial z} dz, \quad \bar{d}_c f = \frac{\partial f}{\partial \bar{z}} d\bar{z}, \tag{2.29}$$

and extended essentially as for real forms to forms on M^n of degree > 0. The Kähler metric is used to define the formally dual operators

$$\delta = *^{-1} d *, \quad \delta_c = *^{-1} d_c *, \quad \bar{\delta}_c = *^{-1} \bar{d}_c *, \tag{2.30}$$

and the corresponding Laplace operator

$$\begin{gathered} \Delta_c = \delta_c d_c + d_c \delta_c, \quad \bar{\Delta}_c = \Delta_c, \\ \Delta = d\delta + \delta d. \end{gathered} \tag{2.31}$$

By "Hodge's lemma," provided the metric is Kähler the real and complex Laplace operators differ only by a constant factor:

[1] An up-to-date survey was written in 1983-1985 and published in Russian in 1986. The work of A. S. Schwarz mentioned above was completed around 1980 and was well-known at that time to certain mathematical physicists (specialists in quantum field theory). Starting from the late 1980s these ideas were developed by Witten and others. This theory is now known as "Topological Quantum Field Theory", and has become an extremely active area of modern topology.

$$\Delta = \text{const.} \times \Delta_c.$$

If follows that for M^n a closed manifold the space of harmonic forms and hence the cohomology groups decompose as direct sums:

$$H^k(M;\mathbb{C}) = \sum_{p+q=k} H^{p,q}(M^n;\mathbb{C}), \tag{2.32}$$

in accordance with the decomposition (2.27). This decomposition depends of course on the metric. Note that the form Ω determined by the metric (see above) has type $(1,1)$:

$$[\Omega] \in H^{1,1}(M^n;\mathbb{C}),$$

and that $[\Omega^n] \neq 0$ since the integral

$$\int_{M^n} \Omega^n$$

is a non-zero multiple of the volume of M^n.

The operation of forming the exterior product of any form w on M^n with the form Ω is usually denoted by L:

$$L(w) = \Omega \wedge \omega,$$

and the dual operator by Λ. *Primitive forms* ω are defined to be those for which $\Lambda\omega = 0$. Every form may be obtained from the primitive ones by means of applications of operators given by expressions in L and Λ. Note the result of Dolbeault:

$$H^{p,q}(M^n;\mathbb{C}) \cong H^p(M^n;\Lambda^q),$$

where here Λ^q denotes the sheaf of germs of holomorphic q-forms on M^n. If the form Ω represents an integral cohomology class, then by Kodaira's theorem the manifold M^n can be algebraically embedded in $\mathbb{C}P^N$ in such a way that the cohomology class of Ω is dual to the hyperplane section (mentioned earlier in connection with its special role in the homology theory of algebraic varieties). Note also the result of Moisheson (mid-1960s) to the effect that a complex manifold endowed with a (not necessarily Hodge) Kähler metric admits an algebraic embedding in $\mathbb{C}P^N$ if the space of meromorphic functions on the manifold has transcendence degree (i.e. largest number of algebraically independent functions) equal to its dimension.

It is a useful observation that algebraic cycles, i.e. subvarieties (possibly with singularities) N^{n-k} of a projective algebraic variety M^n, determine homology classes dual to elements of $H^{k,k}$:

$$D\left[N^{n-k}\right] \in H^{k,k}(M^n;\mathbb{C}) \cap H^{2k}(M^n;\mathbb{Z}). \tag{2.33}$$

According to sharpened version of a conjecture of Hodge every element of the intersection on the right-hand side of (2.33) is realizable in this way by means of a rational linear combination of algebraic cycles. The case $k = 1$ of this conjecture was settled earlier by Lefschetz in stronger form: every class in $H^{1,1} \cap H^2(M^n; \mathbb{Z})$ can be realized (as in (2.33)) by means of the fundamental cycle of an algebraic subvariety $N^{n-1} \cap M^n$ (a so-called "divisor"). For $k > 1$, Thom's theory (see below) shows that obstructions arise even to the realization of elements of $H_{2(n-k)}(M^n; \mathbb{Z})$ as images in M^n of non-singular complex (or even quasi-complex) varieties N^{n-k} (under maps $N^{n-k} \longrightarrow M^n$). We note here also the result of Hironaka of the mid-1960s according to which every variety with singularities may be obtained via contraction from a non-singular variety (and is therefore a continuous image of it).

We shall not discuss in the present survey the original circle of ideas, initiated by Grothendieck in the late 1950s, for formulating a homology theory of projective algebraic varieties over (algebraically closed) fields of characteristic $p > 0$. These ideas depend on elegant use of the category of finite-sheeted, unbranched coverings of open regions of $\mathbb{C}P^N$ (with respect to the Zariski topology, where the closed sets are just the algebraic subvarieties) involving the "étale topology" of Grothendieck. Analogues in the Grothendieck theory, of the Lefschetz formula for the number of fixed points of a smooth self-map $M \longrightarrow M$, applied to the Galois group of the algebraic closure of a finite field, yield, by means of homological algebra along the lines of the Weil-Tate program, number-theoretic asymptotics for the numbers of solutions of diophantine equations modulo p, and the rationality of the mod-p analogue of the Riemann ζ-function (which is considerably simpler than the ordinary ζ-function). This and related questions, including the relevant history and later developments, will doubtless be dealt with in those volumes of the present series devoted to algebraic geometry. What is important for us here is that the Grothendieck homology theory of real varieties defined by polynomials over the integers yields the same information as the ordinary homology theory of the complex variety defined by the same set of polynomials; the situation is more complicated only in mod-p homology where $p > 0$ is the characteristic of the field of interest.

We now return to the theory of ordinary real smooth manifolds. One of the most important areas of application of homology and homotopy theory is the study of global properties of extremals of functionals of paths on M^n with one or the other of two types of boundary conditions:

$$1) \quad S\{\gamma\} = \int_{t_0}^{t_1} L(x, \dot{x}) dt, \quad \gamma = x(t), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \tag{2.34}$$

the extremal problem where the end-points of the paths γ are fixed at x_0, x_1;

$$2) \quad S(\gamma) = \oint_\gamma L(x, \dot{x}) dt, \quad \gamma(t + T) = \gamma(t), \tag{2.35}$$

the periodic variational problem.

The quantity L, the *Lagrangian* of the variational problem, is a real-valued smooth function $L(x, v)$ defined on the points (x, v) of the total space $T(M^n)$ of the tangent bundle of M^n, and is always required to satisfy the positivity condition

$$\frac{\partial^2 L}{\partial v^i \partial v^j}\ \eta^i\eta^j > 0 \quad \text{for all non-zero vectors } \eta. \tag{2.36}$$

The standard examples of Lagrangians arising in geometry or in the classical mechanics of non-charged particles, are as follows:

a) $L = \frac{1}{2}\ g_{ij}v^iv^j = \frac{1}{2}\ |v|^2$, the Lagrangian appropriate to the motion of a particle in a Riemannian manifold M^n; provided the parameter t is natural, i.e. proportional to arc length, the extremals are the parametrized geodesics of M^n;

b) $L = \frac{1}{2}\ g_{ij}v^iv^j - U(x)$, appropriate to the motion of a particle in the Riemannian manifold M^n in the presence of a force field with potential $U(x)$;

c) $L(x, v) = |v| = \sqrt{g_{ij}v^iv^j}$, yields the *length functional*, with the geodesics as extremals;

d) more generally than c), functionals of *Finsler length* type where the Lagrangian satisfies

$$L(x, \lambda v) = \lambda L(x, v), \quad \lambda > 0; \tag{2.37}$$

these occur on submanifolds of Banach spaces, with the induced metric; they are usually required to satisfy the positivity condition

$$l(\gamma) = \int_\gamma L(x, \dot{x})dt > 0. \tag{2.38}$$

If $L(x, -v) = L(x, v)$, the variational problem is said to be *reversible*; this is the case for instance for the length functional c) with respect to a Riemannian metric.

According to the principle of Maupertuis (or Fermat-Jacobi-Maupertuis) the trajectories extremizing functionals of Lagrange type (see b) above) are just the geodesics with respect to the new metric defined in terms of the total energy $E = \frac{1}{2}\ |v|^2 + U(x)$ of the particle by

$$\begin{gathered} \tilde{g}^E_{ij} = 2(E - U)g_{ij}(x), \\ l^E(\gamma) = \int_\gamma \sqrt{2(E - U)g_{ij}\dot{x}^i\dot{x}^j}dt, \quad \dot{x}^j = \frac{dx^j}{dt}. \end{gathered} \tag{2.39}$$

Provided $E > \max_{x\in M^n} U(x)$, this clearly defines an ordinary Remannian metric. The extremals yielded by Fermat's "principle of least time" are essentially the geodesics with respect to a metric of the form

$$g_{ij} = c^{-1}(x)\delta_{ij} \quad \text{in } \mathbb{R}^3,$$

where $c(x)$ is a smooth scalar function (originally the speed of light). (As is well known, Fermat introduced the principle that light rays travel along paths of least time in order to explain the observed bending of light rays in passing from one isotropic medium to another, linking this to the differing speeds of light, then unknown, in the different media.)

In general one considers functionals defined on the *path space* $\Omega(x_0, x_1)$ consisting of all piecewise smooth parametrized paths in M^n joining x_0 to x_1, or (in the periodic problem) on the space $\Omega^+(M^n)$ of of all piecewise smooth, directed, parametrized loops, i.e. maps $S^1 \to M^n$, $f(\varphi + 2\pi) = f(\varphi)$. The manifold M^n is assumed closed. The positivity condition (2.38) ensures that the functionals a), b), c), d) above all satisfy the compactness requirement ("Arzelà's principle") that the subspace of paths γ satisfying $S\{\gamma\} \leq c$ should for each constant c be relatively compact (in the compact-open topology on $\Omega(x_0, x_1)$ or Ω^+).

With respect to the compact-open topology (or a natural distance between paths determined by a Riemannian metric on M^n), the spaces $\Omega(x_0, x_1)$ and Ω^+ are "infinite-dimensional" manifolds. In view of the Fermat-Jacobi-Maupertuis principle, the above examples a), b), c), d) will be essentially covered if we consider only length functionals with respect to a Riemannian metric or of Finsler type on our closed manifold M^n. Assuming in addition the condition (2.36), a Morse theory for such functionals $S\{\gamma\}$ on $\Omega(x_0, x_1)$ (or on Ω^+) analogous to that described above for smooth functions on manifolds, can be formulated: In place of the gradient one uses the "variational derivative" of S with respect to paths, so that *critical paths* are those at which the variational derivative $\delta S/\delta\gamma$ vanishes; it turns out that $\gamma \in \Omega(x_0, x_1)$ is critical precisely if it is a smooth geodesic parametrized by a natural parameter. In terms of the "second variation" $\delta^2 S$ of the functional S one defines a Hessian $d^2 S$ as a certain bilinear form on pairs of smooth vector fields (i.e. elements of $T_\gamma(\Omega)$) along γ in M^n, $\gamma \in \Omega(x_0, x_1)$, and then the index $i(\gamma_0)$ of a critical path γ_0 is the largest dimension of a subspace of $T_{\gamma_0}(\Omega)$ on which $d^2 S$, as a quadratic form on $T_{\gamma_0}(\Omega)$, is negative definite. It was shown by Morse in the 1920s that the index $i(\gamma_0)$ of a critical path $\gamma_0 \in \Omega(x_0, x_1)$ coincides with the number of points p of the curve γ_0, (counting multiplicities and excluding x_1) that are "conjugate" to x_0 in the sense that there exists a non-zero vector field along the arc of γ_0 from x_0 to p vanishing at x_0 and p and satisfying Jacobi's linear differential equation (the Euler-Lagrange equation for the extremals of the functional given by the second variation $\delta^2 S$). In the periodic case, i.e. for $\Omega^+(M^n)$, the number of points conjugate to any particular point on a critical loop provides only a lower estimate for the index. (The rigorous exposition of this theory requires in particular Hilbert's theorem on the existence of a geodesic joining x_0 and x_1.) As in the Morse theory of functions on manifolds, one can show, using the trajectories determined by the "gradient" (and finite-dimensional approximations by means of piecewise geodesic arcs), that the space $\Omega(x_0, x_1)$ is homotopically equivalent to a CW-complex whose cells of dimension i correspond one-to-one with the geodesics of index i from x_0 to x_1 (and similarly for Ω^+). It follows that in the general non-degenerate case (with the above assumptions, including Arzelà's principle), one has in this context also the Morse inequalities

$$m_i(S) \geq b_i(\Omega(x_0, x_1)),$$
$$m_i(S) \geq b_i(\Omega^+), \tag{2.40}$$

where m_i is the number of extremals of index i.

In certain circumstances the first of these inequalities may be used to infer the existence of distinct geodesics from x_0 to x_1. However in the periodic problem a plurality of closed geodesics may be illusory for instance in the sense that they might all be multiples of a simple closed geodesic (i.e. obtained by going round that geodesic a varying number of times). In the case $M^n = S^n$ with $n \geq 3$, it is known that there are at least two geometrically distinct extremals in the periodic problem (A. I. Fet, in the mid-1960s). For any closed manifold M^n with the property that the Betti numbers $b_i(\Omega(x_0, x_1))$ increase at least linearly as $i \to \infty$, it has been shown (by Gromoll and Meyer in the late 1960s) that there exist infinitely many geometrically distinct, periodic extremals.

Working in the context of Poincaré's problem concerning three non-self-intersecting geodesics on the 2-sphere S^2, Lyusternik and Shnirelman extended their inequality (see above) to show that $m(S) \geq cat(X/M^n)$, for arbitrary reversible functionals S on the quotient space X/M^n of the space of non-self-intersecting paths on M^n (or more precisely the completion of this space) modulo the space $M^n \subset X$ of constant paths on M^n. (Here non-degeneracy of the critical points is again not required.) By transferring to the space of non-directed paths and showing that X/S^2 (taking $M^n = S^2$) has the homotopy type of $\mathbb{R}P^2$, they were able to show, around 1920, that for any reversible functional S on X/S^2, the number of critical points, regardless of degeneracy, satisfies

$$m(S) \geq 3,$$

thereby solving Poincaré's problem.[2] In conclusion we note again that this theory applies only to reversible variational problems.

An important chapter of the global calculus of variations is devoted to the study of the length functional and geodesics determined by a metric of Cartan-Killing type on a Lie group or symmetric space (mainly due to Bott in the late 1950s). On the group SU_{2n}, for instance, equipped with the invariant metric determined by the Killing form $\langle X, Y \rangle = \operatorname{Re} \operatorname{Tr}(X\bar{Y}^T)$ on the Lie algebra, the minimal geodesics from $x_0 = I$ to $x_1 = -I$ (I the identity matrix) form a submanifold diffeomorphic to the complex Grassmannian manifold $G^{\mathbb{C}}_{2n,n} \cong U_{2n}/(U_n \times U_n)$, and the minimal geodesics, as critical paths of the action functional a), have index 0, while the non-minimal geodesics (from x_0 to x_1) have index $\geq 2n + 2$. It follows that the inclusion of the space of minimal geodesics ($\cong G^{\mathbb{C}}_{2n,n}$) in $\Omega(x_0, x_1)$ induces an isomorphism between

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[2] A complete and rigorous exposition of the Lyusternik-Shnirelman theory has been published recently; see the survey by I. A. Taimanov in *Russian Surveys*, 1992, v. 2.

the corresponding homotopy groups of dimensions up to and including $2n$. From the homotopy exact sequence of the principal fiber bundle

$$V_{2n,n}^{\mathbb{C}} \longrightarrow G_{2n,n}^{\mathbb{C}}, \quad \text{with fiber } U_n,$$

(see §1 (1.11)), we infer that

$$\pi_j(G_{2n,n}^{\mathbb{C}}) \cong \pi_{j-1}(U_n), \quad 1 \leq j \leq 2n-2. \tag{2.41}$$

Putting these various facts together we obtain (after some work) the result known as *Bott periodicity* for the stable homotopy groups of the unitary group:

$$\pi_j(U_n) \cong \pi_{j-2}(U_n), \quad 2 \leq j \leq 2n-1. \tag{2.42}$$

For the orthogonal and symplectic groups the argument is somewhat more complicated. For the orthogonal groups one obtains 8-fold periodicity:

$$\pi_j(SO_n) \cong \pi_{j-8}(SO_n), \quad 8 \leq j \leq n-2. \tag{2.43}$$

The first seven stable homotopy groups of O_n and U_n are given in the following table:

j	0	1	2	3	4	5	6	7	8	...
$\pi_j(O_n)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	...
$\pi_j(U_n)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...

One has the following isomorphisms for the symplectic group Sp_n:

$$\begin{aligned} \pi_j(O_n) &= \pi_{j-4}(Sp_n), \quad j \leq n-2, \\ \pi_j(Sp_n) &= \pi_{j-4}(O_n), \quad j \leq n-2. \end{aligned} \tag{2.44}$$

In fact the Bott periodicity operator is defined for any CW-complex K in the sense that

$$\begin{aligned} [\Sigma^8 K, O_n] &= [K, O_n] \quad \text{if} \quad \dim K < n-10, \\ [\Sigma^2 K, U_n] &= [K, U_n] \quad \text{if} \quad \dim K < 2n-4, \end{aligned} \tag{2.45}$$

where Σ denotes the suspension and $[X, Y]$ the set of homotopy classes of maps $X \to Y$. (Often the index n is omitted in (2.45) with the understanding that it is sufficiently large.)

Taking the direct limits

$$O = \lim_{n\to\infty} O_n, \quad U = \lim_{n\to\infty} U_n, \quad Sp = \lim_{n\to\infty} Sp_n,$$

we infer from Bott periodicity that

$$\Omega^4 O = Sp, \quad \Omega^4 Sp = O, \quad \Omega^2 U = U,$$

where $\Omega X = \Omega(X, x_0)$ is the space of loops on X based at x_0. (The base point is often left implicit.)

Remark. Without entering into detail we mention the fact (established by several authors) that the non-parallelizability of the spheres S^n for $n \neq 1, 3, 7$ can be deduced from Bott periodicity independently of the results of Adams originally used for this purpose. Thus for all $n \neq 1, 3, 7$, the tangent bundle $\tau(S^n)$ is non-trivial, and moreover for odd n represents (via the classification of SO_n-bundles — see Chapter 3, §6) an element of $\pi_{n-1}(SO_n)$ of order 2. Note that for the unitary group U_n the first non-stable homotopy group is

$$\pi_{2n}(U_n) = \mathbb{Z}_{n!}.$$

On the other hand Adams' result on the non-existence of elements of $\pi_{2n-1}(S^n)$, $n \geq 8$, of Hopf invariant 1 does not seem to follow from Bott periodicity in this way (see Chapter 3, §6). □

We now turn to the cohomology algebras of Lie groups and, more generally, "H-spaces." An *H-space* is a topological space X equipped with a continuous multiplication $x \cdot y = \psi(x, y) \in X$ (i.e. $\psi : X \times X \longrightarrow X$) with respect to which there is a "homotopic identity element," i.e. an element $1 \in X$ such that the maps

$$\psi(1, x) : X \longrightarrow X, \quad \psi(x, 1) : X \longrightarrow X$$

defining multiplication by 1, are both homotopic to the identity map 1_X. Note that H-spaces may be more or less like groups, for instance in the sense that they are "homotopically associative," as in the case of the loop spaces $\Omega(X, x_0)$. The latter, together with Lie groups, furnish the most important examples of H-spaces. For instance it follows from Bott periodicity that the space $BU = \lim_{n\to\infty} BU_n$ has the homotopy type of the loop space $\Omega(U, 1)$: $BU \sim \Omega(U, 1)$, and similarly for BO and BSp, so that these may be regarded as H-spaces as far as their homology or homotopy is concerned. Thus these, as also O, U, Sp, are homotopically associative and commutative, although for finite n, i.e. for the Lie groups O_n, U_n, Sp_n, this is not the case.

We now consider the cohomology algebra of an arbitrary H-space X, with coefficients from any field k. From the natural k-algebra isomorphism

$$H^m(X \times X; k) \cong \sum_{j+q=m} H^j(X; k) \otimes H^q(X; k),$$

and the H-space multiplication ψ on X, one obtains a k-algebra homomorphism

$$\psi^* : H^*(X;k) \to H^*(X;k) \otimes H^*(X;k). \tag{2.46}$$

From the existence of a homotopic identity element it follows that for any $z \in H^*(X;k)$ we may write $\psi^*(z)$ in the form

$$\psi^*(z) = 1 \otimes z + z \otimes 1 + \sum_j z'_j \otimes z''_j, \tag{2.47}$$

where $\dim z'_j$, $\dim z''_j > 0$. The conditions (2.46), (2.47) determine the structure of a Hopf algebra on $H^*(X;k)$. The existence of an algebra homomorphism of the form (2.47) turns out to be a strong restriction on the structure of the algebra $H^*(X;k)$; according to Hopf's theorem, a graded, skew-symmetric, associative algebra

$$H = \sum_{k \geq 0} H^k, \quad \dim H^k < \infty,$$

over a field of characteristic 0 ($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for instance) for which there exists a homomorphism $H \longrightarrow H \otimes H$ of the form (2.47), is a free skew-commutative graded algebra, i.e. satisfies no relations other than skew-commutativity: $zw = (-1)^{ab}wz$, where a, b are the dimensions of z, w. Hence the algebra $H^*(X;k)$ has a set of exterior generators $e_{j_\alpha} \in H^{2j+1}(X;k)$, and a set of polynomial generators $u_{j_\beta} \in H^{2j}(X;k)$, together generating $H^*(X;k)$ without further relations. Since the free polynomial generators u_{j_β} satisfy $u^n_{j_\beta} \neq 0$ for all n, it follows immediately that for finite-dimensional H-spaces X (in particular for Lie groups), the algebra $H^*(X;k)$ is a finitely generated free exterior algebra.

Over fields k of characteristic $p > 0$, it can be shown that there is a set of multiplicative generators of $H^*(X;k)$ in terms of which there are only relations of the form $u^{p^n} = 0$. Furthermore in several cases it is known that although $u^{p^n} = 0$, there is in the "dual algebra" a non-zero element of that dimension. Here the *dual (Pontryagin dual) algebra* to the Hopf algebra $H^*(X;k)$ of an H-space X, is obtained from the direct sum $H_*(X;k)$ of the homology groups (vector spaces) by defining a multiplication on it via the homomorphism

$$\psi_* : H_*(X;k) \otimes H_*(X;k) \longrightarrow H_*(X;k).$$

These results have been obtained by means of close scrutiny of the cohomology of H-spaces over finite fields, initiated by Browder in the early 1960s.

In particular Browder proved a Poincaré-duality law for the homology of finite-dimensional H-spaces. However for a considerable period no nontrivial examples of finite-dimensional H-spaces were known, until in the early 1970s Mislin, Hilton and others found such examples using the "p-localization technique for the category of homotopy types" invented by Quillen and Sullivan around 1970 in connexion with the "Adams conjecture". (The idea behind this technique actually dates back to work of Serre of the first half of the 1950s.) These new H-spaces resemble the homotopy–theoretical p-localizations of certain classical Lie groups and H-spaces for various primes p.

As regards the H–spaces $X = \Omega(Y, y_0)$, it should be pointed out that over a field k of characteristic zero, the Pontryagin ring $H_*(X; k)$ is isomorphic to the enveloping algebra of the Lie–Whitehead superalgebra of the homotopy groups

$$\pi_*(X) \otimes \mathbb{Q}, \quad \pi_i(X) = \pi_{i+1}(Y),$$

where the grading is as in X and the Whitehead (Lie superalgebra) multiplication $[x, y]$ of elements as in Y. To construct the enveloping algebra one takes all elements x_j of some graded basis for the homotopy groups (tensored with $\mathbb{Q}$), adjoins an identity element 1, and considers the algebra of all noncommutative k–polynomials in the variables x, subject to the relations

$$x_i x_j - (-1)^{\dim x_i \dim x_j} x_j x_i = [x_i, x_j].$$

This result is a consequence of theorems of Serre and Cartan.

Example. For the loop space of the sphere S^n we have

$$\begin{aligned} &H^j(\Omega S^n; \mathbb{Z}) = 0, \quad j \neq k(n-1), \\ &H^j(\Omega S^n; \mathbb{Z}) \cong \mathbb{Z}, \quad j = k(n-1),\ k \geq 0. \end{aligned} \tag{2.48}$$

Denoting by v_k a (suitable) generator of the group $H^{k(n-1)}(\Omega S^n; \mathbb{Z})$, one has the following multiplicative relations in $H^*(\Omega S^n; \mathbb{Z})$:

$$\begin{aligned} &1) \quad n = 2k: \qquad v_1^2 = 0, \quad v_{2p} v_{2q} = \binom{p}{p+q} v_{2(p+q)}; \\ &2) \quad n = 2k+1: \quad v_p v_q = \binom{p}{p+q} v_{(p+q)}. \end{aligned} \tag{2.49}$$

The isomorphisms (2.48) and the relations (2.49) have been shown (by Serre) to follow automatically from the Leray cohomology spectral sequence of the appropriate fibrations; however they were first obtained by Morse and Lyusternik around 1930 by considering geodesics on the sphere S^n. □

Any map $\psi : S^n \times S^n \longrightarrow S^n$, can be extended naturally (using a construction of Hopf) to a map

$$f_\psi : S^{2n+1} \longrightarrow S^{n+1},$$

where we consider $S^n \subset S^{n+1}$ and $S^n \times S^n \subset S^{2n+1}$ by means of the following canonical embeddings: S^n is identified with the equator of S^{n+1}:

$$S^{n+1} = D_1^{n+1} \cup D_2^{n+1}, \quad \text{where} \quad S^n = D_1^{n+1} \cap D_2^{n+1};$$

and $S^n \times S^n$ with the following subspace of S^{2n+1}:

$$S^n \times S^n = (D_1^{n+1} \times S^n) \cap (S^n \times D_2^{n+1}),$$

$$\text{where} \quad S^{2n+1} = (D_1^{n+1} \times S^n) \cup (S^n \times D_2^{n+1}).$$

The "Hopf invariant" $h(f,\psi)$ of the map f_ψ turns out to coincide with the product m_1m_2 of the degrees of the maps

$$\psi(s,s_0): S^n \times \{s_0\} \longrightarrow S^n, \ \deg\psi(s,s_0) = m_1,$$

$$\psi(s_0,s): \{s_0\} \times S^n \longrightarrow S^n, \ \deg\psi(s_0,s) = m_2.$$

If S^n is an H-space with multiplication ψ then clearly the degrees $m_1, m_2 = 1$, so that there is a map $f_\psi : S^{2n+1} \longrightarrow S^{n+1}$ of Hopf invariant 1. Hopf showed that such maps can exist only if n has the form $n = 2^q - 1$, and they do exist for $n = 1, 3, 7$. The theorem that maps $S^{2n+1} \longrightarrow S^{n+1}$ with Hopf invariant 1 do not exist for n other than 1, 3, 7, was proved by Adams in the late 1950s.

Theorems of this type — on nonparallelizability of spheres or the above result of Adams (which is much stronger) — yield as a corollary the non-existence of real finite-dimensional division algebras of dimensions other than 2, 4 and 8. (As already noted, this follows alternatively from Bott periodicity, as was shown by Milnor, Kervaire and Atiyah.)

In the early 1980s in the course of investigations of certain classical-mechanical problems such as that of the motion of a rigid body about a fixed point in a gravitational field, or of a rigid body moving freely in an ideal fluid, and also in problems of contemporary mathematical physics (in particular that of giving a detailed topological analysis of Dirac monopoles), there arose a class of multi-valued functionals (for instance as generalized analogues of Dirac monopoles) defined on one or another space of paths or fields with a large number of variables (Novikov). The topology of multi-valued functions and functionals, estimates of the number of their critical points, and the study of the structure of their level sets (in the finite-dimensional case), which we shall now describe, constitute a non-classical analogue of the classical global calculus of variations.

A multi-valued functional with single-valued gradient on a finite- or infinite-dimensional manifold M, is given by a closed 1-form ω ($d\omega = 0$) on M. The multi-valued function S itself is then given by the integral

$$S(x) = \int_{x_0}^{x} \omega, \quad \text{where } x_0 \text{ is a fixed point.}$$

The value $S(x)$ depends, in general, on the path γ connecting x with the point x_0. The multi-valued function S becomes a single-valued one when lifted to some regular infinite-sheeted covering space

$$p: \hat{M} \longrightarrow M, \quad \text{where} \quad p^*\omega = dS. \tag{2.50}$$

Assuming that $H_1(M;\mathbb{Z}) \neq 0$, this covering space has monodromy group isomorphic to $\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}$ (k factors), where $(k-1)$ is called the "irrationality degree" of the form ω. An action of generators $T_1, \dots, T_k$ of the monodromy group $\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}$ on $\hat{M}$ is determined by numbers $\kappa_1, \dots, \kappa_k$, given by the "periods" of the form ω as follows. First, one chooses some basis $\gamma_1, \ldots, \gamma_m \in H_1(M;\mathbb{Z})$, and then the numbers κ_j are given by the integrals (*periods*) of the form ω around the γ_j:

$$\int_{\gamma_j} \omega = \kappa_j, \quad j \leq k,$$
$$\int_{\gamma_j} \omega = 0, \quad j > k. \tag{2.51}$$

(The numbers $\kappa_1, \ldots, \kappa_k$ are linearly independent over $\mathbb{Z}$.) The action of the above monodromy group then is given by

$$S(T_i x) = S(x) + \kappa_i, \quad i = 1, \ldots, k, \tag{2.52}$$

If $\{V_\alpha\}$ is an open cover of the manifold M on each member of which ω is exact: $\omega|_{V_\alpha} = dS^{(\alpha)}$ for some single-valued function $S^{(\alpha)}(x)$ defined on V_α, then on each region of overlap $V_\alpha \cap V_\beta$ the difference $S^{(\alpha)} - S^{(\beta)}$ will be locally constant, i.e. constant on each connected component of $V_\alpha \cap V_\beta$. This affords a canonical method of constructing a multi-valued function S on a manifold M (or any topological space) with single-valued gradient, via an open cover $\{V_\alpha\}$ of M and a corresponding collection of functions $\{S^{(\alpha)}\}$, each $S^{(\alpha)}$ defined and single-valued on V_α, with the property that $S^{(\alpha)} - S^{(\beta)}$ is locally constant on $V_\alpha \cap V_\beta$. Note that if there is a single-valued function S on M such that $S - S^{(\alpha)}$ is constant on each V_α, then the collection $\{S^{(\alpha)}\}$ essentially defines that single-valued function.

If S is a multi-valued function on M with single-valued gradient, then the level sets $S =$ const. are well-defined and determine a "foliation" (possibly with singularities) of M, with "leaves" which may be, in general, non-compact, even if M is compact. The study of such foliations, which are of considerable topological interest, has begun only recently.

As for single-valued smooth functions on a manifold M, so also for multi-valued functions S (as above) is one interested in the critical points x_j : $dS|_{x_j} = 0$. We shall consider, in the infinite-dimensional as well as the finite-dimensional case, only 1-forms ω for which the critical points are all isolated and non-degenerate and have finite Morse index (essentially the number of negative squares in the quadratic form $d^2S|_{x_j}$ on the tangent space at x_j, with respect to appropriate local coordinates). If the monodromy group is $\mathbb{Z}$ with generator γ_1 (or in another words the 1-form ω has zero irrationality degree) and its only period

$$\kappa_1 = \int_{\gamma_1} \omega$$

is an integer (see above), then ω represents an integral cohomology class

$$[\omega] \in H^1(M;\mathbb{Z}). \tag{2.53}$$

This determines a single–valued map

$$\psi = \exp\{2\pi i S\} : M \longrightarrow S^1. \tag{2.54}$$

In this situation the form ω is said to be *quantized.*

Examples. 1. Let $M = M^n$ be a finite-dimensional, closed manifold, and ω a closed 1-form on M. How may one estimate the number of critical points of ω? The above-mentioned classical situation of a quantized form (see (2.53), (2.54)) is especially simple: in this case M^n must have the homotopy type of a fibration over S^1. There are various topological criteria for the existence of maps $\psi : M \longrightarrow S^1$ without critical points, obtained in the 1960s by Browder, Levine, Hsiang and Farrell (see §5 below). If M^n is not of this type, there will be critical points, estimates for which are given below.

2. Let M be the path space $\Omega(N^k, x_0, x_1)$ consisting of the paths in N^k (any closed manifold) from x_0 to x_1, or the space $\Omega^+(N^k)$ of directed loops $\gamma : S^1 \to N^k$. Note that the space $\Omega^+(N^k)$ may be regarded as a fiber bundle over N^k with projection map

$$p : \Omega^+(N^k) \longrightarrow N^k, \quad p(\gamma) = \gamma(0),$$

whose fiber over a point $x \in N^k$ is the loop space $\Omega(N^k, x)$ consisting of the loops beginning and ending at x. There is also the obvious cross-section $\varphi : N^k \longrightarrow \Omega^+(N^k)$, $p\varphi = 1$, where $\varphi(x)$ is the constant loop at x. If N^k is endowed with a Riemannian metric then any closed 2-form Ω on N^k (which may be called a "generalized magnetic field" for $k \leq 3$, or an "electromagnetic field" for $k = 4$) gives rise to the functional

$$S(\gamma) = \int_\gamma |\dot{\gamma}|^2 dt + e \int_\gamma d^{-1}(\Omega) \tag{2.55}$$

(the "action functional"), and the functional (with the same extremals)

$$l^E(\gamma) = \int_\gamma E^{\frac{1}{2}} |\dot{\gamma}| dt + e \int_\gamma d^{-1}(\Omega), \quad E = \text{const.} \tag{2.56}$$

(the "Maupertuis functional"). Here E is the "energy," e the "charge," and γ is from $\Omega(N^k, x_0, x_1)$ or $\Omega^+(N^k)$. Provided N^k is simply-connected and the cohomology class of the form Ω is non-zero:

$$0 \neq [\Omega] \in H^2(N^k; \mathbb{R}),$$

the formulae (2.55), (2.56) define multi-valued functionals on the space $M = \Omega(N^k, x_0, x_1)$ (or $\Omega^+(N^k)$). (If the manifold N^k is not simply-connected, then it can be embedded in the (suitably realized) Eilenberg-MacLane complex $K(\pi, 1)$ where $\pi = \pi_1(N^k)$, in such a way that the cells of $K(\pi, 1) \setminus N^k$ all have dimensions ≥ 3. The induced homomorphism

$$H^2(\pi_1(N^k); \mathbb{R}) = H^2(K(\pi, 1); \mathbb{R}) \longrightarrow H^2(N^k; \mathbb{R})$$

is then a monomorphism, from which it follows that on the subspace $\Omega_0^+(N^k)$ of null-homotopic free loops the functionals (2.55) and (2.56) are single-valued.)

3. Consider a smooth fiber bundle

$$p : E \longrightarrow B,$$

with base B a smooth closed manifold or manifold-with-boundary W^q. Suppose we are given a closed $(q+1)$-form Ω ($d\Omega = 0$) on the total space E, and some single-valued functional $S_0\{\psi\}$ defined on the cross-sections $\psi : B \longrightarrow E$ ($p\psi = 1$) of the fiber bundle. The formula

$$S = S_0\{\psi\} + \int_{(B,\psi)} d^{-1}(\Omega), \tag{2.57}$$

of the general type of (2.55) and (2.56), defines a functional, in general multi-valued, on the space C of cross-sections of the bundle, with prescribed values on the boundary ∂W^q, if there is any boundary. The most interesting case is that where $E = S^q \times G$, G a Lie group. Here the cross-sections are the so-called "chiral fields" $g(x)$, $x \in \mathbb{R}^q$, where $g(x) \longrightarrow g_0$ as $|x| \longrightarrow \infty$. Note that on every compact simple Lie group G there can always be defined a non-trivial two-sided invariant 3-form or 5-form (for instance if $G = SU_n$, $n \geq 3$), appropriate to the cases $q = 2$ or 4 respectively. □

We now explain in more detail how the formula (2.57) (and so also (2.55) and (2.56)) define multi-valued functionals. Let Ω be a closed form (as above) on E, the total space of the fibration $p : E \longrightarrow B$, and $\{V_\alpha\}$ be an open cover of E, with the following two properties:

a) the form Ω is exact on each V_α:

$$\Omega|_{V_\alpha} = d\Phi_\alpha;$$

b) for any smooth cross-section ψ of the fibration $p : E \longrightarrow B$ there should exist an index α such that $\psi(B) \subset V_\alpha$.

Let C be the space of cross-sections of the fibration $p : E \longrightarrow B$. The forms Φ_α determine "local functionals"

$$S^{(\alpha)}(\psi) = \int_{(B,\psi)} \Phi_\alpha$$

on the regions $W_\alpha \subset C$ consisting of those cross-sections contained in V_α. On the regions of overlap we have

$$S^{(\alpha)}(\psi) - S^{(\beta)}(\psi) = \int_{(B,\psi)} (\Phi_\alpha - \Phi_\beta).$$

Since $d(\Phi_\alpha - \Phi_\beta) = 0$ on $V_\alpha \cap V_\beta$, the difference $S^{(\alpha)} - S^{(\beta)}$ is locally constant on $W_\alpha \cap W_\beta$ by Stokes' Theorem, so that the collection $\{S^{(\alpha)}\}$ does determine a well-defined 1-form ω on C.

If, in this context, the form ω is quantized, i.e. $[\omega] \in H^1(C;\mathbb{Z})$, then the value at ψ of the resulting (single-valued) map

$$\exp\{2\pi i S\} : C \to S^1,$$

is called the *Feynman amplitude* of the cross-section ψ. The condition that the form ω be quantized is essential in the quantum theory of such fields with multi-valued action functionals. This represents the procedure known as "topological quantization of coupling constants" in Quantum Field Theory, as formulated by Novikov (1981), Deser-Jackiv-Templeton in the special case of Chern-Simons (1982), and Witten (1983).

In the case of a trivial fibration $E = S^q \times F \longrightarrow S^q$, the form Ω is actually defined on the fiber F, the space C of cross-sections reduces to the space of maps $S^q \longrightarrow F$, and the integrals of the form ω along paths in the domain function space C coincide with integrals of the form Ω on F over appropriate images under the Hurewicz homomorphism $\pi_{q+1}(F) \longrightarrow H_{q+1}(F)$.

For general bundles $E \longrightarrow B$ (as above) the association of integrals of Ω over cycles in E with contour integrals of ω along paths in C, requires individual topological analysis (not usually very difficult) depending on the particular bundle in question and the boundary conditions. We note in conclusion that if the functional (2.57) (or (2.55) or (2.56)) is actually multi-valued (i.e. not single-valued) then on the covering space $\widetilde{C}$ (cf. (2.50)) where $p_1^*\omega = \delta S$ ($p_1 : \widetilde{C} \longrightarrow C$), the functional S takes on all values $(-\infty, \infty)$.

We now turn to the known topological estimates of the number of critical points of multi-valued functions and functionals. We begin with the simplest finite-dimensional case. Let w be a closed 1-form on M^n of irrationality degree zero. We then have a covering space $\hat{M}$ of M^n with monodromy group $\mathbb{Z}$:

$$p : \hat{M} \to M^n, \quad p^*w = dS,$$

where the generating monodromy transformation $T : \hat{M} \longrightarrow \hat{M}$ satisfies

$$S(Tx) = S(x) + \kappa, \quad \kappa \neq 0.$$

By suitably decomposing M^n and $\hat{M}$ as CW-complexes, we can endow the cell-chain complex of $\hat{M}$ with the structure of a $\mathbb{Z}[T, T^{-1}]$-module (see Chapter 3, §5). Denote by K^+ the completion of the ring $\mathbb{Z}[T, T^{-1}]$ in the direction of positive powers of T only, i.e. the ring of formal Laurent series of the form

$$q = \sum_{j \geq -N(q)} a_j T^j, \quad a_j \in \mathbb{Z}.$$

Consider the homology groups of the chain complex of $\hat{M}$ with coefficients in the ring K^+ via the representation (inclusion) $\rho : \mathbb{Z}[T, T^{-1}] \longrightarrow K^+$; these homology groups $H_j(\hat{M}; K^+)$ may then be regarded as K^+-modules. Since K^+ is a principal ideal domain, the groups $H_j(\hat{M}; K^+)$ (which we may write appropriately as $H_j(\hat{M};[\omega])$ to indicate their dependence on $[\omega]$) have well-defined torsion-free ranks (the analogues of the Betti numbers) and torsion ranks over K^+, which we shall denote respectively by

$$b_j(M^n,[\omega]) = rk\ H_j(\hat{M};K^+),$$

and $q_j(M^n,[\omega])$ (for the rank of the torsion submodule). Thus $b_j + q_j$ is the least number of generators of the K^+-module $H_j(\hat{M};K^+)$. Provided the critical points of the multi-valued function S are isolated and all non-degenerate, the following lower bounds for the number $m_i(S)$ (or $m_i(w)$) of critical points of index i can be established (Novikov's inequalities, from the early 1980s):

$$m_i(\omega) \geq b_i(M^n,[w]) + q_i(M^n,[w]) + q_{i-1}(M^n,[w]). \tag{2.58}$$

More recently (in the mid 1980s) Farber has shown by constructing a suitable map $M^n \longrightarrow S^1$, where $\pi_1(M^n) \cong \mathbb{Z}$, $n \geq 6$, that these inequalities are strict in general.

If the irrationality degree $k-1 \geq 1$, so that the covering space $\hat{M}$ has monodromy group $\mathbb{Z}\times\cdots\times\mathbb{Z}$ with more than one factor, then the situation is more complicated. The corresponding ring K_κ^+, depending on the set of periods (2.51): $\kappa = (\kappa_1 : \kappa_2 : \ldots : \kappa_k)$, is defined as the enlargement of the ring

$$\mathbb{Z}[T_1,\ldots T_k, T_1^{-1}\ldots T_k^{-1}]$$

of Laurent polynomials obtained by including all formal Laurent series q in which the coefficient $a_{m_1m_2\ldots m_k}$ of $T_1^{m_1}T_2^{m_2}\cdots T_k^{m_k}$ is zero unless $\sum \kappa_j m_j > -N(q)$ for some positive integer $N(q)$, and further, are such that for each pair of numbers $A < B$ there exist only finitely many non-zero coefficients $a_{m_1\ldots m_k}$ satisfying $A < \sum m_j\kappa_j < B$. The homology groups $H_j(\hat{M};K_\kappa^+)$ should again yield somehow a lower bound for the number of critical points of each index. A variant of this type of completion is the ring $(K_\kappa^+)^{Stable}$ where in addition it is required that the condition $\sum \kappa_j m_j > -N(q)$ hold not just for a particular point $\kappa = (\kappa_1 : \kappa_2 : \ldots : \kappa_k)$ but for some small neighbourhood of it. However since the ring K_κ^+ is no longer a principal ideal domain, the exact picture remains unclear.[3] We conclude by emphasizing the crucial fact underlying the inequalities (2.58) and their possible generalizations to the cases $k \geq 2$, namely that the 1-form w gives rise not just to a chain complex but a complex of K_κ^+-modules with homotopy-invariant homology groups.

We now turn to the analogous problem for functionals of types (2.55) or (2.56) on the function spaces $\Omega(N^n,x_0,x_1)$ and $\Omega^+(N^n)$. As we have seen these functionals determine closed 1-forms $w = \delta S$. However it turns out that for the space $\Omega(N^n,x_0,x_1)$ the homology groups analogous to those used to

[3] Sirokav has proved that the ring $(K_\kappa^+)^{Stable}$ for a generic point κ is an integral domain, so that the Morse inequalities hold in this more general case as well. Recently Pazhitnov and Ranicki have proved further results along these lines and have developed an analogue of Morse theory in this case ("Morse-Novikov theory").

obtain the inequalities (2.58) are all degenerate, so that for the "problem with fixed end-points," i.e. for $\Omega(N^n, x_0, x_1)$, there would appear to be no analogue of the Morse inequalities. On the other hand, for the "periodic problem," i.e. for functionals on $\Omega^+(N^n)$, the situation is different. On a complete Riemannian manifold N^n with Riemannian metric g_{ij}, a functional like, for instance, (2.39) (but allowed to be multi-valued) always has the property that the one-point loops $\psi(N^n) \subset \Omega^+(N^n)$ constitute a submanifold of local minimum points of the functional. In the multi-valued case the complete inverse image

$$p^{-1}(\psi(N^n)) = \bigcup N_j^n, \quad j = (j_1, \ldots, j_k), \quad j_s \in \mathbb{Z},$$

in the appropriate covering space $\hat{M}$ with projection $p : \hat{M} \to M$, always consists of local minimum points of the single-valued function $\hat{l}^E$ on $\hat{M}$ defined by the pullback $p^*w = \delta l^E$. The inclusions

$$N^n \to N_j^n, \ \hat{l}^E(N_0^n) = 0, \ \hat{l}^E N_j^n = \sum_{m=1}^{k} \kappa_m j_m, \tag{2.59}$$

are all homotopic to one another as maps to $\hat{M}$. Assuming $\kappa_1 > 0$, consider a map (homotopy)

$$\Phi : N^n \times I \longrightarrow F, \ I = [a, b],$$

where

$$\Phi(N^n \times \{a\}) = N^n_{(0,\ldots,0)} \subset \hat{M}, \ \ \Phi(N^n \times \{b\}) = N^n_{(1,0,\ldots,0)} \subset \hat{M}.$$

One has

$$0 < \min_{\Phi} \left[\max_{x \in N^n} \hat{l}^E(\Phi(N^n \times I)) \right] < \hat{l}^E(N^n_{(0,\ldots,0)}) = 0. \tag{2.60}$$

Essentially by deforming the map Φ on subsets of the form $z \times I$ where z ranges over the cycles in $H_*(N^n)$, in the direction opposite to the gradient of the functional $\hat{l}^E$, one obtains the following inequalities for the number of non-degenerate critical points in the region $l^E > 0$ (assuming general position):

$$m_{i+1}(l^E) \geq \max_p b_i^{(p)}(N^n), \tag{2.61}$$

where p ranges over all primes (Novikov). These inequalities thus represent the analogues in the "periodic problem" of the Morse inequalities (2.8). It is of interest to observe that the inequalities (2.61) do not involve the homology of the function space $\Omega^+(N^n)$ itself. However because of certain analytic difficulties these inequalities have not been established in more general situations.[4]

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[4]Novikov and Grinevich (1994) established these inequalities for nonzero magnetic fields on the flat 2-torus and on surfaces with negative curvature. A different technique was used by Ginzburg, beginning in the late 1980s, for studying periodic orbits in a magnetic field with fixed period (but not necessarily fixed energy as in the former approach). In particular, he helped to correct certain mistakes in papers of Novikov and Taimanov of the mid-1980s.

The inequality (2.61) applies also to functionals of type (2.38) when l is a single-valued functional on $\Omega^+(N^n)$, but "Arzelà's principle" is violated, in the sense that there exist loops $\gamma \in \Omega^+(N^n)$ for which $l(\gamma) < 0$ (so that the metric constitutes a non-positive analogue of a Finsler-type metric; such metrics do in fact arise in mathematical physics). In this situation one needs to utilize in addition (analogously to above) segments of the form

$$\Phi : [a,b] \longrightarrow \Omega^+(N^n)$$

such that $\Phi(a)$ is a one-point (i.e. constant) loop, and $\Phi(b) = \gamma$ with $l(\gamma) < 0$. One has the following recent theorem of Taimanov:

If for a single-valued functional of Finsler type (see (2.37)) there exist loops $\gamma \in \Omega^+(N^n)$ for which $l(\gamma) < 0$, and $l = 0$ on all constant loops, then there exists a cross-section

$$\psi_1 : N^n \longrightarrow \Omega^+(N^n), \; p\psi = 1,$$

such that

a) $l(\psi_1(N^n)) < 0$, *and*

b) *the cross-section ψ_1 is homotopic to the cross-section ψ of constant loops via a homotopy $\Phi : N^n \times I \to \Omega^+(N^n)$.*

This result thus establishes what might be called the "principle of transfer of cycles to the negative region of values of the functional." The inequality (2.61) can be inferred from this result for such single-valued functionals l, i.e. taking negative as well as positive values. New results of the Morse-Novikov theory on extreme points of locally-regular multi-valued functionals on loop spaces (or on spaces of nonselfintersecting curves) may be found in the survey by I. A. Taimanov in: *Russian Surveys*, 1992, v. 2. [5]

An interesting example (due to Polyakov and Vigman) of a multi-valued functional arises in connection with the "Whitehead formula" for the Hopf invariants of the elements of $\pi_3(S^2)$; we recall the construction of that formula: Let $f : S^3 \longrightarrow S^2$ be a smooth map, and ω a 2-form on S^2 satisfying

$$\int_{S^2} \omega = 1. \tag{2.62}$$

Let v be a 1-form on S^3 satisfying $dv = f^*(\omega) = \overline{\omega}$ say. Then the *Whitehead formula* for the Hopf invariant of f is given by

[5] By the end of the 1980s specialists in symplectic geometry (Floer, Hoffer, A. Salomon, D. McDuff and others) had created a Morse theory for multi-valued functionals on free loop spaces. These functionals do not have, in general, the property of local regularity, i.e. the Morse indices of critical points may be infinite. (These ideas first appeared in Rabinovich's work on periodic orbits of Hamiltonian systems of the early 1980s.) There are applications of Floer's theory to the moduli spaces of connections on 3-dimensional spheres, where the multi-valued functional coincides with the well-known Chern-Simon functional. The latest developments in this theory also involve the rings K^+ of Novikov, defined above.

$$H\{f,\omega\} = \int_{S^3} v \wedge \bar{\omega}. \tag{2.63}$$

and has the properties of:

a) homotopy invariance: $\delta H/\delta f = 0$; and
b) rigidity: $\delta H/\delta w = 0$ (provided (2.62) holds).

We now define a multi-valued functional (determined, as earlier, by a closed 1-form) on the space F of null-homotopic smooth maps $f : S^2 \to S^2$, by

$$S\{f\} = \int_{D^3_+} v \wedge \omega, \tag{2.64}$$

where the map f has been extended from $S^2 = \partial D^3_+$ to the whole disc D^3_+. The corresponding 1-form on F is then furnished by δS.

Beginning with the work of Chen and Sullivan in the late 1970s, various homotopically invariant integrals generalizing (2.63) have been constructed. The most transparent and general version of such integrals (defined by Novikov in the early 1980s), also involving the concept of "rigidity," is important for investigating the associated multi-valued functionals and their quantization. We shall now describe this integral.

Let $A = \sum_{i\geq 0} A^i$ be a skew-commutative differential algebra over a field k of characteristic 0, with $A^0 = k$, and coboundary operator $d : A^i \to A^{i+1}$, satisfying $H^1(A) = 0$. For each positive integer q we construct a free differential extension $C_q(A)$ of A, minimal with respect to the property

$$H^j(C_q(A)) = 0, \ j \leq q, \tag{2.65}$$

as follows: Set $C_q^0 = A$. Let $\{y_{j\alpha}\}$ be a minimal set of multiplicative generators for the cohomology ring in dimensions $j \leq q$. We now introduce corresponding free generators $v_{j-1,\alpha}$, and set

$$C_q^1 = A[\dots, v_{j-1,\alpha}, \dots], \quad dv_{j-1,\alpha} = y_{j\alpha}.$$

Note that the inclusion $A = C_q^0 \longrightarrow C_q^1$ induces the zero homomorphism of the cohomology groups in dimensions $\leq q$. By iterating this construction we obtain a sequence of inclusions

$$A = C_q^0 \subset C_q^1 \subset C_q^2 \subset \cdots \subset C_q(A).$$

We now define the $(q + 1)$st *homotopy group* (relative to k) of the algebra A to be the vector space $H^{q+1}(C^q(A))$:

$$\pi_{q+1}(A) \otimes k = H^{q+1}(C_q(A)). \tag{2.66}$$

If we take $A = \Lambda^*(M^n)$, the algebra of C^∞-forms on a simply-connected manifold M^n, then it can be shown that $\pi_{q+1}(A) \otimes \mathbb{R} \cong \pi_{q+1}(M) \otimes \mathbb{R}$. Hence the infinite portions of the homotopy groups of any simply-connected manifold

M^n may be obtained in a quite elementary manner from the algebra $\Lambda^*(M^n)$ of forms on M^n (although the full verification of the above isomorphism requires the more elaborate Cartan-Serre apparatus of algebraic topology). This result may be regarded as a homotopy-theoretic analogue of de Rham's theorem (see above).

Using (2.66) we can now construct the desired homotopically invariant integrals. Given any smooth map $F : S^{q+1} \times \mathbb{R} \longrightarrow M^n$, we obtain a natural homomorphism

$$\hat{F} : C_q(A) \longrightarrow \Lambda^*(S^{q+1} \times \mathbb{R}), \quad A = \Lambda^*(M^n),$$

commuting with d, by simply taking $\hat{F} = F^*$ on $C_q^0 = A = \Lambda^*(M^n)$ and extending using the fact that all closed forms of degree $\leq q$ on $S^{q+1} \times \mathbb{R}$ are exact. If z is any element of $H^{q+1}(C_q(A))$ represented by a cocycle $\tilde{z}$ say, then $\hat{F}(\tilde{z})$ is a $(q+1)$-form on $S^{q+1} \times \mathbb{R}$ satisfying $d\hat{F}(\tilde{z}) = 0$. We now define our integral by

$$H(t) = H(f_t, \tilde{z}) = \int_{S^{q+1} \times \{t\}} \hat{F}(\tilde{z}), \tag{2.67}$$

where

$$f_t = \hat{F}|_{S^{q+1} \times \{t\}}.$$

Since the form $\hat{F}(\tilde{z})$ is closed we have $dH/dt = 0$, whence the homotopy invariance of the integral (2.67) follows fairly quickly. The definition and investigation of "rigidity" in this context is somewhat more involved.

§3. Smooth manifolds and homotopy theory. Framed manifolds. Bordisms. Thom spaces. The Hirzebruch formulae. Estimates of the orders of homotopy groups of spheres. Milnor's example. The integral properties of cobordisms

In this section we shall first consider some ideas, based on the geometry of manifolds, giving access by elementary means to certain information concerning the homotopy classes of maps of manifolds (in cases other than those examined in Chapter 3 where the homotopy groups were known to be trivial for elementary reasons). In their further development these geometrical ideas become combined with the algebraic techniques described at the conclusion of Chapter 3.

We remind the reader that for smooth manifolds any continuous maps and homotopies can be approximated arbitrary closely by smooth maps coinciding with the original map wherever it happened to be smooth, so that we always assume all maps and homotopies of manifolds to be of smoothness class C^∞ if need be.

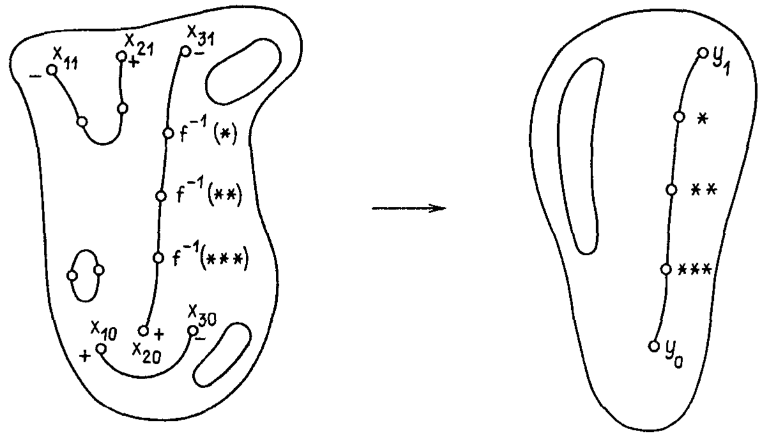


Fig. 4.15

The simplest of homotopy invariants is the *degree* of a map between closed orientable manifolds of the same dimension:

$$f : M^n \longrightarrow N^n,$$

defined as follows: Consider a generic point y (i.e. "regular") in N^n; the Jacobian J_f of f is then non-zero at each point of the complete inverse image $f^{-1}(y) = \{x_1, \ldots, x_k\}$. We define

$$\deg f = \sum_{j=1}^{k} sgn J_f(x_j). \tag{3.1}$$

(Figure 4.15 can be used to illustrate the verification that the degree of f is independent of the choice of the regular point y of N^n, assuming that M^n and N^n are connected. Figures 4.16, 4.17 exemplify the concept.)

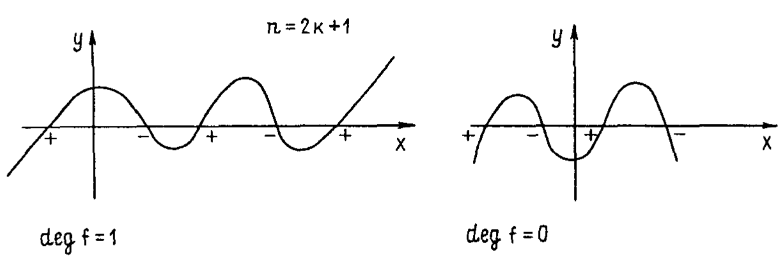


Fig. 4.16

The definition (3.1) of the degree of a map applies also to the following situations:

a) Maps $f : (M^n, W^{n-1}) \longrightarrow (N^n, V^{n-1})$, $(\partial M^n = W^{n-1},\ \partial N^n = V^{n-1})$, where the boundary W^{n-1} of M^n is mapped to the boundary V^{n-1} of N^n. Here the degree of f restricted to the boundary W^{n-1} is equal to its degree on the interior of M^n. (Hence if f defines a diffeomorphism between the boundaries, then as a map between the interiors we shall necessarily have $\deg f = \pm 1$.)

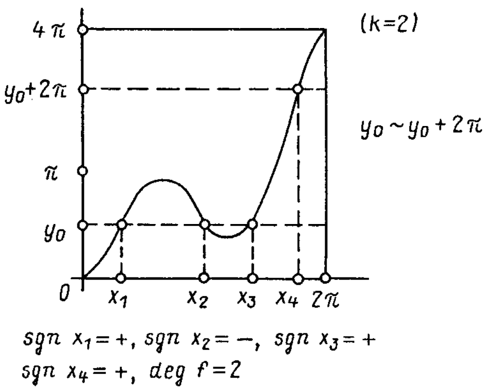


Fig. 4.17

b) Proper maps $f : M^n \longrightarrow N^n$ of open manifolds, i.e. maps with the property that the complete inverse image of any compact set is again compact. Here the degree is invariant under homotopies $F : M^n \times I \longrightarrow N^n$ that are also proper maps. Note that if ω is an n-form on N^n, then one has the formula

$$\int_{M^n} f^*\omega = (\deg f) \int_{N^n} \omega. \tag{3.2}$$

c) The isolated singular points x_0 of a vector field $\eta(x)$ on a manifold M^n: $\eta(x_0) = 0$. On a sphere S_ε^{n-1} about x_0 of sufficiently small radius ε (defined by $|x - x_0| = \varepsilon$) the vector field $\eta(x)$ will be non–vanishing and therefore the following Gauss map is well-defined:

$$\frac{\eta(x)}{|\eta(x)|} : S_\varepsilon^{n-1} \longrightarrow S^{n-1}.$$

The degree of this map is called the *index of the singular point* x_0. If the singular point is non–degenerate, i.e.

$$\det\left(\frac{\partial \eta^i}{\partial x^j}\right)\Big|_{x_0} \neq 0,$$

then the index of the singular point is equal to $sgn \det(\frac{\partial \eta^i}{\partial x^j})|_{x_0}$. (Figure 4.18 shows the possible types of isolated non–degenerate singular points of a vector field on a 2-dimensional manifold.)

For a vector field on a closed manifold M^n, the sum of the indices of the singular points is equal to the Euler characteristic $\chi(M^n)$ (Poincaré, Hopf).

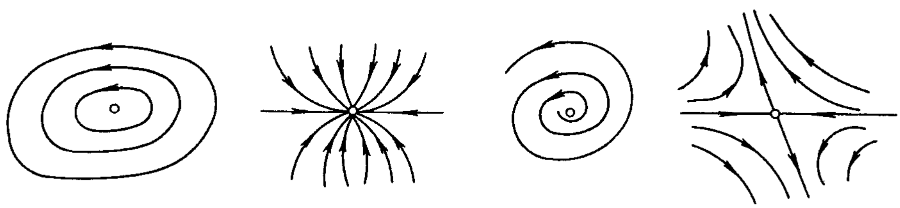


Fig. 4.18

For non–orientable manifolds the degree of a map is defined only modulo 2. As noted above, for connected manifolds the degree is independent of the choice of the point $y \in N^n$. The homotopy invariance of the degree of a map $f : M^n \longrightarrow N^n$ can be seen as follows. Let $F : M^n \times I \longrightarrow N^n$ be any homotopy of f $(F(x,0) = f(x))$, and consider the complete inverse image of a point $\widetilde{y}$ arbitrarily close to y, with the property that at every point of $F^{-1}(\widetilde{y})$ the map F has rank n, as illustrated in Figure 4.19 in the case $M^n = N^n = S^1$. The coincidence of the degrees of the restrictions of F to the base and lid of the cylinder is clear in that figure, and the general case is similar. Hopf's theorem

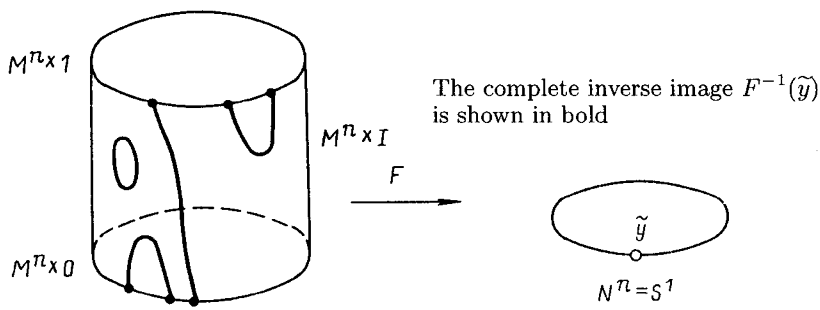


Fig. 4.19

(of the late 1920s) states that for a closed manifold M^n every map $M^n \longrightarrow S^n$ from M^n to the n-sphere is determined to within a homotopy by its degree, so that the degree is a complete homotopy invariant of such maps. Hopf's actual argument was along the lines of the above sketch of the proof of the homotopy invariance of the degree of a map, except for his use of PL-manifolds, i.e. of suitable triangulations of the manifolds. In the more general situation of a simplicial (i.e. PL–) map $f : M^{n+k} \longrightarrow N^n$ of PL–manifolds, the complete inverse image of the interior of a simplex σ^n of N^n of largest dimension will have the form

$$f^{-1}(\text{Int }\sigma^n) = \text{Int }\hat{\sigma}^n \times W^k, \tag{3.3}$$

for some simplex $\hat{\sigma}^n$ of M^{n+k} and submanifold W^k. Following Hopf's idea, it is then natural to seek invariants of the map f using such complete preimages.

Consider a map between spheres:

$$f : S^{n+k} \longrightarrow S^n.$$

Reverting to our use of points in general position, we consider the complete inverse images of two distinct such points $y_1, y_2 \in S^n$; these will be submanifolds

$$W_i^k = f^{-1}(y_i), \quad y_i \in S^n, \ i = 1, 2,$$

$$W_1^k, W_2^k \subset S^{n+k}.$$

In the case where n is even and $k = n-1$, these two submanifolds may be linked in $S^{n+k} = S^{2n-1}$, with "linking coefficient" $\{W_1^{n-1}, W_2^{n-1}\}$ (see Chapter 1 for the definition of the linking coefficient of two loops in a manifold). For odd n the linking coefficient $\{W_1^{n-1}, W_2^{n-1}\}$ is zero. For even n (the case we are considering) the linking coefficient is independent of the pair of regular points $y_1, y_2 \in S^n$, and is invariant under homotopies of f; it is called the *Hopf invariant* of f:

$$h(f) = \{W_1^{n-1}, W_2^{n-1}\}, \quad n = 2q, \qquad f : S^{4q-1} \longrightarrow S^{2q}. \tag{3.4}$$

The Hopf invariant determines the *Hopf homomorphism:*

$$\pi_{4q-1}(S^{2q}) \longrightarrow \mathbb{Z}, \quad f \mapsto h(f),$$

from which it follows that the groups $\pi_{4q-1}(S^{2q})$ are infinite. It was shown later (in the early 1950s) by Serre that all other $\pi_j(S^n)$ are finite; as we saw earlier (in Chapter 3, §7) the methods used here go considerably beyond the purely geometrical.

The switch from PL–maps to smooth ones, leads to the possibility of obtaining a standard method for investigating homotopy groups of spheres using complete preimages of points (Pontryagin, late 1930s). In this approach, one exploits the fact that the inverse image $W^k = f^{-1}(y)$ of a regular point $y \in S^n$ under a map $f : S^{n+k} \longrightarrow S^n$, is a *framed manifold*, i.e. comes with a non–degenerate field $\nu_n(x)$ of vector n-frames determined by f as follows: Let $\phi_1, \dots, \phi_n$ be local co-ordinates in some sufficiently small neighbourhood U of y with y as origin and with linearly independent gradients (with respect to the standard local co-ordinates on S^n); then the complete preimage $W^k = f^{-1}(y)$ is given by the n equations $\widetilde{\phi}_1 = 0, \dots, \widetilde{\phi}_n = 0$, where the $\widetilde{\phi}_i$ are lifts of the

ϕ_i to $f^{-1}(U)$. The gradients $grad\,\widetilde{\phi}_i$ will be linearly independent on W^k, and so determine a field of n–frames on W^k.

Similarly, given a homotopy $F : S^{n+k} \times I \longrightarrow S^n$, the complete inverse image $F^{-1}(y)$ of a regular point $y \in S^n$ is a manifold–with–boundary $V^{k+1} \subset S^{n+k} \times I$ (whose boundary falls into two disjoint parts, one in the base and one in the lid of the cylinder $S^{n+k} \times I$) equipped, much in the same manner as before, with a normal (relative to some Riemannian metric) field $\nu_n(x)$ of vector n-frames in $S^{n+k} \times I$, which on the boundary of V^{k+1} lies in $S^{n+k} \times \{a\}$ or $S^{n+k} \times \{b\}$. (Note that it may be assumed that V^{k+1} approaches the lid and base of $S^{n+k} \times I$ transversely.) The pair $(V^{k+1}, \nu_n) \subset S^{n+k} \times I$ is called a *framed manifold-with-boundary* (or *framed bordism*). Two framed manifolds $(W_1^k, \nu_n^{(1)})$, $(W_2^k, \nu_n^{(2)})$ are said to be *equivalent* if there is a framed bordism (V^{k+1}, ν_n), $V^{k+1} \subset S^{n+k} \times I$, whose respective boundary components are the given framed manifolds. A realization of such an equivalence may be called a *framed bordism* (or *framed cobordism*). It is quite straightforward to show that the equivalence classes of closed framed manifolds are in natural one-to-one correspondence with the elements of the group $\pi_{n+k}(S^n)$.

Before proceeding, we note that the groups $\pi_{n+k}(S^n)$ are called *stable* if $k < n-1$, since for these k they are independent of n; this is immediate from the suspension isomorphism

$$\Sigma : \pi_{n+k}(S^n) \longrightarrow \pi_{n+k+1}(S^{n+1}), \quad k < n-1.$$

More generally, for any CW–complex K the suspension construction yields an isomorphism in homology always (see Chapter 3, §3), and an isomorphism $\Sigma : \pi_{n+k}(K) \xrightarrow{\cong} \pi_{n+k+1}(\Sigma K)$ for $k < n-1$, provided the complex K is $(n-1)$–connected, i.e. if $\pi_j(K) = 0$ for $j \leq n-1$. (This was shown in the late 1930s by Freudenthal, who was also the first to compute the groups $\pi_{n+1}(S^n)$.)

The following facts about framed manifolds are not difficult to establish:

1. Each framed manifold (W^k, ν_n) of dimension $k > 0$ is equivalent to a connected framed manifold. (Figure 4.20 illustrates the construction of a framed bordism (V^{k+1}, ν_n) with $\partial V^{k+1} = W_1^k \cup W_*^k \cup W_2^k$, realizing an equivalence between the disjoint union $(W_1^k \cup W_2^k, \nu_n|_{W_1^k \cup W_2^k})$ and the connected framed manifold $(W_*^k, \nu_n|_{W_*^k})$.)

2. In the case $k = 1$, the only stable invariant of a framed manifold is the homotopy class of its field of frames, which is given by a map $\nu_n : S^1 \longrightarrow SO_n$. Since $\pi_1(SO_n) \cong \mathbb{Z}/2$ for $n \geq 3$, it follows that $\pi_{n+1}(S^n) \cong \mathbb{Z}/2$ for $n \geq 3$.

3. In the case $k = 2$ there is again a single stable invariant of each equivalence class of framed manifolds (W^2, ν_n), but its construction is somewhat more involved. Suppose $n \geq 4$. By statement 1 above we may assume without loss of generality that $W^2 \subset \mathbb{R}^{n+2}$ (or $W^2 \subset S^{n+2}$) is a connected surface (of genus g say) with normal field $\nu_n(x)$. Each class $z \in H_1(W^2; \mathbb{Z}/2)$ may be represented by a simple (i.e. embedded) circle $C \cong S^1$ in $W^2 \subset \mathbb{R}^{n+2}$.

Writing $n(x)$ for the normal vector field to C in W^2, we have a field of normal frames $\nu_{n+1}(x) = (n(x), \nu_n(x))$ to C in $\mathbb{R}^{n+2}$. By statement 2 above there is a unique invariant $\Phi(z)$ say ($\Phi(z) \in \mathbb{Z}/2$) of the framed manifold $(C, \nu_{n+1}(x))$ called the *Arf function.* One has the following "Arf identity":

$$\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2) + z_1 \circ z_2 \quad (\text{modulo } 2), \tag{3.5}$$

where $z_1 \circ z_2$ is the intersection index of the cycles z_1 and z_2. In terms of any canonical basis of cycles $z_1, \dots, z_g, w_1, \dots, w_g$, i.e. satisfying

$$z_i \circ z_j = w_i \circ w_j = 0, \quad z_i \circ w_j = \delta_{ij},$$

the full *Arf invariant* is defined by

$$\Phi = \sum_{i=1}^{g} \Phi(z_i)\Phi(w_i), \quad \Phi \in \mathbb{Z}/2. \tag{3.6}$$

(It is independent of the (canonical) basis.) It turns out that for $n \geq 4$ the quantity Φ is the only stable invariant of framed manifolds (W^2, ν_n), whence it follows that for $n \geq 4, \pi_{n+2}(S^n) \cong \mathbb{Z}/2$ (Pontryagin, in the late 1940s).

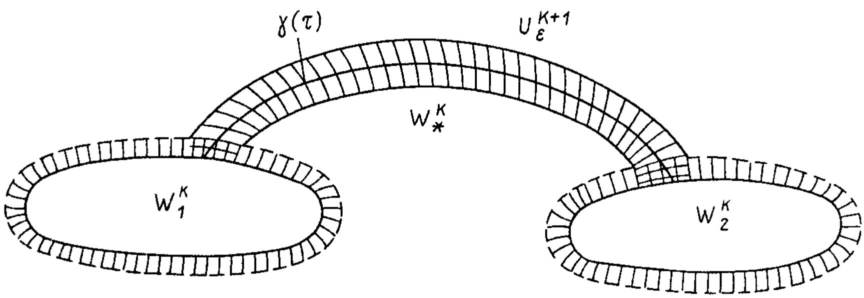


Fig. 4.20

4. There is a complicated theory (devised by Rohlin in the early 1950s) leading to the determination of the groups $\pi_{n+3}(S^n)$. As the first (non–trivial) step, it is shown that every non–trivial framed manifold (W^3, v_n) is bordant to a framed sphere $S^3 \subset \mathbb{R}^{n+3}$, situated in standard fashion in $\mathbb{R}^{n+3}$ (since in the stable range every disposition of S^3 in $\mathbb{R}^{n+3}$ is equivalent to the standard one). It is shown next that the latitude in defining a frame on $S^3 \in \mathbb{R}^{n+3}$ determines a homomorphism (in fact an epimorphism)

$$J : \pi_3(SO_n) \longrightarrow \pi_{n+3}(S^n). \tag{3.7}$$

Since $\pi_3(SO_n) \cong \mathbb{Z}$ for $n > 4$, it now follows that the stable group $\pi_{n+3}(S^n)$ is cyclic. Further analysis reveals that

$$\pi_{n+3}(S_n) \cong \mathbb{Z}/24 \text{ for } n > 4, \quad \pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12, \quad \pi_6(S^3) \cong \mathbb{Z}/12.$$

These results are closely linked to Rohlin's theorem to the effect that for a closed manifold M^4 with the property that $M^4 \setminus \{x_0\}$ is parallelizable, the first Pontryagin class $p_1(M^4)$ is divisible by 48. (Recall that characteristic classes are elements of the integral cohomology groups.)

As a generalization of (3.7) one has that the equivalence classes of framed k–spheres (S^k, ν_n) determine in $\pi_{n+k}(S^n)$ the image of the "Whitehead homomorphism" (see Chapter 3, §8)

$$J : \pi_k(SO_n) \longrightarrow \pi_{n+k}(S^n).$$

It is known that this image is finite cyclic, and that its order (an important quantity) is in the case $k = 4s - 1$ expressible in terms of the Bernoulli numbers. (Milnor and Kervaire produced a lower bound for the order in the late 1950s, and Adams an upper bound in the mid-1960s.) □

These results represent the limit of what can be discovered about the homotopy groups of spheres using geometrical methods; further progress on this problem involves heavy use of the methods of homotopy theory (see Chapter 3, §7).

The theory of bordisms and cobordisms may be regarded as developing naturally out of the above–described geometrical ideas, providing a bridge to algebraic techniques for settling questions about smooth manifolds. The classical *bordism groups* (or *cobordism groups*) are defined as follows: Two smooth closed manifolds W_1^k, W_2^k of the same dimension k, are said to be *equivalent* (or *cobordant*) if their sum (i.e. disjoint union) is diffeomorphic to the boundary of a compact smooth manifold N^{k+1}:

$$W_1^k \cup W_2^k \cong \partial N^{k+1}.$$

For each $k \geq 0$ the equivalence (or cobordism) classes of closed k–manifolds form a group Ω_k^O under the operation defined in the obvious way in terms of the sum operation on manifolds. The direct sum

$$\Omega_*^O = \sum_{k \geq 0} \Omega_k^O \tag{3.8}$$

then forms a graded ring,[6] the *classical bordism ring*, with multiplicative operation determined by the direct product of manifolds. It is in fact an algebra over $\mathbb{Z}/2$.

Examples. 1. $\Omega_0^O \cong \mathbb{Z}/2$ (the scalars).

2. $\Omega_1^O = 0$.

[6] There is an another standard notation, namely $\mathfrak{N}_*$, for the bordism ring Ω_*^O.

3. $\Omega_2^O \cong \mathbb{Z}/2$. (This follows from the classification theorem for closed surfaces. Note that $\mathbb{R}P^2$ represents the non–zero element, i.e. $\mathbb{R}P^2$ is not the boundary of any compact 3–manifold. Since $\chi(\mathbb{R}P^2) = -1$, this follows from the following general fact:
If a closed manifold is a boundary: $M^n = \partial W^{n+1}$, then its Euler–Poincaré characteristic is even.
This is in turn a consequence of the equality

$$\chi(W^{n+1} \cup_{M^n} W^{n+1}) = 2\chi(W^{n+1}) - \chi(M^n),$$

where $W^{n+1} \cup_{M^n} W^{n+1}$ is the "double" of W^{n+1} (obtained by taking two copies of W^{n+1} and identifying their boundaries), and the fact that any odd–dimensional closed manifold has zero Euler-Poincaré characteristic).

3. $\Omega_3^O = 0$ (Rohlin, early 1950s). □

Restricting ourselves to oriented manifolds, we obtain analogously the *orientable bordism groups* Ω_k^{SO} and the *orientable bordism ring*

$$\Omega_*^{SO} = \sum_{k \geq 0} \Omega_k^{SO}.$$

Examples. 1. $\Omega_0^{SO} \cong \mathbb{Z}$ (the scalars).

2. $\Omega_1^{SO} = 0 = \Omega_2^{SO}$ (obviously).

3. $\Omega_3^{SO} = 0$ (Rohlin, early 1950s).

4. $\Omega_4^{SO} \cong \mathbb{Z}$ (Rohlin and Thom, both in the early 1950s). Note that the bordism class $[\mathbb{C}P^2]$ generates Ω_4^{SO}. □

If a closed manifold M^n is a boundary, i.e. represents zero in the bordism group Ω_n^O, then all of its "Stiefel–Whitney numbers" (see below) are zero. In the late 1940s Pontryagin established the following basic facts: If a closed 4k-dimensional manifold M^{4k} is the boundary of an orientable manifold then in addition its "Pontryagin numbers" are all zero. The *Stiefel–Whitney characteristic numbers* of M^n are the values taken on the fundamental class $[M^n]$ by homogeneous polynomials over $\mathbb{Z}/2$ of degree n in the Stiefel–Whitney characteristic classes. (To obtain the *Pontryagin characteristic numbers* of M^{4k} one evaluates homogeneous polynomials over $\mathbb{Z}$ of degree k in the Pontryagin characteristic classes.)

The main method for computing the bordism groups is based on their connexion with homotopy theory; we shall now describe this connexion.

Definition 3.1 The *Thom space* T_η of a vector bundle η (or any fibre bundle with fibre $\mathbb{R}^n$) is the quotient space $E/\Delta E$, where ΔE is the subspace consisting of the vectors in the fibres of length ≥ 1. (Provided the base B of η is a compact space one may take the one–point compactification of the total space E, as the resulting space is homotopically equivalent to the Thom space T_η.)

For each $j \geq 0$ there is a natural isomorphism, the *Thom isomorphism* (valid in modulo 2–cohomology if the bundle is not orientable):

$$\begin{gathered} \phi : H^j(B) \longrightarrow H^{j+n}(T_\eta), \quad j \geq 0, \\ \phi(z) = z \cdot u_n, \quad u_n = \phi(1), \end{gathered} \tag{3.9}$$

where $\phi(1)$ is the cohomology class of T_η taking the value 1 on the fibres, the *Thom class*. One has the following formula of Thom (from the early 1950s) relating the Stiefel–Whitney characteristic classes $w_j \in H^j(B; \mathbb{Z}/2)$ of the vector bundle η and the Steenrod squares Sq^j:

$$w_j(\eta) = \phi^{-1} Sq^j \phi(1). \tag{3.10}$$

Applied to the tangent bundle $\tau(M^n)$ of a smooth closed manifold M^n, this formula leads to a proof of the homotopy invariance of the Stiefel–Whitney classes $w_j(M^n)$ of M^n. By exploiting the fact that the tangent bundle $\tau(M^n)$ is isomorphic to the normal bundle to the diagonal $\Delta \subset M^n \times M^n$, in conjunction with Thom's formula (3.10), one may obtain explicit expressions for the classes $w_j(M^n)$ in terms of the Steenrod squares Sq^i acting on the cohomology ring $H^*(M^n; \mathbb{Z}/2)$ (Wu's formulae, of the early 1950s). To obtain Wu's formulae one solves the equation (for each $i = 1, \cdots, [n/2]$)

$$Sq^i(z) = z \cdot \kappa_i, \tag{3.11}$$

over the group $H^{n-i}(M^n; \mathbb{Z}/2)$ for the unknowns κ_i (for all $z \in H^{n-i}(M^n; \mathbb{Z}/2)$). We note that the equation (3.11) has a solution since any homomorphism $H^{n-i}(M^n; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$ (and so in particular the homomorphism

$$Sq^i : H^{n-i}(M^n; \mathbb{Z}/2) \longrightarrow H^n(M^n; \mathbb{Z}/2) \cong \mathbb{Z}/2)$$

is obtained via Poincaré duality by multiplying the elements of H^{n-i} by a single i–dimensional class. Wu's formulae have the following form:

$$\begin{gathered} Sq(w) = \kappa, \\ w = 1 + w_1 + \ldots + w_n, \\ \kappa = 1 + \kappa_1 + \ldots + \kappa_{[n/2]}, \\ Sq = 1 + Sq_1 + Sq_2 + \ldots . \end{gathered} \tag{3.12}$$

These formulae yield, for instance, the result that all orientable 3–manifolds are parallelizable.

If a manifold M^n is almost parallelizable, i.e. if M^n with one point removed is parallelizable, then $w_i = 0$ for $i < n$, and from this together with (3.12) one may deduce that if $n = 4k$ then in $H^{2k}(M^n; \mathbb{Z}/2)$ every square is zero: $z^2 \equiv 0$ (modulo 2). This implies that the scalar square $(z^2, [M^{4k}]) = \langle z, z\rangle$ is even for every

$$z \in H^{2k}(M^{4k}; \mathbb{Z})/(\text{Torsion}).$$

Since the quadratic form $\langle z, z\rangle$ is even and unimodular, it then follows that the signature $\tau(M^{4k})$ is divisible by 8. The *signature* $\tau(M^{4k})$ is defined as the difference in the numbers of positive and negative squares in the (diagonalized) quadratic form $\langle x, y\rangle$ given by the intersection index of elements $x, y \in H_{2k}(M^{4k}; \mathbb{Z})$ (or equivalently by $\langle x, y\rangle = (xy, [M^{4k}]), x = Dx, y = Dy$), although this quadratic form is non-degenerate only for closed manifolds. It can be shown that for closed manifolds the signature is invariant under bordisms: if $M^{4k} = \partial W^{4k+1}$ for some orientable manifold–with–boundary W^{4k+1}, then $\tau(M^{4k}) = 0$. (To prove this one first observes that if two $2k$–cycles in M^{4k} are null–homologous in W^{4k+1}, then their intersection index is zero, so that $\langle\ ,\ \rangle$ is zero on Im i^* where i^* is the inclusion homomorphism $H^{2k}(W^{4k+1}; \mathbb{Q}) \longrightarrow H^{2k}(M^{4k}; \mathbb{Q})$. The desired conclusion then follows by showing that the dimension of Im $i^* \subset H^{2k}(M^{4k}; \mathbb{Q})$ is exactly half the dimension of $H^{2k}(M^{4k}; \mathbb{Q})$.) From this fact important conclusions follow; for instance that the group Ω_4^{SO} is isomorphic to $\mathbb{Z}$ with generator $[\mathbb{C}P^2]$. In §1 above we saw that the first Pontryagin class p_1 of $\mathbb{C}P^2$ satisfies

$$(1 + p_1) = (1 + u^2)^3 = 1 + 3u^2 \in H^*(\mathbb{C}P^2; \mathbb{Z}),$$

where u is a generator of $H^2(\mathbb{C}P^2; \mathbb{Z}) \cong \mathbb{Z}$. It follows that $p_1(\mathbb{C}P^2) = 3$. Since $\tau(\mathbb{C}P^2) = 1$ (this can be obtained directly from knowledge of the structure of $H^*(\mathbb{C}P^2; \mathbb{Q})$), and since $[\mathbb{C}P^2]$ generates Ω_4^{SO}, it follows that

$$\tau(M^4) = \frac{1}{3}p_1, \tag{3.13}$$

for every orientable closed 4–dimensional manifold M^4 (since $3\tau(M^4) - p_1$ is an identically zero linear form on $\Omega_4^{SO} \cong \mathbb{Z}$). (This result is due to Rohlin and Thom in the early 1950s.) Generalization of the formula (3.13) to higher dimensions requires computation of the groups $\Omega_j^{SO} \otimes \mathbb{Q}$; we shall be considering this problem in the sequel.

The signature $\tau(M^{4k})$ of a manifold–with–boundary M^{4k} has the property of *additivity*: if M_1^{4k}, M_2^{4k} are two manifolds–with–boundary and $V_1, V_2 \cong V$ are diffeomorphic connected components of their respective boundaries, then for the manifold obtained by identifying V_1 and V_2 one has (Rohlin and Novikov in the late 1960s):

$$\tau(M_1^{4k} \cup M_2^{4k}) = \tau(M_1^{4k}) + \tau(M_2^{4k}). \tag{3.14}$$

We remark that this additivity property is possessed also by the Poincaré–Euler characteristic of even–dimensional manifolds M^{2k}, but by no other characteristic.

For any group $G \subset O_m$, the Thom space T_η of a universal vector bundle with fibre $\mathbb{R}^m$ and base BG will be denoted by $M(G)$. More generally any representation $\rho : G \longrightarrow O_m$, not necessarily one–to–one, determines a Thom space; for instance the double cover

$$\rho : Spin_m \longrightarrow SO_m,$$

determines the Thom space $M(Spin_m)$. If $G \subset SO_m$, one speaks of the *orientable Thom space* $M(G)$. In the case $G = \{1\} \subset SO_m$, we obtain the m–sphere: $M(G) \cong S^m$. By (3.9) we have the following Thom isomorphisms:

$$\text{a)} \qquad \phi : H^j(BG;\mathbb{Z}) \longrightarrow H^{m+j}(M(G);\mathbb{Z}), \quad G \subset SO_m,$$

$$\phi(z) = z \cdot u_m, \quad u_m = \phi(1);$$

$$\text{b)} \quad \phi : H^j(BG;\mathbb{Z}/2) \longrightarrow H^{m+j}(M(G);\mathbb{Z}/2), \quad G \subset O_m,$$

$$\phi(z) = z \cdot u_m, \quad u_m = \phi(1).$$

Classical cobordism theory concentrates (as we have seen) on the cases $G = O_m$ or SO_m. Further developments of that theory are based on the following facts (established by Thom):

(i) *An element $z \in H_{n-m}(M^n;\mathbb{Z}/2)$ is representable by a closed submanifold $W^{n-m} \subset M^n$, precisely if there exists a map $f : M^n \longrightarrow M(O_m)$ such that*

$$f^*(u_m) = Dz \in H^m(M^n;\mathbb{Z}/2),$$

where $u_m = \phi(1)$ is the Thom class (see 3.9) of the Thom space $M(O_m)$, and D is the Poincaré-duality isomorphism.

(ii) *An element $z \in H_{n-m}(M^n;\mathbb{Z})$, M^n orientable, is representable by an orientable submanifold $W^{n-m} \subset M^n$ if and only if there is a map $f : M^n \longrightarrow M(SO_m)$ such that*

$$f^*(u_m) = Dz \in H^m(M^n;\mathbb{Z}).$$

(iii) *The bordism groups $\Omega_k^O, \Omega_k^{SO}$ are canonically isomorphic to the corresponding "stable" homotopy groups of $M(O_m)$ and $M(SO_m)$:*

$$\begin{aligned} \Omega_k^O &\cong \pi_{m+k}(M(O_m)), \quad k < m-1, \\ \Omega_k^{SO} &\cong \pi_{m+k}(M(SO_m)), \quad k < m-1, \end{aligned} \tag{3.15}$$

(Thom, early 1950s).

Note that if we replace SO_m by the trivial group $\{1\}$ in (3.15) we obtain the isomorphism established by Pontryagin (in the late 1930s) between

$\pi_{m+k}(M(\{1\})) = \pi_{m+k}(S^m)$ and the group Ω_k^{fr} of bordisms of framed manifolds, $k < m - 1$.

Statements (i), (ii) and (iii) above were established using the "theorem on transversal–regularity" (or t–"regularity"), according to which any map $g : M^n \longrightarrow X^q$ of smooth manifolds can be approximated arbitrary closely by a map f that is "t-regular" on any prescribed submanifold $Y^{q-m} \subset X^q$. (A map f is *t-regular* on Y^{q-m} if for every point y of the complete inverse image $f^{-1}(Y^{q-m}) \subset M^n$, the image of the tangent space to M^n at y (under the induced map of tangent spaces) together with the tangent space to Y^{q-m} at $f(y)$, spans the whole of the tangent space to X^q at $f(y)$.) This condition ensures that the complete preimage $f^{-1}(Y^{q-m})$ is a smooth submanifold of M^n of dimension $n - m$:

$$f^{-1}(Y^{q-m}) = N^{n-m} \subset M^n,$$

moreover such that the restriction map $N^{n-m} \longrightarrow Y^{q-m}$ extends naturally to a map of the respective normal bundles with fibre $\mathbb{R}^m$.

To obtain the representability of cycles by submanifolds (Statements (i) and (ii)) one takes $M(O_m)$ (or $M(SO_m)$) in the role of X^q, and $BO_m \subset M(O_m)$ (or $BSO \subset M(SO_m)$) in the role of Y^{q-m}; the submanifold representing a cycle is then $f^{-1}(BO_m)$ (or $f^{-1}(BSO_m)$). To obtain Statement (iii) one takes $X^q = S^q$.

Remark. The spaces $M(O_n)$ and $M(SO_n)$ together with the natural maps

$$M(O_n) \longrightarrow M(O_{n+1}), \qquad M(SO_n) \longrightarrow M(SO_{n+1})$$

define *spectra* (as objects in the appropriate category), denoted by MO and MSO respectively. The "homotopy groups of the spectra" MO and MSO coincide with the above "stable" homotopy groups:

$$\pi_k MO = \pi_{m+k}(M(O_m)), \quad k < m - 1,$$

$$\pi_k MSO = \pi_{m+k}(M(SO_m)), \quad k < m - 1.$$

The spectra MU, MSU, $MSpin$ and MSp are defined in a similar manner. □

A natural geometric interpretation of the homotopy groups of $M(G)$, $G \subset O_m$, is as follows: Define a *G–manifold* to be the normal bundle ν_m on a submanifold W^k of S^{m+k} (or of $\mathbb{R}^{m+k}$) equipped with a G–structure. Two G-manifolds $W_1^m, W_2^m \subset S^{m+k}$ are said to be *G-bordant* if there exists a G–manifold $V^{m+1} \subset S^{m+k} \times I$, $I = [a, b]$, normal to the boundaries:

$$\partial V = W_1^m \cup W_2^m,$$

$$W_1^m \subset S^{n+k} \times \{a\}, \quad W_2^m \subset S^{n+k} \times \{b\}$$

(analogously to the framed bordism introduced earlier).

It turns out that the equivalence classes of G–manifolds are in natural one–to–one correspondence with the elements of the homotopy group $\pi_{m+1}(M(G))$.

By restricting attention to manifolds $M^k \subset S^{m+k}$ having normal bundles with prescribed structure group, one is led to the following bordism rings corresponding to the classical Lie groups:

Ω_*^O, Ω_*^{SO} (the classical bordism rings, introduced in the early 1950s);

$\Omega_*^{fr}, \Omega_q^{fr} \cong \pi_{q+n}(S^n)$; $G = \{1\}$ (bordisms of framed manifolds, introduced in connexion with the study of the homotopy groups of spheres, in the late 1930s);

$\Omega_q^U, \Omega_q^{SU}$; U_k, $SU_k \subset SO_{2k}$ (the unitary and special unitary bordism groups);

Ω_q^{Sp}; $Sp_k \subset SO_{4k}$ (the symplectic bordism groups);

Ω_q^{Spin}; $\rho : Spin_k \longrightarrow SO_k$ (the spinor bordism groups).

Investigations into the structure of the G–bordism rings Ω_*^G for $G = U$, SU, Sp, $Spin$, and their further generalizations, were initiated around 1960 (by Milnor and Novikov). For the classical groups G, the G–structure of the stable normal bundle of a manifold is determined by the G–structure of the tangent bundle, so that for $G = U$ for instance, a complex (or quasi–complex) manifold M^{2k} of real dimension $2k$, determines a bordism class of Ω_{2k}^U:

$$[M^{2k}] \in \Omega_{2k}^U.$$

A generic polynomial equation of degree $n+1$ (in $n+1$ homogeneous complex co–ordinates) defines in $\mathbb{C}P^n$ an SU–manifold M^{2n-2} representing a bordism class

$$[M^{2n-2}] \in \Omega_{2n-2}^{SU}.$$

In the case $n = 3$ one obtains in this way the *Kummer surface* K^4 (of 4 real dimensions), which is almost parallelizable (i.e. the complement of a point is parallelizable), simply–connected, and has the following invariants:

$$\begin{gathered} \chi(K^4) = c_2(K^4) = 24, \\ \tau(K^4) = 16, \quad p_1(K^4) = 48, \quad c_1(K^4) = 0. \end{gathered} \tag{3.16}$$

As noted in connexion with the definition of a characteristic class of a G–bundle, given in §1 above, each such characteristic class is uniquely determined by an element ψ of $H^*(BG)$, where BG is the base of the universal G–bundle. For a closed manifold M^k the *characteristic numbers* of M^k (already mentioned above) are then the values $\psi([M^k])$ ($\psi \in H^k(BG)$) taken by characteristic classes of dimension k on the fundamental homology class $[M^k]$. For $G \subset SO_m$ we have (as noted above) the Thom isomorphism

$$\phi : H^k(BG;\mathbb{Q}) \cong H^{k+m}(M(G_m);\mathbb{Q}),$$

$$\phi(z) = z \cdot u_m.$$

On the other hand by the Cartan–Serre theory (see Chapter 3, §7), for $k < m-1$ the Hurewicz homomorphism determines an isomorphism

$$\pi_{m+k}(M(G)) \otimes \mathbb{Q} \cong H^{k+m}(M(G);\mathbb{Q}).$$

From these two isomorphisms we conclude that the characteristic numbers provide a complete collection of linear forms on the homotopy groups

$$\pi_{m+k}(M(G)) \otimes \mathbb{Q}.$$

The characteristic numbers may be viewed as invariants of G-bordism classes for any $G \subset SO_m$ provided $k < m-1$. It follows that the G–bordism classes of Ω_*^G are determined up to torsion by the characteristic numbers.

As a corollary of Statement (ii) of Thom's theorem, it can be shown that for $G = SO_m$, given any cycle $z \in H_{n-m}(M^n;\mathbb{Z})$, there is an integer $\lambda \neq 0$ such that the element λz may be represented by a submanifold (this in fact holds even in non–stable dimensions).

For the simplest Lie groups (i.e. of small dimension) the Thom spaces $M(G)$ are as follows:

$$G = SO_1 = \{1\}, \quad M(SO_1) = S^1;$$

$$G = O_1 = \mathbb{Z}/2, \quad M(O_1) = \mathbb{R}P^\infty;$$

$$G = SO_2 = S^1 = U_1, \quad M(SO_2) = M(U_1) = \mathbb{C}P^\infty.$$

We see from this that the spaces $M(SO_1), M(O_1)$ and $M(SO_2) = M(U_1)$ are respectively the Eilenberg–MacLane spaces $K(\mathbb{Z},1), K(\mathbb{Z}/2,1)$ and $K(\mathbb{Z},2)$. It follows that all cycles in $H_{n-1}(M^n;\mathbb{Z}), H_{n-1}(M^n;\mathbb{Z}/2)$ and $H_{n-2}(M^n;\mathbb{Z})$ are realizable as submanifolds.

The ring $\Omega_*^{SO} \otimes \mathbb{Q}$ has as generators the bordism classes $[\mathbb{C}P^{2i}]$ (the groups $\Omega_j^{SO} \otimes \mathbb{Q}$ being trivial for $j \neq 4k$). As a linear basis for $\Omega_{4k}^{SO} \otimes \mathbb{Q}$ one may take the set of bordism classes of the form

$$[\mathbb{C}P^{2i_1}] \times \cdots \times [\mathbb{C}P^{2i_s}], \quad i_1 + \ldots + i_s = k.$$

The Pontryagin numbers provide a complete set of invariants for the ring $\Omega_*^{SO} \otimes \mathbb{Q}$. The signature $\tau(M^{4k})$ is also a bordism invariant; in fact τ is a ring homomorphism $\Omega_*^{SO} \longrightarrow \mathbb{Z}$ with value 1 on the generators $[\mathbb{C}P^{2i}]$:

$$\tau(M_1^{4k}) \cdot \tau(M_2^{4l}) = \tau(M_1^{4k} \times M_2^{4l}),$$
$$\tau(\mathbb{C}P^{2i}) = 1. \tag{3.17}$$

How is the signature $\tau(M^{4k})$ expressed in terms of the Pontryagin numbers of M^{4k}? For $k = 1$, i.e. on the group Ω_4^{SO}, we have the Thom–Rohlin formula $\tau = \frac{1}{3}p_1$, mentioned above. From the formulae given in §1 above in the course of our exposition of characteristic classes, one obtains the following formulae for the Pontryagin classes of the manifolds $\mathbb{C}P^4$ and $\mathbb{C}P^2 \times \mathbb{C}P^2$:

$$\text{For} \quad \mathbb{C}P^4, \quad p = 1 + p_1 + p_2 = (1 + u^2)^5,$$

where u is a generator of $H^2(\mathbb{C}P^4; \mathbb{Z}) \cong \mathbb{Z}$; hence

$$p_1 = 5u^2, \quad p_2 = 10u^4, \quad p_1^2 = 25u^4.$$

$$\text{For} \quad \mathbb{C}P_1^2 \times \mathbb{C}P_2^2, \quad p = 1 + p_1 + p_2 = (1 + u_1^2)^3(1 + u_2^2)^3,$$

where u_i generates $H^2(\mathbb{C}P_i^2; \mathbb{Z})$; hence

$$p_1^2 = 18u_1^2u_2^2, \quad p_2 = 9u_1^2u_2^2.$$

The Pontryagin numbers for these manifolds are shown in the following table:

	τ	p_1^2	p_2
$\mathbb{C}P^4$	1	25	10
$\mathbb{C}P^2 \times \mathbb{C}P^2$	1	18	9

Hence we infer the following expression for the signature τ in terms of the Pontryagin numbers (on the group Ω_8^{SO}):

$$\tau = \frac{1}{45}(7p_2 - p_1^2). \tag{3.18}$$

The general formula of this type, given by Hirzebruch in the mid–1950s, is as follows:

$$\tau(M^{4k}) = \langle L_k(p_1, \ldots, p_k), [M^{4k}] \rangle, \tag{3.19}$$

where the polynomial $L_k(p_1, \cdots, p_k)$ is given by the homogeneous part of degree k of the series

$$L = 1 + L_1 + L_2 + \cdots + L_k + \cdots = \prod_{i=1}^{N} \frac{u_i}{\tanh u_i}, \qquad N >> k,$$

expressed as a polynomial in the first k elementary symmetric functions

$$p_j = \sum_{i_1 < \cdots < i_j} u_{i_1}^2 \cdots u_{i_j}^2$$

This is a particular case of the general method (noted in §1 in connexion with the discussion of characteristic classes given there) of obtaining multiplicative expressions in the Pontryagin charateristic classes by using a single formal power series

$$Q(u) = 1 + Q_1 u^2 + Q_2 u^4 + \cdots .$$

For the Hirzebruch formula above one takes the formal series

$$L(u^2) = \frac{u}{\tanh u}. \tag{3.20}$$

The values taken on the $\mathbb{C}P^m$ by any multiplicative characteristic Q defined in terms of the Pontryagin or Chern numbers on $\Omega_*^{SO} \otimes \mathbb{Q}$ or Ω_*^U respectively, determine the series $Q(u)$ completely: If one takes the formal series (a "generic series" for the multiplicative characteristic Q)

$$g_Q(u) = \sum_{n \geq 0} Q(\mathbb{C}P^n) \frac{u^{n+1}}{n+1}$$

with formal inverse $g_Q^{-1}(u)$: $g_Q^{-1}(g_Q(u)) \equiv u$, the following general formula (due to Novikov in the late 1960s) can be established (related to the theory of formal groups — see Chapter 3, §9):

$$\frac{u}{Q(u)} = g^{-1}(u). \tag{3.21}$$

In the case of the signature τ, the Hirzebruch polynomials $L_k(p_1, \ldots p_k)$ are determined by the signature of the complex projective spaces:

$$\tau(\mathbb{C}P^{2k}) = 1, \quad \tau(\mathbb{C}P^{2k+1}) = 0.$$

We now define one more important invariant, the *Todd genus* T, determined by its values on the $\mathbb{C}P^k$:

$$T(\mathbb{C}P^k) = 1, \quad j \geq 0,$$

and by the generic series $g_T(u)$:

$$g_T(u) = \sum_{n \geq 0} \frac{u^{n+1}}{n+1} = -\log(1-u),$$

$$g_T^{-1} = 1 - \exp(-u), \tag{3.22}$$

$$T(u) = \frac{u}{1-e^{-u}}, \quad T(M^n) = \langle \prod_j \frac{u_j}{1-e^{-u_j}}, [M^n] \rangle.$$

By using certain general categorical properties of complex algebraic varieties and holomorphic fiber bundles, one can deduce from the Hirzebruch signature formula (3.19) the *Riemann-Roch-Hirzebruch theorem,* as follows. Let ξ be a holomorphic vector bundle over a complex algebraic variety M^n (of complex dimension n), and denote by $\mathcal{F}_\xi$ the sheaf of germs of holomorphic cross–sections of ξ. Define a characteristic χ by

$$\chi(M^n, \xi) = \sum_{j \geq 0} (-1)^j \dim H^j(M^n; \mathcal{F}_\xi).$$

This invariant $\chi(M^n, \xi)$ can be shown to be given by the following *Riemann-Roch-Hirzebruch formula:*

$$\chi(M^n, \xi) = \langle (ch\ \xi) \cdot T(M^n), [M^n] \rangle, \tag{3.23}$$

where $ch\ \xi$ is the Chern character of ξ, and $T(M^n)$ is as in (3.22), with $n = 2k$. In the case where the vector bundle ξ is one-dimensional and trivial, one obtains the *Todd genus*, known also as the "holomorphic Euler characteristic" of the variety M^n. In the classical situation (the case of a complex curve M^1) a holomorphic vector bundle ξ is determined by the equivalence class of divisors $D = \sum n_i x_i$, where $x_i \in M^1$ and the n_i are integers, so that the sheaf $\mathcal{F}_\xi$ is denoted instead by $\mathcal{F}_D$. The group $H^0(M^1; \mathcal{F}_D)$ then consists of the functions having no poles outside the divisor D, and satisfying

$$(f) + D \geq 0,$$

where $(f) = \sum m_k y_k$, $m_k \leq 0$ is the divisor of the poles of the function f. The classical Riemann-Roch theorem asserts that

$$\dim H^0(M^1; \mathcal{F}_D) - \dim H^1(M^1; \mathcal{F}_D) = n(D) - g + 1,$$

$$n = \sum n_j, \quad g = \text{genus of the curve } M^1.$$

This theorem may be cast in a different form by means of the equality

$$\dim H^1(M^1; \mathcal{F}_D) = \dim H^1(M^1; \mathcal{F}_{K-D}),$$

where K is a divisor of the zeros and the poles of meromorphic forms.

The Todd genus (see (3.22) above) turns out, for reasons to be given below, to be integral-valued on the whole bordism group Ω^U_{2k}. In the late 1950s Hirzebruch and Atiyah proved several "integrality theorems", useful in many calculations. The most important of these is as follows: From the above formula $T = \dfrac{u}{1 - e^{-u}}$ for the Todd genera we have

$$T(u) = e^{u/2} \frac{u/2}{\sinh u/2}.$$

Set

$$A(u^2) = \frac{u/2}{\sinh u/2} = e^{-u/2}T(u). \tag{3.24}$$

The characteristic determined (as described above) by this series:

$$\begin{gathered} A(p_1, \dots, p_k, \dots) = 1 + A_1 + \dots + A_k + \dots, \\ A_1 = -\frac{p_1}{24}, \quad A_2 = \frac{1}{45 \cdot 2^7}(-4p_2 + 7p_1^2), \end{gathered} \tag{3.25}$$

is called the *A-genus*. It turns out that provided $w_1(M^{4k}) = 0$, $w_2(M^{4k}) = 0$, the number $\langle A_k, [M^{4k}] \rangle$ is an integer for all k, and even for k odd. It follows that for all k, A_k defines an integer-valued linear form on the group Ω_{4k}^{Spin} of spinor bordisms. (Note that a manifold admits a spinor structure if and only if it is orientable ($w_1 = 0$) and also $w_2 = 0$.)

To give further applications of these integrality theorems we need to recall one of the most important consequences of Bott periodicity of the homotopy groups $\pi_{j-1}(SO)$, $\pi_{j-1}(U)$ (which classify oriented and complex bundles over the sphere S^j). The generator of the group $\pi_{4k}(SO)$ corresponds to a "basic" oriented vector bundle η^{4k} over S^{4k} with Chern character (see Chapter 3, §8) $ch\ \eta^{4k} = a_k\mu$, where $\langle \mu, [S^{4k}] \rangle = 1$ and

$$a_k = \begin{cases} 1 & \text{for even} \quad k \geq 2, \\ 2 & \text{for odd} \quad k \geq 1. \end{cases}$$

The Chern character of the corresponding "basic" U-bundle $\eta_{\mathbb{C}}$ over S^{4k} is $ch\ \eta_{\mathbb{C}} = \mu$. The Chern and Pontryagin classes of $\eta_{\mathbb{C}}$ are as follows:

$$\begin{gathered} p_k(\eta_{\mathbb{C}}) = a_k \cdot (2k-1)!\mu, \\ c_k(\eta_{\mathbb{C}}) = a_k \cdot (k-1)!\mu. \end{gathered} \tag{3.26}$$

(Note that the formulas (3.26) follow from the inductive definition of the Chern character:

$$c_k = (k-1)!\ ch_k + f(c_1, \dots, c_{k-1}).)$$

Remark. If M^n is an almost parallelizable manifold with stable normal bundle ν_M, then it is not difficult to show that there exists a "normal" map $f : M^n \longrightarrow S^n$ of degree one (i.e. mapping the fundamental class $[M^n]$ to $[S^n]$ in such a way that $f^*(\xi) \cong \nu_M$, where ξ is some stable vector bundle over the sphere S^n). This construction shows that Pontryagin (and all other) characteristic classes of almost parallelizable manifolds may be expressed in terms of characteristic classes of vector bundles over spheres. We note also that the class $p_k(M^{4k})$ is the only possibly nontrivial Pontryagin class. □

From this remark it follows that the stable normal (or tangent) bundle of any almost parallelizable manifold M^{4k} is determined by a multiple of the "basic" bundle η over the sphere S^{4k}. In particular, the only nontrivial Pontryagin class $p_k(M^{4k})$ has the form:

$$p_k(M^{4k}) = \lambda_M \cdot a_k \cdot \mu \cdot (2k-1)!\ ,$$

$$\langle \mu, [M^{4k}] \rangle = 1,$$

for some integer λ_M. Let $\lambda_{\min}$ denote the least number λ_M over all almost parallelizable manifolds M^{4k}. This number $\lambda_{\min}$ determines the order of the previously mentioned group $J\pi_{4k-1}(SO) \subset \pi_{4k-1}(S^n)$, $n > 4k$, which may be realized by means of framings of the standard sphere $S^{4k-1} \subset \mathbb{R}^{n+4k-1}$:

$$\lambda_{\min} = |J\pi_{4k-1}(SO)|. \tag{3.27}$$

The A-genus of such a manifold M^{4k} is given by

$$\begin{aligned} A_j(M^{4k}) &= 0 \quad \text{for} \quad j < k, \\ A_k(M^{4k}) &= a_k \frac{\alpha_k}{\beta_k} \lambda_M (2k-1)!\ , \end{aligned} \tag{3.28}$$

so by integrality of the A-genus the number $a_k \dfrac{\alpha_k}{\beta_k} \lambda_M (2k-1)!$ must be even for odd k. It follows that $|J\pi_{4k-1}(SO)|$ is divisible by the denominator of the fraction $B_k/4k$ where B_k is the kth Bernoulli number, since

$$\frac{\alpha_k}{\beta_k} = \frac{B_k}{2 \cdot (2k)!}, \quad A_k(M^{4k}) = \frac{\alpha_k}{\beta_k} p_k + \dots.$$

(This result was proved by Milnor and Kervaire in the late 1950s.)

Example a) $k = 1$. Here we have $A_1 = -\dfrac{p_1}{24}$, $a_1 = 2$. Since the number $\lambda_{\min} \cdot \frac{2}{24}$ is even, it follows that the order of the group $|J\pi_3(SO)|$ is divisible by 24. (Recall that in fact $\pi_3(SO) \cong \mathbb{Z}/24$.) Since $A_1(M^4)$ is even and $\tau = p_1/3$, the signature must therefore be divisible by 16 (Rohlin, in the early 1950s). □

Remark. Algebraic considerations involving the intersection index as a quadratic form, in particular its even-valuedness and unimodularity, yield only that the signature is divisible by 8. □

Example b) $k = 2$. here we have $a_2 = 1$ and

$$A_2 = \frac{1}{45 \cdot 2^7}(7p_1^2 - 4p_2),$$

so that $\lambda_{\min} \cdot \frac{1}{240}$ must in this case be integral. Hence $|J\pi_7(SO)|$ is divisible by 240. □

Remark. The stable homotopy groups $\pi_{n+j}(S^n)$ have been computed using methods of Cartan-Serre-Adams type, significantly beyond $j = 7$, thanks to the efforts of several authors (in particular, Toda). The formula (3.27) for the order of $J\pi_{4k-1}(SO)$ for all k represents an important result in manifold theory; in particular it provides a lower bound for the order of the stable homotopy groups of spheres. This "geometric" approach to the computation of the stable homotopy groups of spheres may be combined with algebraic methods (the Cartan-Serre method and the Adams spectral sequence) only in the framework of extraordinary cohomology theories. □

We mention one more application of the signature in manifold theory of fundamental importance. By means of the signature formula, non-trivial smooth structures on manifolds have been found, first on the sphere S^7 (by Milnor in the late 1950s). The construction is as follows:

Consider a 4-dimensional vector bundle η of Hopf type over the sphere $B = S^4$. The corresponding spherical bundle $S(\eta)$ has fiber S^3, and structure group SO_4. We assume that

$$\chi_4(\eta) = \mu, \quad \langle \mu, [S^4] \rangle = 1, \tag{3.29}$$

where μ is a generator of $H^4(S^4; \mathbb{Z})$. This condition implies that the total space of the bundle with fiber S^3 is homotopy equivalent to S^7; we shall denote this total space by M^7_α, where α is defined in terms of μ and the first Pontryagin class of η by

$$p_1(\eta) = \alpha\mu, \quad \langle \mu, [S^4] \rangle = 1.$$

Denote by N^8_α the total space of the corresponding disk bundle $D(\eta)$ with fiber D^4. Note that $\partial N^8_\alpha = M^7_\alpha$. It is not difficult to show that α must be even: $\alpha = 2k$. The quaternionic Hopf bundle whose spherical bundle is the standard Hopf fibration $S^7 \longrightarrow S^4$ corresponds to the case $k = 1$. Milnor gave an explicit construction of these bundles for all even α, at the same time defining a Morse function on each such manifold M^7_α with exactly two critical points (a maximum point and a minimum point). From this it follows that each M^7_α is homeomorphic (even piecewise linearly) to the standard sphere S^7. Consider now the PL-manifold

$$\bar{N}^8_\alpha = N^8_\alpha \cup_{S^7} D^8.$$

Assuming that $\bar{N}^8_\alpha$ is a smooth manifold, we compute its first Pontryagin class and its signature:

$$p_1(\bar{N}^8_\alpha) = p_1(N^8_\alpha) = \alpha = 2k; \quad \tau(\bar{N}^8_\alpha) = \tau(N^8_\alpha) = 1. \tag{3.30}$$

Hence by the Hirzebruch formula,

$$p_2(\bar{N}^8_\alpha) = \frac{45\tau + p_1^2}{7} = \frac{45\tau + (2k)^2}{7},$$

which is not an integer when $k = 2$ (i.e. $\alpha = 4$). We infer that the manifold $\bar{N}_4^8$ is not smooth, so that M_4^7 is not diffeomorphic to S^7.

It follows from the topological invariance of the Pontryagin numbers that in fact the manifold $\bar{N}_4^8$ is not homeomorphic to any smooth manifold; it is clear, however, that $\bar{N}_4^8$ is a PL-manifold. Further development of the theory of smooth structures on spheres and other manifolds will be discussed in the next section.

We now return to cobordism theory and the problem of realizing cycles by submanifolds. We have seen that the computation of bordism rings via the Thom-Pontryagin construction leads to the study of homotopy properties of the Thom spaces of the universal vector G-bundles (for $G = O$, SO, U, SU, Sp, $Spin$, and others). Adams' method, on the other hand, involves the calculation of the cohomology ring $H^*(M(G); \mathbb{Z}/p)$ and the action on it of the Steenrod operations, i.e. the stucture of the cohomology ring as a module over the Steenrod algebra (see Chapter 3, §6). Except in the case of $H^*(M(Spin); \mathbb{Z}/2)$, the structure of these modules may be inferred from the observation that the embedding $i : BG_n \longrightarrow M(G_n)$ of the base space as the zero section, induces a monomorphism (of modules over the Steenrod algebra)

$$i^* : H^*(M(G_n); \mathbb{Z}/p) \longrightarrow H^*(BG_n; \mathbb{Z}/p)$$

onto a prime principal ideal generated by $i^*(u_n)$, where u_n is the Thom class in $H^*(M(G_n); \mathbb{Z}/p)$. This implies, in particular, that the product operator

$$z \longmapsto i^*(u_n) \cdot z = i^*(z \cdot \phi(1)) = i^*\phi(z) \tag{3.31}$$

has zero kernel in the appropriate cohomology group of $M(G_n)$.

For $G_n = O_n$, SO_n one obtains:

$$\begin{gathered} i^*(u_n) = w_n, \\ H^*(M(G_n); \mathbb{Z}/2) \cong w_n \cdot H^*(BG_n; \mathbb{Z}/2). \end{gathered} \tag{3.32}$$

Note that if $G_n = U_n$, SU_n then $n = 2k$, and if $G_n = Sp_n$ then $n = 4k$. One has the following isomorphisms:

$$\begin{aligned} H^*(M(SO_{2k}); \mathbb{Z}/p) &\cong \chi_{2k} \cdot H^*(BSO_{2k}; \mathbb{Z}/p), && p > 2, \\ H^*(M(U_{2k}); \mathbb{Z}/p) &\cong c_k \cdot H^*(BU_{2k}; \mathbb{Z}/p), && p \geq 2, \\ H^*(M(SU_{2k}); \mathbb{Z}/p) &\cong c_k \cdot H^*(BSU_{2k}; \mathbb{Z}/p), && p > 2, \\ H^*(M(Sp_{4k}); \mathbb{Z}/p) &\cong i^*(u_{4k}) \cdot H^*(BSp_{4k}; \mathbb{Z}/p), && p > 2. \end{aligned} \tag{3.33}$$

Recall (see §1 above) that the cohomology rings of the BG_n can be described in terms of symbolic generators (namely the symmetric polynomials in the generators of $H^*(T^n)$ where T^n is a maximal torus of G_n: in $t_1, \dots, t_n$ mod 2 for $G_n = O_n$, SO_n, in terms of which $w_n = t_1 \cdots t_n$, and in the 2-dimensional generators $u_1, \dots u_k$ in other cases, in terms of which $\chi_{2k} = c_k = u_1 \cdots u_k$). The isomorphism (3.32) implies that the cohomology ring $H^*(M(O_n); \mathbb{Z}/2)$ is in dimensions $\leq 2n$ a free module over the Steenrod algebra $\mathcal{A}_2$ with the non-dyadic monomials

$$\mathfrak{p} = \sum t_{i_1}^{q_1} \cdots t_{i_s}^{q_s},$$

$$q = \sum q_j < n, \quad q_j \text{ not of the form } 2^r - 1,$$

as generators, and the action of these generators on the Thom class $w_n = t_1 \cdots t_n$ is ordinary multiplication of polynomials:

$$\mathfrak{p}(w_n) = \mathfrak{p} t_1 \cdots t_n \in H^{n+q}(M(O_n); \mathbb{Z}/2), \quad q = \sum q_j.$$

(This is a theorem of Thom.) This result allows (using the Adams spectral sequence) the complete computation of the homotopy groups of the Thom spaces $M(O_n)$ (in dimensions $\leq 2n$), or, equivalently, all stable homotopy groups of the spectrum MO, which can be identified (via the Thom-Pontryagin construction) with the bordism ring $\Omega_*^O = \mathfrak{N}_*$. As an immediate application of this computation one has the following theorem:

Any cycle of dimension $\leq n/2$ in the homology (mod 2) of a manifold M^n may be realized by a closed submanifold, and, in all dimensions, as the image of a closed manifold. □

For $p > 2$ and $G = SO$, U, SU, Sp the $\mathcal{A}_p$-modules $H^*(M(G); \mathbb{Z}/p)$ turn out to be direct sums of copies of the one-dimensional $\mathcal{A}_p$-module $\mathcal{A}_p/(\beta)$ (where β is the Bockstein operator and (β) the two–sided ideal generated by β). The module $\mathcal{A}_p/(\beta)$ is given by the relation $\beta(z) \equiv 0$.

Remark. The latter statement also holds for $H^*(M(U); \mathbb{Z}/2)$, with the Bockstein operator β replaced by the operation Sq^1. □

The structure of the modules $H^*(M(G); \mathbb{Z}/2)$ (as $\mathcal{A}_2$-modules) for $G = SO$, SU, Sp and $Spin$ is somewhat different. They are also direct sums of one-dimensional $\mathcal{A}_2$-modules, but the summands vary: For $G = SO$ they are free or with the single relation $Sq^1(v) = 0$ (where v is a generator); for $G = SU$ with the same relation $Sq^1(v) = 0$ together with the identical relation $Sq^2(z) \equiv 0$ for every element z; for $G = Sp$ (the most difficult case) there are the two identical relations $Sq^1(z) \equiv 0$, $Sq^2(z) \equiv 0$ (for all z); and finally for $G = Spin$ there are summands with the two relations $Sq^1(v) = 0$, $Sq^2(v) = 0$ (where v is a generator of this $\mathcal{A}_2$-module). (These results are due to Milnor and Novikov (around 1960), and in the case of $Spin$ to Anderson, Brown and Peterson in the mid-1960s.)

It should be noted that there is a product structure (induced by the direct product of manifolds) on all of the above bordism rings. This "geometric" product structure determines an algebraic "product" on the $\mathcal{A}_p$-modules $\mathcal{M} = H^*(M(G); \mathbb{Z}/p)$ via the diagonal homomorphism

$$\Delta : \mathcal{M} \longrightarrow \mathcal{M} \otimes_{\mathbb{Z}/p} \mathcal{M},$$

which in turn determines a multiplication on the the cohomology groups

$$\mathrm{Ext}^{*,*}_{\mathcal{A}_p}(\mathcal{M}, \mathbb{Z}/p).$$

After certain algebraic obstacles have been overcome, the Adams spectral sequence (see Chapter 3, §6,7) yields up the structures of the bordism rings Ω_*^{SO}, Ω_*^U (Milnor, Novikov). It should be mentioned that the 2-torsion of the groups Ω_*^{SO}, Ω_*^{SU} had been calculated ealier using different, purely geometric ideas (by Rohlin and Wall in the late 1950s in the case of Ω_*^{SO}, and by Conner and Floyd in the mid–1960s, for Ω_*^{SU}; however the combination of these geometric methods with those of homological algebra turned out to be feasible only in the framework of the generalized (extraordinary) cohomology (cobordism) theories $\Omega_U^*(\cdot)$ and $\Omega_O^*(\cdot)$.

The results are as follows: None of the bordism rings Ω_*^G (with G as before) has p-torsion for $p > 2$, and in the case Ω_*^U for $p \geq 2$. The rings $\Omega_*^G \otimes \mathbb{Z}/p$ and $\Omega_*^G \otimes \mathbb{Q}$ are polynomial for $p > 2$. The ring Ω_*^U is polynomial (over the integers) in even-dimensional generators, one of each even dimension. On these generators $[M^{2n}]$ the Newton polynomials $\tilde{c}_n = n!\, ch_n$ take the following values:

$$\langle \tilde{c}_n, [M^{2n}] \rangle = \begin{cases} 1 & \text{for} \quad n \neq p^i - 1, \\ p & \text{for} \quad n = p^i - 1, \end{cases} \qquad p \text{ prime}, \quad [M^{2n}] \in \Omega_{2n}^U.$$

Since $\langle \tilde{c}_n, \mathbb{C}P^n \rangle = n + 1$, it follows that only the elements $[\mathbb{C}P^{p-1}]$ satisfy the above condition. All other generators $[M^{2n}]$ may be constructed out of the Milnor manifolds $H_{r,t}$, which represent the cycles $\mathbb{C}P^r \times \mathbb{C}P^{t-1} + \mathbb{C}P^{r-1} \times \mathbb{C}P^t$ in the product $\mathbb{C}P^r \times \mathbb{C}P^t$. Linear combinations of the elements $[H_{r,t}]$ provide a full set of multiplicative generators of the ring Ω_*^U.

The following theorem (Novikov, early 1960s) on the representability in the stable dimensions $j < n/2$ of integral cycles $z \in H_j(M^n; \mathbb{Z})$ by oriented submanifolds (or, in any dimension, by continuous images of manifolds) is a consequence of results concerning the stable homotopy structure of the complexes $M(SO_m)$ (or the spectrum MSO):

There exists an odd integer λ such that the cycle λz can be so represented; moreover if for all $k \geq 1$, $p > 2$, there is no p–torsion in the homology groups $H_j(M^n; \mathbb{Z})$ of dimensions $j = 2k(p-1) - 1$, then in fact λ may be taken to be 1.

Remark. If there is p-torsion in the indicated groups, then the factor λ may be taken as the product

$$\lambda = \prod_{p>2} p^{\left[\frac{j}{2(p-1)}\right]},$$

which is the formula for the denominator of the general coefficient in Hirzebruch's polynomial

$$L_m(p_1, \ldots, p_m), \quad \text{when } m = \left[\frac{j}{4}\right]. \qquad \square$$

In the late 1960s Buchstaber constructed examples showing that this result cannot in general be improved. At the same time he showed that in the absence of p-torsion in certain intermediate groups, the result *can* be improved.

§4. Classification problems in the theory of smooth manifolds. The theory of immersions. Manifolds with the homotopy type of a sphere. Relationships between smooth and PL-manifolds. Integral Pontryagin classes

The general formulation of the problem of the homotopy classification of immersions is as follows:

Given two manifolds M^n and N^k, where $k > n$, characterize the classes of regularly homotopic immersions $M^n \longrightarrow N^k$.

The simplest case of interest is that where $M^n = S^n$ and $N^k = \mathbb{R}^k$. Consider the following two function spaces of immersions:

a) The space $X^k_{n,s}$ of immersions of the disc D^n equipped with a normal field ν_s of dimension s, into $\mathbb{R}^k$, where $n + s < k$, under the condition that for each admissible immersion each point s_0 of the boundary ∂D of the disc should have a neighbourhood on which the immersion is "normalized", in the sense that its image is essentially a region of a standard n–dimensional plane in $\mathbb{R}^k$ equipped with a standard constant normal s-dimensional field of frames in $\mathbb{R}^k$.

b) The space $Y^k_{n,s}$ of immersions of the sphere S^n equipped with a normal field ν_s, in $\mathbb{R}^k$, $n + s < k$, with the property that each immersion is "normalized" on some neighbourhood U_x of each point s_0 of S^n, i.e. the image of the neighbourhood U_x is a region of a standard n–dimensional plane in $\mathbb{R}^k$ equipped with a standard constant normal s-dimensional frame field in $\mathbb{R}^k$.

There is an obvious map

$$X^k_{n,s} \longrightarrow Y^k_{n-1,s+1},$$

defined by taking the boundary $S^{n-1} = \partial D^n$ of the disc together with an additional 1–dimensional field normal to S^{n-1} and tangential to the interior of D^n. By an observation of Smale (made in the late 1950s) this map is the projection map of a Serre fibration. Since therefore the fibers are all homotopically equivalent, it is not difficult to see that in fact they have the homotopy type of $Y^k_{n,s}$. Since, further, the total space $X^k_{n,s}$ is contractible: $\pi_j(X^k_{n,s}) = 0$ for

$j \geq 0$, we obtain an isomorphism between the appropriate homotopy groups of the fiber and base:

$$\pi_j(Y^k_{n,s}) \cong \pi_{j+1}(Y^k_{n-1,s+1}). \tag{4.1}$$

Now classifying the immersions of S^n in $\mathbb{R}^k$, simply means identifying the path–connected components of $Y^k_{n,0}$, i.e. identifying $\pi_0(Y^k_{n,0})$. From the sequence of isomorphisms obtained from (4.1) by letting j vary, we have immediately

$$\pi_0(Y^k_{n,0}) \cong \pi_n(Y^k_{0,n}). \tag{4.2}$$

Since $Y^k_{0,n}$ is just the Stiefel manifold with points the non–degenerate fields of n–frames on $\mathbb{R}^k$:

$$Y^k_{0,n} = V^{\mathbb{R}}_{n,k},$$

we infer *Smale's theorem:*

$$\pi_0(Y^k_{n,0}) \cong \pi_n(V^{\mathbb{R}}_{n,k}) \cong \pi_n(SO_k/SO_{k-n}).$$

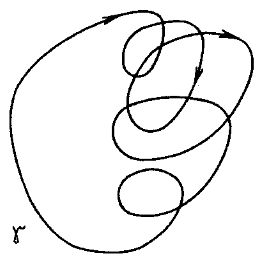
The particular case $k = 2n$ of this result is the classical theorem of Whitney on the classification of immersions $S^n \longrightarrow \mathbb{R}^{2n}$. (The case $n = 1$ of Whitney's theorem is illustrated in Figure 4.21 a); the invariant of the homotopy class of an immersion is in this case given alternatively by $1/2\pi \oint k\ ds = n$, the integral of the curvature of the immersed circle over the whole circle.) As another non–trivial consequence of Smale's theorem we have that all immersions $S^2 \longrightarrow \mathbb{R}^3$ are regularly homotopic, since $\pi_2(V^{\mathbb{R}}_{2,3}) \cong \pi_2(SO_3) = 0$; this is by no means obvious even for simple examples of immersions: compare the immersion $S^2 \longrightarrow \mathbb{R}P^2 \longrightarrow \mathbb{R}^3$ with the standard embedding $S^2 \longrightarrow \mathbb{R}^3$.

Smale's theorem generalizes without great difficulty to the situation of immersions $S^n \longrightarrow M^k$ of the sphere S^n in any manifold M^k; here the result is that the homotopy classes of immersions are completely determined (i.e. in one–to–one correspondence with) the elements of the homotopy group $\pi_n(E_{n,k})$, where $E_{n,k}$ is the total space of the bundle of non–degenerate fields of tangential n–frames on the manifold M^k. Already in the case $k = 2$, for the surface M_g (sphere–with–g–handles) the fundamental group $\pi_1(E_{1,2})$ of the manifold of "linear elements" (i.e. fields of 1–dimensional subspaces of the tangent space of M_g) is non–trivial; it is given by generators $a_1, b_1, \cdots, a_g, b_g, c$ with the defining relations

$$\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = c^{2g-2}, \quad ca_i = a_i c, \quad cb_i = b_i c. \tag{4.3}$$

(The case $g = 0$ is illustrated in Figure 4.21 (b).)

In the early 1960s the ultimate generalization of Smale's theorem was established by Hirsch, who showed that for any two manifolds M^n, N^k $(k > n)$ each class of immersions $M^n \longrightarrow N^k$ is uniquely determined by the ordinary homotopy class of maps of tangent bundles induced by these immersions. The proof makes crucial use of the fact that, by definition of an immersion, no tangent n-frame at any point of M^n becomes degenerate under the map of tangent spaces induced by an immersion $f : M^n \longrightarrow N^k$. Consider the



(a) $S^1 \longrightarrow \mathbb{R}^2$, $V^{\mathbb{R}}_{1,2} \sim S^1$,

$\pi_1(S^1) \cong \mathbb{Z}$, $\frac{1}{2\pi}\oint_\gamma kds = n$

(b) $S^1 \longrightarrow S^2$, $E_{1,2} \cong SO_3$, $\pi_1(SO_3) \cong \mathbb{Z}/2$; thus on going over from $\mathbb{R}^2$ to the sphere S^2, the invariant in (a) becomes reduced modulo 2

Fig. 4.21

manifold $G_n(N^k)$ whose points are the n-dimensional planes (subspaces) of the tangent spaces at the points of N^k. The immersion f induces a map $\widetilde{f} : M^n \longrightarrow G_n(N^k)$, which is covered by a map of principal bundles with structure group $GL_n(\mathbb{R})$; it is the homotopy classes of these bundle maps that determine the classes of immersions.

A theory concerned with closed manifolds homotopy equivalent to spheres was developed by Milnor and Kervaire around 1960. In order to describe

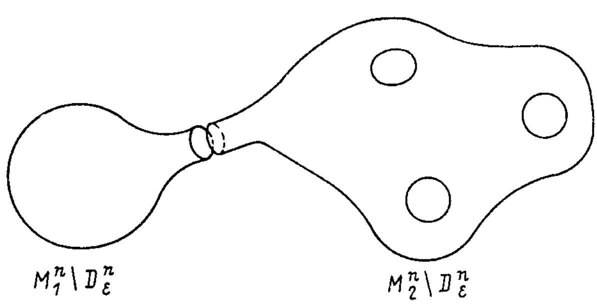


Fig. 4.22

this theory we need the construction called a "connected sum" $M_1^n \# M_2^n$ of manifolds M_1^n, M_2^n, with respect to which the set of (diffeomorphism classes of) smooth manifolds of each dimension n forms a commutative semigroup: From each of the manifolds M_1^n, M_2^n a small disc is removed:

$$\sum_j (x^j_\alpha)^2 < \varepsilon^2 \quad \text{on } M_1^n,$$

$$\sum_j (y^j_\beta)^2 < \varepsilon^2 \quad \text{on } M_2^n,$$

in terms of local co-ordinates on appropriate charts $U_\alpha \subset M_1^n$, $V_\beta \subset M_2^n$, and then the resulting boundaries (each diffeomorphic to the standard sphere S_ε^{n-1}) are identified (taking orientation into account if the manifolds are oriented) by means of the standard diffeomorphism; the resulting smooth manifold is the *connected sum* $M_1^n \# M_2^n$ (see Figure 4.22).

Example. We have an exhaustive list of closed 2-manifolds (closed surfaces):

$$M_g^2 = \underbrace{T^2 \# \cdots \# T^2}_{g \text{ times}}, \quad T^2 \text{ the torus},$$

$$N_{1,g}^2 = K^2 \# \underbrace{T^2 \# \cdots \# T^2}_{g-1 \text{ times}}, \quad K^2 \text{ the Klein bottle},$$

$$N_{2,g}^2 = \mathbb{R}P^2 \# \underbrace{T^2 \# \cdots \# T^2}_{g \text{ times}}.$$

Hence T^2, K^2 and $\mathbb{R}P^2$ are generators for the semigroup of all closed surfaces, with defining relations

$$\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# T^2 \cong \mathbb{R}P^2 \# K^2,$$

$$K^2 \# T^2 \cong K^2 \# K^2.$$

(Figures 4.24, 4.25, 4.26 illustrate these relations.)

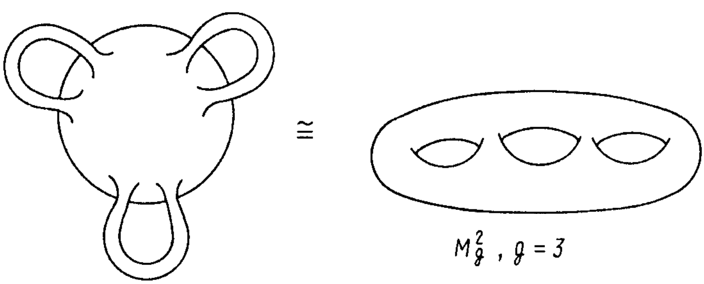


Fig. 4.23. The sphere-with-g-handles M_g^2 ($g = 3$ in the figure)

Remark. The orientable closed surfaces form a free monoid on the single generator T^2. It would seem that the closed orientable 3-manifolds also form a free monoid; however for closed orientable 4–manifolds this is certainly no longer the case. Note incidentally that of the closed surfaces, the $N^2_{2,g}$, $g \geq 0$, all represent the non–zero cobordism class of Ω_2^O, while all the other surfaces represent the zero class. □

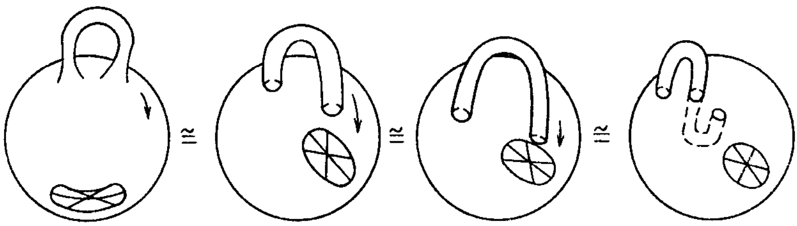


Fig. 4.24. The process of deforming a handle so that it goes from the outside to the inside

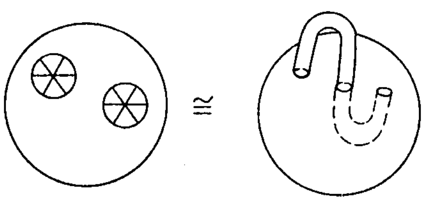


Fig. 4.25

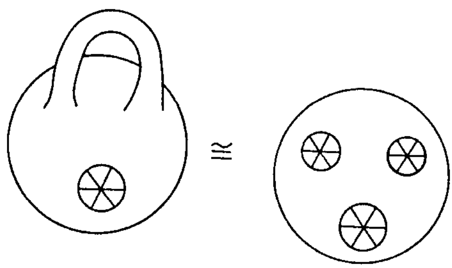


Fig. 4.26

Returning now to the topic of homotopy spheres, we first observe that the oriented h–cobordism classes of homotopy n–spheres form a group (denoted by Θ^n) under the operation of taking the connected sum. (See §3 above for the definition of h–cobordism.) Here the zero class consists of the oriented manifolds M^n bounding contractible manifolds–with–boundary W^{n+1}:

$$M^n = \partial W^{n+1}, \quad W^{n+1} \sim 0.$$

The inverse of (the class of) an oriented homotopy n–sphere M^n_+ is (the class of) the same manifold M^n_- with the opposite orientation. To see this, i.e. to see that the connected sum $M^n_+ \# M^n_-$ is h–cobordant to S^n consider the manifold–with–boundary W^{n+1} with $\partial W^{n+1} = M^n_+ \# M^n_-$, constructed as follows (see Figure 4.27): From the cylinder $M^n_+ \times I$ remove the subset $D^n_\varepsilon \times I$ where $D^n_\varepsilon \subset M^n_+$ is a small open ball of radius ε; after appropriate "smoothing of corners" one is left with a smooth manifold–with–boundary

$$W^{n+1} = (M^n_+ \times I) \setminus (D^n_\varepsilon \times I) = (M^n_+ \setminus D^n_\varepsilon) \times I,$$

for which $\partial W^{n+1} = M^n_+ \# M^n_-$, and which is moreover contractible.

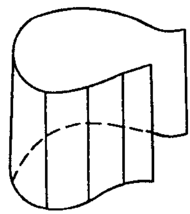


Fig. 4.27. $M^n_+ \# M^n_- \sim 0$, $M^n \sim S^n$ (homotopy sphere)

Smale's important "h–cobordism theorem", stating that every h–cobordism W^{n+1} between simply–connected manifolds M_1^n, M_2^n of dimension $n \geq 5$ is diffeomorphic to $M_1^n \times I$ (whence M_1^n is diffeomorphic to M_2^n), implies that for each $n \geq 5$ the group Θ^n provides an exact classification of the homotopy n–spheres, and hence of the distinct smooth structures definable on the topological n–sphere.

Of particular interest is the subgroup (denoted by bP^{n+1}) of Θ^n consisting of the h–cobordism classes of boundaries of compact parallelizable manifolds–with–boundary. Milnor has shown how to construct, corresponding to each even–dimensional, unimodular, integer matrix (a_{ij}) (as in (4.4)), a parallelizable manifold-with-boundary P^{4k} with boundary a homotopy sphere M^{4k-1}, for which $H_{2k}(P^{4k};\mathbb{Z})$ is a free abelian group with a basis of cycles having matrix of intersection indices (a_{ij}), and $\tau_j(P^{4k}) = 0$ for $j < 2k$. The minimal signature of such a manifold is 8: $\tau_{\text{min}}(P^{4k}) = 8$; here is a matrix $A = (a_{ij})$ yielding such a manifold:

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \tag{4.4}$$

Note that the manifold P^{4k} is a *Spin*-manifold (it is oriented and the Stiefel class $w_2(P^{4k}) = 0$). Thus a necessary condition for the boundary of P^{4k} to be the ordinary $(4k-1)$-sphere S^{4k-1}, is the integrality of the $\hat{A}$-genus of the manifold

$$P^{4k} \cup_{\partial P^{4k}} D^{4k}$$

(see §3 above), given by

$$\hat{A}[P^{4k} \cup_{\partial P^{4k}} D^{4k}] = \tau(P^{4k}) \cdot \frac{1}{2^{2k+1}(2^{2k-1}-1)}.$$

Hence in this situation the signature $\tau(P^{4k})$ must in fact be divisible by

$$2^{2k+1}(2^{2k-1}-1).$$

Using the fact that the signature is divisible by 8, one has that the order of the group bP^{4k} is divisible by

$$2^{2k-2}(2^{2k-1}-1).$$

(Recall that for even k the $\hat{A}$-genus is likewise even, so that in fact for k even the order of the group bP^{4k} is divisible by $2^{2k-1}(2^{2k-1}-1)$.)

For large enough N ($N > n+1$) every homotopy sphere $M^n \subset \mathbb{R}^{n+N}$ has trivial normal bundle ν. (For this one uses the classification of normal fields of N-frames on M^n via elements $[\nu]$ of $\pi_n(SO_N)$, in conjunction with Bott periodicity for SO, the fact that $p_k(M^n) = 0$ for $n = 4k$ (a consequence of the Hirzebruch signature formula and $\tau(S^n) = 0$), and a result of Adams of the early 1960s implying that the classifying element $[\nu] = 0$ also in the cases where $\pi_n(SO_N) \cong \mathbb{Z}/2$.) In view of this property, homotopy n-spheres admit framings ν_N, and the natural one-to-one correspondence (see §3 above) between framed normal bundles on n-manifolds in $\mathbb{R}^{n+N}$ and the elements of $\pi_{N+n}(S^N)$ (whereby framed manifolds of the form (S^n, ν_N) correspond to the subgroup $J(\pi_n(SO_N))$, the image under the Whitehead homomorphism — see §3 above) then yields a homomorphism

$$\mu : \Theta^n \longrightarrow \pi_{N+n}(S^N)/J\pi_n(SO). \tag{4.5}$$

The kernel of this homomomorphism coincides with the subgroup bP^{n+1} of classes of homotopy n-spheres bounding parallelizable manifolds-with-boundary. A theorem of Milnor and Kervaire asserts that the homomorphism μ is an epimorphism provided n is not of the form $4k+2$, while for $n = 4k+2$ the image under μ has index at most 2. The kernel bP^{n+1} is cyclic, and in fact trivial for even n. It is known further that for n of the form $4k+1$, the subgroup $bP^{4k+2} \subset \Theta^{4k+1}$ has order at most 2, that $bP^{\infty} \cong \mathbb{Z}/28$, and that the order of bP^{4k} increases as $k \to \infty$.

A detailed technique for investigating homotopy spheres has been developed, involving "killing" the appropriate homotopy groups of framed manifolds by means of "Morse surgery", in order to reduce them to framed homotopy spheres. (This technique was used, in the case $n = 2$, by Pontryagin in the late 1940s in connexion with his calculation of the groups $\pi_{n+2}(S^n)$; see §3 above.) The given framed manifold is first replaced by an equivalent connected one, and this in turn by an equivalent simply–connected framed manifold, and so on; difficulties requiring non–trivial analysis arise only in dimension $[n/2]$. For odd n no invariants emerge in this dimension, although the proof of this is non–trivial. For n of the form $4k$, since the signature of a framed manifold is zero by Hirzebruch's formula, one may always by means of elementary Morse surgeries eliminate the $n/2$–dimensional homotopy also, provided $n > 4$, the cycles of dimension $n/2$ being in this case realizable as embedded $n/2$–spheres, as required for Morse surgery. For n of the form $4k+2$ the framed manifold M^{4k+2} can also be reduced to a manifold M_1^{4k+2} satisfying $\pi_j(M_1^{4k+2}) = 0$ for $j \leq 2k$. Here, although the elements of $H_{2k+1}(M_1^{4k+2}) \cong \pi_{2k+1}(M_1^{4k+2})$ are, as in the previous case, realizable by means of embedded spheres $S^{2k+1} \subset M_1^{4k+2}$, these spheres may have non–trivial normal bundles in M_1^{4k+2}, corresponding to the non–trivial element $[\nu]$ of the first non–stable homotopy group $\pi_{2k}(SO_{2k+1})$:

$$[\nu] \in \pi_{2k}(SO_{2k+1}) \cong \mathbb{Z}/2, \quad k \neq 0, 1, 3.$$

(These groups were encountered earlier following the discussion of Bott periodicity in §2 above, where it was remarked that it is their non–triviality that gives rise to the non–parallelizability of spheres of dimensions $\neq 1, 3, 7$, and the non–existence of division algebras of dimensions $\neq 2, 4, 8$.) Thus for all $k \neq 0, 1, 3$, to each element

$$z \in H_{2k+1}(M^{4k+2}; \mathbb{Z}) \cong \pi_{2k+1}(M^{4k+2})$$

there corresponds an element

$$\Phi(z) = [\nu] \in \pi_{2k}(SO_{2k+1}) \cong \mathbb{Z}/2,$$

the *Arf function*. The following "Arf identity"

$$\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2) + z_1 \circ z_2 \pmod 2$$

holds for the Arf function Φ. For $k = 0, 1, 3$, the Arf function may be defined alternatively in terms of framed normal bundles on spheres $S^{2k+1} \subset M^{4k+2} \subset \mathbb{R}^N$. (This approach was taken in §3 above, in the case $k = 0$.) Much as described in §3 (see (4.8) there), the Arf function determines the *Arf–invariant*

$$\Phi : \pi_{N+4k+2}(S^N) \longrightarrow \mathbb{Z}/2.$$

The "Arf–invariant problem" then consists in determining this homomorphism for the various k. It is nontrivial in the cases $k = 0, 1, 3$, on the stable groups $\pi_{N+2}(S^N)$, $\pi_{N+6}(S^N)$, $\pi_{N+14}(S^N)$. In the late 1950s Kervaire showed that it is trivial when $k = 2$; this result then enabled him to construct a 10–dimensional PL–manifold not homotopy equivalent to any smooth manifold, and to show that $bP^{10} \cong \mathbb{Z}/2$ ($bP^{10} \subset \Theta^9$). For $k = 0, 1, 3$ we have $bP^2 \cong bP^6 \cong bP^{14} \cong \mathbb{Z}/2$. It was then shown by Kervaire and Milnor that $\Phi = 0$ precisely if $bP^{4k+2} \cong \mathbb{Z}/2$. Somewhat later, in the mid–1960s, it was proved by Anderson, Brown and Peterson that in fact the Arf–invariant Φ is zero in all dimensions of the form $n = 8k + 2$; in their proof they used the idea (due to Brown and Novikov in the early 1960s) of extending the Arf–invariant Φ to a homomorphism of the SU–cobordism group:

$$\pi^s_{8k+2} = \pi_{8k+2+N}(S^N) \longrightarrow \Omega^{SU}_{8k+2} \xrightarrow{\Phi} \mathbb{Z}/2.$$

The image of the homotopy groups of spheres in Ω^{SU}_* turns out to be describable in quite simple terms, and it is this which yields the result.

For n of the form $8k + 6$, this approach via ordinary cobordism theory does not succeed. However by constructing a special cobordism theory which maximally extends the Arf–invariant, Browder was able to show (in the late 1960s) that the Arf–invariant Φ is zero for all n not of the form $2^s - 2$, and moreover that for $n = 2^s - 2$ it is non–zero if and only if the elements

$$h^2_{s-1} \in \mathrm{Ext}^{2^s,2}_{\mathcal{A}_2}(\mathbb{Z}/2, \mathbb{Z}/2)$$

are cycles with respect to all the Adams differentials (therefore representing nontrivial elements of the group $\pi_{N+2^s-2}(S^N)$). In particular, in the cases $s = 5, 6$, the elements h^2_4, h^2_5 are in fact cycles with respect to all Adams differentials, so that $bP^{30} = 0$, $bP^{62} = 0$.[7]

There is another naturally occuring group Γ^n, related to Θ^n, obtained as a quotient of the group of connected components of the space $Diff^+(S^{n-1})$ of orientation–preserving diffeomorphisms $f : S^{n-1} \longrightarrow S^{n-1}$ under composition, by factoring by the normal subgroup of those diffeomorphisms f that extend to diffeomorphisms ψ of the disc D^n bounded by S^{n-1}:

$$\psi : D^n \longrightarrow D^n, \quad \psi|_{S^{n-1}} = f.$$

We thus have the following sequence of homomorphisms, whose composite we denote by $\bar{\mu}$:

$$\bar{\mu} : \pi_0(Diff^+(S^{n-1})) \longrightarrow \Gamma^n \longrightarrow \Theta^n \xrightarrow{\mu} \pi_{n+N}(S^N)/\mathrm{Im}\, J.$$

(Here the homomorphism $\Gamma^n \longrightarrow \Theta^n$ is defined by associating with each diffeomorphism $f : S^{n-1}_1 \longrightarrow S^{n-1}_2$, the homotopy n–sphere obtained by identifying the boundaries of two discs D^n_1, D^n_2 via that diffeomorphism.)

It follows from results of Smale that $\Gamma^n \cong \Theta^n$ for $n \geq 5$, since for these n all homotopy spheres M^n can be constructed from discs D^n_1, D^n_2 by identifying their boundaries by means of a diffeomorphism $f : S^{n-1}_1 \longrightarrow S^{n-1}_2$. The triviality of the groups Γ^1, Γ^2 is easy. That $\Gamma^3 = 0$ follows from Smale's result (of the late 1950s) according to which $Diff^+(S^2)$ is homotopy equivalent to SO_3. That $\Gamma^4 = 0$ follows from the (highly non–elementary) result of Cerf (of the mid–1960s) asserting that $\pi_0(Diff^+(S^3)) = 1$. On the other hand it follows from Milnor's example (see §3 above) that $\pi_0(Diff^+(S^6)) \neq 1$. Putting these results together (with others) one has:

$$\Gamma^1 = \Gamma^2 = \Gamma^3 = \Gamma^4 = \Gamma^5 = \Gamma^6 = 0$$

$$\Gamma^7 \cong \mathbb{Z}/28, \quad \Gamma^n = \Theta^n, \quad n \neq 3.$$

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[7]The latter result is due to Barrat, Mahowald and J. D. S. Jones; they proved that h^2_5 is a cycle of all Adams differentials by direct computation in the Adams spectral sequence. It is not known (as of 1994) whether or not the elements h^2_j are cycles for $j \geq 6$.

The fiber bundle projection

$$p : Diff^+(D^q) \longrightarrow Diff^+(S^{q-1}), \tag{4.6}$$

and the inclusion map

$$\kappa : Diff^+(D^q, S^{q-1}) \longrightarrow Diff^+(S^q),$$

are defined in the natural way. The fibre F_q of the fibre bundle (4.6) is the group of diffeomorphisms fixing the boundary: $F_q = Diff^+(D^q, S^{q-1})$. One has the usual induced homomorphisms of homotopy groups:

$$\partial : \pi_j(Diff^+(S^{q-1})) \longrightarrow \pi_{j-1}(Diff^+(D^q, S^{q-1})),$$

$$\kappa_* : \pi_{j-1}(Diff^+(D^q, S^{q-1})) \longrightarrow \pi_{j-1}(Diff^+(S^q)).$$

Composing $2j$ times, we obtain a homomorphism

$$\lambda = (\kappa_* \circ \partial) \circ \cdots \circ (\kappa_* \circ \partial),$$

$$\lambda : \pi_j(Diff^+(S^{q-1})) \longrightarrow \pi_0(Diff^+(S^{q+j-1})).$$

For the composite of λ with the Milnor–Kervaire homomorphism $\bar{\mu}$ (see above):

$$\bar{\mu} : \pi_0(Diff^+(S^{q-1})) \longrightarrow \pi_{q+N}(S^N)/\mathrm{Im}\ J\pi_q(SO_N),$$

the following formula can be shown to hold in the ring of stable homotopy groups of spheres (or in the cobordism ring Ω_*^{Fr} of framed manifolds) for all elements $a \in \pi_0(Diff^+(S^{N-1}))$, $b \in \pi_q(SO_N)$:

$$\begin{aligned} \bar{\mu}(\lambda(aba^{-1})) = \bar{\mu}(a) \circ J(b) \quad (\mathrm{mod} J\pi_{q+N}(SO_N)), \\ aba^{-1} \in \pi_q(Diff^+(S^{N-1})). \end{aligned} \tag{4.7}$$

In certain dimensions the structure of the ring

$$\Omega_*^{Fr} = \sum_{j \geq 0} \pi_{N+j}(S^N)$$

is such that the right–hand side of (4.7) turns out to be non–zero (for appropriate a, b); this occurs for instance when $q = 1, n = 8$ — see the table at the end of §7 of Chapter 3. It can by inferred from this that for such N the component of the identity of the space $Diff^+(S^{N-1})$ is not contractible to the subgroup SO_N (Novikov, in the early 1960s).[8]

The general classification theory of closed, simply–connected–manifolds of dimension $n \geq 5$, and then of the non–simply–connected ones, depends on certain important, although elementary, properties of maps

[8]Proved slightly later also by Milnor.

$$f : M_1^n \longrightarrow M_2^n$$

of degree one between closed manifolds. Since for such maps (of degree one) the homomorphism $f_* D^{-1} f^* D$ (where D is the Poincaré–duality isomorphism) is the identity map on the homology of M_1^n, that homology admits a standard decomposition

$$H_*(M_1) \cong (D^{-1} f^* D)(H_*(M_2)) \oplus \operatorname{Ker} f_*, \tag{4.8}$$

which is orthogonal with respect to the intersection of cycles (and analogously for the cohomology of M_1^n). The direct summand $\operatorname{Ker} f_*$ of the homology of M_1^n is closed under the Poincaré–duality operator, and in general behaves as if it itself represented the homology of some manifold. Thus if the kernel $\operatorname{Ker} f_*^{(\pi_j)} \subset \pi_j(M_1^n)$ of the induced map $f_*^{(\pi_j)}$ of the jth homotopy group is trivial for $j < k-1$, then for the kth such kernel the analogue of Hurewicz' theorem is valid:

$$\operatorname{Ker} f_*^{(\pi_k)} \cong \operatorname{Ker} f_*^{(H_k)}, \tag{4.9}$$

and the induced homomorphisms between the respective homotopy groups:

$$f_* : \pi_i(M_1^n) \longrightarrow \pi_i(M_2^n),$$

are isomorphisms for $i < k$. This result is true also for non–simply–connected manifolds provided the homology groups are considered as $\mathbb{Z}[\pi]$–modules where $\pi_1(M_1^n) \cong \pi_1(M_2^n) \cong \pi$. Under the same assumption as before, namely that the kernels of the induced homomorphisms of the π_i should be trivial for $i < k-1$, one has the isomorphism

$$\operatorname{Ker} f_*^{(H_k)} \cong \operatorname{Ker} f_*^{(\pi_k)} \otimes_{\mathbb{Z}[\pi]} \mathbb{Z} \tag{4.10}$$

(provided the induced homomorphism of the fundamental groups π_1 has trivial kernel).

The classification theory of manifolds of given homotopy type was developed by Novikov in the early 1960s, and of all homotopy types of closed smooth manifolds by Browder and Novikov (also in the early 1960s). In the course of solving these classification problems one encounters the situation where a homotopy type of manifolds is represented by a particular CW-complex X. In the most general context, as formulated by Browder, X need not be a manifold and may have geometric dimension different from the given n, and it is required only that there exist a fundamental class $[X^n] \in H_n(X^n; \mathbb{Z})$ such that the cap operation $a \longrightarrow a \cap [X^n]$ defines an appropriate Poincaré isomorphism from homology to cohomology; CW–complexes with this property are called *Poincaré complexes.* It turns out that a Poincaré complex X admits a unique stable "normal spherical bundle" ν_X which behaves like the usual stable normal bundle of a manifold, and in the case where X is a smooth (or PL-) manifold ν_X is just the stable homotopy class of its standard normal bundle. If there exists a homotopy equivalence $f : M^n \longrightarrow X$, where M^n is a manifold, then there must exist a vector bundle ξ over X which is homotopy equivalent to the stable normal spherical bundle ν_X (if $g : X \longrightarrow M^n$ is a homotopy inverse of f, then $\xi = g^*\nu_M$). Thus the classification problems above lead to the consideration of a Poincaré complex X together with a vector bundle ξ over X which is homotopy equivalent to the stable normal spherical bundle ν_X. However when X is a smooth closed manifold, the vector bundle ξ is homotopy equivalent to the ordinary normal bundle ν_X, and the problem of determining the latitude in such vector bundles ξ can be solved effectively (Novikov, early 1960s).

Assuming that there is a vector bundle ξ over a given Poincaré complex X, a degree one map $f : M^n \longrightarrow X$ is called *normal* if the normal bundle ν_M is equivalent (as a vector bundle) to $f^*\xi$, i.e there is a commutative diagram:

$$\begin{array}{ccc} \nu_M & \xrightarrow{\hat{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array} \tag{4.11}$$

In this situation the kernel Ker f_* behaves with respect to both homology and homotopy as in the case of parallelizable manifolds, which circumstance permits the use of a natural analogue of the Milnor–Kervaire technique to reduce the kernels of the induced maps of homotopy groups by means of Morse surgeries. For a given manifold (or, more generally, Poincaré complex) X^n and a given vector bundle ξ over X^n, one defines *normal bordism classes* of normal maps $f : M^n \longrightarrow X^n$, $f^*\xi = \nu_M$, covered by bundle maps $\hat{f} : \nu_M \longrightarrow \xi$, where ν_M is the stable normal bundle of $M^n \subset \mathbb{R}^{n+N}$, as follows: A *normal bordism* is a normal map g covered by a map $\hat{g}$ of bundles, as indicated:

$$\begin{array}{ccc} \nu_N & \xrightarrow{\hat{g}} & \xi \times \varepsilon^0 \\ \downarrow & & \downarrow \\ N^{n+1} & \xrightarrow{g} & X^n \times I \end{array} \tag{4.12}$$

where the manifold $N^{n+1} \subset \mathbb{R}^{n+N} \times I$, $I = [a, b]$, approaches the boundary of $\mathbb{R}^{n+N} \times I$ transversely, and itself has boundary

$$\partial N^{n+1} = M_a^n \cup M_b^n,$$

$$M_a^n \subset \mathbb{R}^{n+N} \times \{a\}, \quad M_b^n \subset \mathbb{R}^{n+N} \times \{b\},$$

on each component of which there is given a normal map of degree one. We denote the resulting set of bordism classes by $\mathcal{N}(X^n, \xi)$. It can be shown without difficulty that the set $\mathcal{N}(X^n, \xi)$ can be realized as the subset of those elements α of the homotopy group $\pi_{n+N}(T\xi)$ of the Thom space of the bundle ξ, homologous to the fundamental class $[X^n]$ (see above), i.e. such that

$$H(\alpha) = \phi[X^n] \in H_{n+N}(T\xi) \cong \mathbb{Z}, \tag{4.13}$$

where

$$H : \pi_j(T\xi) \longrightarrow H_j(T\xi; \mathbb{Z})$$

is the Hurewicz homomorphism, and $\phi : H_n(X^n; \mathbb{Z}) \longrightarrow H_{n+N}(T\xi; \mathbb{Z})$ is the Thom isomorphism. Cartan–Serre theory shows that the stable group $\pi_{n+N}(T\xi)$ has the form

$$\pi_{n+N}(T\xi) = \mathbb{Z} \oplus \bar{\mathcal{N}}(X^n, \xi) \tag{4.14}$$

where $\bar{\mathcal{N}}(X^n, \xi)$ is finite abelian; with respect to this decomposition the above normal bordism classes of normal maps of degree one are represented as follows:

$$\alpha \in \mathcal{N}(X^n, \xi), \quad \alpha = 1 + a, \quad a \in \bar{\mathcal{N}}(X^n, \xi). \tag{4.15}$$

In the situation where $X^n = M^n$ is a smooth manifold and ξ is its stable normal bundle ν_M ($M^n \subset \mathbb{R}^{n+N}$), the identity element 1 of $\pi_{n+N}(T\xi)$ corresponds to the pair of identity maps

$$\begin{array}{ccc} \nu_M & \xrightarrow{id} & \nu_M \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{id} & M^n \end{array}$$

and the following assertions are valid:

1. *Suppose $f_1 : M_1^n \longrightarrow M^n$, $f_2 : M_2^n \longrightarrow M^n$ are maps of degree one which together with the induced maps of normal bundles determine one and the same normal bordism class. If in addition each of the maps f_1, f_2 is a homotopy equivalence, then there exists a homotopy n-sphere $\Sigma^n \in bP^{n+1}$ such that*

$$M_1^n \cong \Sigma^n \# M_2^n. \tag{4.16}$$

It follows that for $n \geq 5$ the manifolds M_1^n and M_2^n are homeomorphic, with the punctured manifolds obtained by removing a point from each actually diffeomorphic, and that for even $n > 5$, M_1^n and M_2^n are in fact diffeomorphic (Novikov).

2. *The following result applies to an arbitrary fixed vector bundle ξ over M^n: For odd $n \geq 5$ each normal bordism class $\alpha \in \mathcal{N}(M^n, \xi)$ is realized by some homotopy equivalence $f : M' \longrightarrow M^n$. The following condition on the Pontryagin classes of the vector bundle ξ:*

$$L(p_1(\xi), \ldots, p_k(\xi)) = \tau(M^n)$$

is both necessary and sufficient for this to be true also in the case $n = 4k$ (Browder, Novikov).

From the first result one obtains immediately the corollary that for each given homotopy type there exist only finitely many pairwise non–diffeomorphic closed, simply–connected n–manifolds ($n \geq 5$) with the same Pontryagin classes $p_j \in H^{4j}(M^n; \mathbb{Q})$. Invoking the topological invariance of the Pontryagin classes, one infers from this in turn the finiteness of the number of distinct smooth structures that exist on each closed, simply–connected topological manifold of dimension $n \geq 5$. Since this theory is valid also for PL–manifolds, one obtains also the finiteness of the number of distinct PL–structures definable on any such topological manifold.

Remark. Note in this connexion that in the analogue of the first result for PL–manifolds, one obtains an incidental strengthening of the conclusion, namely to the effect that the manifolds M_1^n and M_2^n will be in fact PL–homeomorphic, in view of the fact that all homotopy n–spheres are PL–homeomorphic to S^n. □

An exact classification of the diffeomorphism classes of manifolds satisfying the hypotheses of Result 1 above (i.e. of the same homotopy class, and with maps to M^n whose extensions to the stable normal bundles define the same normal bordism class) is obtained by taking the set $\mathcal{N}(M^n, \nu_n)$ of elements of the form $1 + a \in \pi_{N+n}(T\nu_n)$ modulo the subgroup $\mathcal{G}(M^n, \nu_n)$ of those elements representing homotopy classes of automorphisms of degree one, i.e. of self–maps $f : M^n \longrightarrow M^n$ of degree one, preserving the stable normal bundle: $f^*\nu_n = \nu_n$. Thus the diffeomorphism classes of manifolds satisfying the hypotheses of the above Result 1 are in natural one–to–one correspondence with the cosets of

$$\mathcal{N}(M^n, \nu_n)/\mathcal{G}(M^n, \nu_n) \tag{4.17}$$

There is a smaller group $\mathcal{J}(M^n, \nu_n)$, a subgroup of the automorphism group $\mathcal{G}(M^n, \nu_n)$, consisting of those automorphisms of the bundle ν_n inducing the identity map on the base M^n. The orbit set

$$\mathcal{N}(M^n, \nu_n)/\mathcal{J}(M^n, \nu_n) \tag{4.18}$$

is relevant to the following question: Given a normal homotopy equivalence $f : M_1^n \longrightarrow M^n$ of degree one, when can f be deformed to a diffeomorpism (more precisely, to a diffeomorphism up to forming the connected sum with a Milnor sphere Σ^n from the group bP^{n+1})? The answer is that a normal bordism defined by such a map f is so deformable if and only if its projection to the coset (4.18) is the same as the projection of the distinguished element (represented by the identity map $M^n \longrightarrow M^n$). Futhermore if $f_1 : M_1^n \longrightarrow M^n$ and $f_2 : M_2^n \longrightarrow M^n$ are normal maps of degree one (representing normal bordism classes), then they correspond to the same coset in (4.18) if and only if $f_1 f_2^{-1}$ is homotopic to a diffeomorphism (modulo bP^{n+1}).

In the situation where the given vector bundle ξ is the stable normal bundle of $M^n \subset S^{N+n}$, $\xi = \nu_n$, the Thom space $T\nu_n$ has particular geometric significance: For a suitably small ε, let U_ε be the ε–neighbourhood of the manifold $M^n \subset S^{N+n}$; then by definition we have

$$T\nu_M = S^{N+n}/(S^{N+n}\backslash U_\varepsilon),$$

the quotient space of S^{N+n} obtained by identifying to a point the complement of the neighbourhood U_ε (here $N >> n$). The associated natural map $S^{N+n} \longrightarrow T\nu_n$ thus restricts to the identity map on U_ε; it has degree one and represents (in $\pi_{n+N}(T\nu_n)$) the manifold M^n. For n of the form $n = 4k+2$ it turns out that the elements of $\pi_{n+N}(T\nu_n)$ of the form $1 + a$ where a ranges over a certain subgroup $\mathcal{B}$ of $\mathcal{N}(M^n, \nu_n)$ of index 1 or 2, are precisely those corresponding to n–manifolds of the same homotopy type as M^n. (The spaces $T\nu_n$ are often useful for carrying out calculations for particular manifolds.)

The above–described theory has been generalized to the situation of manifolds–with–boundary by Golo and Wall (in the mid–1960s).

Around the mid–1960s several authors[9] demonstrated the usefulness of the concept of "S–duality", which has a more general categorical character. Spanier and G.W. Whitehead defined the *S-dual* DK of a CW–complex K embedded in a sphere S^m of sufficiently high dimension, to be simply the complement $S^m\backslash K$; thus the operator D acts on CW–complexes K as follows:

$$D : K \mapsto S^m\backslash K.$$

In particular the S–dual of a sphere is a sphere (up to homotopy equivalance). On (homotopy classes of) maps $f : K \longrightarrow L$ the S–dual operator reverses the arrows (this is clear if f is an embedding, and the general case is reduced to this by means of the mapping cylinder):

$$Df : DL \longrightarrow DK,$$

$$[K, L]_s = [DL, DK]_s,$$

where $[\ ,\ ]_s$ denotes the set of stable homotopy classes of maps. It was observed by Atiyah in the early 1960s that the S–dual of the Thom space of the stable normal bundle of a closed manifold M^n is the Thom space of a trivial vector bundle ε^q over M^n:

$$D(T\nu_M) = \Sigma(M^n_+) = T\varepsilon^q,$$

where $M^n_+ = M^n \cup \{*\}$ (the disjoint union with the one–point space). The map $f : S^{N+n} \longrightarrow T\nu_M$ turns out to have as S–dual the map $Df : T\varepsilon^q \longrightarrow S^q$ from the Thom space of the trivial vector bundle ε^q to S^q, of degree one on each fibre, varying with the point of the base as parameter.

[9] A. Schwarz, Novikov, Sullivan.

Denote by SG_q the semigroup of maps $S^q \longrightarrow S^q$ of degree one. We have, corresponding to each normal bordism class $\alpha \in \mathcal{N}(M^n, \xi)$, a map from the base manifold M^n to the semigroup SG_q:

$$\tilde{Df} : M^n \longrightarrow SG_q, \quad (q \longrightarrow \infty).$$

The image under this map in the quotient SG_q/SO then constitutes an obstruction to the existence of a diffeomorphism between the various manifolds M_1^n in the class α. Note that as $q \longrightarrow \infty$, the homotopy properties of SG_q (taking account of its structure as an H-space) stabilize. The natural embeddings give rise to a spectrum

$$SG_q \subset SG_{q+1} \subset \cdots$$

with limit SG, say. We also have embeddings

$$SG \supset SPL \supset SO,$$

where SPL denotes the group of germs of origin-fixing and orientation-preserving PL-automorphisms of a disc of sufficiently high dimension. The induced monomorphism $\pi_j(SO) \longrightarrow \pi_j(SG)$ of homotopy groups coincides with the Whitehead homomorphism J (see §3 above) once one makes the identification afforded by the isomorphism

$$\pi_j(SG) = \pi_{N+j}(S^N), \quad N > j+1.$$

One gives meaning to the universal base BSG for spherical bundles (i.e. with spherical fibres) by defining equivalence of such bundles in terms of bundle maps commuting with the projections, and of degree one on the fibres. Although the space SG_q clearly has the homotopy type of $\Omega^q(S^q)$, the connected component of the identity of the q-dimensional loop space, so that

$$\pi_j(\Omega^q(S^q)) \cong \pi_{q+j}(S^q) \cong \pi_j(SG),$$

the multiplicative structures of these two spaces are different. Note that the space BSG does not have the homotopy type of any $(q-1)$-loop space $\Omega^{q-1}(Y)$.

This theory transfers to the context of PL-manifolds without significant change. The tangent and stable normal bundles of a manifold, equipped with the group of germs of origin-fixing PL-automorphisms of discs of unspecified dimensions, are called, following Milnor, *microbundles*. A theorem proved by several authors in the early 1960s asserts that a PL-manifold admits a compatible smooth structure if and only if the stable normal (or tangent) microbundle reduces to a vector O-bundle, $O \subset PL$ (or vector SO-bundle, $SO \subset SPL$, in the orientable case). The relative homotopy groups $\pi_j(SPL, SO)$, representing obstructions to the reducibility of a PL-microbundle to a smooth one, are isomorphic to the respective groups Γ^{j-1}:

$$\pi_j(SPL, SO) = \pi_j(SPL/SO) = \Gamma^{j-1}. \tag{4.19}$$

From this isomorphism one obtains the obstructions to the existence of a smoothness stucture in a neighbourhood of the j–dimensional skeleton of a PL–manifold M as the cohomology classes in $H^j(M;\Gamma^j)$. The theory of PL-manifolds was developed starting from the 1930s by many authors (Newman, Whitehead, Zeeman, Munkres, Mazur, Hirsch and others).

On a small open region of a PL–manifold where the tangent bundle is trivial, a "local" smooth structure can always be defined, and it is this fact which allows the natural transference to the context of PL–manifolds of the entire above–described technical apparatus of the classification theory of smooth manifolds of a given homotopy type, since the Morse surgeries eliminating the *kernels of the appropriate homotopy groups* (see above) *are carried out over* small parallelizable regions.

However one needs to take into account the fact that in general a PL–vector bundle has more automorphisms acting identically on the base than a smooth (or SO-) bundle. Hence in the approach via S–duality the obstructions to the existence of PL–isomorphisms between manifolds are represented by homotopy classes of maps

$$M^n \longrightarrow SG/SPL,$$

rather than maps to SG/SO. In 1966 it was shown by Sullivan and Wagoner that the groups $\pi_j(SG/SPL)$ (or the isomorphic groups $\pi_j(G/PL)$) have simple structure and obey a nice 4–periodicity:

$$\pi_j(SG/SPL) = \pi_j(G/PL) = \begin{cases} \mathbb{Z} & j = 4k \\ 0 & j = 4k+1 \\ \mathbb{Z}/2 & j = 4k+2 \\ 0 & j = 4k+3. \end{cases}$$

Subsequently, using the homotopy structure of the space SG/SPL,[10] and its connexion with Bott periodicity, together with other properties, Sullivan has been able to establish certain important general facts, however up to the present (1994) complete proofs of Sullivan's results have not appeared in print.

As we have mentioned, the homotopy groups $\pi_j(SPL/SO)$ are isomorphic to the groups Γ^{j-1}. The results above imply that that the inclusion homomorphism

$$\pi_j(SO) \longrightarrow \pi_j(SPL)$$

is in fact an isomorphism for $j \leq 6$. For $j = 7$ one has that the image of $\pi_7(SO)$ in $\pi_7(SPL)$ has order divisible by 7. The latter result may be established without difficuty by going over to the inclusion $BSO \longrightarrow BSPL$ of the bases of the universal SO– and SPL–bundles, and using the fact that the Pontryagin class of a PL–manifold may be fractional with denominator 7 (see §3). From this divisibility result one may infer the following corollary:

[10]The homotopy properties of the spaces G/PL, SG/SPL are completely described in the book by Madsen, Milgram, *Classifying Spaces in Surgery and Cobordism of Manifolds, Princeton University Press, 1979, Annals of mathematics studies, no. 92.*

There exists a CW-complex M (a smooth manifold) having 7-torsion in its cohomology group $H^8(M;\mathbb{Z})$, and two SO-vector bundles η_1, η_2 over M, such that

(i) $p_2(\eta_1) \neq p_2(\eta_2)$, *and*

(ii) *the vector bundles η_1, η_2 are PL-isomorphic.*

This result implies that the torsion parts of integer Pontryagin classes are not in general PL-invariant (Milnor–Kervaire, early 1960s).

As early as the 1940s it had been established (by Whitney) that provided $n \neq 2$, under suitable, simple conditions an immersion $S^n \longrightarrow M^{2n}$ can be regularly deformed into a smooth embedding. This fails in the exceptional case $n = 2$; this exceptional case underlies the cardinal technical difficulties of the theory of simply-connected 4-dimensional manifolds. From the end of the 1950s throughout the 1960s various topologists (Haefliger, Stallings, Levine, and others) developed a theory of embeddings of simply-connected manifolds (starting with homotopy spheres) into Euclidean spaces, using, among other things the entire apparatus described above for classifying manifolds. We shall not however expound the results of this theory here; a separate essay in the present series is to be devoted to this topic, and also to the deep theory of 3-manifolds and knots in $\mathbb{R}^3$.

§5. The role of the fundamental group in topology. Manifolds of low dimension ($n = 2, 3$). Knots. The boundary of an open manifold. The topological invariance of the rational Pontryagin classes. The classification theory of non-simply-connected manifolds of dimension ≥ 5. Higher signatures. Hermitian K-theory. Geometric topology: the construction of non-smooth homeomorphisms. Milnor's example. The annulus conjecture. Topological and PL-structures

The fundamental group plays a singularly important role in topology; it is involved in all of the technical apparatus of the subject, and likewise in all applications of topological methods. In fact for low-dimensional manifolds (i.e. of dimension 2 or 3) the fundamental group underlies essentially all non-trivial topological facts. For instance, as the reader will recall, the classification of the self-homeomorphisms of a closed surface M^2, its homotopy and isotopy classes (i.e. classes of homotopic homeomorphisms), all reduce, according to Nielsen, to consideration of the automorphism group of $\pi_1(M^2)$ taken modulo the subgroup of inner automorphisms, and so ultimately to composites of "elementary" automorphisms. For the orientable surface M_g^2 of genus g, the automorphism group of $\pi_1(M_g^2)$, presented as usual in terms of $2g$ generators and a single relation by

$$a_1, b_1, \ldots, a_g, b_g; \quad \prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1,$$

is generated by the following *elementary automorphisms* (Dehn, 1920s):

$$\begin{aligned}
\alpha_i: &\quad b_i \mapsto b_i a_i; \quad b_k \mapsto b_k, \quad k \neq i, \quad a_q \mapsto a_q; \quad 1 \leq i \leq g; \\
\beta_i: &\quad a_i \mapsto a_i b_i; \quad a_k \mapsto a_k, \quad k \neq i, \quad b_q \mapsto b_q; \quad 1 \leq i \leq g; \\
\gamma_i: &\quad b_i \mapsto a_{i+1}^{-1} b_i a_i; \\
&\quad b_{i+1} \mapsto b_{i+1} b_i a_i^{-1} b_i^{-1} a_{i+1}; \quad b_q \mapsto b_q, \quad q \neq i, i+1; \qquad (5.1) \\
&\quad a_{i+1} \mapsto a_{i+1}^{-1} b_i a_i b_i^{-1} a_{i+1} b_i a_i^{-1} b_i^{-1} a_{i+1}; \\
&\quad a_k \mapsto a_k, \quad k \neq i+1, \quad 1 \leq i \leq g-1; \\
\delta: &\quad a_j \mapsto b_{g+1-j}; \quad b_j \mapsto a_{g+1-j}.
\end{aligned}$$

Note that $\alpha_i, \beta_i, \gamma_i$ preserve orientation, while δ reverses it.

The fundamental group also determines the theory of knots, i.e. of embeddings $S^1 \hookrightarrow S^3$; the *knot group* determined by such an embedding is defined as $\pi_1(S^3 \backslash S^1)$. This group has a distinguished element a, namely the class of the loop S^1_ε obtained by moving the embedded circle S^1 a small distance ε in a direction normal to $S^1 \subset S^3$ at each point, in such a way that the resulting circle S^1_ε has zero linking coefficient with the original knot $S^1 \subset S^3$. The boundary of the tubular ε–neighbourhood of the knot $S^1 \subset S^3$ is a torus T^2, and it can be shown that for non–trivial knots the inclusion

$$T^2 \hookrightarrow (S^3 \backslash S^1)$$

determines in canonical fashion a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with canonically defined generators (one of which is the above–defined element a). It is likely that the knot group together with this canonically defined subgroup ($\cong \mathbb{Z} \oplus \mathbb{Z}$) with distinguished basis, affords a complete invariant of the knot. In particular the group of a knot is isomorphic to $\mathbb{Z}$ if and only if the knot is trivial, i.e. the embedding $S^1 \hookrightarrow S^3$ is isotopic (i.e. deformable by means of self–diffeomorphisms of S^3) to the trivial embedding (Papakyriakopoulos' theorem, proved in the late 1950s).

However difficulties of a group–theoretical kind prevent one from obtaining by these means an algorithm for determining whether or not a given knot is trivial. Such an algorithm was constructed by Haken in the early 1960s using different ideas, however with the drawback that it is an algorithm only "in principle", in the sense that the vast number of steps in the algorithm rule it out for practical purposes.

There are various other knot–invariants in use (the most practically useful of which is the "Alexander polynomial") all ultimately determined by the group of the knot. It is convenient for some purposes to consider invariants occuring in the $\mathbb{Z}[\mathbb{Z}]$–module homology (see Chapter 3, §5) of the canonical $\mathbb{Z}$–covering $\widehat{U} \longrightarrow U$ of the knot complement $S^3 \backslash S^1 = U$.

There is a class of relatively simple knots for which the knot complement U is a fibre bundle over S^1 with fibre a punctured suface of genus g (the *genus* of the knot):

$$U \longrightarrow S^1, \quad F \cong M_g^2 \backslash \{*\}.$$

(The *genus* of an arbitrary knot is the least genus of a surface in U having the knot as boundary.) It can be shown in the case of a knot whose complement U is a fibre bundle (as above), that the knot is defined by an algebraic equation in $\mathbb{C}^2 \supset S^3$:

$$f(z, w) = 0, \quad |z|^2 + |w|^2 = \varepsilon > 0,$$

where $f(z, w)$ is a polynomial satisfying $f_z = f_w = 0$ at $z = w = 0$; in general such an equation determines a link in S^3, consisting of several embedded circles, but often the solution set is connected and so represents a knot $S^1 \subset S^3$. In terms of f the fiber bundle U is given essentially by

$$(z, w) \mapsto \arg f = \phi, \quad f = \rho e^{i\phi}, \quad \rho^2 = \varepsilon.$$

By a theorem of Papakyriakopoulos (of the late 1950s), if M^3 is any manifold for which $\pi_2(M^3) \neq 0$, then there exists an embedded 2–sphere $S^2 \subset M^3$ not homotopic to zero. It follows that for the complement $S^3 \backslash S^1$ of a knot, one has $\pi_i(S^3 \backslash S^1) = 0$ for $i \geq 2$, so that all homotopy invariants are, in principle, determined by π_1.

Every finitely presented group (i.e. presentable by means of a finite number of generators and relations) can be realized by means of a standard construction as the fundamental group $\pi_1(K)$ of a finite CW–complex of dimension ≤ 2, or as the fundamental group $\pi_1(M)$ of a manifold M of dimension ≤ 4.

From results of P. S. Novikov, Adyan, and Rabin (of the early and mid–1950s) on the algorithmic undecidability of various questions in the theory of finitely presented groups (in particular the question of whether or not a given arbitrary presentation presents the trivial group) it is not difficult to infer the algorithmic undecidability of the question as to whether or not an arbitrary given finite CW–complex of dimension ≥ 2 is homotopically equivalent to some standard simply–connected CW–complex, and likewise whether or not an arbitrary given closed manifold of dimension ≥ 4 is homotopically equivalent to some standard simply–connected manifold. This observation was made by A. A. Markov (in the late 1950s), who also showed (this is more difficult) that it is also not algorithmically decidable for every manifold of dimension ≥ 4 whether or not it is homeomorphic (or PL–homeomorphic, or diffeomorphic) to a standard simply–connected manifold. Contractibility of a CW–complex is an algorithmically unrecognizable property in dimensions

≥ 3, and the question of whether or not a closed manifold of dimension $n \geq 5$ is homeomorphic to the n–sphere S^n is also undecidable; these undecidability results were established by S. P. Novikov in the early 1960s using homological algebra; it is likely that the latter question remains undecidable also for $n = 4$. Note that, on the other hand, the homeomorphism problem for finite 2–dimensional CW–complexes and the problem of deciding whether or not a given 3–manifold–with–boundary is homeomorphic to a contractible manifold, are in all likehood algorithmically soluble. The latter problem reduces, modulo calculation of the genus of the boundary of the 3–manifold–with–boundary, to that of recognizing the homotopy type of S^3. The simplest algorithmic problems of the theory of closed 3–manifolds and knots are, it would seem, also soluble, although, as the problem of recognizing the trivial knot shows (see above), notwithstanding the existence of an algorithm, a problem may still be intractable in practice. For the recognition problem for the sphere S^3 there is an algorithm solving it in the class of Heegaard diagrams of genus $g = 2$ (Birman, Hilden, in the late 1960s).[11]

It is possible that for some 3–dimensional problems there are efficient, sufficiently fast, algorithms, for instance for recognizing the trivial knot or the sphere S^3, in the sense that, although not always strictly applicable, they nonethetheless give a quick resolution of the problem "practically always"; such algorithms would for practical purposes be more useful than those applying always but only "in principle". It is of interest to mention in this regard a numerical experiment (recorded in the biophysical literature) carried out in connexion with an investigation of the properties of, for instance, substances possessing long, closed molecules ("catenated") in the configuration of knots and links (Frank–Kamenetskiĭ and others, in the 1970s); similar structures have been encountered also in other areas of physics. In conclusion we note that up to a large number of crossings knots can be effectively and uniquely recognized through their Alexander polynomials.[12]

The Poincaré conjecture to the effect that S^3 is the only simply–connected, closed 3–manifold, remains unproven. In the late 1950s Milnor showed that from the above–mentioned theorem asserting the existence of a homotopically

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[11] In his recent article *The solution to the recognition problem for S^3*, Haifa, Israel, May 1992, H. Rubinstein claims to have constructed an algorithm for recognizing the 3–dimensional sphere. The author of the present survey does not know if the arguments in this article have all been verified.

[12] In the late 1980s V. Jones discovered remarkable new polynomial invariants of knots, allowing the solution of certain classical problems of knot theory. Here the idea derives from representations of equivalence classes of knots by conjugacy classes in different groups equivalent via "Markov moves", and representations of braid groups arising in the Yang-Baxter theory of mathematical physics, yielding precise solutions of certain 2-dimensional models of statistical and quantum physics. Following Jones' work several topologists, algebraists, and mathematical physicists have developed the theory further. (See for example Kauffman, Louis H., *"Knots and physics"*. Singapore: World Scientific, 1994. Series on knots and everything or the lectures of Dror Bar-Natan *"Lectures on Vassiliev invariants"*). This theory has now been reformulated in the more modern setting of "topological quantum field theory", invented by A. Schwarz and Witten. (See also the Appendix.)

non–trivial, embedded 2–sphere $S^2 \subset M^3$ in manifolds M^3 with $\pi_2(M^3) \neq 0$, one can deduce the uniqueness, up to homotopy equivalence, of the decomposition of a 3–dimensional closed manifold as a connected sum (see §4 above) of 3–manifolds of the following three elementary types:

Type I: those with π_1 finite;

Type II: those with $\pi_2 = 0$;

Type III: $M^3 = S^2 \times S^1$.

Moreover if the Poincaré conjecture is true then this decomposition is topologically unique.

Various Type I manifolds can be constructed by factoring by the action of finite subgroups of SU_2 or SO_4 acting freely (i.e. in such a way that no non–trivial elements fixes any point) on S^3. Similarly, manifolds of Type II may be obtained as the orbit spaces of actions of discrete groups G of motions of 3–dimensional Lobachevskian space L^3, $G \subset O_{3,1}$, acting freely and with compact fundamental region. Over the period from the mid–1960s till the early 1980s, V. S. Makarov produced several infinite series of such manifolds; however these examples have not yet been studied in detail.[13] An extensive program of investigation of 3–dimensional manifolds undertaken by Thurston several years ago, is complete only for the rather narrow class of Haken manifolds. It has issued in the "Geometric Conjecture", which, if true, would have as corollary the following elegant result: If a closed 3-manifold M has $\pi_2(M) = 0$ and $\pi_1(M)$ without non-trivial abelian subgroups other than $\mathbb{Z}$, then M is homotopy equivalent to a compact 3-manifold of constant negative curvature, so that its fundamental group is isomorphic to a discrete group of isometries of Lobachevskian 3-space, with compact fundamental domain. Relevant to this program is the conjecture that if $\pi_1(M^3)$ is infinite and $\pi_2(M^3) = 0$, then the manifold M^3 is homotopy equivalent to a 3–manifold of constant negative curvature, and therefore obtainable as the orbit space of the action of a discrete group of motions of L^3.

We now turn to problems concerning the topology of higher–dimensional manifolds (of dimension $n \geq 5$). The problem of the topological invariance of the Pontryagin classes $p_k \in H^{4k}(M^n; \mathbb{Q})$ of rational or real cohomology, i.e. of the invariance of integrals of the classes p_k over cycles under homeomorphisms (assuming the Pontryagin classes represented as differential forms, perhaps expressed in terms of the curvature tensor relative to some Riemannian metric), would on the face of it seem to be not at all related to the fundamental group; in fact in solving the problem one may assume without

[13]In the mid-1980s Fomenko and Matveev constructed, with the help of a computer, nice families of closed 3-manifolds with constant (normalized) negative curvature, using Matveev's "complexity theory" for 3-manifolds. One of these manifolds turned out to have volume less than that of a certain 3-manifold conjectured by Thurston to be minimal.

loss of generality that the manifold M^n is simply-connected. Thus at first glance the (affirmative) solution of the problem (by Novikov in the mid-1960s) by resorting to the consideration of non-simply-connected toroidal regions and techniques specific to non-simply-connected manifolds, appears artificial. However no alternative proof has hitherto been found. Moreover the device of introducing toroidal regions in order to reduce problems concerning homeomorphisms to auxiliary problems concerning the smooth or PL-topology of non-simply-connected manifolds, has undergone substantial development in the first place by Kirby as a means for proving the so-called "Annulus Conjecture" (finally settled by Siebenmann, Hsiang, Shaneson, Wall and Casson in the late 1960s), and has subsequently been further refined to a tool for solving topological problems about purely topological manifolds and homeomorphisms between them (by Kirby, Siebenmann, Lashof and Rothenberg).

Apart from the signature formula, there exist for simply-connected manifolds essentially no other homotopy-invariant relations among the Pontryagin classes and numbers. (By theorems of Thom and Wu concerning the classes w_i and p_k respectively, the only possible homotopy invariants are certain characteristic numbers modulo 2 (for the classes w_i) and modulo 12 (for the classes p_k), and certain other characteristic numbers. The fact that the rational Pontryagin classes p_k are not in general homotopy-invariant is a relatively straightforward consequence (as was observed by Dold in the mid-1950s) of the Serre-Rohlin theorem on the finiteness of the groups $\pi_{n+3}(S^n)$ for $n \neq 4$. The simplest examples are afforded by the family of SO_3-vector bundles with base S^4 and fibre S^2, each determined by a class $p_1 \in H^4(S^4;\mathbb{Z})$ ($\cong \mathbb{Z}$) (or by an element of $\pi_3(SO_3) \cong \mathbb{Z}$). It can be inferred from the finiteness of the image under the homomorphism

$$J : \pi_3(SO_3) \longrightarrow \pi_6(S^3) \cong \mathbb{Z}_{12},$$

that the total spaces of these vector bundles fall into only finitely many homotopy types (determined by the class of the bundle to within "fiberwise homotopy equivalence", as defined by Dold). Since in fact there are infinitely many such bundles with different Pontryagin classes p_1, it follows that at best only $p_1 (\text{mod } 48)$ may be a homotopy invariant in the class of manifolds obtained as total spaces of these bundles. Theorems of Browder and Novikov (see §4 above) show that even for a particular simply-connected manifold there are no homotopy-invariant relations among the Pontryagin classes apart from the Hirzebruch-Thom-Rohlin formula for the signature.

Thom, Rohlin and Schwarz showed in the late 1950s that the rational Pontryagin classes p_k are invariant under PL-homeomorphisms, and on this basis were able to propose the following combinational definition of those classes. For each cycle $z \in H_{4k}(M^n;\mathbb{Q})$ realizable as a submanifold $M^{4k} \subset M^n$ with trivial normal bundle

$$M^{4k} \times \mathbb{R}^{n-4k} \hookrightarrow M^n,$$
$$i_*([M^{4k}]) = z, \tag{5.2}$$

we set (cf. the Hirzebruch formula (3.19))

$$\langle L_k(p_1, \ldots, p_k), z \rangle = \tau(M^{4k}). \tag{5.3}$$

General categorical properties may now be invoked to show that the values of $L_k(p_1, \ldots, p_k)$ at such cycles z determine its values on all of $H_{4k}(M^n; \mathbb{Q})$. Now from the form of the Hirzebruch polynomials, we have

$$L_k = \frac{2^{2k}(2^{2k-1} - 1)B_k}{(2k)!} p_k + \widetilde{L}_k(p_1, \ldots, p_{k-1}), \tag{5.4}$$

where B_k is the kth Bernoulli number, whence it follows that the class p_k may be expressed in terms of $L_1, \ldots, L_k$:

$$p_k = p_k(L_1, \ldots, L_k). \tag{5.5}$$

On the basis of this "signature" definition of the classes p_k one can give a combinatorial treatment of their properties. If M^n is a PL–manifold and $f : M^n \longrightarrow S^{n-4k}$ a simplicial map of sufficiently fine triangulations, then the complete inverse image $f^{-1}(x)$ of any interior point x of a simplex σ^{n-4k} of S^{n-4k} of maximal dimension, has the form

$$f^{-1}(x) = M^{4k}, \quad f^{-1}(\text{Int } \sigma^{n-4k}) = \text{Int } \sigma^{n-4k} \times M^{4k},$$

where M^{4k} is a PL–submanifold of M^n. We now define

$$\langle L_k(p_1, \ldots, p_k), z \rangle = \tau(M^{4k}),$$

where z is the element of $H_{4k}(M^n; \mathbb{Q})$ represented by the PL-submanifold $M^{4k} \subset M^n$, and thence we define the Pontryagin classes p_k as before, using the formula

$$p_k = p_k(L_1, \ldots, L_k).$$

As earlier this can be shown to determine p_k on $H_{4k}(M^n; \mathbb{Q})$, and so affords a PL–invariant definition of the rational Pontryagin classes p_k. We see that underlying this definition is the natural and simple analogue for PL–maps of the property of transversal regularity for smooth maps.

Note that this definition is non–local, by contrast with the smooth case, where the classes p_k are expressed to terms of the curvature tensor. The existence of a local PL–representation of the classes p_k has been proved (by I.M. Gel'fand, Losik and Gabrielov in the mid–1970s), but so far an effective (i.e. constructive) local definition is lacking.

We now sketch the proof of the topological invariance of the rational Pontryagin classes of a manifold M^n. To begin with, note that different smooth (or PL–) structures on a single continuous manifold M^n must, by definition of homeomorphism, all possess the same underlying collection of open sets of M^n (each of which is a smooth (or PL–) manifold with respect to each of the various superimposed smooth or PL–structures on M^n). The rational Pontryagin classes p_k of a manifold M^n, or, more precisely, their integrals over the individual cycles of M^n, turn out to be representable as homotopy invariants of certain open "toroidal" regions of M^n and their covering spaces, and from this one infers their topological invariance for the whole manifold M^n. (The above definition of the p_k via the signature is used.)

Thus, to begin, consider a fixed embedding

$$\mathbb{R} \times T^{n-4k-1} \hookrightarrow \mathbb{R}^{n-4k}$$

of a neighbourhood of a torus, and a corresponding "toroidal" region of M^n

$$U^n = M^{4k} \times T^{n-4k-1} \times \mathbb{R} \hookrightarrow M^n. \tag{5.6}$$

With respect to a smooth (or PL–) structure on M^n, this region need not be a smooth direct product; what is important here is that the region is homotopy equivalent to the manifold $M^{4k} \times T^{n-4k-1}$, and that the signature $\tau(M^{4k})$ may be expressed as a homotopy invariant of this region or of its covering spaces in terms of their cohomology rings. It may be assumed without loss of generality that the manifolds M^n, M^{4k} are simply–connected, and that $k > 1$, $n < 8k$. By using methods of differential topology it may be shown that for a manifold with free abelian fundamental group the following assertion is valid: If $W^m, m \geq 6$, is such a manifold (i.e. $\pi_1(W^m)$ is free abelian) on which there is defined a free continuous or smooth action of the group $\mathbb{Z}$ with the property that the orbit space $W^m/\mathbb{Z}$ is a closed manifold with the homotopy type of a fibre bundle $V^m \longrightarrow S^1$ with base S^1, then the manifold W^m is diffeomorphic to the direct product of a closed manifold and the real line:

$$W^m = N^{m-1} \times \mathbb{R}.$$

(In fact for this conclusion it suffices that the manifold W^m with free $\mathbb{Z}$-action, be homotopy-equivalent to a finite CW–complex, and that the orbit space $W^m/\mathbb{Z}$ be compact; moreover the condition on $\pi_1(W^m)$ may be weakened to the requirement that the Grothendieck group $K_0(\pi_1)$ (see below) be trivial: $K_0(\pi_1) = 0$.)

From this lemma the topological invariance of the rational Pontryagin classes is deduced as follows: We first apply the lemma to the region U^n (i.e. we take $W^m = U^n$, $m = n$) endowed with any smooth structure, taking $\mathbb{Z}$ to act in the natural way:

$$T : (x, t) \longrightarrow (x, t + 1),$$

where $x \in M^{4k} \times T^{n-4k-1}$, $t \in \mathbb{R}$. Since $\pi_1(M^{4k}) = 0$, we have that $\pi_1(U^n)$ is free abelian, so that by the above lemma there is a diffeomorphism

$$U^n \cong V^{n-1} \times \mathbb{R}, \tag{5.7}$$

where V^{n-1} is homotopy equivalent to $M^{4k} \times T^{n-4k-1}$. Since the direct decomposition (5.7) is smooth, we have for the Pontryagin classes

$$p_j(U^n) = p_j(V^{n-1}), \quad j = 1, 2, \ldots.$$

Consider the $\mathbb{Z}$–covering space of V^{n-1} determined by the $\mathbb{Z}$–covering of the torus:

$$T^{n-4k-2} \times \mathbb{R} \xrightarrow{\mathbb{Z}} T^{n-4k-1},$$

$$\hat{V}^{n-1} \xrightarrow{\mathbb{Z}} V^{n-1};$$

here $\hat{V}^{n-1}$ has the homotopy type of $M^{4k} \times T^{n-4k-2}$. Applying the procedure once again, this time with $\hat{V}^{n-1}$ in the role of W^m, we obtain

$$\begin{gathered} \hat{V}^{n-1} \cong V^{n-2} \times \mathbb{R}, \quad V^{n-2} \sim M^{4k} \times T^{n-4k-2}, \\ p_j(V^{n-2}) = p_j(V^{n-1}) = p_j(U^n), \quad j \geq 1. \end{gathered} \tag{5.8}$$

Iterating this argument we finally arive at

$$\begin{gathered} \hat{V}^{4k+1} \cong V^{4k} \times \mathbb{R}, \\ p_j(V^{4k}) = p_j(V^{4k+1}) = \cdots = p_j(U^n), \quad j \geq 1, \end{gathered} \tag{5.9}$$

where V^{4k} is a smooth, closed manifold, homotopy–equivalent to M^{4k}. The signature formula gives

$$\tau(M^{4k}) = L_k(M^{4k}) = L_k(V^{4k}) = \langle L_k(p_1, \ldots, p_k), [M^{4k}] \rangle,$$

where the Pontryagin classes are defined in terms of any smooth structure on the manifold U^n. From this the topological invariance of the rational Pontryagin classes quickly follows.

For $n \geq 8k$ the above argument yields a smooth embedding $V^{4k} \times \mathbb{R}^{n-4k} \hookrightarrow U^n$, endowed however with a possibly different smooth structure, whence one obtains a diffeomorphism

$$V^{4k} \times \mathbb{R}^{n-4k} \longrightarrow M^{4k} \times \mathbb{R}^{n-4k},$$

where the latter manifold is endowed with any smooth structure. Following soon after the appearance of the above proof, Siebenmann was able to show by means of a more careful analysis of the argument that this conclusion holds also in the non–stable dimensions $n < 8k$.

For non–simply–connected closed manifolds there are non–trivial homotopy–invariant relations between integrals of the rational Pontryagin classes over certain cycles; the simplest examples of such relations are as follows:

Example 1. Let $n = 4k+1$ and $z \in H_{4k}(M^n)$. Consider a $\mathbb{Z}$–covering space projection $p : \hat{M}^n \longrightarrow M^n$ such that $p_*\pi_1(\hat{M}_n)$ consists just of those cycles γ having zero intersection index with the cycle z:

$$p_*\pi_1(\hat{M}^n) \circ z = 0.$$

There is a cycle $\hat{z} \in H_{4k}(M^n)$ definable in homotopically invariant fashion,

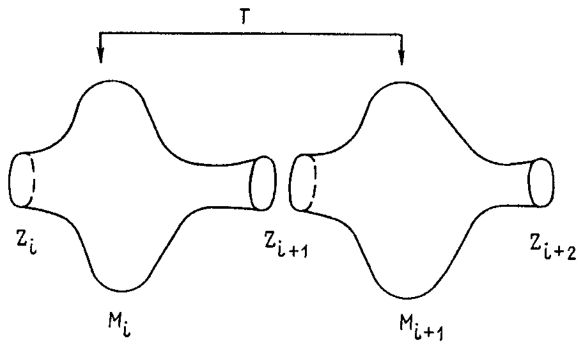


Fig. 4.28

such that $p_*\hat{z} = z$. In the cohomology ring $H^*(\hat{M}^n;\mathbb{Q})$ we have the bilinear form defined on $H^{2k}(\hat{M}^n;\mathbb{Q})$ by

$$\langle x,y\rangle = \langle xy,\hat{z}\rangle = \langle y,x\rangle, \tag{5.10}$$

which has finite–dimensional carrier (i.e. the subspace annihilating every element has finite codimension). Denoting by $\tau(\hat{z})$ the signature of this bilinear form, one has the following formula:

$$\langle L_k(p_1,\ldots,p_k),z\rangle = \tau(\hat{z}). \tag{5.11}$$

Example 2. If a smooth manifold V^{4k+q} has the homotopy type of a product of the form $M^{4k}\times T^q$:

$$V^{4k+q} \sim M^{4k}\times T^q,$$

then the following formula (due to Novikov in the mid–1960s) may be established by means of the technique used to prove the topological invariance of the classes $L_k(p_1,\ldots,p_k)$:

$$L_k(V^{4k+q}) = \tau(M^{4k}). \qquad \square$$

The general conjecture concerning the "higher signatures" consists in the following: As was shown already by Hopf (in the early 1940s) every CW–complex K has a distinguished set of cohomology classes determined completely by $\pi_1(K)$, namely the image under the induced homomorphism

$f^* : H^*(K(\pi, 1)) \longrightarrow H^*(K)$ (with any coefficients) of the canonical map $f : K \longrightarrow K(\pi, 1)$. (Hopf described these in the 2–dimensional case.) An algebraic characterization of the homotopy types of the spaces $K(\pi, 1)$ and their cohomology $H^*(K(\pi, 1))$ (with respect to arbitrary coefficients, including $\mathbb{Z}[\pi]$–modules) was given by Eilenberg and MacLane in the mid–1940s; they then defined the *cohomology* $H^*(\pi)$ *of the group* π, as being given by $H^*(K(\pi, 1))$. (Slightly later, but independently, D. K. Faddeev also gave definitions of these algebraic objects, motivated by considerations from algebraic number theory.) In fact this concept lay at the base of the homological algebra extended by Cartan, Eilenberg, Serre, Grothendieck and others to the context of modules and more general abelian categories. (One such generalization — to the category of A–modules –was used in §6, 7, 9 of Chapter 3 above, as part of the algebraic apparatus of stable homotopy theory.)

Of course one has the corresponding class of cycles (of a closed manifold M^n) given by

$$Df^* H^*(\pi; \mathbb{Q}) \subset H_*(M^n; \mathbb{Q}).$$

The "conjecture concerning the higher signatures" is then that for each cycle z in $Df^* H^*(\pi; \mathbb{Q})$ of dimension $4k$ the integral of the Pontryagin–Hirzebruch class over that cycle:

$$\langle L_k(p_1, \ldots, p_k), z \rangle,$$

should be homotopically invariant. For $n = 4k + 1$ and arbitrary $\pi = \pi_1(M^{4k+1})$ the distinguished cycles $z \in H_{4k}(M^{4k+1}; \mathbb{Q})$ of codimension one are precisely those for which the formula (5.11) holds with respect to the bilinear form defined by (5.10), and intersections of such cycles also have this property. (In the case that π is free abelian all cycles are of this form.) The conjecture concerning higher signatures for such intersections of cycles of codimension one, was settled by Rohlin in the case of intersections of two cycles (in the second half of the 1960s), and then (at the end of the 1960s) by Kasparov, Hsiang and Farrell for intersections of any number of cycles:

$$z = D(y_1 \wedge \cdots \wedge y_k), \quad y_j \in H^1(M^n; \mathbb{Q}).$$

An analytic proof of this theorem using the theory of elliptic operators was obtained in the early 1970s by Lusztig, who also established the conjecture for certain cycles in the case where π is a discrete subgroup of the group of motions of a symmetric space of constant negative curvature. For abelian π he made use, for the first time, of a family of elliptic complexes associated with finite–dimensional representations (and their characters) of the group; for non–abelian π he was able to exploit infinite–dimensional "Fredholm" representations $\rho_i : \pi \longrightarrow Aut\, \mathcal{H}_i$ of π in rings of unitary operators on Hilbert spaces $\mathcal{H}_i$. (A pair of unitary representations ρ_1, ρ_2, together with a Noetherian operator $F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, is called a *Fredholm representation* if $F\rho_1 - \rho_2 F$ is a compact operator; cf. end of Chapter 1.) It turns out that, by exploiting analogues of formulae of Atiyah–Hirzebruch–Singer type for the index of elliptic operators with coefficients in a Fredholm representation, constructed using the geometry of compact manifolds of positive curvature to define quantities like "the signature with coefficients from a representation", the higher-signatures conjecture can be fully established for all groups π realizable as discrete groups of motions, with compact fundamental region, of symmetric spaces (Mishchenko, early 1970s). This result was generalized by Solov'ev and others in the 1970s; it now appears that these methods have evolved to the extent of yielding a proof of the conjecture for all discrete subgroups of Lie groups (Kasparov, in the early 1980s). [14]

It is interesting to note in this connexion that the above mentioned "Fredholm representations" were first introduced by Atiyah in the late 1960s for the algebras $C^*(M^n)$ of complex-valued functions, in the algebra of bounded operators on Hilbert space; in this context a *Fredholm representation* is a triple $\rho_1, \rho_2, F : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, as defined above, with in addition $\rho_i(f) = \rho_i(f^+)$. Every pseudodifferential elliptic operator D of order $m = 0$ determines a Fredholm representation (ρ_1, D, ρ_2), where $\rho_i(f)$ is the operation of multiplication by f on the sections of bundles η_i:

$$D = F : \Gamma_s(\eta_1) \longrightarrow \Gamma_s(\eta_2), \quad m = 0.$$

Since $Df - fD$ is a compact operator, the triple (ρ_1, D, ρ_2) is indeed a Fredholm representation. It was shown by Kasparov, Douglas, and others, in the first half of the 1970s, that Fredholm representations of $C^*(X)$ for compact spaces X provide a basis for constructing by analytic means $K_*(X)$-theory as a homological theory. By these means Kasparov was also able to formulate an intersection theory for manifolds $(X = M^n)$.

Among the topological questions concerning simply-connected manifolds arising in connexion with the development of the techniques of the classification theory outlined in §4 above, the following one has come in for special attention: Under what conditions does a smooth open manifold $W^m, m \geq 6$, admit a simply-connected boundary, i.e. under what conditions is W^m realizable as the interior of a smooth manifold-with-boundary $\bar{W}^m$ with simply-connected boundary $\partial\bar{W}^m = V^{m-1}$ $(\pi_1(V^{m-1}) = \{1\}$, $W^m = \text{Int } \bar{W}^m)$? If it is assumed that the purely continuous version of this problem, i.e. for the class of topological manifolds-with-boundary, is settled in some sense, then the problem as formulated can be shown to reduce to the construction of a diffeomorphism between a manifold M^n homeomorphic to the direct product of a certain closed *topological* manifold with the real line, and the product of

[14] In this connexion many deep results have been obtained over the last decade using both topological and analytical methods. In particular Novikov's conjecture has been established for the "hyperbolic groups" introduced by Gromov, and for certain other classes of groups, by Cohn, Pedersen, Gromov, Rosenberg and several others. Some beautiful applications of cobordism theory to the theory of manifolds with positive scalar curvature have been found and elaborated on by Gromov, Lawson, Kreck and Stolz. Here, in the non-simply-connected case, characteristic numbers analogous to the higher signatures, but with the L-genus replaced by the A-genus, play an important role.

a *smooth* closed manifold with the real line. If on the other hand the solution of the continuous version is not assumed beforehand, then the following conditions are imposed instead: the open manifold W^m should have only finitely many ends, and be simply-connected at infinity in the direction of each of these ends, and furthermore the homotopy type at infinity in the direction of every end should be finite. The theorem of Browder–Levine–Livesay asserts that under these assumptions (once made precise) the open manifold W^m is the interior of a compact manifold–with–boundary $\bar{W}^m$ with each component of its boundary simply–connected:

$$V^{m-1} = \bigcup_j V_j^{m-1},$$

$$\pi_1(V_j^{m-1}) = 0.$$

In certain cases the conditions at infinity on W^m may be replaced by global restrictions on W^m. Of especial interest among such restrictions are those of an algebraic character, for instance the condition that W^m have the homotopy type of a finite CW-complex, and admit a free action of a discrete group G of smooth (or continuous) transformations with compact fundamental region. It was precisely this sort of modification of the above problem concerning the boundary of an open manifold W^m (in the particular case $G = \mathbb{Z}$, $\pi_1(W^m)$ free abelian) that proved to be of technical importance in the proof of the topological invariance of the rational Pontryagin classes (see above).

It seems that at the same time as the Browder–Levine–Livesay theorem was being proved, Siebenmann was independently investigating the non–simply–connected version of that theorem; however, although publication of his results was delayed over a protracted period, it has emerged (if one goes by his work published in the late 1960s) that he considered only a restricted case of the problem, namely that of the representability of a smooth manifold in the form $V \times \mathbb{R}$.

In the non–simply–connected case an obstruction to the representability of a manifold W in the form $W = V \times \mathbb{R}$, where V is closed and of dimension ≥ 5, is given by an element of the Grothendieck group $K_0(\pi)$, $\pi = \pi_1(V)$, which by definition (cf. Chapter 3, §8) consists of the stable classes of finite–dimensional, projective $\mathbb{Z}[\pi]$–modules η_i (i.e. direct summands of free modules), where η_1, η_2 are said to be *equivalent* if

$$\eta_1 \oplus N_1 \cong \eta_2 \oplus N_2$$

for some finite–dimensional free modules N_1, N_2. (For the ring $C^*(X)$ of functions on a compact space X, the Grothendieck group $K_0(C^*(X))$ consists of classes of modules of cross–sections of non–trivial vector bundles over X (the free modules corresponding in this case to the trivial bundles), and coincides with $K^0(X, *)$.) Taking into account the involutory operation in the group ring $\mathbb{Z}[\pi]$ defined by inversion in the group π, one can associate naturally with each $\mathbb{Z}[\pi]$–module η the dual $\eta^* = \mathrm{Hom}_A(\eta, A)$, $A = \mathbb{Z}[\pi]$, which is projective if η is projective. Taking the dual defines an involution on $K_0(\pi)$:

$$\eta \mapsto \eta^*, \quad \eta \in K_0(\pi), \quad \eta^{**} = \eta.$$

If W is known to have the homotopy type of a finite CW–complex, then it can be shown that

$$\eta + (-1)^n \eta^* = 0$$

in the group $K_0(\pi)$, $\pi = \pi_1(W)$.

At around the same time Wall found an obstruction to homotopy finiteness of CW–complexes L (under the assumption that L is embeddable as a retract in some homotopically finite CW–complex X, i.e. that there exists a map $f : X \longrightarrow L$, $L \subset X$, such that $f|_L = 1_L$). This obstruction is also given by an element of $K_0(\pi)$, $\pi = \pi_1(L)$. It turns out that for a manifold W, Wall's obstruction to homotopy finiteness of W is just $\eta \pm \eta^*$ where η is the Novikov–Siebenmann obstruction to the existence of a smooth direct decomposition $W = V \times \mathbb{R}$. In the case $\pi \cong \mathbb{Z}/p$, p prime, the group $K_0[\pi]$ may be interpreted as the group of Kummer classes of ideals of the algebraic number field $\mathbb{Q}[\sqrt{p}]$, and has been computed in detail, along with the duality operator $*$, in the context of algebraic number theory. Using these results, Golo, in the second half of the 1960s, constructed a series of examples of specific open manifolds W not decomposable in the form $V \times \mathbb{R}$.

The general problem of the realizability of a given open manifold W as the interior of a compact manifold–with–boundary (where the boundary is no longer necessarily simply–connected) was considered by Brakhman (in the early 1970s) under the assumption that W is a regular covering space of some closed manifold. The special case where the manifold W is homotopically equivalent to a manifold of the form $M^q \times T^{n-q}$ ($n \geq 5$), with $\pi_1(M^q)$ free abelian, was considered earlier. (If M^q is a torus, then of course the universal covering space of this manifold is $\mathbb{R}^n$.)

As indicated earlier, for closed manifolds V^n with the homotopy type of a torus, $V^n \sim T^n$, the Pontryagin classes $p_i(V^n)$ are all trivial. By using Adams' theorem (of the early 1960s) on the injectivity of the Whitehead homomorphism

$$J : \pi_j(SO) \longrightarrow \pi_{N+j}(S^N), \quad N > j + 1,$$

for $j \neq 4k - 1$, together with the homotopy equivalence of the suspension ΣT^n with a bouquet of spheres, it can be shown that all homotopy tori are stably parallelizable (Novikov, in the mid-1960s). Further applications of homotopy tori will be described below.

Recall (from Chapter 3, §5) that in connexion with the concept of "simple homotopy type", as defined by Whitehead, elements of $K_1(\pi)$ arise as obstructions to simple homotopy equivalence between CW–complexes known to be homotopy equivalent in the ordinary sense. For manifolds this obstruction (actually in the Whitehead group $K_1(\pi)/\pm\pi$) arises in connexion with the generalization to the non–simply–connected case of Smale's theorem on the triviality of h–cobordisms between simply–connected manifolds of dimension

≥ 5. An h–cobordism W^{n+1} between non–simply–connected manifolds may be non–trivial even for $n \geq 5$; even if W^{n+1} is contractible onto one of its boundary components ($\pi_j(W^{n+1}, V^n) = 0$ for $j \geq 0$), there may nonetheless be an obstruction to triviality, given by an element

$$\alpha(W^{n+1}, V^n) \in Wh(\pi) = K_1(\pi)/\pm\pi. \tag{5.12}$$

In fact for each $n \geq 5$ every element α of $Wh(\pi)$ is realizable as such an obstruction, and the vanishing of the obstruction $\alpha(W^{n+1}, V^n)$ is necessary and sufficient for triviality of the h-cobordism: $W^{n+1} = V^n \times I$ (Mazur, first half of the 1960s; the proof was brought to completion by Barden and Stallings in the mid–1960s).

The appropriate algebraic definition of K_2 for associative rings with an identity element was discovered by Milnor and Steinberg in the second half of the 1960s, and the higher analogues K_j were constructed around the turn of that decade by several authors (Quillen, Volodin, Gersten, Karoubi, Villamayor); the equivalence of the various definitions proposed was established somewhat later. The topological realizations of these groups (as considered by Wagoner), especially of K_2, are of great interest; however we shall not pursue the topic here, not least because this line of investigation remains far from complete.

It is noteworthy that for the Laurent extension $A[t, t^{-1}]$ of a ring A, besides the obvious projectors

$$K_0(A[t, t^{-1}]) \rightleftharpoons K_0(A),$$

$$K_1(A[t, t^{-1}]) \rightleftharpoons K_1(A),$$

determined by the natural ring epimorphism $A[t, t^{-1}] \longrightarrow A$ and the inclusion $A \longrightarrow A[t, t^{-1}]$, there is a non-trivial projector B (the *Bass projector*, discovered in the mid–1960s):

$$K_0(A) \xrightarrow{B} K_1(A[t, t^{-1}])$$

$$K_0(A) \xleftarrow{\bar{B}} K_1(A[t, t^{-1}]),$$

satisfying $\bar{B}B = 1$, with the kernel of $\bar{B}$ given explicitly. The construction of this projector is as follows: If η is a projective A-module, then $\eta + \bar{\eta}$ is a free module for some $\bar{\eta}$. The infinite direct sum

$$M = \sum_{n > -\infty}^{n < +\infty} t^n(\eta + \bar{\eta}) = \sum_n (\eta_n + \bar{\eta}_n)$$

is then a free $A[t, t^{-1}]$–module. We define $B(\eta)$ as the automorphism of the A–module M commuting with t, t^{-1}, given by

$$B(\eta) : \begin{cases} \eta_n \longrightarrow \eta_{n+1}, \\ \bar{\eta}_n \longrightarrow \bar{\eta}_n. \end{cases} \tag{5.13}$$

The definition of $\bar{B}$ is as follows: Each automorphism λ of the free $A[t, t^{-1}]$-module F_N of rank N is, after being multiplied by a sufficiently large power n of t, representable by a matrix over $A[t]$ relative to some basis $e_1, \dots, e_N$ of F_N. Denoting by F_N^+ the "positive part" of F_N:

$$F_N^+ = \left\{ \sum a_j e_j \mid a_j \in A[t] \right\},$$

we set

$$\bar{B}(\lambda) = F_N^+ / t^n \lambda = \eta. \tag{5.14}$$

It is straightforward to verify the projectivity of the module η, and then that $\bar{B}B = 1$ on $K_0(A)$.

It follows that one always has

$$K_1(A[t, t^{-1}]) \cong K_0(A) \oplus K_1(A) \oplus P. \tag{5.15}$$

In the case $A = \mathbb{Z}[\pi]$ with π free abelian, Bass showed that $P = 0$. Since for such π ($\cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$) every finitely-generated $\mathbb{Z}[\pi]$-module has a free acyclic resolution of finite length, and $K_0(\pi) = 0$, one obtains by induction Bass' theorem for free abelian π:

$$Wh(\pi) = K_1(\pi) / \pm \pi = 0.$$

It follows that in the situation of free abelian fundamental group the above mentioned invariant of non-simply-connected manifolds is trivial.

A natural continuation of this sort of problem is that of representing a manifold W^n as a smooth fiber bundle over the circle S^1 ($W^n \longrightarrow S^1$), on the assumption that the given manifold W^n has the homotopy type of such a bundle. If $\pi_1(W^n) \cong \mathbb{Z}$ and $n \geq 6$, and the fiber of the fibration is simply-connected, then the problem has an affirmative solution (Browder, Levine, Livesay, in the mid-1960s). In the more general situation, where the fiber is not simply-connected, obstructions arise involving both $K_0(\pi)$ and $K_1(\pi)$, where π is the fundamental group of the fiber given as the kernel Ker f_* of the homomorphism $f_* : \pi \longrightarrow \mathbb{Z}$, induced from the appropriate map $f : W^n \longrightarrow S^1$, a candidate for being a smooth projection. (The solution of the problem was obtained in the second half of the 1960s by Hsiang and Farrell.)

We turn now to the general classification theory of non-simply-connected manifolds of dimension $n \geq 5$. This theory follows closely the Browder-Novikov scheme for classifying the simply-connected manifolds of dimension $n \geq 5$ (see §4 above); however rather than yielding a final classification theorem, the theory reduces the problem to the existence of certain algebraic obstructions, which we shall now describe. All of the Browder-Novikov apparatus (described in §4) of normal maps of degree one of closed manifolds, and of bordisms between them, carries over to the non-simply-connected case. Invoking the non-simply-connected analogue of Hurewicz' theorem, concerning the kernels of the homomorphisms induced from normal maps $f : M_1 \longrightarrow M$ of degree one, one carries out Morse surgeries transforming such a map into one inducing isomorphisms between the respective homotopy groups up to and including dimension $[n/2] - 1$. For even n $(= 2k)$ analysis of the homotopy kernel in dimension k (in terms of universal covering manifolds) reveals that this kernel is a finitely-generated free stable $\mathbb{Z}[\pi]$-module, on which there is defined a π-invariant scalar product given by the intersection index of cycles lifted onto the universal covering manifold:

$$\langle z, w \rangle = \sum_{\sigma \in \pi} (z \circ \sigma w) \in \mathbb{Z}[\pi]. \tag{5.16}$$

Thus this scalar product takes its values in $\mathbb{Z}[\pi]$, on which there is an involution $u \mapsto \bar{u}$, where

$$u = \sum_i a_i g_i, \quad a_i \in \mathbb{Z}, \quad g_i \in \pi, \quad \bar{u} = \sum_i a_i g_i^{-1}, \tag{5.17}$$

with the usual properties of an involution:

$$\overline{uv} = \bar{v}\bar{u}, \quad \bar{\bar{u}} = u. \tag{5.18}$$

Remark. Note that for non-orientable manifolds the definition of the involution needs to be adjusted as follows:

$$u \mapsto \bar{u} = \sum_i (sgn\ g_i) a_i g_i^{-1}, \tag{5.19}$$

where $sgn\ g_i = -1$ if the action of g_i is orientation-reversing and $sgn\ g_i = +1$ otherwise. □

The scalar product (5.16) is Hermitian for k even and co-Hermitian for k odd:

$$\begin{gathered} \langle uz, w \rangle = u\langle z, w \rangle, \quad \langle z, w \rangle = \pm\langle w, z \rangle, \\ \langle z, uw \rangle = \langle z, w \rangle \bar{u}, \end{gathered} \tag{5.20}$$

and is non-degenerate in the sense that relative to a free basis $\{e_i\}$ for the $\mathbb{Z}[\pi]$-module $\mathcal{M} = \operatorname{Ker} f_*^{(\pi_k)}$ $(n = 2k)$, the matrix

$$B = (b_{ij}) = (\langle e_i, e_j \rangle)$$

is invertible. (Recall that we are already in the situation where $\operatorname{Ker} f_*^{(\pi_j)} = 0$ for $j < k$.) In invariant form (i.e. expressed without reference to a basis) this non-degeneracy means that the scalar product $\langle\ ,\ \rangle$ determines an isomorphism

$$h : \mathcal{M} \longrightarrow \mathcal{M}^* = Hom_A(\mathcal{M}, A), \quad A = \mathbb{Z}[\pi],$$

given by

$$u \mapsto h(u), \quad \langle h(u), v \rangle = \langle u, v \rangle. \tag{5.21}$$

(Note that the above-mentioned involution allows $\mathcal{M}^*$ to be endowed with the structure of a $\mathbb{Z}[\pi]$-module.) The scalar product has the property of "evenness", which in matrix terminology (given that the module $\mathcal{M}$ is free) means that the above matrix B is expressible in the form:

$$B = V \oplus (-1)^k V^+,$$

where $V^+ = \overline{V}^T$, T denoting the operation of taking the transpose. For odd k there also exist $\mathbb{Z}/2$-invariants (analogous to the Arf invariant) which we shall not, however, describe here. If M^n is neither a smooth nor PL-manifold but only a CW-complex whose $\mathbb{Z}[\pi]$-homology and cohomology satisfy Poincaré duality, then the module $\mathcal{M} = \operatorname{Ker} f_*^{(\pi_k)}$ will be merely a projective $\mathbb{Z}[\pi]$-module rather than stably free. In this case one may proceed geometrically, adding more generators by attaching handles of index k in order to make a stably free module out of the projective module $\mathcal{M}$. The obstruction then appears as before in the Hermitian and co-Hermitian groups $K_0^h(\mathbb{Z}[\pi])$, $K_0^{sh}(\mathbb{Z}[\pi])$. In this context the role of the trivial element is taken by the module (denoted by H_g) having a canonical basis of cycles whose behavior with respect to the scalar product is analogous to that of the standard cycles on the surface of genus g with respect to the intersection index:

$$\begin{gathered} a_1, \ldots, a_g, \quad b_1, \ldots, b_g, \\ \langle a_i, a_j \rangle = 0, \quad \langle b_i, b_j \rangle = 0, \quad \langle a_i, b_j \rangle = \delta_{ij} = \pm \langle b_j, a_i \rangle. \end{gathered} \tag{5.22}$$

Under the operation of forming the direct sum of modules, with the relations $H_g \sim 0$ for all g imposed, one obtains the following analogues of the Grothendieck groups:

$K_0^h(\mathbb{Z}[\pi])$ k even, the Hermitian case;

$K_0^{sh}(\mathbb{Z}[\pi])$ k odd, the co-Hermitian case,

where $\mathbb{Z}[\pi]$ is equipped with the involution (5.17) (or (5.19) in the nonorientable situation).

The algebraic formulation of the obstruction theory to Morse surgeries was given in the case $n = 2k$ by Novikov and Wall, in the mid–1960s, and was extended to odd n by Wall at the end of the 1960s. It turns out that the obstructions reduce to elements of the groups $K_1^h(\mathbb{Z}[\pi])$ (k even), and $K_1^{sh}(\mathbb{Z}[\pi])$ (k odd), constructed on the analogy of the group $K_1(\pi)$, using stable classes of automorphisms of the canonical modules defined by (5.22), preserving the scalar product. The Whitehead relations are imposed on these classes (i.e.

direct sums of such automorphisms are identified with their corresponding composites) yielding commutativity together with a further relation, conveniently formulated as follows: any automorphism $\lambda : H_g \longrightarrow H_g$ for which $\lambda(a_j) = a_j$, $j = 1, \dots, g$, is set equal to the trivial one. These groups of obstructions to surgeries, defined via automorphisms of free modules over the ring $\mathbb{Z}[\pi]$, are called the *Wall groups* of π, denoted by $L_n(\pi)$:

$$L_0 \cong K_0^h, \quad L_1 \cong K_1^h, \quad L_2 \cong K_0^{sh}, \quad L_3 \cong K_1^{sh}. \tag{5.23}$$

As noted earlier, we shall not enter here a into discussion of the $\mathbb{Z}/2$–structure of these groups and the associated analogues of the Arf invariant.

There is the alternative approach (along the lines of that described prior to the preceding paragraph) to the definition of the groups K_0^h, K_1^h, K_0^{sh}, K_1^{sh} via projective modules equipped with a non–degenerate Hermitian or co-Hermitian scalar product (with the requisite property of "evenness" and the structure carried by the Arf invariant) determining an isomorphism $h : \mathcal{M} \xrightarrow{\cong} \mathcal{M}^*$, and with values in a ring with an involution. (In the above topological context the ring was $\mathbb{Z}[\pi]$ with the involution $u \to \bar{u}$ as in (5.17) or (5.19).) The groups $\tilde{K}_0^h$ and $\tilde{K}_0^{sh}$ are defined as usual in terms of direct sums of such modules, with the relations equating the projective modules of the form $M = N \oplus N^*$ $(N^{**} \cong N)$ with the trivial one, imposed:

$$\begin{gathered}\langle N, N\rangle = \langle N^*, N^*\rangle = 0, \\ N \oplus N^* \sim 0, \quad h : N \oplus N^* \longrightarrow (N \oplus N^*)^*,\end{gathered} \tag{5.24}$$

where the isomorphism h is given by the pair of canonical isomorphisms:

$$h : \begin{cases} N^* \xrightarrow{\cong} N^* \\ N \xrightarrow{\cong} N^{**} \end{cases}. \tag{5.25}$$

The algebraic definition of $\tilde{K}_1^h$ and $\tilde{K}_1^{sh}$ is based not on automorphisms (as above) but rather on the concept of a *Lagrangian submodule* $L \subset N \oplus N^*$ of a canonical projective module $N \oplus N^*$, defined by the conditions that $h|_L = 0$ and that L be a direct summand satisfying

$$M = N \oplus N^* \sim L \oplus L^*, \quad L^{**} = L. \tag{5.26}$$

The group operation is given by the direct sum of pairs (M, L), with those pairs taken to be trivial for which the projection $M \to N$ $(N^* \to 0)$ restricts to an isomorphism $L \to N$. The stable equivalence classes of pairs (M, L) then constitute the groups $\tilde{K}_1^h$ and $\tilde{K}_1^{sh}$, the *Novikov-Wall groups.*

Without considering the possible variants or entering into any detail, we note that the higher analogues $\tilde{K}_i^h$ and $\tilde{K}_i^{sh}$ can be defined for any ring A with an involution $u \to \bar{u}$ $(\overline{uv} = \bar{v}\bar{u})$ in both the free and projective cases. It turns out that, for instance, $\tilde{K}_2^h = \tilde{K}_0^{sh}$ and $\tilde{K}_2^{sh} = \tilde{K}_0^h$.

An algebraic theory of the groups $K_i^h \otimes \mathbb{Z}[\frac{1}{2}]$, $K_i^{sh} \otimes \mathbb{Z}[\frac{1}{2}]$ was formulated by Novikov (in the late 1960s) in the category of rings with involution, and included an algebraic construction of analogues of the Bass projectors (see above):

$$K_i^h(A[t,t^{-1}]) \underset{B}{\overset{\bar{B}}{\rightleftarrows}} K_{i+1}^h(A), \quad \bar{B}B = 1. \tag{5.27}$$

The groups $K_i^h \otimes \mathbb{Z}[\frac{1}{2}]$ furnish a periodic (of period 4) homology theory in the category of rings with involution, in which the Hermitian analogues of the Bass projectors (5.27) subsume the co–Hermitian case in view of the isomorphisms

$$K_j^h \cong K_{j+2}^{sh}, \quad K_j^{sh} \cong K_{j+2}^h. \tag{5.28}$$

Note in this connexion the isomorphism (modulo tensoring with $\mathbb{Z}[\frac{1}{2}]$)

$$K_{j+1}^h(A[t,t^{-1}]) \cong K_j^h(A) \oplus K_{j+1}^h(A), \tag{5.29}$$

contrasting with the corresponding situation in ordinary K–theory where there is a non-trivial kernel N.

The above *Hermitian K-theory*, as formulated by Novikov, uses analysis of geometric realizations of the above algebraic objects together with analogues of concepts of Hamiltonian formalism arising in connexion with analytical and quantum mechanics, as applied intensively by Maslov in the early 1960s to the construction of short–wave asymptotics in the theory of hyperbolic equations (and developed further by many authors). Once translated into an algebraic context, the standard terminology of symplectic geometry and Hamiltonian formalism turns out to be extremely useful; in particular it clarifies the ideas behind the algebraic constructions (and sometimes even suggests the appropriate ideas), and makes more explicit the mechanism underlying the algebraic analogues of Bott periodicity. Of course once the algebraic theory has been precisely formulated, it ceases to be dependent on its source in the analogy with Hamiltonian formalism.

It was shown by Mishchenko (around 1970) that tensoring by $\mathbb{Z}[\frac{1}{2}]$ eliminates the differences between the various proposed Hermitian K-theories. At the same time he constructed a homotopy invariant τ^π of an orientable manifold M^j, given by an element of $K_j^h(\mathbb{Z}[\pi])$ (where $\pi = \pi_1 M^j$), analogous to the signature, and determining a homomorphism from each bordism group of the Eilenberg-MacLane space $K(\pi,1)$ to the corresponding Hermitian K–theory:

$$\tau^\pi : \Omega_j^{SO}(K(\pi,1)) \longrightarrow K_j^h(\mathbb{Z}[\pi]) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

For normal maps $f : M_1^n \longrightarrow M^n$ of degree one (figuring centrally in the general classification theory of manifolds of dimension ≥ 5), obstructions to

Morse surgeries are given by the formula

$$\tau^\pi(M_1^n) - \tau^\pi(M^n) \in K_n^h(\mathbb{Z}[\pi]) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Remark. Note that even for simply–connected manifolds the calculation of such obstructions without dividing by 2 was impossible owing to a difficulty involving the Arf invariant. □

Novikov's conjecture on the higher signatures concerns the existence of a purely algebraic analogue of the Chern character:

$$ch : K_*^h(\mathbb{Z}[\pi]) \otimes \mathbb{Q} \longrightarrow H_*(\pi; \mathbb{Q}),$$

with the property that for a closed manifold M^n the following formula should hold:

$$\langle z, ch \circ \tau^\pi(M^n) \rangle = \langle L_k(M^n), D\phi^*(z) \rangle. \tag{5.30}$$

Here $\pi = \pi_1(M^n)$, $\phi : M^n \longrightarrow K(\pi, 1)$ is the canonical map, $D\phi^*(z) \in H_{4k}(M^n; \mathbb{Q})$, and L_k is the Hirzebruch polynomial in the Pontryagin classes of M^n. It would appear that this formula, if true for any group π, would embrace all possible relations between the Pontryagin classes for a given homotopy type of the manifold M^n. In all cases where the general conjecture on the higher signatures has been established, the existence of such an algebraic analogue of the Chern character (satisfying (5.30)) has also been shown.

In the second half of the 1960s it was observed by I. M. Gel'fand and Mishchenko that for rings $C^*(X)$ of complex–valued functions on compact spaces X, one has

$$K_0^h(C^*(X)) \cong K_{\mathbb{C}}^0(X).$$

In fact such an isomorphism exists also for K_1, and therefore for all the K_*^h of such function algebras with involution (the involution on $C^*(X)$ being given by $f \to \bar{f}$). In the case $X = T^n$, use of the Fourier transform yields the following isomorphism modulo tensoring by $\mathbb{Z}[\frac{1}{2}]$:

$$K_*^h(\mathbb{C}[\mathbb{Z} \times \cdots \times \mathbb{Z}]) \cong K_*^h(C^*(T^n)) \quad (\cong K_{\mathbb{C}}^*(T^n)). \tag{5.31}$$

Hence the ordinary Chern character of a vector bundle over T^n affords an algebraic analogue of the Chern character for $\pi = \mathbb{Z} \times \cdots \times \mathbb{Z}$ (n times), although this method of obtaining such an analogue is obviously not algebraic.

The algebraic definition of the analogues of the Bass projectors in K_*^h-theory for the rings $C^*(X)$, yields a suspension isomorphism and an approach to Bott periodicity different from that afforded by the ordinary K-theory of Atiyah and others. Note that the presence of an imaginary i ($i^2 = -1$) in the ring $C^*(X)$ allows the Hermitian theory to be transformed to the co-Hermitian, so that for algebras A with such an an imaginary element we have $K_*^h(A) = K_*^{sh}(A)$, and the periodicity is shortened to 2.

By sorting out the various algebraic constructions one arrives at the significance of periodicity from the point of view of Hamiltonian formalism; however this theory (as conceived by Novikov in the late 1960s) concerns the tensor product $K_*^h \otimes \mathbb{Z}[\frac{1}{2}]$. A complete formalization of Hermitian algebraic K-theory (called *algebraic L-theory*), avoiding tensoring by $\mathbb{Z}[\frac{1}{2}]$, was achieved by Ranicki in the first half of the 1970s, by recasting the theory over the integers; for a group ring $\mathbb{Z}[\pi]$ the corresponding groups are denoted by $L_n(\pi)$. These groups were studied by several people throughout the 1970s and into the 1980s (Pedersen, Beck, Sharpe). In the first half of the 1970s Wall and Beck developed detailed techniques for computing these objects in the situation where π is finite and the modules are free, and actually carried out the calculations for various particular groups.

On the other hand although the full algebraic theory was not available until the end of the 1960s, it was possible earlier on to obtain complete information about the groups $L_n(\pi)$ in certain special cases (although noneffectively) without having to hand the details of the algebraic definition, for instance when π is free abelian, $\pi = \mathbb{Z} \times \cdots \times \mathbb{Z}$. In the case $\pi = \mathbb{Z}$ the theory was formulated by Browder (in the mid–1960s) as a geometric obstruction theory to reducibility to the simply–connected situation. In this form the theory was then extended to the more general case $\pi = \mathbb{Z} \times \cdots \times \mathbb{Z}$ by Shaneson (in the late 1960s). Independently of the various algebraic definitions of the groups $L_n(\pi)$, there exists an isomorphism

$$L_n(\pi \times \mathbb{Z}) \cong L_n(\pi) \oplus L_{n+1}(\pi), \tag{5.32}$$

which can be established geometrically, and this isomorphism allows the structure of the groups $L_n(\mathbb{Z} \times \cdots \times \mathbb{Z})$ to be inferred (non-effectively) from that of the groups $L_n(1) = L_n$, which are as follows:

$$L_n \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0 \pmod 4, \\ \mathbb{Z}/2 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

(These isomorphisms were already known from simply-connected surgery theory.) Knowledge of the groups $L_n(\mathbb{Z}\times \cdots \times \mathbb{Z})$ leads to a complete classification of the homotopy n-tori, i.e. of the n–manifolds having the homotopy type of the torus T^n, $n \geq 5$. In particular, it turns out that such a manifold always has a finite-sheeted covering manifold diffeomorphic to the standard torus T^n.

Remark. We note that all surgery obstructions (i.e. the elements of the groups $L_{n+1}(\pi)$) are realized geometrically as invariants distinguishing two manifolds in the same homotopy class as the given n-manifold. The construction is as follows: If two normal maps $f_i : M_i^n \longrightarrow M^n$ are homotopy equivalences and lie in the same normal bordism class, then there exists a bordism W^{n+1} with boundary the disjoint union of M_1^n and M_2^n ($\partial W^{n+1} = M_1^n \cup M_2^n$) together with a normal map $F : W^{n+1} \longrightarrow M^n \times I$, whose restrictions to the boundary components M_1^n and M_2^n are f_1, f_2 respectively. Denote by $p : M \times I \longrightarrow M$ the projection on the first factor. Under the above condition the map $F_1 = f_1^{-1} \circ p \circ F$ affords a normal retraction $W^{n+1} \longrightarrow M_1^n$ onto the boundary component M_1^n (see §4 above). Starting with a given manifold M_1^n one can realize geometrically by means of such bordisms W^{n+1} every element of the surgery obstruction group, just as in the simply–connected case. Thus one may construct (starting with M_1^n) such a bordism W^{n+1} corresponding to any prescribed element of the group $L_{n+1}(\pi) \cong K_{n+1}^h(\pi)$. □

For homotopy tori there is a subgroup A_n of finite index:

$$A_n \subset L_{n+1}(\mathbb{Z} \times \cdots \times \mathbb{Z}),$$

whose elements are realized by bordism manifolds W_{n+1} with diffeomorphic boundary components M_1^n, M_2^n, $\partial W^{n+1} = M_1^n \cup M_1^n$. This theorem implies (via Novikov's result that all homotopy tori are stably parallelizable) the finiteness of the number of different homotopy tori in each normal bordism class (Wall, Siebenmann, Hsiang, Shaneson, Casson, end of the 1960s).

Since, by a theorem of Shaneson, the groups

$$L_{n+1}(\mathbb{Z} \times \cdots \times \mathbb{Z}) \cong K_{n+1}^h(\mathbb{Z} \times \cdots \times \mathbb{Z})$$

are determined (modulo a finite group) by the higher signatures, the result on the finitness of the number of homotopy tori in each normal bordism class, also follows from the non–simply–connected analogues of the Hirzebruch formula for the higher signatures, in much the same way Milnor's theorem on the finiteness of the groups bP^{4k} follows from the ordinary Hirzebruch formula. The argument here is as follows. Consider the group of vector bundles ξ_N over the torus T^{n+1} with the property that $J(\xi_N) = 0$ and $\xi_{T^n} = 0$ (i.e. the bundle ξ_N is a stably homotopy trivial one over the torus T^{n+1} and its restriction ξ_{T^n} to the n-torus $T^n \subset T^{n+1}$ is the trivial vector bundle). Following the general construction of a normal bordism (see §4 above), one obtains a map

$$S^{N+n+1} \longrightarrow T\xi_N,$$

and a normal map of degree one:

$$f : M^{n+1} \longrightarrow T^{n+1}, \quad f^*(\xi_N) \cong \nu_M,$$

where M^{n+1} is a smooth closed manifold realizing an element α of the group

$$L_n(\pi) \subset L_n(\pi \times \mathbb{Z}), \quad \pi = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}},$$

determined uniquely (mod 2^n) by the Chern character $D \circ ch\, \xi_N \in H_*(T^{n+1}; \mathbb{Q})$. Thus the construction above provides a normal map of degree one:

$$\begin{array}{ccc} \nu_M & \xrightarrow{\hat{f}} & \xi_N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & T^{n+1} \end{array}$$

trivial over $T^n \subset T^{n+1}$:

$$f^{-1}(T^n) \cong T^n \subset M^{n+1}, \quad \nu_M|_{T^n} = f^*\xi_N|_{T^n} = 0.$$

By cutting M^{n+1} along the embedded torus T^n one then obtains the desired manifold realizing the element α, and with each of its two boundary components diffeomorphic to T^n.

Given two CW-complexes or manifolds one can often distinguish them, i.e. show that they are not homeomorphic (or not PL-homeomorphic, or not diffeomorphic), by means of algebro–topological invariants. However if all conceivable invariants coincide for these two spaces, then one is forced to resort finally to seeking some means or other of actually constructing a homeomorphism between them. It is interesting to consider the various methods that have been employed to this end over the last two or three decades. In differential topology the approach to this problem has relied either on producing an appropriate map directly by analytic means, or else on using completely finitistic procedures involving careful selection and analysis of sequences of attachments of handles (for example, by means of Morse surgeries, when the topology of the level sets of a Morse function alters as its value passes through a critical point). In carrying out sequences of Morse surgeries, their effects on homotopy, the algebraic obstructions to realizing them that arise, and various reductions and simplifications that occur (for instance the mutual cancellation of a pair of handles) are all investigated. These basic techniques, which have been in use for a very long time, underwent intensive elaboration in the 1960s, in parallel with the remarkable developments in the topology of manifolds of that era. We shall not however consider them further here; they have been mentioned at various points earlier on in this book. What other tools does topology have at its disposal appropriate to the direct investigation of homeomorphisms? In low dimensions ($n \leq 3$) every compact family of homeomorphisms of manifolds can be approximated by PL-homeomorphisms, and by diffeomorphisms, and the whole group of self–homeomorphisms of a manifold has the same local and global homotopy properties as the group of PL–homeomorphisms (Moise, in the early 1950s), and even the group of self–diffeomorphisms. However in higher dimensions this is certainly no longer the case. Among the best–known of the early results relevant to this sort of question, obtained by elementary visual means, is that of Mazur and Brown (of around 1960): *Given a "flat" embedding of a sphere, i.e. an embedding of a whole neighbourhood:*

$$F : S^{n-1} \times I \longrightarrow S^n, \quad I = [-1, 1],$$

the image $F(S^{n-1} \times \{0\})$ of the "middle" sphere $S^{n-1} \times \{0\}$ bounds a closed disc $D^n \subset S^n$: $\partial D^n = F(S^{n-1} \times \{0\})$ (the "generalized Schoenflies conjecture"). Following hard on the appearence of the Brown-Mazur theorem, the "Annulus Conjecture" was formulated (by various people):

Given two non-intersecting flat embeddings F_1 and F_2 of S^{n-1} in S^n:

$$F_1(S^{n-1} \times \{0\}) \cap F_2(S^{n-1} \times \{0\}) = \emptyset,$$

is it true that the closed region of S^n between them is homeomorphic to an annulus $S^{n-1} \times I$?

For $n \leq 3$ this conjecture holds for elementary reasons, and for $n \geq 6$ it holds under the assumption of smoothness by virtue of Smale's h-cobordism theorem. In fact, in higher dimensions, the Annulus Conjecture can be established without difficulty under the assumption that the embeddings F_1, F_2 are smooth at at least one point. (We shall return to the discussion of the Annulus Conjecture below.)

We now describe another result of Mazur (from the early 1960s) which, although it concerns smooth manifolds, uses entirely elementary techniques, and allows one to establish that a certain two CW-complexes known to be combinatorially inequivalent, are nonetheless homeomorphic (Milnor, also in the early 1960s). The theorem is as follows:

If $f : M_1^n \longrightarrow M_2^n$ is a normal homotopy equivalence between two manifolds M_1^n and M_2^n, i.e. $f^*(\nu_N^{(2)}) = \nu_N^{(1)}$, where $\nu_N^{(i)}$ denotes the normal bundle of M_i^n with respect to an embedding $M_i^n \subset \mathbb{R}^{N+n}$, $N \geq n+1$, then the direct products

$$M_1^n \times \mathbb{R}^N \quad \textit{and} \quad M_2^n \times \mathbb{R}^N$$

are diffeomorphic.

The idea of the proof is as follows: Consider embeddings

$$M_1^n \times D^N \hookrightarrow M_2^n \times \mathbb{R}^N \quad \text{and} \quad M_2^n \times D^N \hookrightarrow M_1^n \times \mathbb{R}^N \tag{5.33}$$

respectively "approximating" the homotopy equivalence $f : M_1^n \longrightarrow M_2^n$, and a homotopy inverse $g : M_2^n \longrightarrow M_1^n$ ($f \circ g \sim 1$, $g \circ f \sim 1$). The normal bundles of $M_2^n \subset M_1^n \times \mathbb{R}^N$ and $M_1^n \subset M_2^n \times \mathbb{R}^N$ (where the inclusions are obtained by restricting the above embeddings to $M_1^n \times \{0\}$ and $M_2^n \times \{0\}$) are trivial in view of the assumption $f^*(\nu_N^{(2)}) = \nu_N^{(1)}$. From the pair of embeddings (5.33) we then obtain an expanding sequence of regions

$$M_1^n \times D_1^N \subset M_2^n \times D_2^N \subset M_1^n \times D_3^N \subset \cdots ,$$

where the radii of the balls $D_i^N \subset \mathbb{R}^N$ increase, all of the embeddings

$$M_1^n \times D_{2i+1}^N \hookrightarrow M_1^N \times \mathbb{R}^N$$

are standard, and the embeddings of $M_2^n \times D_{2i}^N$, $M_2^n \times D_{2i+2}^N$, etc. represent successive extensions and are all isotopic. The assertion of the theorem now follows readily.

We now describe Milnor's example. Note first that every 3-dimensional orientable, closed manifold is parallelizable. Let $L_{q_1}^3$, $L_{q_2}^3$ be two lens spaces with fundamental group isomorphic to $\mathbb{Z}/p$ and invariants $q_1, q_2 \in \mathbb{Z}^*$, satisfying $q_1 = \lambda^2 q_2 \bmod p$. It is a classical result that this condition guarantees the homotopy equivalence of the two lens spaces, but that they may be chosen so as to have different Reidemeister torsion, which implies that they are not PL-homeomorphic (See Chapter 3, §5). By Mazur's theorem the manifolds

$$L^3_{q_1} \times \mathbb{R}^N \quad \text{and} \quad L^3_{q_1} \times \mathbb{R}^N$$

are diffeomorphic for some N, so that the Thom spaces (which are CW-complexes) of the trivial bundles

$$K_1 = \left(L^3_{q_1} \times D^N\right) / \left(L^3_{q_1} \times S^{N-1}\right),$$

$$K_2 = \left(L^3_{q_2} \times D^N\right) / \left(L^3_{q_2} \times S^{N-1}\right)$$

are also homeomorphic. However they are not combinatorially equivalent (Milnor). To see this one first observes that the boundary of a combinatorial neighbourhood of the singular point of K_i has the form $L^3_{q_i} \times S^{N-1}$, and then that the relative Reidemeister torsion of K_i (with respect to the singular point) is well-defined and equal to the ordinary Reidemeister torsion of the lens space $L^3_{q_i}$. The result now follows.

Thus the *Hauptvermutung* is false in general for CW-complexes of dimension $3+N$. Since N can be reduced to 3, this applies in fact to CW-complexes of dimension ≥ 6.

We now return to the topic of (not necessarily smooth) homeomorphisms. Recall that in the early 1960s Stallings established, independently of Smale and Wallace, a continuous version of the higher-dimensional Poincaré conjecture, showing that a PL-manifold homotopy equivalent to the sphere S^n ($n \geq 5$) is homeomorphic to the sphere S^n. As part of the proof he showed that every "locally flat" embedding $S^n \subset S^{n+k}$ can be reduced, by means of a homeomorphism of the sphere S^{n+k}, to a standard embedding provided $n \geq 3$, $k \geq 3$. (In the case $k = 2$ one requires also that $\pi_1(S^{n+2} \setminus S^n) \cong \mathbb{Z}$.) An embedding $F : S^n \longrightarrow S^{n+k}$ is called *locally flat* if for each point x of S^n there is a neighbourhood D^n_x such that the restriction of F to that neighbourhood extends to an embedding

$$D^n_x \times I \longrightarrow S^{n+k}, \quad I = [-1, 1],$$

where D^n_x is identified with $D^n_x \times \{0\}$.

Technically more complex is the proof of the following theorem of Chernavskiĭ (of the mid 1960s):

The group of self-homeomorphisms of any closed topological manifold, or of $\mathbb{R}^N$, is locally contractible.

In the space of all self-maps of a given manifold endowed with the C^0-topology the homeomorphisms are "unstable", in contrast with the diffeomorphisms, which with respect to the C^1-topology form an open subset in the space of all smooth self-maps. However a diffeomorphism arbitrarily close to the identity map with respect to the C^0-topology, may still be very complex, and extremely difficult to deform to the identity map within the class of homeomorphisms; as will be indicated below, this fact has played an important role in topology. Chernavskiĭ constructs such isotopies of a diffeomorphism to the identity map using essentially elementary, but extremely complicated techniques.

At the end of the 1960s Kirby found an approach to the proof of the Annulus Conjecture via reduction to the situation of smooth (or PL–) homotopy tori, and almost immediately thereafter the crucial problems concerning homotopy tori associated with this approach were solved by Wall, Siebenmann, Hsiang, Shaneson and Casson. We shall now describe Kirby's idea.

In the theory of homeomorphisms $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ the concept of a *stable homeomorphism* is important; this is a homeomorphism $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ that is a composite $h = h_1 \circ \cdots \circ h_k$ of homeomorphisms h_i each of which restricts to the identity on some non-empty open region U_i. Diffeomorphisms and PL-homeomorphisms are stable, as are homeomorphisms approximable by PL-homeomorphisms. If a homeomorphism agrees with a stable one on some open non-empty region, then it must itself be stable. It therefore makes sense to speak of the pseudogroup of stable homeomorphisms between open regions of $\mathbb{R}^n$ and thus of the class of *stable manifolds* determined by this pseudogroup, where the co–ordinate transformations are restricted to being stable (Brown, Glück and others, in the early 1960s). From the stability of orientation–preserving homeomorphisms of $\mathbb{R}^n$, the Annulus Conjecture for spheres and for $\mathbb{R}^n$ follows relatively easy. We shall now sketch the proof of stability.

In the early 1960s Connell made the useful observation that a homeomorphism $h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for which the distance between x and its image $h(x)$ is bounded above ($|h(x) - x| < M$) is stable. From this it follows that all homeomorphisms of a torus are stable; for if a homeomorphism $h : T^n \longrightarrow T^n$ fixes a point and is homotopy equivalent to the identity map, then the universal covering map $\hat{h} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies

$$|\hat{h}(x) - x| < M \quad \text{for all } x \in \mathbb{R}^n,$$

whence it follows that in fact $h : T^n \longrightarrow T^n$ is stable. The general case of a homeomorphism h of T^n is reduced to this special case by composing h with suitable affine transformations. There is a resemblance between Kirby's strategy for proving the Annulus Conjecture, and the proof of the topological invariance of the Pontryagin classes, in that at a certain point in Kirby's scheme a "toroidal region" of $\mathbb{R}^n$ is introduced, however somewhat differently: one considers an immersion $\phi : T^n \setminus \{*\} \longrightarrow \mathbb{R}^n$, which, although such immersions exist for any open, parallelizable n-manifold, may here be constructed by elementary direct means. (In fact its precise form is unimportant.) Different homeomorphisms $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ induce potentially different smooth structures on the image of $T^n \setminus \{*\}$ in $\mathbb{R}^n$ by mapping this region to other regions of $\mathbb{R}^n$.

Now every smooth open manifold M^n of dimension $n \geq 6$, with a neighbourhood of infinity homeomorphic to $S^{n-1} \times \mathbb{R}$, is realizable as the interior of a smooth manifold–with–boundary whose boundary has the homotopy type of S^{n-1} (see the earlier discussion of the problem of Browder-Levine-Livesay). Hence by using the result of Smale and Wallace that all homotopy spheres of dimension $n \geq 5$ are PL-homeomorphic to S^n (see above), we may compactify such an image region

$$\overline{T^n \setminus \{*\}}$$

with its new smooth structure (induced from a homeomorphism $g : \mathbb{R}^N \longrightarrow \mathbb{R}^N$) by means of a single point $*'$ to obtain a homotopy torus V^n endowed with a PL-structure (Wall, Hsiang, Shaneson, Siebenmann, Casson):

$$V^n \setminus \{*'\} \cong \overline{T^n \setminus \{*\}}.$$

If the new closed manifold V^n is PL–isomorphic to T^n then we have a homeomorphism $T^n \longrightarrow T^n$, and since such homeomorphisms are, as observed above, always stable, it follows that the original homeomorphism $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is also stable. The proof is completed by showing that it suffices to construct a PL-homeomorphism between some finite-sheeted covering spaces $\hat{V}^n \longrightarrow \hat{T}^n$, rather than between V^n and T^n. Hence all self-homeomorphisms of $\mathbb{R}^n$ are stable, and the Annulus Conjecture follows.

Using results about smooth or piecewise linear homotopy tori, it can be shown that for $n \geq 5$ there exists a PL-homeomorphism $h : T^n \longrightarrow T^n$ not isotopic to the identity map, with obstructions to such isotopies, in general, lying in $\mathbb{Z}/2$. (In fact there is a PL–homeomorphism not even "pseudo–isotopic" to the identity map.) Lifting such an h to a PL–homeomorphism $\hat{h}$: $\hat{T}^n \longrightarrow \hat{T}^n$ of a covering space with an odd number of sheets, we can, provided the number of sheets is large enough, deform the latter PL–isomorphism (via a topological isotopy) to a self–homeomorphism arbitrary close, in the C^0-topology, to the identity map, but still not PL–isotopic to it in view of the fact that the obstruction to such a PL–isotopy (with values in $\mathbb{Z}/2$) prevails in odd–sheeted covering spaces. However by Chernavskiĭ's theorem, since the homeomorphism $\hat{h}$ is C^0-close to the identity map, it must in fact be topologically isotopic to it. The upshot of this delicate discrimination between PL– and topological homeomorphisms is another counterexample to the *Hauptvermutung*, and an example of a topological manifold not admitting a compatible PL–structure (Kirby and Siebenmann, end of the 1960s).

Thanks to the work of several people (Lees, Lashof, Rothenberg, at the end of the 1960s), it is now known that such properties of closed topological manifolds of dimension ≥ 5 as those of admitting a compatible PL– or smooth structure, are equivalent to the reduction of the structure group TOP (roughly speaking, the group of germs of homeomorphisms of an open disc) of the tangent microbundle, to its subgroup $PL \subset TOP$ or to O (even in non–stable dimensions of these bundles). These investigations, together with those of Kirby and Siebenmann, yield the following structures of the relative homotopy groups $\pi_i(TOP/PL)$:

$$\pi_i(TOP/PL) \cong \begin{cases} 0 & \text{for } i \neq 3, \\ \mathbb{Z}/2 & \text{for } i = 3. \end{cases}$$

Knowledge of these groups renders the results of the above authors concerning triangulability (i.e. admissibility of a PL–structure) effective. An upper bound for the number of distinct PL–structures definable on a topological manifold M^n (where $n \geq 5$ for closed manifolds, and $n \geq 6$ for manifolds–with–boundary), is afforded by the order of the group $H_3(M^n; \mathbb{Z}/2)$. Note that here simple–connectness is not involved. (Complete proofs of all of the results of this theory depends on some results of the Sullivan theory, which, as noted earlier, is as yet incomplete.)

It is noteworthy that the fact that $\pi_3(PL/TOP) \cong \mathbb{Z}/2$, which alone distinguishes PL-structures on manifolds from continuous ones, is ultimately a consequence of Rohlin's theorem to the effect that the signature τ of every almost parallelizable manifold M^4 is divisible by 16, while in all dimensions of the form $4k > 4$ there exist almost parallelizable closed PL–manifolds with $\tau = 8$ (Milnor; see §3 above).

Noteworthy among the more recent successful constructions of homeomorphisms are the following two:

1. *The direct construction of a homeomorphism*

$$\Sigma^2 M^3 \longrightarrow S^5$$

from the double suspension of any 3-dimensional manifold M^3 *satisfying* $H_1(M^3; \mathbb{Z}) = 0$ *(Edwards, in the late 1970s). From this there follows the existence of a triangulation of* S^5 *with respect to which it is not a* PL*-manifold, and which is not combinatorially equivalent to the standard triangulation.*

2. *The direct (and not exceptionally complicated) construction of a homeomorphism between any smooth 2–connected closed 4–manifold and the sphere* S^4 *(Freedman, around 1980). On the other hand with respect to diffeomorphisms the 4–dimensional analogue of the Poincaré conjecture remains open. (Note that in dimension 4* PL*– homeomorphisms are in this context equivalent to smooth.)*

Concluding Remarks

In this survey of the ideas and methods of topology far from all topics have been considered. Little has been mentioned here of 3–dimensional topology and the theory of Kleinian groups as developed by Thurston and others, or of the developments in 4–dimensional topology due to Donaldson, Freedman, and others. Neither have we surveyed here the qualitative theory of foliations: notably the theorems of Reeb, Haefliger and Novikov on analytic foliations and and on foliations with compact sheet; the techniques for establishing the existence of foliations and other geometric structures on open manifolds (Gromov), of foliations of codimension one of odd-dimensional spheres (Lawson, Tamura, and others), and of compact manifolds of arbitrary dimensions (Thurston); the theory of characteristic classes of foliations (Bott, Godbillon, Vey, Bernshtein, and others); the classification theory of foliations (Haefliger and others); the cohomology theory of Lie algebras of vector fields on manifolds (I. M. Gel'fand, Fuks); the theory of singular points of complex hypersurfaces (Milnor, Brieskorn); real algebraic curves and surfaces (Gudkov, Arnol'd, Rohlin, Kharlamov, Kirilov, Viro).

We have also omitted from our survey such areas of topology as the theory of embeddings of manifolds and higher–dimensional knots (created by several people, beginning with Whitney and including Haefliger, Stallings and Levine), as also the theory of "typical singularities of maps and and functions" (Whitney, Pontryagin, Thom, Boardman, Mather, Arnol'd).

The author of the present essay has preferred — and this is in full conciousness — to omit discussion of topological results of a general categorical and abstract nature, nothwithstanding their usefulness and even necessity as far as the intrisic logic of topological concepts is concerned.

There is also missing from the survey a summary of the homology theory of general spaces, in particular of subsets of $\mathbb{R}^n$, developed in the 1920s by several authors, including P. S. Alexandrov, Čech, Pontryagin, Kolmogorov, Steenrod, Chogoshvili, Sitnikov, Milnor, among others.

There is also missing a discussion of the topological properties of manifolds with various differential–geometric properties (global geometry).

Finally, it has not proved possible to discuss the substantial applications of topology that have been made over recent decades to real physical problems, and have transformed the apparatus of modern mathematical physics. We hope that this lack will be made good in other essays of the series.

Appendix
Recent Developments in the Topology of 3-manifolds and Knots

§1. Introduction: Recent developments in Topology

The present survey of topology was in fact written in 1983-84 and published (in Russian) in 1986. Over the intervening decade several very beautiful new ideas have appeared in the pure topology of 3-manifolds and knots, and it is to these that the present appendix is devoted. It should however be noted that impressive developments have occurred also in other areas of topology: in symplectic topology, with the creation of "Floer homology" and its subsequent development by several mathematicians (Gromov, Eliashberg, Hoffer, Salamon, McDuff, and others); in the theory of SO_2-actions on smooth manifolds, including analysis on loop spaces (Witten, Taubes), originating in quantum mechanics, and "elliptic genera" (Oshanine, Landweber) allowing formal groups and other techniques of complex cobordism theory to be applied to the theory of such actions (already mentioned in the body of this survey in connection with work carried out by the Moscow school around 1970); in the so–called "topological quantum field theories" (whose beginnings were also mentioned above in connection with the construction of Reidemeister-Ray-Singer torsion via a functional integral, by A. Schwarz in 1979-80) as developed by Witten in the late 1980s following a suggestion of Atiyah, and by others in the 1990s. The present author is of the opinion that these conformal and topological quantum field theories constitute a new kind of analysis on manifolds, hitherto remaining, however, without rigorous foundation. Nonetheless these new methods have led to the discovery (by physicists such as Candelas and Vafa, among others, with subsequent more-or-less rigorous justification by various mathematicians) of subtle features of the structure of rational algebraic curves on certain symplectic manifolds and algebraic varieties (for instance on "Calaby-Yau" or "toroidal" manifolds).

In a similar manner deep facts about the topology of moduli spaces have been obtained by Kontzevich using the technique of "matrix models" borrowed from statistical mechanics, by means of which physicits (Gross, Migdal, Brezin, Kazakov, Douglas, Shenker, and others) were led in 1989-90 to the discovery of some beautiful results, in part topological, in string theory, finding in particular a connection between that theory and the celebrated integrable models of soliton theory.

Thus over the last two decades (since the discovery of instantons in the mid-1970s by Belavin, Polyakov, Schwarz and Tyupkin, mentioned in the body of this survey), several very interesting developments in topology have been initiated by quantum field physicists. One outcome of this has been that some first-rank physicists have gone from doing research in physics proper to working on problems of abstract mathematics using the approach and mathematical techniques of quantum field theory familiar from their training in physics. Although the latter technique is standard among physicists, it is scarcely known to the community of pure and applied mathematicians, partly because some of the basic notions and techniques of the theory, although certainly mathematical in nature (contrary to the opinion of many mathematicians that they belong to physics), are not easily amenable to rigorous mathematical treatment. In fact the current rigorous version of quantum field theory formulated in terms of 20th century pure mathematical analysis, although highly non-trivial from the viewpoint of functional analysis, seems rather impoverished by comparison with its physical prototype. The interaction of topology and algebraic geometry with modern quantum physics over the last two decades has proved extremely fruitful in pure mathematics; it is also to be hoped that ultimately it will bear fruit also outside pure mathematics.

§2. Knots: the classical and modern approaches to the Alexander polynomial. Jones-type polynomials

We shall sketch here some of the ideas of modern knot theory, inaugurated in the mid-1980s by the discovery of the "Jones polynomial". We begin by recapitulating the relevant basic notions of knot theory.

We shall consider only classical *knots*: such an object is a smooth, closed nonselfintersecting curve K in Euclidean space $\mathbb{R}^3$ or in the 3-sphere S^3, with everywhere nonzero tangent vector. A *link* is a union of such curves, pairwise nonintersecting:

$$K = \bigcup K_i, \quad K_i \cap K_j = \emptyset \quad \text{for} \quad i \neq j.$$

By projecting a knot or link orthogonally onto a plane (or "screen") in the direction of some suitable vector η, we obtain the *diagram* of the knot or link, consisting of a "generic" collection of plane curves (where by *generic* we mean that the tangent vector is everywhere non-zero, all intersections ("crossings") are transversal, and there are no triple intersection points); the diagram should also include, for each crossing, the information as to which of two curve segments is "above" the other relative to the screen. The diagram is said to be *oriented* if each component curve of the knot or link is directed, otherwise *unoriented.*

Classical knot theory is concerned with the space $S^3 \setminus K = M$, an open 3-manifold. There is a natural embedding of the torus T^2 in M, namely as the boundary of small tubular neighbourhood of the knot K. Similarly, for a link we obtain a disjoint union of 2-tori in M.

The principal topological invariant of a knot K is the fundamental group $\pi_1(M)$ of the complement M of K, with distinguished subgroup the natural image of $\pi_1(T^2)$, $T^2 \subset M^2$, with the obvious standard basis. The classical theorem of Papakyriakopoulos of the 1950s asserts that a knot is equivalent to the trivial one if and only if $\pi_1(M)$ is abelian. It was shown by Haken in the early 1960s that there is an algorithm for deciding whether or not any knot is equivalent to the trivial knot. However, while it appears to have been established (by Waldhausen and others in 1960s and 1970s) that two knots are topologically equivalent if and only if the corresponding fundamental groups with labelled abelian subgroups are isomorphic, the existence of an appropriate algorithm for deciding such equivalence remains an open question.

The complexity of the knot goup $\pi_1(M)$ has led to the search for more effectively computable invariants to distinguish knots and links. The first such non-trivial invariant to be discovered was the *Alexander polynomial.* We shall now give an alternative elementary definition of this invariant in terms of the oriented diagram of the knot or link, following early articles of Alexander himself. (This definition was rediscovered by Conway around 1970.) This approach to the definition is based on the very strong additive property of Alexander polynomials with respect to an operation eliminating crossings. (The Jones, HOMFLY, and Kauffman polynomials have similar properties — see below.) Consider the three oriented diagrams of Figure A.1, differing from one another only in a small neighbourhood of a single crossing. The first

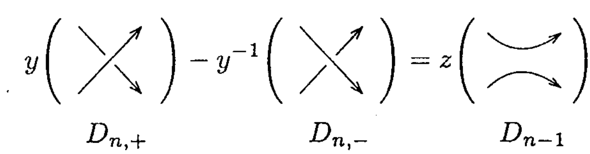


Fig. A.1. Defining the Alexander and HOMFLY polynomials

two diagrams $D_{n,\pm}$ have exactly n crossings and identical projections on the screen; moreover their oriented diagrams coincide exept for one crossing. The third diagram D_{n-1} has $n-1$ crossings (one fewer then the other two), being obtained from either $D_{n,\pm}$ by replacing a neighbourhood of the crossing at which these differ by the configuration indicated in Figure A.1, in natural agreement with the orientation. The Alexander polynomials corresponding to the respective diagrams are related as follows:

$$zP_{D_{n-1}}(z) = P_{D_{n,+}}(z) - P_{D_{n,-}}(z).$$

This will serve as a defining condition once supplemented by the conditions that the Alexander polynomial should be zero for any diagram with 2 or more connected components, and should be equal to 1 for the trivial one-component diagram (i.e. for a simple closed curve).

This defines the Alexander polynomial as a polynomial in the variable z; the standard version is then obtained by means of a substitution:

$$A(t, t^{-1}) = P(z), \quad z = t^{1/2} - t^{-1/2},$$

yielding finally the Alexander polynomial $A(t, t^{-1})$, well-defined up to multiplication by a power of t.

The *HOMFLY polynomial* $H(x, y)$ is defined using the same three diagrams but with the general condition

$$xH_{D_{n-1}}(x, y) = yH_{D_{n,+}}(x, y) - y^{-1}H_{D_{n,-}}(x, y),$$

together with the condition that $H = 1$ for the trivial knot. Note that the most general formula of this type, involving three variables, can be normalized to this form by multiplying by a suitable factor.

Setting $y = 1$, $x = z$ yields the Alexander polynomial. Putting $x = y^{1/2} - y^{-1/2}$ yields the celebrated *Jones polynomial* $J(y)$, of which, therefore the HOMFLY polynomial is a natural generalization. We shall call all of these polynomials (including the "Kauffman polynomial" defined below) *Jones-type polynomials.*

We shall now describe Kauffman's method, based on unoriented diagrams, of constructing polynomial invariants constituting a different generalization of the Jones polynomials. For four unoriented diagrams $D_{n,\pm}$, $D_{n-1,1}$, $D_{n-1,2}$ differing only in a small neighbourhood of one crossing, as depicted in Figure A.2, we first define a 2-variable *state polynomial* Q via the following recurrence

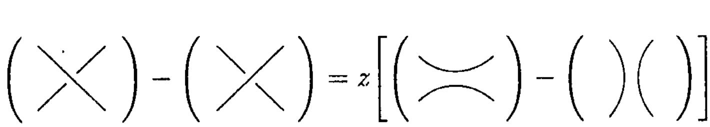


Fig. A.2. Defining the state polynomial and the Kauffman polynomial

relation:

$$Q_{D_{n,+}} - Q_{D_{n,-}} = z(Q_{D_{n-1,1}} - Q_{D_{n-1,2}}),$$

together with

$$Q_{D_0} = \mu = \frac{(a - a^{-1})}{z} + 1$$

for the trivial one-component diagram D_0, and the requirement that Q change by a factor $A^{\pm 1}$ with each Reidemeister move (see Figure A.3). The *Kauffman polynomial* of a diagram D_n is then defined by

$$K_{D_n} = a^{-w(D_n)} Q_{D_n}(a, z),$$

where the "writhe number" $w(D)$ of an oriented knot or link diagram D is defined as the algebraic sum of signs ± 1 attached to the crossings according

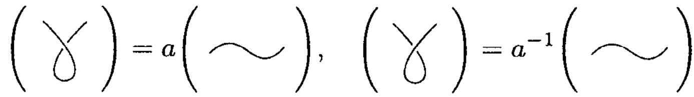


Fig. A.3. The first Reidemeister move and the state polynomial

to the orientation of the 2-frame determined by each crossing, with the first vector of the frame given by the (directed) upper branch (see Figure A.4). (Thus the orientation of the diagram D is significant only for the number

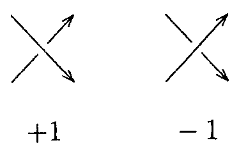


Fig. A.4. The writhe number, determined by the orientations of the crossings

$w(D)$.) This completes the definition of the Kauffman polynomial, an invariant of the knot or link in question.

For any oriented diagram D the Jones polynomial of D is then given by

$$J_D(t) = \frac{K_D(z = t^{-1/4} - t^{1/4},\ a = t^{3/4})}{t^{1/2} - t^{-1/2}}.$$

On the other hand by a theorem of Jaeger the Kauffman polynomial of a diagram can be expressed as a linear combination (with appropriate coefficients) of HOMFLY polynomials of certain diagrams associated with the given diagram.

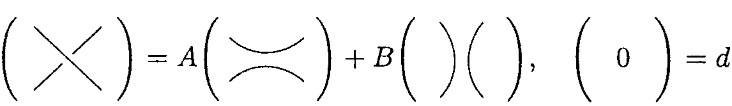


Fig. A.5. Defining the bracket polynomial and the Jones polynomial

Kauffman also defines a polynomial S_{D_n} via the recurrence relations (see Figure A.5)

$$S_{D_n}(A, B; d) = AS_{D_{n-1,a}} + BS_{D_{n-1,b}},$$

with $S_{D_0} = d$ for the trivial diagram D_0. Upon substituting

$$B = A^{-1}, \quad -d = A^2 + A^{-2},$$

one obtains the Jones polynomial in the form

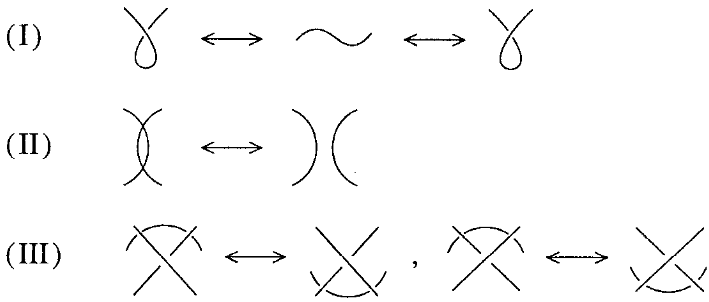


Fig. A.6. The Reidemeister moves

$$J(A^4, A^{-4}) = (-A)^{-3w(D)} S(A, B; d), \quad t = A^{-4}.$$

(The "bracket" polynomial S will appear later on in the definition of the Turaev-Reshetikhin invariant of 3-manifolds using Kauffman's approach.) Thus Kauffman provides two different ways of arriving at the Jones polynomial.

The topological invariance of all of the above polynomials can be established in a purely combinatorial manner using "Reidemeister moves", which are elementary changes (of three types) of diagrams, realizing equivalences of knots and links (see Figure A.6).

Thus one has merely to check that each polynomial remains unaffected by Reidemeister moves; the appropriate recurrence relation may then be used to calculate the polynomial. (It should be noted that the identification of these elementary moves was non-trivial in the era of Reidemeister and Markov (the 1930s and 1940s), but that the introduction of the concept of "generic properties" by Whitney, Pontryagin and Thom (from about 1935 till the 1950s) simplified the problem to the point of becoming an exercise for students: "Consider a generic deformation and decompose it into elementary topological moves.")

In older treatments of knot theory (in particular in "Modern Geometry", Part II, by Dubrovin, Fomenko and the present author) the definition of the Alexander polynomial is couched in terms of the somewhat unintuitive operation of "differentiation in the group ring". It has been pointed out (by L. Alania in the author's Moscow seminar) that this "differentiation" can be arrived at via the following topological route: Starting with the oriented diagram of the knot or link K on the plane, one calculates in the standard manner a presentation of the group $\pi_1(M)$ of the knot ($M = S^3 \setminus K$), obtaining one generator for each edge of the diagram (see Figure A.7) and a pair of relations for each crossing. Since one relation of each such pair simply equates the pair of generators corresponding to the edges forming the upper branch of the crossing, the presentation reduces immediately to the standard one involving the same number of generators and relations. The 2-complex L with exactly

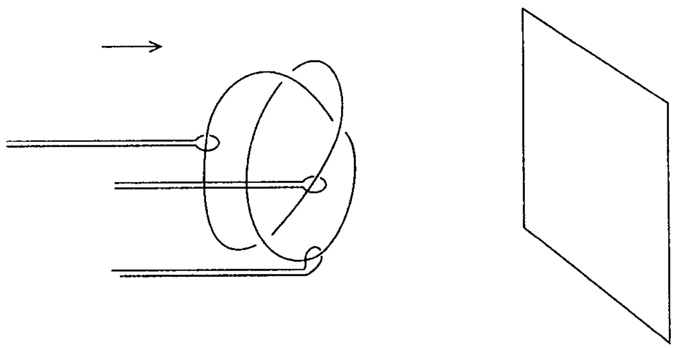


Fig. A.7. Calculating the fundamental group

one 0-cell, and with 1-cells labelled by generators and 2-cells labelled by the relations, is then a deformation retract of M. Lifting to the universal cover we obtain a boundary operator on a complex of free $\mathbb{Z}[\pi_1]$-modules, which takes the form of a square matrix with entries from this group ring, and it is this matrix which is related to the above-mentioned "differentiation", as follows. Denoting the generators by a_i and relators by r_j, one defines the operator ∂_{a_i} by

$$\partial_{a_i}(a_j) = \delta_{ij},$$
$$\partial_{a_i}(bc) = \partial_{a_i}(b) + b\partial_{a_i}(c);$$

the matrix in question then has entries q_{ij} given by

$$q_{ij} = \partial_{a_i}(r_j).$$

Mapping each generator a_i to t we obtain a complex of modules over the ring of integer Laurent polynomials, with boundary operator the corresponding square matrix now with Laurent polynomials as entries. The determinant of this matrix turns out to be zero, and the highest common factor of its cofactors, after multiplication by a suitable power of t, turns out to be just the Alexander polynomial $A(t)$.

The homological treatment of the Alexander polynomial by Milnor in the 1960s, formulated in terms of the torsion of the first homology module over the Euclidean ring $\mathbb{Q}[t, t^{-1}]$ of the $\mathbb{Z}$-covering of L, may be given along these lines. A note by the author in Soviet Math. Doklady contains the following more attractive exposition. Consider all one-dimensional representations $\rho : \pi_1(M) \longrightarrow \mathbb{C}$, and the homology $H_i^\rho(M; \mathbb{C})$ with local coefficients in such a representation. In fact we may take M to be any manifold, replace $\mathbb{C}$ by $\mathbb{C}^n$, and consider more generally the space $\mathrm{Rep}_n(\pi_1)$ of all representations of π_1 in $GL_n(\mathbb{C})$, an algebraic variety over $\mathbb{Z}$. The rank $b_i(\rho)$ of the homology group $H_i^\rho(M; \mathbb{C}^n)$ then defines an integer-valued function on this algebraic variety, which is constant almost everywhere (i.e. outside certain "jumping" algebraic subvarieties $W_j \subset \mathrm{Rep}_n(\pi_1)$ over $\mathbb{Z}$); the value of $b_i(\rho)$ jumps on each such subvariety, jumps further on their pairwise intersections, and so on.

In the particular case we were considering, i.e. where $M = S^3 \setminus K$, K a knot, and $n = 1$, we obtain the variety $\mathrm{Rep}_1(\pi_1) = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$. One has the following result:

For $i = 1$ the jumping subvarieties constitute a finite set of points which turn out to be just the roots of the Alexander polynomial.

For a link with k components, we have $H_1(M) = \mathbb{Z}^k$ and $\mathrm{Rep}_n(\pi_1) = (\mathbb{C}^*)^k$, and one obtains corresponding jumping subvarieties, and a "generalized Alexander polynomial" in k variables. A modern elementary combinatorial treatment of the multi-variable Alexander polynomial was given a few years ago by H. Murakami.

The algebraic geometry in this picture is homotopy invariant, but may be complicated, and even in the case $M = S^3 \setminus K$, K a link, is not well understood.

The present author had conjectured that one should be able to extract the Jones and HOMFLY polynomials directly from the structure of the jumping subvarieties in the case $n = 2$. However a few years ago Le Tu, in Moscow, investigated these subvarieties in the interesting case of the 2-bridge knots, and discovered that their structure is far more complex than expected. Thus the problem of finding a classical algebraic-topological treatment of the Jones polynomial remains open.

We now give a third definition of the Alexander, Jones, and HOMFLY polynomials, involving the braid groups B_n. We recall the definition of these groups:

Consider the group with generators σ_i, $i \in \mathbb{Z}$, and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

The nth *braid group* B_n is then the subgroup generated by n consecutive generators $\sigma_{k+1}, \ldots, \sigma_{k+n}$. There is thus a natural embedding of B_n in B_{n+1}. (It is more standard to take B_n to be the subgroup generated by $\sigma_1, \ldots, \sigma_n$, i.e. to take $k = 0$.)

By a theorem of Markov (Jr.) of the 1940s, the set of equivalence classes of knots and links is in natural one-to-one correspondence with the classes of braids under the equivalence relation determined by the following two *elementary operations (Markov moves)*:

1. Conjugation of any braid $a \in B_n$ by any element $c \in B_n$: $a \sim cac^{-1}$.

2. Multiplication of any braid $a \in B_n$ by $\sigma_{n+1}^{\pm 1} \in B_{n+1}$: $a \sim a\sigma_{n+1}^{\pm 1}$.

Two braids are then *equivalent* if one can be transformed into the other by means of a finite sequence of elementary Markov moves.

The idea behind this correspondence is both simple and natural. Consider the circle $S^1 \subset \mathbb{R}^3$ given by $x^2 + y^2 = 1$, $z = 0$, and a tubular neighbourhood T of small radius: $\mathbb{R}^3 \supset T \supset S^1$. A *transverse knot* or *link* is then defined to be a simple closed curve (or collection of such) in T with tangent vector everywhere transverse to the 2-balls orthogonal to the circle. A braid is then obtained from a transverse knot or link by cutting the tubular neighbourhood orthogonally to the circle at any point. It was in fact Alexander who first showed that any knot or link in $\mathbb{R}^3$ is isotopic to a transversal one; Markov corrected a deficiency in the argument.

Any linear representation ρ of the nth braid group B_n in $GL_n(A)$, where A is any commutative and associative ring, has character (trace function) invariant under the first elementary Markov operation above:

$$\chi_\rho(a) = \mathrm{Tr}[\rho(a)], \quad a \in B_n.$$

To obtain such a representation to start with (for all n simultaneously) one may try to represent the generators σ_i, $i \in \mathbb{Z}$, by matrices satisfying the above defining relations. Jones obtained a class of representations in this way using the technique for solving certain integrable models of statistical physics and quantum field theory; thus he was able to obtain from the Yang-Baxter equation a special "local, translation-invariant" class of representations of all braid groups simultaneously.

Let V be a vector space and

$$S : V \otimes V \longrightarrow V \otimes V$$

a linear transformation with the property that on the tensor product $V \otimes V \otimes V$ of three copies of V, we have

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23},$$

where S_{ij} coincides with S on the tensor product of the ith and jth copies of V in $V \otimes V \otimes V$, and restricts to the identity map on the remaining copy. (Note that in the theory of solvable models of statistical mechanics one may find the Yang-Baxter equation written more generally for a map of the form $R = \tau S$, the composite of $S : V_i \otimes V_j \longrightarrow V_i \otimes V_j$ with the permutation map $\tau : v_i \otimes v_j \longrightarrow v_j \otimes v_i$, where now the factors of $V_i \otimes V_j \otimes V_k$ may be different, in the form

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.)$$

Defining

$$\rho(\sigma_i) = S_{i,i+1},$$

we obtain a representation ρ of the infinite braid group by linear transformations of the infinite tensor product of countably many copies of V, restricting to a representation ρ_n of the nth braid group B_n by linear transformations of the space $V \otimes V \otimes \ldots \otimes V$ ($(n+1)$ factors). Thus the image $\rho(\sigma_i)$ of each generator σ_i is determined by its action "locally" on $V_i \otimes V_{i+1}$ (since it acts identically on all other factors), and is "translation-invariant" in the sense that for all i, $\rho(\sigma_i)$ acts in the same way as S acts on $V \otimes V$.

(One may easily imagine variants of this construction using different (periodic) partitions of $\mathbb{Z}$; however to date no examples of interest have been found.)

A representation ρ is said to have the *Markov property* if there exists a constant A independent of n such that for every n and every braid $a \in B_n$ one has

$$Tr[\rho_n(a)]A = Tr[\rho_{n+1}(a\sigma_{n+1})].$$

Then according to Jones, given such a representation ρ, the quantity

$$A^{-w(a)}Tr[\rho_n(a)], \quad a = \sigma_{i_1}^{j_1} \dots \sigma_{i_m}^{j_m} \in B_n,$$

where $w(a) = \sum_{k=1}^{m} j_k$, affords a topological invariant.

We shall give below some important examples of such representations yielding as invariants the Alexander, Jones and HOMFLY polynomials. However before doing so we give a direct geometric explanation of the connection between knots and solutions of the Yang-Baxter equation. Let R_{cd}^{ab} and $\overline{R}_{k\ell}^{ij}$ be two solutions of the Yang-Baxter equation, where the indices all run over the same set I indexing a basis for the vector space V. Given a diagram D, one labels each crossing with a copy of the solution R or $\overline{R}$ according as the crossing is "positive" or "negative" in the earlier sense. Each incoming edge is labelled with the appropriate lower indices (cd or kl) and each outgoing edge with the appropriate upper indices. On carrying out a total tensor contraction of the product of all Rs and $\overline{R}$s associated with all crossings of the diagram, one obtains a number $\langle D \| R, \overline{R} \rangle$ depending on the diagram D and the solutions R and $\overline{R}$.

The number $\langle D \| R, \overline{R} \rangle$ gives a topological invariant of the corresponding knot or link under two more additional conditions. First, the number $\langle D \| R, \overline{R} \rangle$ is invariant with respect to the second and the third Reidemeister moves provided the following "channel unitarity" and "cross-channel unitarity" equations hold:

$$\overline{R}_{ij}^{ab} R_{cd}^{ij} = \delta_c^a \delta_d^b;$$

$$R_{jb}^{ia} \overline{R}_{ic}^{jd} = \delta_c^a \delta_d^b.$$

Secondly, let $R, \bar{R}$ satisfy the restrictions

$$R_{ad}^{ab} = A\delta_d^b, \bar{R}_{ad}^{ab} = A^{-1}\delta_d^b.$$

Then the number $A^{-w(D)}\langle D \| R, \overline{R} \rangle$ is invariant with respect to the first Reidemeister move and therefore affords a topological invariant.

(Conditions of this type, together with the associated terminology, come from the interpretation by Yang, Zamolodchikov, and others, of the entities $R, \overline{R}$ in terms of the "factorizable scattering amplitude" in integrable models of 2-dimensional quantum field theory. The "Baxter-type" integrable 2-dimensional lattice models of statistical mechanics furnish different interpretations. Note however that in both cases the actual Yang-Baxter solutions are much more complicated, depending on an additional "spectral parameter".)

It is convenient to introduce "cup" and "cap" matrices M^{ab} and M_{ij} satisfying

$$M^{ai}M_{ib} = \delta^a_b.$$

The pair of solutions of the Yang-Baxter equation related by

$$\overline{R}^{ab}_{cd} = M_{ce}R^{ea}_{dh}M^{hb},$$

then satisfy both channel and cross-channel unitarity. These matrices arise from the diagrams obtained from diagrams of braids by eliminating a single crossing; they are associated with the extrema of the Morse function on the xz-plane which projects the diagram on the z-axis (see Figure A.8). (Here we are assuming the standard definition of an n-braid as consisting of n nonintersecting and nonself-intersecting curve segments within the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$, starting at fixed points $P_1, \ldots, P_n$ in the base ($z = 0$) and ending at the points $Q_1, \ldots, Q_n$ vertically above these in the lid ($z = 1$)). The diagram of such an n-braid is then the projection of the braid on the xz-plane with the nature of the crossings indicated (as for knots). The corresponding *closed* braid is constructed by joining P_1 to Q_1, P_2 to $Q_2, \ldots, P_n$ to Q_n in the obvious way by standard unknotted curve segments outside the cylinder (see Figure A.9). Given the diagram D of a closed braid B, one

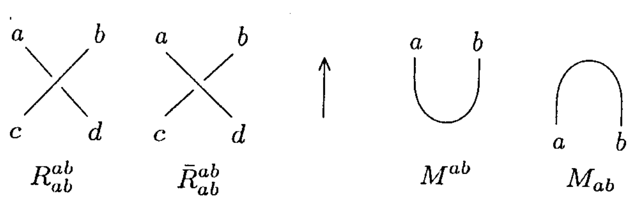


Fig. A.8. Defining R-matrices and cup and cap matrices

defines the *Markov trace* by

$$\tau(D) = Tr[\eta \otimes \eta \otimes \ldots \otimes \eta\rho_n(B)],$$

where ρ is the representation obtained from the solution R of the Yang-Baxter equation, and

$$\eta^b_a = M_{ai}M^{bi}.$$

Consider the explicit solution R given by

$$R^{ab}_{cd} = AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d,$$

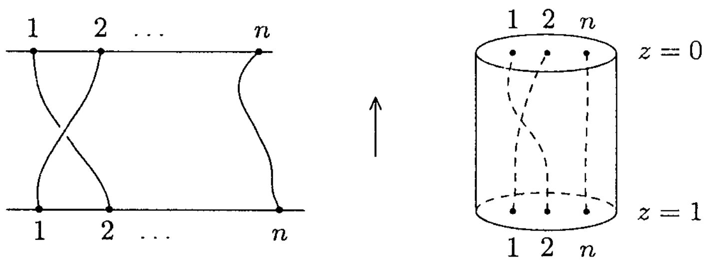


Fig. A.9. The diagram of a closed braid on the plane. A closed braid in the cylinder

where $i^2 = -1$, the matrix M is of the form

$$M = \begin{pmatrix} 0 & iA \\ -iA^{-1} & 0 \end{pmatrix},$$

and the index set $I = (+, -)$. The diagram D of the closed braid B in question is considered here to be unoriented. However D has a natural orientation determined by the direction of the z-axis in the (x, z) plane. This yields the formula

$$J(A^4, A^{-4}) = (-A)^{-3w(D)}\tau(D)$$

for the Kauffman-Jones polynomial discussed above.

In the case of oriented diagrams one has an explicit solution R given by the formula

$$R^{ab}_{cd} = \left(\left(q - q^{-1}\right)[a < b] + q[a = b]\right)\delta^a_c\delta^b_d + [a \neq b]\delta^a_d\delta^b_c,$$

where $[P]$ is defined to be 1 if the proposition P is true and 0 if P is false, and the index set $I = I_n = (-n, -n+2, \dots, n)$. The associated solution $\overline{R}$ may be obtained by replacing q by q^{-1} and a, b by $-a, -b$ in this formula. These Yang-Baxter solutions yield the following topological invariant of a given knot or link with oriented diagram D:

$$P^{(n)}_D = \left(q^{(n+1)}\right)^{-w(D)} \langle D \| R, \overline{R} \rangle.$$

The HOMFLY polynomial may then be obtained by taking the index set I of cardinality $n + 1$ (where $n \geq 1$) and by making the substitution $y = q^{n+1}$. In the case $n = 1$ the substitution $q = t^{1/2}$ gives the Jones polynomial. Note that the substitution $q = 1$ does not yield any new information.

Consider now "tangles", i.e. oriented diagrams D_1 in the strip $0 \leq z \leq 1$ with one input edge and one output edge, ending at the lines $z = 0$ and $z = 1$ respectively. For the index set $I = (-, +)$ we always label both the input and output edges with the index $+$. One then obtains the Alexander polynomial for the closed oriented diagram D in the plane by cutting it along an (arbitrarily chosen) external edge.

Following Jaeger and Kauffman, one may obtain the Alexander polynomial from the following R-matrix: taking the index set $I = (+1, -1)$, one sets

$$R = (q - q^{-1})X_{+-} + qX_{++} - q^{-1}X_{--} + Y,$$

and then obtains $\overline{R}$ by substituting q^{-1} for q and X_{-+} for X_{+-} ; here the tensors X_{sp} are given by

$$(X_{sp})^{ab}_{cd} = \begin{cases} 1 & \text{if } a = c = s \text{ and } b = d = p, \\ 0 & \text{otherwise,} \end{cases}$$

and the tensor Y by

$$Y^{ab}_{cd} = [a \neq b]\delta^a_d\delta^b_c,$$

where $a, b, c, d = \pm 1$.

The Alexander polynomial may also be obtained from the "Burau representation" of the braid group on the direct sum of one-dimensional spaces $\mathbb{R}_i = \mathbb{R}$, given by

$$\rho(\sigma_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

here 1 denotes the identity matrix of arbitrary size and T is the 2×2 matrix

$$T = \begin{bmatrix} 0 & t \\ 1 & (1-t) \end{bmatrix}$$

acting on $\mathbb{R}_i \oplus \mathbb{R}_{i+1}$. The Alexander polynomial is then given by

$$A_{D_a} = (\det)^*(\rho(a) - 1_n),$$

where D_a is the relevant closed braid obtained from the braid a, and $(\det)^*$ denotes any $(n-1)$-minor of the matrix $\rho(a) - 1_n$. As obtained by these means the Alexander polynomial is well-defined up to multiplication by $\pm t^m$, m any integer.

It is worthwhile pointing out the following interesting identity for the exterior powers of the linear transformation M:

$$\det(M - \lambda 1_n) = \sum_{j=0}^{j=n} Trace((-\lambda)^k \Lambda^k M);$$

this leads to a (non-Markov) trace, and to Yang-Baxter solutions R, $\overline{R}$ essentially equivalent to those used above in constructing the Alexander polynomial using tangles.

There are three nice algebraic objects that have turned out to be very useful in the modern theory of knots and 3-manifolds: the *Hecke algebra*, the *Temperley-Lieb* (or *Jones*) *algebra*, and the *Birman-Wenzl algebra.* They are defined as follows:

A *Hecke algebra* is generated by the elements σ_i, $i \in \mathbb{Z}$, satisfying the braid relations (see above), together with relations of the form

$$\sigma_i - \sigma_{i-1} = z,$$

where z may be either a number or in the centre of the algebra.

The *Temperley-Lieb algebra* (TL) over the ring $\mathbb{Z}[A, A^{-1}]$ is given by generators v_i, $i \in \mathbb{Z}$, satisfying the relations

$$v_i v_j = v_j v_i, \quad |i-j| > 1,$$
$$v_i v_{i\pm 1} v_i = v_i.$$

Here A may be taken as some number. There is an associated *abstract trace*, i.e. a linear map

$$f : TL \to \mathbb{Z}[A, A^{-1}],$$

satisfying $f(uv) = f(vu)$ for all u, v. Note also that one obtains a representation ρ of the braid group in TL in the general form

$$\rho(\sigma_i) = av_i + b,$$

for appropriate numbers a, b.

The *Jones algebra* is the same as TL, however considered rather as given in terms of generators $e_i = \delta^{-1} v_i$, where $-\delta = A^2 + A^{-2}$; the e_i are projectors: $e_i^2 = e_i$.

The *Birman-Wenzl algebra* is also generated by the σ_i, with the additional relations

$$\sigma_i - \sigma_i^{-1} = z(v_i - 1).$$

This is easily seen to be realizable as a subalgebra of TL.

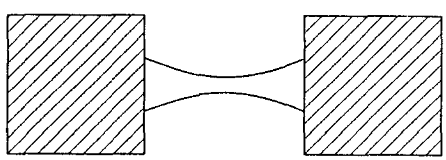
From braids and the Yang-Baxter equation the theory of "quantum groups" has been constructed (by Drinfeld, Jimbo, Sklyanin, Kulish, Faddeev, Takhtadjan, Reshetikhin, Voronovich, and others). This represents a pleasing systematization of the behaviour of the most familiar solutions of the Yang-Baxter equation in terms of certain algebras (shown by Drinfeld to be Hopf algebras equipped with a nice additional structure called a "quantum double". (In his terminology this is "quantum deformation" of the enveloping algebra of a semisimple Lie algebra.)

This striking terminology derives, in fact, merely from the property that to a first approximation this deformation is determined by a certain Poisson bracket on the Lie group, so that, on the face of it, it appears to represent some sort of quantization of the group as of a phase space. (Such brackets had made an appearance earlier in connexion with certain (not all) integrable models of the theory of solitons.) However this procedure does not in fact represent any quantization of the group or Lie algebra as objects measuring the symmetry of some physical system to be quantized.

Several examples show that this very appealing construction is more like a "discretization", in a certain subtle, nontrivial sense, than a "quantization" as this term is usually employed in the context of symmetry in analysis (and in fact similar nontrivial "q-deformations of analytic calculus" can be traced back to the 19th century). However the term "quantum group" sounds impressive, and lends a, perhaps somewhat spurious, air of excitement to the topic. (A first-rate quantum physicist at the Landau Institute who had himself discovered many nontrivial solutions of the Yang-Baxter equation, once told the present author (in the late 1980s) that he did "not know what 'quantum group' means, but it sounds very attractive". It seems however that he himself had in effect already applied quantum groups to the discretization of the Schrödinger operator in a magnetic field.)

It seems likely that the technique of quantum groups can be used to improve some of the known results concerning the above polynomial invariants. However hitherto nothing along these lines has been achieved aside from a systematization of known results, although the approach continues to look promising.

The Jones polynomial has led to some beautiful new results. One such result is as follows. In the late 19th century the physicist Tait, following a suggestion of either Maxwell or Thompson (Lord Kelvin), began to study knots. Having compiled a table of the simplest nontrivial knots, he observed in particular various properties of the "alternating" knots in his table, i.e. those where undercrossings alternate with overcrossings as one traces out the oriented diagram of the knot. Using the Jones polynomial it was shown by Kauffman and Murasugi in the late 1980s that, provided such an oriented diagram is irreducible (i.e. is not obtainable by joining two nontrivial diagrams together by means of a pair of nonintersecting line segments (see Figure A.10)), then the number of crossings is a topological invariant of the knot. (In fact the number of crossings of an irreducible alternating diagram is equal to the difference between the largest and smallest powers of t occuring in the Jones polynomial $J(t,t^{-1})$.)



$$K_1 \# K_2 = K$$

Fig. A.10. A reducible diagram

Although this had never been considered a major question of knot theory, it is significant that the methods of classical topology had seemed inadequate for settling it. This (and other results) provide evidence of the great importance of the discovery of the Jones polynomial.

We mention in passing also the following nice property of the Alexander polynomial (known for some time): For a *ribbon knot* (i.e. one bounding an immersed 2-disc in $\mathbb{R}^3$) there exists a polynomial $f(t)$ such that

$$A(t,t^{-1}) = f(t)f(t^{-1}).$$

§3. Vassiliev Invariants

The additive properties of the Alexander and Jones polynomials have a very attractive interpretation in terms of "Vassiliev invariants". The theory of these invariants has appeared over the last 5 years or so; chief among its exponents are V. Vassiliev, D. Bar-Natan, J. Birman , X-S. Lin, M. Kontzevich.

Consider the space F_k of all immersions G_k of the circle S^1 in $\mathbb{R}^3$ with the property that the image has exactly k double points (crossings), $P_i, \overline{P}_i \in S^1$, $i = 1, \ldots, k$, at each of which the image curve meets itself transversely. By perturbing the image near one such intersection point we obtain a pair G^+_{k-1}, G^-_{k-1} of immersions of S^1 each with $(k-1)$ crossings, and therefore belonging to the space F_{k-1} (see Figure A.11). Consider a topological invariant

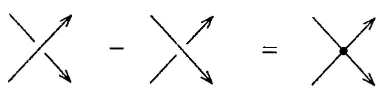


Fig. A.11. Defining the Vassiliev derivative

f_k of the space F_k, i.e. any function from the set of components of F_k to some abelian group A:

$$f_k : \pi_0(F_k) \to A.$$

Thus in particular when $k = 0$ we obtain precisely the topological invariants of knots. We call such an invariant f_k a *Vassiliev derivative* of some invariant f_{k-1} if the following equation holds:

$$f_{k-1}(G^+_{k-1}) - f_{k-1}(G^-_{k-1}) = f_k(G_k).$$

A topological invariant of knots is then called a *Vassiliev k-invariant* if its $(k+1)$st Vassiliev derivative is zero.

By a result of Bar-Natan, all coefficients of the Alexander polynomial are Vassiliev invariants. A few years ago Bar-Natan, Lin, and Birman were also able to deduce from the additive properties of the HOMFLY polynomial that after performing the substitution.

$$y = \exp(Nx),\ x = \exp(\frac{z}{2}) - \exp(-\frac{z}{2}),$$

the coefficients of the HOMFLY polynomial as a series in z are Vassiliev invariants.

There is an attractive formula due to Kontzevich expressing all Vassiliev invariants analytically in terms of multiple integrals, assuming that the knot or link diagram comes with some generic Morse function (for instance the projection of the planar diagram on the y-axis).

There is also a purely combinatorial characterization of all possible Vassiliev invariants, which, however, is at present computationally impractical for calculating specific invariants. (In fact there are at present no known specific Vassiliev invariants other than the above-mentioned coefficients of the Alexander and HOMFLY polynomials of knots.)

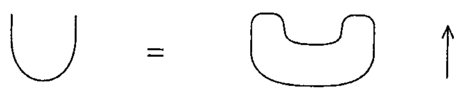


Fig. A.12. The diagram U

This combinatorial description is as follows. Define a *chord diagram* to be a circle with $2k$ distinct points labelled $P_j, Q_j, j = 1, 2, \ldots, k$, marked on it. (In fact only the cyclic order of the points around the circle will be significant here.) We now impose the following relations on the free abelian group freely generated by all chord diagrams (with $2k$ points):

1. A chord diagram is set equal to zero if it contains a pair P_j, Q_j of points not "linked" with any other pair, i.e. if for any i one has $P_j < P_i < Q_j$ in the cyclic order round the circle, then also $P_j < Q_i < Q_j$.

2. Given any $(k-2)$ pairs $P_j, Q_j,\ j = 1, \ldots, k-2$, of points and three further points A_1, A_2, A_3 on the circle, all distinct, one may construct a chord diagram with $2k$ points as follows: First replace one of the three points A_1, A_2, A_3 by two adjacent points (close to the chosen A_i) labelled P_{k-1}, P_k, and relabel one of the remaining points Q_{k-1} and the other Q_k; since this relabelling can be carried out in two distinct ways, there are two chord diagrams, C_+ and C_- say, obtainable in this way; thus if A_1 is the point first chosen, then C_+ and C_- are given by

$$C_+ : A_2 = Q_{k-1},\ \ A_3 = Q_k,$$
$$C_- : A_2 = Q_k,\ \ A_3 = Q_{k-1}.$$

One then imposes the relations ensuring that the difference $C_+ - C_-$ is independent of the choice of the first point of the triple A_1, A_2, A_3.

The resulting quotient group B_k then has encoded in it all Vassiliev k-invariants.

The graded sum

$$B = \sum_{k \geq 0} B_k,$$

can be given the structure of a graded Hopf ring by defining multiplication via the connected sum of chord diagrams along arcs disjoint from the marked points. (That this multiplication is well-defined is a consequence of the above defining relations of the B_k.)

We now describe the formula of Kontzevich from which it follows easily that each element of the B_k does indeed correspond to a Vassiliev invariant. Let K be an oriented knot in $\mathbb{R}^3$ given via co-ordinates $z = z(t)(= x(t) + iy(t)), t$. (Here t may be regarded as a generic Morse function on K.) Consider the following sum (of "Kniznik-Zamolodchikov monodromy"):

$$Z(K) = \sum_m \frac{1}{(2\pi i)^m} \int \cdots \int_{t_{\min} < t_1 \ldots t_n < t_{\max} \| AS\{\text{pairs } z_j, z_j^1\}} (-1)^{N\downarrow} \bigwedge_j d\log(z_j - z_j^1) D_N;$$

here the integral is taken with respect to the variables t_s. For each value of t_s one has the set of points $(z(t_s), t_s)$ on the knot K, and from this set one chooses (the choice being indicated by AS) m_s pairs

$$(z_j(t_s), t_s),\ (z_j^1(t_s), t_s),\ j = 1, \ldots, m_s,$$

where $\sum_s m_s = m$. The associated chord diagram is then just the circle with this collection of pairs of points. The symbol $N \downarrow$ denotes the number of descending chords with respect to t. Writing $s = s_K$ for the number of minimum points of the knot K, and $\gamma := Z(U)$ where U is as in Figure A.12, one has the following *formula of Kontzevich* :

$\gamma^{-s} Z(K)$ is a topological (Vassiliev) invariant.

§4. New topological invariants for 3-manifolds. Topological Quantum Field Theories

We shall now describe some applications of these ideas to the construction of new topological invariants of 3-manifolds. These invariants were discovered by Witten as natural non-abelian generalizations of the Schwarz representation of the Reidemeister-Ray-Singer invariant, in the form of a functional integral of "Chern-Simons" type, i.e. a multi-valued action functional on the space of gauge-equivalent classes of Yang-Mills fields (i.e. differential-geometric connexions on principal G-bundles where G is any compact Lie group (e.g. $G = SU_2$)).

There is a "partition function" defined on the class of 3-manifolds of the following form (provided this can be made sense of):

$$Z(M^3) = \int DA(x) \exp\{ikS(A)\}\,,\ i^2 = -1.$$

Here k is an integer by virtue of the "topological quantization of the coupling constants" (as formulated for multi-valued functionals by the present author in 1981, by Deser-Jackiv-Templeton in the special case of the Chern-Simons action (1982), and by Witten in 1983), which in turn is a consequence of the requirement that the "Feynman amplitude" $\exp\{ikS\}$ should be single-valued. Note that the CS-action has the following local form:

$$S(A) = Tr(\partial_i A_i - \partial_j A_j + \frac{2}{3}A^3).$$

Using a Heegaard decomposition of the 3-manifold along some surface, Witten developed a beautiful "Hamiltonian" approach to topological quantum field theory, enabling him to reduce the problem of defining and calculating the quantity $Z(M^3)$ to certain problems of 2-dimensional conformal field theory on this surface. A perturbation series for $Z(M^3)$ in the variable $\frac{1}{k} \to \infty$, constructed by Axelrod and Singer (and in a special case by Kontzevich) yielded some interesting topological quantities; the properties and possible applications of these has, however, hitherto not been investigated.

The problem of giving an exact, rigorous, nonperturbative, purely topological definition of this invariant was solved by Reshetikhin and Turaev. The theory of this type of invariant has been further developed by several people, including Viro, Lawrence, and Lickorish. We shall now describe some of these invariants, as elaborated in the work of Lickorish of the late 1980s. (Our account of this work is largely taken from Kauffman's book "Knots and Physics".)

One can obtain any 3-manifold M^3_L by performing surgery along a suitable link or knot L in S^3. This realization of a 3-manifold was used by Lickorish, Zieschang and the present author in the early 1960s in the construction of a non-singular 2-foliation on an arbitrary closed 3-manifold, starting from the Reeb fibration of S^3 and performing a "tubulization" operation along some suitable transversal knot or link, i.e. along some appropriate closed braid in $S^1 \times D^2 \subset S^3$. It seems likely that Dehn had already been aware of the possibility of obtaining an arbitrary 3-manifold via surgery along a knot or link in S^3; the reduction to closed braids is certainly due to Alexander (see above). At some time in the 1980s it was pointed out by Kirby that the latitude in this construction can be characterized via two standard elementary operations figuring in the multi-dimensional surgery theory used by Smale and Wallace in the proof of the n-dimensional Poincaré conjecture for $n > 4$. The two operations in question are as follows (where we now assume that a link comes with a "frame" - see below):

1. The first Kirby operation consists in the addition of a further unknotted component to the link, separated from the original link by a "wall" $\mathbb{R}^2 \subset \mathbb{R}^3$, and with frame of linking number ± 1 (see below).

2. The second Kirby operation is as follows: Choose two components L_i, L_j, and denote by L_i^1 the parallel shift of one of them (L_i). Join L_i and L_j by a Jordan arc equipped with a normal frame compatible with normal frames on L_i and L_j not meeting the other components. Form the "connected sum" $\overline{L}_j$ of L_i and L_j along this framed arc, analogous to the "connected-sum" construction of the theory of framed surgeries (as for instance in the calculation of $\pi_3(S^2)$ according to Pontryagin's scheme). Finally, replace the pair L_i, L_j by the new pair $L_i^1, \overline{L}_j$ (the remaining components being left as before).

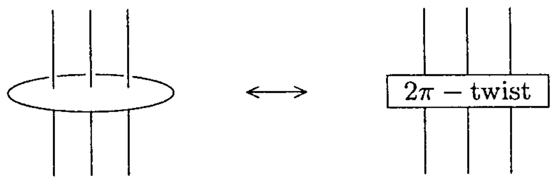


Fig. A.13. A 2π–twist

(Instead of the second operation, 3-manifold theorists often prefer to use a combination of the two operations. Thus for instance if the link has the form of an unknotted framed circle with linking number ± 1 (see below) bounding a disc, together with a number of other components all intersecting this disc transversely, then one removes this unknotted component and gives a "2π-twist" to the others (see Figure A.13).) As mentioned above, in the present context the given knot or link is assumed to come equipped with a frame (i.e. a normal field of unit vectors in $\mathbb{R}^3$). A full topological invariant of such a frame is afforded by the linking number of the original curve (component) S^1 with its shift a small distance in the direction of the normal field. The *canonical normal frame* is the one for which this linking number is zero: $\{S^1, S^1_+\} = 0$. A surgery along a closed Jordan curve S^1 in a 3-manifold M^3, with trivial normal bundle $T(S^1) = S^1 \times D^2 \subset M^3$, is determined completely by the linking number of a framing of the curve: the shift S^1_+ is to be null-homotopic after the surgery, in the new manifold M^3_L (and if the surgery is along a link, then the same applies to each component of the link).

For a framed link L one has a *linking matrix*, a symmetric $n \times n$ matrix, where n is the number of components. The *index* $i(L)$ of the link is then defined to be the number of strictly negative entries in the diagonalization of the linking matrix.

A *blackboard frame* is the normal vector field to a planar diagram of a link L, obtained by means of a generic plane projection. A shift of the knot or link L a small distance in the direction normal to the plane of projection yields a *parallel* knot or link. If D is any diagram of the link, we shall understand that the same diagram D with the n components (numbered $1, 2, \ldots, n$ in some order) labelled with integers $i_1, \ldots, i_n$, represents a new link with i_q components parallel to the qth component (labelled i_q), and close to one another. Thus this new link has altogether $i_1 + \ldots + i_n$ components.

For each fixed positive integer r write $I_r = \{0, 1, \ldots, r-2\}$, and denote by c an arbitrary function from the set of components of the link (numbered $1, 2, \ldots, n$) to I_r. Given any numbers $\lambda_0, \lambda_1, \ldots, \lambda_{r-2}$, we define a number $\langle\langle L\rangle\rangle$ (where L denotes the link in question) by

$$\langle\langle L\rangle\rangle = \sum_c \lambda_{c(1)} \times \ldots \times \lambda_{c(n)} S_{D_L}(A, A^{-1}; \delta);$$

here the summation is over all such functions c, $\delta = -A^2 - A^{-2}$ (as in Kauffman's construction of the Jones polynomial-see earlier), and D_L denotes the diagram of the link L.

The formula

$$TR(M_L^3) = \delta^{-i(L)} \langle\langle L\rangle\rangle$$

then gives a topological invariant of the 3-manifold M_L^3 (the *TR* or *Turaev-Reshetikhin invariant*) provided $r \geq 3$, $A = \exp\{i\pi/2r\}$, and the numbers λ_q satisfy the overdetermined linear system

$$\delta^j = \sum_{i=0}^{i=r-2} \lambda_i T_{i+j}, \ j \geq 0,$$

where $T_j (j \in \mathbb{Z})$ denotes the quantity $S_{D_j}(A, A^{-1}; \delta)$ determined by the diagram D_j consisting of j distinct parallel copies of the elementary diagram with just one crossing (see Figure A.14). Although of considerable intrinsic interest, this invariant has not so far found use in the solution of any topological problem.

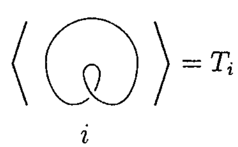


Fig. A.14. The elementary diagram and its parallel copies

The important notion of a "topological quantum field theory" (TQFT) has emerged in recent years, after it began to be realized that TQFT represents more than the mere construction of certain topological invariants of knots and manifolds via functional integration or the more rigorous combinatorial approach. We shall now describe an axiom system for TQFT, devised by Segal and Atiyah. (This system may perhaps be regarded as analogous to the well-known Eilenberg-Steenrod system of axioms for homology, which proved to be particularly useful in connexion with generalizations of homology such as K-theory and complex cobordism theory.)

An n-dimensional TQFT is postulated to be a category of closed, oriented $(n-1)$-dimensional manifolds V, each equipped with a Hilbert space H_V (usually finite-dimensional). The morphisms correspond to the oriented n-manifolds W with boundary V partitioned into two parts (the "inside" and "outside"):

$$\partial W = V = V_1 \cup V_2,$$

where V_1, V_2 carry the induced orientation. Any such cobordism W determines a linear map

$$f_W : H_{V_1} \to H_{\overline{V}_2},$$

(where the bar indicates the opposite orientation) which may be considered as an element of $H_{V_1 \cup V_2}$:

$$f_W \in H_{V_1 \cup V_2} = H_{V_1} \otimes H^*_{\overline{V}_2},$$

in view of the axiom

$$H_V = H^*_{\overline{V}};$$

(here the asterisk indicates the adjoint space). Clearly for a cylinder with the natural orientation, the map f_W will be the identity map. Thus a morphism is determined by an element $f_W \in H_V$ for any manifold W with boundary V.

If the manifold V happens to be the boundary of two manifolds W_1, W_2, we shall have two associated elements

$$f_{W_1}, f_{W_2} \in H_V.$$

We then have that the number $\langle f_{W_1}, f_{W_2}\rangle$ is a topological invariant of the closed manifold $W_1 \cup W_2$.

In classical topology there were only two examples of such invariants, namely the Euler characteristic of even-dimensional manifolds, and the signature of manifolds-with-boundary of dimension $4k$; the "Additivity Lemma", proved by Rohlin and the present author in the mid-1960s, shows that the latter example satisfies the axiom system. In these examples the spaces H_V are all one-dimensional, and the morphisms amount to multiplication by $\exp(a\tau(W))$, where τ is the signature (or the Euler characteristic in the simpler case). However we now have many new examples where the dimensions of the spaces H_V exceed 1; these represent highly nontrivial TQFTs, and yield the above invariants.

In the case $n = 2$ any cobordism can be achieved by iterating the elementary "trousers", i.e. the 2-manifold with boundary consisting of three circles $V_1, V_2, \overline{V}_3$, on which there is a Morse function definable with exactly one critical point and constant on each of the two parts $V_1 \cup V_2$ and $\overline{V}_3$ of the boundary. As a morphism in the appropriate TQFT this surface determines a tensor C^i_{jk} on the space H_V of the circle $V = S^1$; in fact its components are just the structural constants of a certain "Frobenius algebra", i.e. a commutative and associative algebra with identity element, and with a scalar product

$\eta_{ij} = \langle \ , \ \rangle$, satisfying $\langle fg, h\rangle = \langle f, gh\rangle$. The scalar product η_{ij} corresponds to the cylinder $S^1 \times I$ with boundary circles carrying the same orientation.

This construction leads to the description of the values of the "correlation functions" (or the vacuum expectations for products of fields in terms of TQFT, which are constant by definition) as follows. Let $\{e_i\}$ be a basis of the corresponding Frobenius algebra, then one can derive the following properties:

(a) the 0-loop correlation function of pairs is equal to η_{ij};
(b) the 0-loop correlation function of triples is equal to

$$C_{ijk} = \eta_{is} C^s_{jk};$$

(c) any 0-loop correlation function may be expressed in the form

$$\langle e_i e_j \dots e_k, 1\rangle;$$

(d) any k-loop correlation function may be expressed in the form

$$\langle e_i e_j \dots e_k, H^k\rangle,$$

where $H^k = \eta^{ij} e_i e_j$ is the appropriate element of the Frobenius algebra under consideration.

In the most interesting cases one has families of such theories (i.e. families of Frobenius algebras) depending on several parameters, and here the "axiomatization" of such families becomes very important. (A suitable axiom system was formulated by Witten, Dijgraaf, and the Verlinde brothers.) It should be noted that the space of parameters satisfies certain "soliton-type dispersionless integrable hierarchies" or "hierarchies of hydrodamic type" (in the sense of Dubrovin and the present author, involving the Hamiltonian formalism determined by a certain flat metric on the space of parameters).

Remark. There are some aesthetically very pleasing cases when the space H_V, $V = S_1$, turns out to be the even part of the cohomology ring of certain Calaby-Yau manifolds, and the associated Frobenius algebras correspond to certain "quantum deformations" of the cohomology ring. A rigorous treatment of this case, at least in part, has been given by Yau, Kontzevich, Manin, Tian, Ruan, Piunikhin, Givental, Salamon, MacDuff and others, using techniques from symplectic topology and algebraic geometry and topology. □

Following Witten, Dijgraaf, the Verlinde brothers, Dubrovin, Krichever and others, we shall describe an axiom system for the particular case of the 2-dimensional TQFT in the absence of a gravitational field. This approach involves a space with co-ordinates t^i, and a function $F(t)$ satisfying

$$\eta^{ks} \partial_i \partial_j \partial_k F(t) = C^s_{ij}(t), \tag{4.1}$$

where the metric η is constant in the t-coordinates, and $C(t)$ is made up of the structural constants of a family of time-dependent Frobenius algebras (associated with the same metric) at each value of t. We also have the equations

$$\eta_{ij} = \partial_1 \partial_i \partial_j F(t), \tag{4.2}$$

since the first basic vector corresponds to the identity in these Frobenius algebras. The equations (4.1), (4.2) are called the *associativity equations.* (Note that the equations (4.1) are the most difficult part of this system to study.) These provide a geometric description of the so called *Frobenius geometry* of the space with the co-ordinates t^i.

The following aesthetically very pleasing description of the Frobenius geometry is due to Dubrovin. Assuming the metric η is such that the tensor $\eta_{is}C^s_{jk}$ is symmetric, consider the operators $\partial_i - zC^j_{ik}$ depending on the parameter z and defining a connection on our space. It turns out that the curvature of this connection is zero provided the associativity equations (4.1), (4.2) hold (signifying that the Frobenius algebras $C(t)$ are associative and the tensor field C_{ijk} coincides with the third derivative of some function $F(t)$ which may be interpreted as the "free energy" for the corresponding TQFT).

Following the scheme of Dijgraaf and Witten (1991) one may use flat coordinates $h_{j,m}(t)$ (as formal series in the variable z) to define the "dispersionless hierarchy" and so–called "0-loop gravitation continuation" (in this TQFT), as follows:

$$h_j = \sum_{m=0}^{\infty} h_{j,m}(t), \quad j = 1, 2, \dots, \quad m \geq 0, \quad h_{j,0} = t^i \eta_{ij},$$

$$H_{j,m} = \int h_{j,m}(t(X))\, dX.$$

Here the $H_{j,m}$ may be interpreted also as commuting Hamiltonians (of the hydrodynamic type) for this hierarchy, with "times" $T^{j,m}$, and $X = T^{1,0}$, $t^j = T^{j,0}$. The Hamiltonian formalism here is determined by the flat metric η in the sense of Dubrovin and Novikov (early 1980s; corresponding examples may be found in the book *Modern Geometry, II,* by Dubrovin, Fomenko, Novikov). In particular, the "free energy" 0-loop was defined there as a special solution $\overline{F}(X,T)$. The partial derivatives with respect to the variables $T^{j,m}$ of the solution $\overline{F}(X,T)$ coincide with the "0-loop correlation functions" in this TQFT. The following initial conditions are imposed on the solution $\overline{F}(X,T)$:

$$\overline{F}(X,T) = F(t),$$

where $T^{j,m-1} = 0$ for $m > 1$, $T^{j,0} = t^j$ and $X = t^1$;
the condition for a "dispersionless string":

$$\partial_X \overline{F} = \sum T^{j,m} \partial_{T^{j,m-1}} \overline{F} + \frac{1}{2} \eta_{ij} T^{i,0} t^{j,0};$$

the 0-loop relation for the corresponding correlation functions:

$$\partial^3_{(i,m),(j,n),(k,p)}\overline{F} = (\partial^2_{(i,m-1),(l,0)}\overline{F}\ \eta^{ls}(\partial^3_{(s,0),(j,n),(k,p)}\overline{F}).$$

There are some simple cases when it is natural to conjecture the existence of a "k-dimensional loop gravitation extension" for all values of k. Here the dimension $n = 1$, and the "free energy" function satisfies the celebrated KdV hierarchy with the Schrödinger potential u having initial value $u(X) = (\partial_X)^2 F = X$, and vanishing for all other "times".

It turns out that the Schrödinger potential may easily be obtained as a formal power series with integral coefficients, in the variables T^m, $T^0 = X$. The coefficients of this formal power series are equal to the Chern numbers of certain holomorphic vector bundles over the corresponding moduli spaces of algebraic curves (of arbitrary genus and with certain points labelled); this result is due to Kontzevich (1992), who also proved the above conjecture of Witten.

A nice exposition of the differential-geometrical approach to the subject may be found in the recent lectures "Geometry of 2–dimensional topological field theories" (SISSA preprint, 1994) given by Dubrovin.

We conclude with a few more words on the associativity equations for the function $F(t)$: we note that in the case $\eta_{11} = 0$, $\eta_{12} = 1$, $\eta_{22} = 0$, the function $F(t)$ (for $n = 2$) has the form:

$$F(t) = \frac{1}{2}t_1^2 t_2 + f(t_2),$$

where f is an arbitrary function. There are very interesting 2-dimensional examples of TQFT in this case which are significant for physics.

Since this topic is quite recent, not all of the interesting observations that have been made concerning it have been given rigorous proof. It remains an intensely active area of research. The present author is not aware of the existence of a good survey article on the subject.

For this brief survey the book "Knots and Physics" by Kauffman, and lectures of Bar-Natan on Vassiliev invariants were the main sources; in addition a number of recent papers and the above–mentioned lectures by Dubrovin were consulted. Most of the remaining material was gleaned by the author from private conversations, lectures and seminars.

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The items listed below have not been cited in the text. This bibliography is intended as a supplement to the text — likewise making no pretensions as to completeness — by means of which readers may study in more basic and detailed fashion the areas of topology and its applications surveyed in the text. The books and articles which the author considers methodologically the best — in the pedagogical sense that, as it seems to him, they expound the important parts of topology and adjacent disciplines in a way that is especially intuitive, clear and convenient for the reader — are indicated by an asterisk $^{(*)}$.

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