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Topology in the 20th century: a view from the inside

S. P. Novikov

Foreword

The editors of the Russian academy's new *Great Russian Encyclopaedia* asked me for an article to be entitled "Algebraic Topology". On starting to write, I made a free improvisation, presenting the knowledge accumulated over a lifetime. I did not want to curtail my treatment in an artificial way, although the title itself imposes cuts of many important developments growing out from topology into other areas of mathematics. With one thing and another, the article turned out something like 10 times longer than could be included in the new encyclopaedia. I made a selection of the first few pages of what I had written for the requested encyclopaedia article. A preliminary version of the complete text was published in the Steklov Institute publication *Current Problems of Mathematics* (vol. 4).[1] At the recommendation of many friends and colleagues, I decided that it would be useful to publish a slightly polished version in *Uspekhi*. The first three parts below make up the article "Algebraic topology" in the *Great Russian Encyclopaedia*.

Algebraic topology

Algebraic topology (A.T.) is the area of mathematics that arose to study those properties of geometric figures and maps between them that are invariant under continuous deformations (or homotopies); here *geometric figures* are interpreted widely, to include any objects for which continuity makes sense. The aim of A.T. is, in principle, the complete enumeration of such properties; the name A.T. itself arises from the determinative role played by algebraic notions and methods in solving problems in this area. The most basic classes of objects whose properties are studied in A.T. are complexes (or polyhedra): simplicial complexes, cell complexes or other complexes; manifolds: closed or open manifolds, manifolds with boundary. Manifolds in turn subdivide into smooth (or differentiable), analytic, complex analytic, piecewise linear, and finally purely continuous topological manifolds; twisted products (fibre bundles) and their sections. The basic types of maps considered in A.T. are arbitrary continuous maps, piecewise linear maps or differentiable maps, together with subclasses of these: homeomorphisms, and in particular continuous, piecewise linear or smooth homeomorphisms (diffeomorphisms); embeddings of one object into another, and also immersions (or local embeddings). An important notion of A.T. is that of deformation; maps between some classes of objects are subject to deformations. The basic types of deformations are homotopies between (continuous, smooth

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or piecewise linear) maps, that is, arbitrary deformations of (continuous, smooth or piecewise linear) maps; isotopies (continuous, smooth or piecewise linear), that is, deformations of homeomorphisms, embeddings or immersions under which at every moment of the deformation the map remains a homeomorphism, embedding or immersion. The main internal problems of A.T. are the classification of manifolds up to (continuous, smooth or piecewise linear) homeomorphism, the classification of embeddings and immersions up to isotopy (or regular homotopy), the classification of general continuous maps up to homotopy. An important role in solving these problems is played by the classification of complexes and manifolds up to homotopy equivalence, or up to so-called homotopy type.

The wealth of ideas introduced by topology and by the great mathematicians who worked in topology placed it in the centre of world mathematics from the mid 20th century. For example, between 1950 and 2002, a total of 44 Fields Medals were awarded at World Congresses to active young mathematicians under 40 years of age and recognised as the most outstanding. Among these were Serre (1954), Thom (1958), Milnor (1962), Atiyah (1966), Smale (1966), Novikov (1970), Quillen (1978), Thurston (1982), Donaldson (1986), Freedman (1986), Witten (1990), Vaughan Jones (1990), Kontsevich (1998), whose central mathematical contributions during those years relate to topology; also Kodaira (1950), Grothendieck (1966), Mumford (1974), Deligne (1978), Yau (1982) and Voevodsky (2002), whose work is at the crossroads of the ideas of topology, algebraic geometry and homological algebra. As one can see from this, from 1954 to 1970 around one half of all the Fields Medals were awarded to topologists, and these were influential in many other areas of mathematics. Although one is aware of one or two vexing cases when an outstanding mathematician of the highest order and in the right age group did not win the Fields Medal, on the whole the above statistics reflects correctly the position of topology at the summit of mathematics during the period in question.

Historical sketch: topological ideas before the 19th century

A naive understanding of coarse topological distinctions between 3-dimensional geometric figures was already present in distant antiquity. It is obvious to anyone that the number of holes or handles of a 3-dimensional domain (or figure) does not change if the figure is bent without tearing and without selfcrossing. The complexity of knots formed by ropes on ships already attracted the attention of the ancient Greeks. However, the first topological observations taking the form of precise mathematical relations and theorems only arose with Euler in the 18th century: the number of vertices minus the number of edges plus the number of faces of a convex polyhedron is equal to 2; as Poincaré was to discover later, the analogous alternating sum is a topological invariant of any complex. The problem of three houses and three water wells is similar: prove that it is impossible to join three houses to three wells with paths that do not cross one another. In modern language, this is a graph (a 1-dimensional complex) having 6 vertices and 9 edges; it cannot be embedded in the plane without crossing itself.

Topological ideas in the 19th century

Up to the 19th century, observations such as these were only in the nature of toys, like original olympiad problems generated by a game of pure wit – similar to many problems of number theory. In the second quarter of the 19th century the situation changed: Gauss came to a series of non-trivial topological observations after analysing experiments of Faraday, where humanity first witnessed electromagnetic phenomena. In particular, Gauss discovered the so-called *linking number* of two disjoint closed curves in three space, which does not change under deformations without crossings. It was Gauss who posed the question of constructing an exact theory of such properties. The term "topology" arose in the work of his student Listing. Riemann contributed greatly to the topological theory of 2-dimensional manifolds ("Riemann surfaces"). In general, 2-dimensional topology arose essentially as a key aspect of the then new complex analysis – in the works of Cauchy in the plane, and in the works of Abel, Jacobi and Riemann on 2-dimensional manifolds with non-trivial topology. A series of topological observations were due to physicists: Kelvin was interested in knots. His starting point was his discovery of closed vortex lines in fluid mechanics having curious properties, and he wanted to apply knots to the classification of atoms (we know now that this idea turned out to be a delusion). His student Tait was the first to start on a systematic development of the theory of knots, and he stated interesting conjectures in the late 19th century that have only recently been proved. Maxwell noticed the following relation between the number of critical points of functions of different indices: for an isolated island the number of sinks minus the number of saddles plus the number of summits equals 1. This is a distant precursor of the idea of "Morse theory". Poincaré began to apply topological ideas systematically to analyse the qualitative behaviour of trajectories of dynamical systems, especially in the theory of plane systems he created. It was he who isolated topology as a separate branch of mathematics, which he called "Analysis Situs". The simplest and most fundamental topological characteristics turned out to generalise the number of holes and handles of the space, domain or manifold under study: these are the kth Betti numbers, that is, the number of independent k-dimensional cycles in a definite sense. According to the legend of the topologists, Betti himself was a chemist, and so these characteristics and their subsequent generalisations came to be called "homology". Poincaré gave a topological classification of 2-dimensional manifolds. He introduced the crucial topological invariant – the *fundamental group* of a space, consisting of homotopy classes of closed paths starting and ending at one common point – and constructed the topological theory of *covering spaces*. He discovered the *Poincaré duality law*, which asserts that for a closed n-dimensional manifold the Betti numbers of some definite type in dimensions k and $n-k$ are equal. The problem of classifying 3-dimensional manifolds met with more serious difficulties: up to recent times no one has succeeded in proving that every simply connected 3-manifold (that is, with fundamental group equal to $\{1\}$) is homeomorphic to the sphere. This is the Poincaré conjecture.

From the start of the 20th century the subject became known as topology. In the 1930s, when algebraic methods acquired decisive significance, thanks to Lefschetz, the subject became known as algebraic topology.

Topology in the 20th century: a sketch of the period to 1945

During the 20th century, a whole series of major mathematicians contributed to the development of algebraic topology and its applications. From the philosophical article that Poincaré wrote in 1912, one sees clearly that he considered the central problem of Analysis Situs (Topology) at the time as working out a topologically invariant notion of dimension – aiming to establish rigorously that our space is 3-dimensional. His text makes clear that what he had in mind was the invariance of dimension under continuous homeomorphisms, since under smooth assumptions the question becomes trivial. A year later, in 1913, Brouwer proved that the dimension of a manifold is invariant under continuous homeomorphisms; he discovered the degree of a map between spheres, and the theorem that a map from the disc to itself has a fixed point. In the early 20th century Dehn extended the ideas of 3-dimensional topology a long way. Starting from the fundamental group, he created the combinatorial theory of finitely presented groups, stating its main algorithmic problems: the problem of identity of words, conjugacy and isomorphism. At the time no one had any doubt that these questions were algorithmically determinable.

In the 1920s Alexander proved the topological invariance of homology (the Betti numbers). Homology groups turned out to be coarser invariants than homotopy type, discovered by Alexander. Vietoris, P. Alexandrov and Čech extended homology theory to general spaces. An important idea due to Emmy Noether allowed the complete algebraisation of homology theory. Lefschetz created a deep theory of homology of complex projective algebraic varieties, and carried over its ideas to topology, especially the idea of intersection of cycles. Elie Cartan, starting from Poincaré's ideas and Riemannian geometry, introduced differential forms and a homology theory based on them ("tensorial homology theory"). His conjecture that this theory is equivalent to usual homology theory was proved in the 1930s by his student de Rham, and this theory became known as "de Rham theory". In the 1920s Morse discovered the topological methods of the calculus of variations, and the topological Morse inequalities involving the number of critical points ("Morse theory"). Around 1930 Lyusternik and Shnirel'man developed an extremely original version of Morse theory for degenerate critical points, which is especially deep in the 2-dimensional case. They proved Poincaré's conjecture on three geodesics: on a Riemann surface homeomorphic to the 2-sphere, there always exists at least three non-selfintersecting closed geodesics. Alexander, Lefschetz, Morse and Lyusternik–Shnirel'man's most important results were not rigorously proved mathematically. Alexander's theory of the homotopy invariance of singular homology theory was first rigorously established in the 1940s by Eilenberg and others, Morse theory was made rigorous in the early 1950s after the development of cell complexes and transversality, and Lefschetz's homology theory of algebraic varieties in the 1950s; Lyusternik–Shnirel'man's results on three closed geodesics were only made rigorous in the 1990s.

The development of algebraic topology became even more intensive in the 1930s. Hurewicz introduced the idea of homotopy groups (groups of homotopy classes of maps from the sphere into a space, with the image of one point fixed). Even earlier, Hopf had discovered the homotopy classification of maps from an n-dimensional manifold into the n-sphere: the degree of a map determines its homotopy class. He subsequently discovered that spheres have non-trivial higher homotopy groups (the "Hopf invariant"). The problem of computing the homotopy groups of spheres was to become one of the central problems in algebraic topology – Pontryagin and Freudenthal began the process of solving it. In particular Pontryagin came from this to the *cobordism problem*: what are the algebraic topological conditions for a closed manifold to be the boundary of a manifold with boundary? In dimension two, already the projective plane is not the boundary of anything.

Algebraising the different versions of duality laws such as Poincaré duality for the homology of manifolds (Alexander duality and Lefschetz duality) led to Pontryagin's theory of characters of continuous groups, and also to the notion of "cohomology", which is dual to homology (Kolmogorov and Alexander, 1935). Later in the 1940s, duality acquired a deep interpretation in the context of stable homotopy theory ("Spanier–Whitehead duality"). Cohomology turned out to be a supercommutative graded ring: the Kolmogorov–Alexander multiplication operation, with mistakes in the definition corrected by Čech and Whitney in 1936–37, were conceptually borrowed from Lefschetz' intersection theory in algebraic geometry and from Elie Cartan's theory in tensor analysis; it turned out to be exceptionally important in algebraic topology.

Important progress was also realised in low-dimensional topology: Nielsen and Magnus created a theory of surface automorphisms, and began new chapters in combinatorial group theory based on topological results developed by van Kampen. Alexander and Reidemeister developed the theory of knots and links of closed curves in 3-dimensional space. Important invariants of knots such as the Alexander polynomial were discovered, and Reidemeister created his combinatorial model of knot theory (Reidemeister moves for knot and link diagrams in the plane). Emil Artin's theory of braid groups appeared at this time; it appears that he gave a simple computation for the fundamental group of spaces of polynomials with no repeated roots (without mentioning it). There also appeared a reduction of knot theory to braids (the model of Alexander and the younger Markov, the rigorous justification of which was only published in the 1980s). Reidemeister discovered a famous combinatorial invariant of a 3-manifold, called "Reidemeister torsion", which he used in solving the classification problem for lens spaces; this quantity is not homotopy invariant. To deal with this phenomenon, Whitehead created the theory of "simple homotopy type", which lies over the fundamental group. He thereby demonstrated for the first time a deep interference between the fundamental group and homology theory, and at the same time extended the notion of the determinant of a linear map. This led to the theory of the generalised determinant of matrices over rings, subsequently interpreted and developed by Whitehead in the 1940s (Dieudonné came independently to related ideas in pure algebra, but without the topological applications), and later to deep discoveries in analysis.

One of the principal general mathematical achievements of topology in the 1930s was the creation of the theory of differentiable manifolds and their embeddings in Euclidean space on the basis of ideas of transversality of smooth maps (Whitney). The new method allowed the construction of a completely rigorous analytic and geometric approach to the construction and computation of topological invariants (for example, the theory of intersection of cycles), based on transversality and reducing to general position, which was invaluable from the point of view of the analysis of the future. Pontryagin developed this approach in order to compute the first non-trivial k-dimensional homotopy groups of n-spheres. (For $k = n+1$ and $n+2$; Pontryagin got the case $k = n+2$ wrong, not observing the "Arf-invariant".

He corrected this mistake in 1950.) This method leads to the problem of cobordism of k-dimensional manifolds, as we mentioned above.

At the end of the 1930s, Whitney, Stiefel, Pontryagin and Chern discovered extremely important topological invariants of smooth and complex manifolds – the characteristic classes, which unify topology, Riemannian geometry and complex analysis. These are the Stiefel–Whitney classes in all dimensions in cohomology modulo 2, and the Pontryagin classes in $4k$-dimensional integral cohomology of a smooth manifold; the Chern classes lie in the $2k$-dimensional integral cohomology of a complex manifold. There is one basic class of each stated type in each of the dimensions indicated; one also considers any polynomial in the basic classes to be a characteristic class in the cohomology ring. By a theorem of Pontryagin, a closed manifold that occurs as the boundary of a manifold with boundary has the property that all the characteristic residues modulo 2 of Stiefel–Whitney classes are zero; and, in the orientable case, all the Pontryagin numbers are zero (that is, all the characteristic classes of the dimension of the manifold). This fails, for example, for the projective plane.

Results concerning the problem of fixed points of maps from a manifold to itself had an important influence on the development of topology and neighbouring disciplines. One example is Lefschetz's formula for the algebraic number of such fixed points; another is Smith theory, which studies the topological invariants of the set of fixed points of a transformation of finite order. The deep algebraic methods developed for this eventually led to the rise of non-trivial homological algebra: it became necessary to figure out an enormous number of relations between different versions of homology or cohomology of different objects. Already here there arose the simplest operations on cohomology with coefficients in a finite field, that do not reduce to the structure of the cohomology ring – the Bockstein coboundary operations.

Algebraic topology in the 20th century: 1945–55

By the mid-1940s, the theory of fibred spaces, and the theory of characteristic classes described above were already in place, together with the homological theory of obstructions to the construction and extension of maps and sections of fibrations. Subsequently, non-trivial cohomological operations also turned up, that do not reduce to multiplication – the squares and powers of Pontryagin and Steenrod, and the Massey operations. Of particular importance were the Steenrod operations in cohomology modulo p for all primes $p \geq 2$, together with the simplest Bockstein operations. These are linear operators that act on the cohomology of all spaces, and commute with continuous maps (as one says, "endomorphisms of the cohomology functor"). In 1945, in the course of solving Hopf's problem of computing the cohomology of aspherical spaces (that is, spaces with contractible universal cover), Eilenberg and Mac Lane laid the foundations of what was called "homological algebra" – the theory of group cohomology. D. Faddeev arrived at similar ideas independently but a little later from considerations of pure algebra and algebraic number theory. Eilenberg and Mac Lane also constructed an algebraic model ("Eilenberg–Mac Lane complexes") for spaces having all but one homotopy group zero. This circle of ideas was completed around 1950 by Postnikov, who constructed the generalisation of this model to arbitrary spaces (the "Postnikov system").

In principle, the Postnikov system gives complete information on the homotopy type of complexes and manifolds. In the important special case of simply connected 4-dimensional manifolds, Pontryagin and Whitehead solved this problem without Postnikov systems in the late 1940s. Hopf established a remarkable connection between certain quantities in the homotopy groups of spheres (the Hopf invariants) and a celebrated problem of algebra: whether there exist real division algebras in dimension $n \neq 1, 2, 4$ and 8, where the real and complex numbers, the quaternions and the Cayley numbers are known. Using the algebra of Steenrod operations, Hopf proved that there are no such algebras for $n \neq 2^q$, but he did not succeed in proving this for $q > 3$. The algebraic relations between the Steenrod operations for $p = 2$ discovered by Adem in the late 1940s only give the result for $n \neq 2^q$.

At the time, the situation was as follows: topologists had discovered a large number of non-trivial algebraic quantities and laws relating them: the fundamental group, the higher homotopy groups as modules over it (or even the Lie–Whitehead superalgebra they form), the homology groups and the cohomology ring, characteristic classes of manifolds and fibre bundles, Steenrod operations and much more besides. How were all these to be computed? In certain cases one succeeded in computing the homology of spaces using a triangulation, or a subdivision as a cell complex, a new idea originally introduced in the 1940s to give foundations to Morse theory. In other cases one succeeded, say, in computing the homology of Lie groups and loop spaces, using the calculus of variations (Morse theory); sometimes one could use de Rham theory, where already Elie Cartan used the theory of Lie groups for similar purposes for homogeneous spaces, reducing the calculation of homology to invariant closed forms. Thus already in the 1940s, computing homology seemed to be practicable, while at the same time the problem of computing homotopy groups appeared in a series of cases – for spheres, say – to be of transcendental difficulty.

From the second half of the 1940s to the early 1950s, Leray, Serre, Henri Cartan, Thom and Borel created a circle of remarkable algebraic methods and applied them successfully in computing the homology of fibred spaces, based on the so-called "Leray spectral sequences". This outstanding discovery led to what is now called non-trivial homological algebra, and became its fundamental method, allowing us to compute in a regular way quantities that, as we just said, had seemed up to then to be transcendental in topology. This computation proceeds via a series of auxiliary generalised "Serre fibrations", where Leray spectral sequences work at each successive step. Serre created a method allowing one to compute all the homotopy groups of spheres, ignoring finite parts: these are all finite, except for the special cases of homotopy groups of dimension $4n - 1$ for the $2n$-sphere, where the Hopf invariant appears. The general Cartan–Serre theorem completes the same computation for all those spaces whose loop spaces have known cohomology ring over a field of characteristic 0. To develop methods for computing the finite parts of homotopy groups, Serre (in the case $p = 2$) and Henri Cartan (for $p > 2$) computed all the Steenrod algebras of cohomology operations mentioned above. These algebras are generated by Steenrod operations and Bockstein operations. Their elements are linear operators acting on the cohomology of all spaces and commuting with continuous maps. The relations holding between them are the Adem–Cartan relations, and only these.

Already at the end of the 1940s it became clear that homology theory is completely determined by a simple collection of functorial properties (the "Eilenberg–Steenrod axioms"), which include

(1) "homotopy invariance";
(2) "functoriality", that is, covariance under maps of spaces;
(3) the "long exact sequence" of a pair;
(4) the "excision axiom", reflecting the local nature of homology; and finally
(5) "normalisation" – for a point, all homology groups are zero except in dimension zero.

Homotopy groups form a more complicated non-local functor, for which the excision axiom does not hold; although the analogy between them, called "Eckmann–Hilton duality", is very useful in many cases. The Steenrod algebra consists of the endomorphisms of the cohomology functor, whose classification leads, by a lemma of Serre, to the computation of the cohomology of Eilenberg–Mac Lane complexes. This problem was solved within the required bounds by Serre (for $p = 2$) and by Henri Cartan (for $p > 2$).

Borel carried out a series of deep computations for homogeneous spaces and established deep algebraic properties of spectral sequences. In 1954 he introduced the important notion of a "Hopf algebra": by a theorem of Hopf, the cohomology algebra over a field of characteristic zero of any space with a continuous multiplication having a unit is a free supercommutative algebra, that is, the tensor product of an exterior algebra by a polynomial algebra. As Borel showed, Hopf's proof is based on purely algebraic properties of these algebras, and he formally named this class of algebras "Hopf algebras", thus generalising Hopf's theorem to fields of finite characteristic, for which a Hopf algebra may not be free. The significance for topology of the theory of Hopf algebras over a finite field and homological algebra over them became clear rapidly. Hopf algebras entered into pure algebra much later; they subsequently became generally recognised also in quantum mathematical physics, and in analysis.

Developing the ideas of transversality and Pontryagin's method, Thom reduced the study of the problem of cobordism (described above) in a beautiful way to that of computing homotopy groups of auxiliary "Thom spaces" (in Pontryagin's special case, this is the sphere). According to Thom's scheme, the problem of realising cycles by smooth submanifolds also reduces to these, and also the Steenrod problem of realising cycles as the images of a smooth manifold. In the unoriented case Thom gave a pretty solution of these problems using Serre's method. The converse of Pontryagin's theorem mentioned above also holds: the vanishing of all the Stiefel–Whitney residue classes is sufficient for a smooth closed manifold to be a boundary; all cycles modulo 2 are realised as the image of manifolds. In the orientable case Thom established that the converse of Pontryagin's theorem only holds up to taking multiples. The Steenrod problem is likewise only solvable for multiples of a cycle. By this method of "Thom complexes", using Steenrod squares for $p = 2$, Thom proved the homotopy invariance of the Stiefel–Whitney classes, and Wu computed them completely effectively using Thom's method (Wu's formula).

It is interesting to compare this with the deepest of the results computing homotopy groups of spheres and cobordisms, obtained by Rokhlin using Pontryagin's direct geometric method: he established at the start of the 1950s that for $n \geq 5$, the $(n+3)$-dimensional homotopy group of the n-sphere is isomorphic to a finite cyclic group of order either 12 or 24, but he erroneously asserted at first that the order is 12, until the appearance of Serre's paper. Rokhlin also proved that any 3-dimensional manifold is the boundary of a 4-dimensional manifold (that is, is cobordant to 0), and that any 4-dimensional smooth oriented closed manifold is cobordant to a finite union of complex projective planes. Rokhlin discovered a new and very important invariant of oriented cobordism – the so-called "signature" of a manifold (that is, the difference of the number of positive and negative squares in the intersection matrix of $2k$-cycles on a $4k$-dimensional oriented manifold). It follows from Rokhlin's results for $k = 1$ and Thom's for all $k \geq 1$ that the signature can be expressed as a linear function of the Pontryagin numbers.

In 1954 Hirzebruch obtained a beautiful formula for the signature of a manifold in terms of the Pontryagin numbers, and he applied this result in a brilliant way to prove the general Riemann–Roch theorem in algebraic geometry. The significance of the signature formula for many purposes appeared prominently in the subsequent development of mathematics. Rokhlin only published complete proofs of his theorems at the end of the 1950s; a direct geometric deduction of the fact that the above-mentioned homotopy group of the n-sphere is of order 24 rather than 12, was only published in the 1970s, when it became important for low-dimensional topology. An analysis of the reasons leading to the order 24 rather than 12 led Rokhlin to an example of a 2-cycle on a simply connected 4-manifold that cannot be realised as a smoothly embedded 2-sphere. He did not publish this result himself: it was mentioned with a reference to Rokhlin by Milnor and Kervaire in the early 1960s. The problem of the optimal smooth realisation of a 2-cycle on a simply connected 4-manifold as a surface of the smallest genus subsequently became famous as the "Thom problem", and was solved only in the 1990s by completely different methods (coming from quantum field theory). The finiteness of homotopy groups of spheres reduces under Dold's scheme to the corollary that the Pontryagin classes are not homotopy invariants of closed manifolds (Dold, 1956). The finiteness of the $(n+3)$-dimensional homotopy groups of the n-sphere is already enough for this result. For simply connected manifolds there are no homotopy invariant relations between the integrals of the Pontryagin classes over cycles, other than Hirzebruch's signature formula. For non-simply connected manifolds the *Novikov conjecture* (see below) describes all the homotopy invariant integrals. It is proved in many cases.

This period 1945–55 of grandiose conceptual advances thus resulted in the creation of a unique assembly of methods, that permitted A.T. to reach its highest level at the very centre of world mathematical thought. It was able to maintain this level for about 15 years, up until the early 1970s. We now proceed to describe this period of highest flowering of classical A.T.

Algebraic topology from 1956 to 1970

The start of the new period is marked by Milnor's famous discovery in 1956 of non-standard smooth structures on the 7-sphere. Consider as an analogy the well-known Hopf fibration, which represents the 3-sphere as fibred over the 2-sphere in circles. In the 7-dimensional case the base is the 4-sphere, the fibre the 3-sphere and the total space is a 7-sphere according to its homotopy type. The theory of fibre bundles shows that, in contrast to the case of the Hopf fibration, there are an infinite series of fibrations of this type; they are distinguished by their first Pontryagin class, which the usual Hopf fibration did not have. All these 7-spheres are boundaries of an 8-manifold in an obvious way, namely, the fibre bundle over the 4-sphere with fibre the 4-disc; these have distinct Pontryagin classes, but the same cohomology ring. If we assume that the boundary is the usual 7-sphere, we can glue in a disc to obtain a series of smooth manifolds having signature 1, but distinct first Pontryagin classes. Let us compute the second Pontryagin class using Hirzebruch's formula, in terms of the signature and the first Pontryagin class: remarkably, in a series of cases we obtain a fraction with denominator $1/7$, although the Pontryagin number of a smooth closed manifold is always an integer. Therefore, the boundary cannot be the usual 7-sphere. Milnor constructed a function on this manifold having 2 critical points (a minimum and a maximum), from which it follows that it is homeomorphic to the usual 7-sphere (even piecewise linear homeomorphic to it); but not diffeomorphic! The closed 8-manifolds obtained in this way are not homeomorphic to any smooth manifold. Admittedly, the complete proof of this assertion follows from the topological invariance of the integrals of Pontryagin classes over cycles, which was only established later in 1965 by Novikov; but already in 1960 Kervaire exhibited a 10-dimensional piecewise linear manifold that is not homotopy equivalent to a smooth manifold, using his newly discovered Arf invariant. Thus following Milnor's discovery it was understood that not all manifolds have a smooth structure, and the smooth structure may be non-unique.

Thom, Rokhlin and A. Schwarz (1957) proved that the integrals of Pontryagin classes over cycles are invariant under piecewise linear transformations, based on Hirzebruch's formula and the piecewise linear analogue of transversality. Milnor and Kervaire developed the classification theory of manifolds having the homotopy type of the sphere up to h-cobordism; here two manifolds are h-cobordant if together they bound a manifold having two ends that contracts to either of them. They reduced this problem to the theory of homotopy groups of spheres using Pontryagin's method. At the same time Smale proved that every simply connected h-cobordism in dimension ≥ 6 (having boundary of dimension ≥ 5) is trivial, and the two ends are diffeomorphic. He (and independently Stallings and Wallace) proved the *generalised Poincaré conjecture*, that every manifold of dimension ≥ 5 having the homotopy type of the sphere is piecewise linear homeomorphic to the sphere. Smale constructed a Morse function on any simply connected manifold of dimension ≥ 6 having the smallest possible number of critical points, the "Morse–Smale number", expressed in terms of the rank and torsion of the homology groups. The critical points are all non-degenerate. Smale and Hirsch also gave the complete solution of the problem of classifying smooth immersions (that is, embeddings with self-crossing), based on a beautiful idea of Smale, by reducing to computations of homotopy groups. Their result is quite non-trivial already for the 2-sphere in 3-space. Haefliger obtained deep results concerned with the theory of smooth embedding of manifolds into higher-dimensional Euclidean space. Even before Smale's work, around 1958, the younger Markov used results of P. Novikov, Adyan and Rabin in the algorithmic theory of finitely presented groups to prove that the question of whether a manifold of dimension ≥ 4 is homotopy equivalent (or homeomorphic or diffeomorphic) to a certain simply connected manifold (having large second Betti number) is algorithmically undecidable. It is curious that in the course of the proof one must establish an important special case of Smale's lemma on "cancelling handles", although at the time topologists did not realise this. Markov never published a complete proof of his theorems. Subsequently S. Novikov, using Smale's lemma and his construction of "universal central extensions of a group" that kill the second cohomology group of the group, established that the property of being an n-sphere is algorithmically undecidable in dimension ≥ 5. This result was proved in 1962, although it was only published in the 1970s.

The early 1960s saw the construction of "Browder–Novikov theory", which allows us to classify up to diffeomorphism the simply connected manifolds of dimension ≥ 5 homotopy equivalent to a given manifold (S. Novikov, 1961), and distinguish the homotopy types of closed manifolds among all simply connected complexes (Browder, 1962). These independent pieces of work solved different problems, but they turned out to be extremely close in their method, which became generally accepted. In this a fundamental role is played by a topological analogue of birational map – degree-1 maps between closed manifolds, having remarkable algebraic topological properties. The classification of manifolds reduces to the computation of a single homotopy group of the "Thom space of the normal bundle". Because of this, Browder–Novikov theory is closely related to the ideas of cobordism theory. It follows from Browder–Novikov theory, in particular, that there are only a finite number (easily bounded from above) of simply connected manifolds of general homotopy type having the same integrals of the Pontryagin classes over cycles (Novikov's theorem).

This theory is also easily carried over to the classification of piecewise linear manifolds and to manifolds with boundary. Extending it to all non-simply connected manifolds of dimension ≥ 5 was accomplished for a series of special problems and particular cases by Browder, Novikov and Shaneson, and in the general case by Wall around 1970. Levine extended these methods to the theory of embeddings during the same period. Non-simply connected manifolds lead to a series of extremely interesting algebraic phenomena, related to what is known as "algebraic K-theory" and its Hermitian analogue, where already in 1970 Novikov pointed out the significance of symplectic ideas. For finite fundamental groups the methods of algebraic number theory were actively applied during the 1970s. For infinite groups, the methods of functional analysis were applied, starting with works of Lusztig, Mishchenko and Kasparov in the 1970s and 1980s. A considerable number of topologists contributed to the development and application of these methods.

The development of algebraic and other methods of topology to compute homotopy groups from 1956 onwards was also extremely intensive. E. Brown proved the algorithmic computability of Postnikov systems of simply connected complexes. The algorithmic decidability of homotopy type among finite simply connected complexes is a recent result of Weinberger and Nabutovsky (1998). They used the theorems of Wilkerson and Sullivan on the arithmetic nature of groups of homotopic automorphisms of finite simplicial complexes (1975–76) and deep results of the 1980s from the theory of algebraic groups.

As Milnor indicated in 1957, Steenrod algebras are Hopf algebras. This is the starting point of the non-trivial theory of Hopf algebras in topology – up to this point, they were only considered as an axiomatisation of the cohomology rings of

Lie groups or of spaces with a multiplication and unity (such as the loop space) – that is, of H-spaces. In 1958 Bott, starting from the computation of Morse indexes of geodesics on Lie groups and homogeneous spaces, noticed a remarkable periodicity phenomenon: the space of paths on the unitary group with starting point 1 and endpoint -1 is homotopy equivalent up to large dimension to the complex Grassmannian variety; it is known in turn that the loop space of the latter is homotopy equivalent up to large dimension to the unitary group. Thus a remarkable 2-periodicity phenomenon holds for the "stable" homotopy groups of the unitary groups U_n (that is, the groups of dimension $< 2n$, independent of n): these groups only depend on the dimension modulo 2. Computations lead to 8-periodicity for the stable homotopy groups of the orthogonal group O_n (that is, in dimension less than $n-1$). The discovery of "Bott periodicity" allowed the bulk of computations related to stable vector bundles to be made effective.

From Bott periodicity it is not hard to deduce the theorem on the non-existence of real division algebras of dimension $n = 2^q$ for $q > 3$, which was mentioned above as one of the famous problems of algebra that reduces to topology (Milnor, Kervaire, Atiyah, Hirzebruch, Bott, 1959). However, we discuss a different solution to the same problem by Adams in 1959, based on different methods. After the creation by Serre and others of methods to compute homotopy groups, a number of topologists began to apply them systematically and to perfect them. Adams, Toda, Mahowald and others were among the first to start developing this, and obtained many valuable results. In 1958 Adams discovered a remarkable method of investigating stable homotopy groups (for $(n-1)$-connected spaces, these are the homotopy groups of dimension less than $2n-1$). For the n-sphere, the kth stable homotopy group depends only on the difference $k-n$, and is non-trivial for $k-n \geq 0$. The direct sum of all the stable groups of spheres is a supercommutative graded ring with respect to superposition of maps of spheres to spheres. The "Adams spectral sequence" was discovered to compute the stable homotopy groups; its second term is expressed in terms of homological algebra for cohomology as a module over the Steenrod algebra. Adams solved the problem of the Hopf invariants (and with it the problem of division algebras) based on this method for $p = 2$. The scientific youth of the author of these lines took place precisely here, in 1958–59, developing this method for $p > 2$ to find "long superpositions" of spheres, perfecting the computations with Hopf algebras, introducing for this original analogues of the Steenrod algebra in the homology of Hopf algebras over a finite field.

In 1960 Milnor and Novikov applied Adams' method to the problem of computing the oriented cobordism ring; they introduced the unitary (complex) analogue of cobordisms, and its special unitary, spin and symplectic analogues. Milnor, as Western topologists testify, arrived at these ideas somewhat earlier, but first wrote and published only the first part of it in 1960, containing a proof using Adams' method of the theorem on non-existence of odd torsion in oriented cobordism and the non-existence of torsion in complex cobordism. Milnor's original geometric idea for proving that these rings are polynomial rings (which he communicated privately in 1961) was left to a second part, which was never written up. Much later other Western authors (Stong and Hattori, 1965) published a proof realising Milnor's idea (but without any mention of him: they wrote only that this was the first computation of the unitary cobordism ring not using the Adams spectral sequence). Novikov, formalising the Hopf property of certain cohomology modules over the Steenrod algebra as a "co-algebra", used the multiplicative properties of the Adams spectral sequence for universal Thom spaces and computed the ring structure as an elementary corollary of algebraic arguments. He obtained a string of results on the special unitary cobordism ring, and also on the symplectic cobordism ring (1962). These latter rings are still not computed at present, although the second term of the Adams spectral sequence for them was found in the 1970s by Ivanovskii and Vershinin; Botvinnik and Kochman obtained a series of results in the 1970s and 1980s, but the problem is very complicated. Spin cobordisms are easier, and were computed by Brown and Peterson in 1965; they successfully applied cobordism, solving the well-known problem of the Arf invariant in the theory of smooth structures on spheres in dimension $8k+2$. The deepest result here was obtained by Browder (1970), but the problem is still open in dimension 2^k-2 for $k>5$. Rokhlin in 1958 developed a geometric method for the study of 2-torsion in oriented cobordism, but he made an error. The definitive result was obtained by Wall (1960). Novikov also proved that the Steenrod problem is solvable in the oriented case for any k-cycle multiplied by some odd number. This number can be taken to be 1 if some homology groups do not have special odd torsion in the oriented case (and similarly in the unitary case, without the oddness condition).

The development of the algebraic methods of topology and homology theory during this period took on extremely interesting aspects, especially after Grothendieck's work that found a completely new understanding of fundamental theorems in algebraic geometry such as the Riemann–Roch theorem. He introduced in particular the K-functor, which Atiyah and Hirzebruch carried over into topology. In particular, the purely topological version of K-theory replaces even-dimensional cocycles with stable classes of unitary bundles. These groups (and the ring) are $\mathbb{Z}_2$-graded, and they satisfy all the Eilenberg–Steenrod axioms, except for "normalisation": the K-functor of a point is non-trivial and consists of the ring of Laurent polynomials in a symbolic 2-dimensional variable – the Bott periodicity operator and its inverse over the integers. Objects of this type, similar to cohomology but without normalisation, began to be called "extraordinary cohomology theories" (or "generalised cohomology theories"). Their axiomatisation and very general properties were systematised by the second Whitehead (G.W. Whitehead) in the early 1960s. Concretely, K-theory is effectively computable because of Bott periodicity. The role of cohomology operations in K-theory is played by representations of unitary groups or special linear combinations of these – for example, exterior powers. Here the substantial theory was developed first by Atiyah, who pointed out the possible topological applications. He also pointed to cobordism theory as an extraordinary homology and cohomology theory, and established a series of its important properties. For example, Atiyah duality for Thom complexes turned out to be important in the Browder–Novikov theory of classification of manifolds, reducing it to universal spaces, as several authors observed in 1965–66.

The famous work of Atiyah and Singer (1963) successfully applied K-theory to the well-known problem of computing the index of elliptic differential operators, that goes back to Fritz Noether and N. I. Muskhelishvili in the 1930s for singular integral operators on discs, developed by a series of Soviet mathematicians.

This was formulated by Gel'fand in 1961 as a general mathematical problem and quickly solved by Atiyah and Singer. The significance of these ideas for analysis is hard to exaggerate. Invariants of an elliptic operator are successfully formulated in K-theory: the Pontryagin classes of a manifold enter into the index formula.

In 1962 Adams constructed linear combinations of the exterior powers as cohomology operations in K-theory (the "Adams operations") having remarkable properties. Using them Adams solved a number of well-known problems of topology: he proved Toda's conjecture that the maximal number of linearly independent vector fields that can be constructed on the n-sphere is the well-known number arising from infinitesimal rotations; he gave the upper bound for important subgroups of the stable homotopy groups of spheres, that had previously been prettily bounded from below by Milnor in terms of Bernoulli numbers starting from the same formulas of Hirzebruch type. The lower and upper bounds coincided in the final analysis for these special subgroups. Complete proofs were only published several years later (1965–66).

Adams stated the "Adams conjecture" on the orders of finite Abelian groups of Dold classes of fibrewise homotopy equivalent stable fibre bundles of spheres, which are important in topology, and also posed the problem of finding the analogue of the Adams spectral sequence to compute stable homotopy groups in K-theory in place of ordinary cohomology. The latter problem turned out not to be solvable within the framework of K-theory. In 1966–67 Novikov realised this program, replacing K-theory by complex cobordism. Oriented and unitary (complex) cobordisms had already been used effectively to study fixed points of smooth maps of finite order and to complete the computation of special unitary cobordisms (Conner and Floyd, 1964–65). They introduced the analogues of Chern classes of vector bundles, taking values in unitary (complex) cobordism, in place of ordinary homology. This is extremely natural – already in the early 1950s Gamkrelidze studied Chern cycles as linear combinations of algebraic subvarieties. Recall that in the cobordism groups, cycles are represented by maps of manifolds of one type or another.

In 1966 Novikov and Landweber introduced the analogues of Steenrod powers in complex cobordism. The algebra they generate is a Hopf algebra over the integers, and is called the Landweber–Novikov algebra. The role of scalars (or the cohomology of a point) in cobordism theory as a cohomology theory is played by the Milnor–Novikov complex cobordism ring. According to a theorem of Novikov, all the cohomology operations in this theory are obtained by applying the Landweber–Novikov algebra and then multiplying through by the ring of "scalars" (as above). However, these two rings do not commute. Their commutation relations (the Novikov formulas) complete the description of the analogue of the Steenrod algebra in this theory. As Buchstaber remarked later (in 1978), these are two dual Hopf algebras with a curious integral structure. Already in the 1990s Novikov and Buchstaber studied the resulting general construction in the theory of Hopf algebras, calling it the "operator double". L. Faddeev and Semenov-Tian-Shansky, arriving at this at the same time from other ideas, called it the "Heisenberg double".

The construction of the "Adams–Novikov spectral sequence" to compute stable homotopy groups of spheres and other spaces led to a large number of results, and a string of topologists developed these ideas. We note in particular the formal group of geometric cobordisms introduced by Novikov and Mishchenko in 1967 to compute the analogue of the first Chern class of the tensor product of two complex bundles of rank 1; many applications of this were discovered, for example, to construct analogues of the Adams operations and Chern classes (Novikov), and subsequently the Chern–Dold character (Buchstaber). In 1969 Quillen identified this formal group with Lazard's universal formal group in pure algebra, and successfully applied this theory to compute a series of topological quantities – in particular, for the effective computation of a projector from cobordism theory onto its direct summand (Brown–Peterson theory) over the ring of p-adic integers, which is extremely useful in computing the stable homotopy groups of spheres by the method of Adams and Novikov.

Developing the ideas of the theory of formal groups led Novikov and Buchstaber to the notion of two-valued formal group (1971). Buchstaber extensively developed the theory of multivalued formal groups during the 1970s. In the late 1960s–early 1970s Novikov, Buchstaber, Gusein-Zade and Krichever successfully applied these methods to compute the invariants of fixed points of smooth actions of cyclic groups and of the circle on manifolds. Significantly later, in the late 1980s, remarkable topological relations for actions of the circle were discovered by quantum field theorists, in particular Witten. Here the topologists Ochanine and Landweber were also involved, unifying the new ideas with the classical theory of complex cobordism from the early 1970s. The special role of the so-called elliptic genera of the characteristic classes and their "rigidity" property became apparent. In the last decade in the same direction, a large number of curious relations between the characteristic classes of Chern and Pontryagin were uncovered, especially for special unitary cobordisms (in the new theory, these began to be known as "Calabi–Yau manifolds"). In these relations, only the nature of the cobordism theory plays a role, and not at all the metric of Calabi–Yau type.

The possibilities of cobordism theory are far from exhausted. One naturally expects that in singularity theory, as developed by Arnol'd and his school, cobordism should play an active role in place of ordinary cohomology.

Continuous homeomorphisms in topology

The problem of continuous homeomorphisms plays a special role in topology. Whereas diffeomorphisms are stable among all smooth maps in the corresponding topology or homotopy equivalences, continuous homeomorphisms are unstable among all continuous maps: the condition of being bijective disappears under a small random perturbation, if the derivatives are not controlled (or do not exist). This makes the problem of continuous homeomorphisms especially difficult. It turns out in the final analysis that for smooth closed simply connected manifolds, homeomorphism is equivalent to homotopy equivalence in dimensions 2, 4 and 5, and probably also 3 (the Poincaré conjecture); for $n = 4$ this is Freedman's theorem (1984), and for $n = 5$ they are even diffeomorphic, as Novikov established in 1964. However, the picture changes for $n > 5$: the principal quantities that distinguish homeomorphism from homotopy type are the integrals of the Pontryagin classes over cycles; the fact that these are topological invariants was established by Novikov in 1965 (see below).

Topology in the 20th century for a long time started out from the presumption that all its essential laws are invariant under continuous homeomorphisms, even though the definitions of important quantities always proceed using additional structures: from the time of Poincaré combinatorial technique (such as subdivision into simplexes and representing the space as a simplicial complex) was the foundation for the rigorous definition of homology and the fundamental group, and later also for Reidemeister–Whitehead torsion. The use of smooth structure became possible after Whitney's discoveries of the 1930s: the characteristic classes of Stiefel–Whitney and Pontryagin were introduced as invariants of the smooth structure – either in terms of integrating some definite expressions in the Riemann metric over cycles, or in terms of cycles of singularities of vector or tensor fields, or through the analogue of the Gauss map. From the very beginning of the existence of topology as an exact science, built on combinatorial foundations, the assertion arose that simplicial complexes that are continuously homeomorphic (in particular, piecewise linear manifolds) are in fact combinatorially equivalent (piecewise linear equivalent); this is the "Hauptvermutung", the basic conjecture of combinatorial topology. A similar conjecture was presumed to be true also for smooth manifolds that are continuously homeomorphic (namely, that they are diffeomorphic), until this was refuted by Milnor in 1956 (see above). In effect, Milnor uncovered the difference between smooth and piecewise linear structures: there are different smooth manifolds that are piecewise linear isomorphic but not diffeomorphic – in dimension ≥ 7. There are piecewise linear manifolds in dimension ≥ 8 that do not admit a smooth structure: some manifolds of this type can even be defined in projective space by one complex equation having a single isolated singularity.

The obstruction theory created by several authors in 1958–65 (Thom, Munkres, Hirsch and others) lies in cohomology with values in the finite Abelian groups of diffeomorphisms of a sphere modulo those that extend to the disc, which coincide with the groups of smooth structures on spheres, discovered by Milnor and Kervaire in all dimensions except 4. The missing obstruction group on 4-stems was found by Cerf in 1966; it equals zero. That is, modulo a finite number, the two categories – the smooth and the piecewise linear – are equivalent for manifolds of dimension ≥ 5. The topological conjectures turned out to be "almost correct" in this case. Even some of the torsion parts of the Pontryagin classes turn out not to be piecewise linear invariant for higher-dimensional manifolds in integral cohomology (for example, the 7-torsion of the second Pontryagin class), although their integrals over cycles are combinatorially invariant, as indicated above. The theory of piecewise linear manifolds thus appears to be very similar to the smooth theory in its methods and results, with corresponding use of the analogue of the idea of transversality, the analogues of vector bundles, and so on. The Browder–Novikov classification theorem can also be easily carried over to piecewise linear manifolds, replacing vector bundles by the so-called piecewise linear microbundles of Milnor.

What about purely continuous homeomorphisms? In small dimensions any continuous homeomorphism can be approximated by a piecewise linear one, and even by a diffeomorphism. For 3-dimensional manifolds this was proved by Moise in the early 1950s, and this is the strongest result that can be obtained here by direct elementary methods. Around 1960 M. Brown and Mazur proved a higher-dimensional analogue of the Schönflies theorem in 3 dimensions: every embedded $(n-1)$-sphere in n-dimensional Euclidean space bounds an n-dimensional disc, provided that the boundary of the sphere is "tame", that is, it looks locally like a direct product with an interval (already in 3 dimensions, Alexander's "wildly embedded" horned sphere shows that this does not always hold). Mazur also gave a beautiful and elementary proof (1960) that two closed manifolds that are of the same homotopy type, including the homotopy type of the tangent vector bundle, become diffeomorphic to each other after multiplication by a high-dimensional Euclidean space – although of course, they may themselves not be diffeomorphic. The difference between them washes away on multiplying by a high dimensional Euclidean space. As Milnor showed, applying these arguments to lens spaces, a refutation of the Hauptvermutung follows from this: if homotopy equivalent lens spaces (these are always parallelisable in 3 dimensions) have different Reidemeister determinant, then the Thom spaces of the equivalent bundles over them are combinatorially inequivalent, but homeomorphic. For example, for trivial bundles the Thom complexes are obtained from direct products of the lens space by an n-dimensional disc by shrinking the whole boundary to a point; this is a singularity. The version of the Reidemeister torsion "modulo a singular point" is a combinatorial invariant, which coincides with the Reidemeister invariant of the lens space; thus these homeomorphic complexes have different combinatorial classes. These arguments lead to examples of complexes in dimension ≥ 6. Moreover, Milnor stated the conjecture that the double suspension over any 3-dimensional manifold that is a homology sphere is always homeomorphic to the 5-sphere. This conjecture was proved 15 years later in the 1970s by Edwards, by a direct construction of a homeomorphism. His geometric method rests on the construction of "resolutions (nice approximations) of wild embeddings" developed by Shtan'ko. Needless to say, this shows that the 5-sphere has extremely complicated triangulations, for which it is not even a piecewise linear manifold. These are of course not combinatorially equivalent to the ordinary 5-sphere with its standard triangulation.

However, how do things stand with the most important invariants of smooth and piecewise linear manifolds? All the counterexamples discussed so far are not piecewise linear manifolds. As we have said, homology and homotopy groups are homotopy invariant. The Stiefel–Whitney classes are also homotopy invariants of closed manifolds, so that they are not capable of distinguishing homeomorphism and homotopy type. The integrals of Pontryagin classes over cycles are not homotopy invariants in general, as we pointed out above.

In 1964–65 Novikov discovered a completely new approach to the study of continuous homeomorphisms, that allowed him to prove the topological invariance of the integrals of the Pontryagin classes over cycles; that is, they are preserved by purely continuous homeomorphisms of smooth and piecewise linear manifolds. The basis of this approach is the localisation of the Pontryagin–Hirzebruch integral at special "toroidal" domains of manifolds, with the subsequent development and the use of the far-reaching technique of differential and algebraic topology for the study of manifolds of toroidal type with a free Abelian fundamental group. For example, in a particular case we need to prove that the Pontryagin classes of a manifold of toroidal type are zero.

This approach is conceptually akin to Grothendieck's "étale topology", which from the late 1950s to the early 1960s used the category of covers over open domains in the Zariski topology, organised into the étale topology, to give the right homology of algebraic varieties by their realisation over fields of finite characteristic. All the subsequent deep results of the topology of continuous homeomorphism use this approach or its subsequent development. From the invariance of Pontryagin classes it follows that a simply connected closed manifold of dimension ≥ 5 has only a finite number of smooth or piecewise linear structures. The development of this method allowed also to establish the truth of the Hauptvermutung for simply connected manifolds of dimension ≥ 5, provided that there is no 2-torsion in its 3-dimensional homology (Sullivan, 1967). It is curious that he did not notice the latter restriction on 3-dimensional homology at first, and it was pointed out to him by Novikov and Browder. This is undoubtedly an indication that these results were at the time in an extremely raw state. We should observe that Sullivan's theory is still not yet written up completely rigorously, even though a series of its components were proved and published later by different authors: the necessary results from homotopy theory were obtained by Madsen and Milgram in the 1980s using different ideas, and a series of other components can be established based on the technique developed in the 1980s by Quinn. However, a completely written up unified proof of Sullivan's theory is still not yet available in the literature.

Weaker theorems were proved more simply and in complete form: the Hauptvermutung for 3-connected manifolds (Lashof and Rothenberg, 1968) and even for simply connected manifolds for which the third homology group is trivial modulo 2 (Casson, 1969, published much later). All these results on continuous homeomorphisms rest on the toroidal construction introduced by Novikov. In 1968 Kirby proposed a modification of the toroidal construction and gave a beautiful application of it to the famous Annulus Problem: prove that any domain in Euclidean space bounded by two "tame" spheres is homeomorphic to a cylinder (that is, the product of an $(n-1)$-sphere and a closed interval). For the smooth case this follows from Smale's theorems in dimension $n \geq 6$, but the purely continuous case turned out to be hard. Kirby's method reduces this question to the theory of smooth structures on manifolds having the homotopy type of an n-torus, but the homotopy tori are required to satisfy more than in the theory of Pontryagin classes: we need to prove not just that they are all parallelisable, but that the differences between them wash out on passing to a sufficiently big many-sheeted cover. The proof of this was completed by Siebenmann in all dimensions ≥ 5. Kirby's method led to a series of strong results: resting on Chernavskii's theorem on the local contractibility of the group of continuous homeomorphisms of a manifold, in the late 1960s Kirby and Siebenmann used the whole accumulated body of methods in topology to show that the Hauptvermutung is false for higher-dimensional piecewise linear manifolds. In their work the invariant that distinguishes two manifolds takes values in the 2-torsion of the 3rd homology. It arises from the so-called "Rokhlin double", the difference between 12 and 24 in the computation of the order of the $(n+3)$rd stable homotopy group of the n-sphere, where Rokhlin committed the mistake mentioned above, which is deep in nature. Based on the results of Farrell and Hsiang (1968), in the 1970s Kirby and Siebenmann constructed a classification theory for continuous manifolds. However, in contrast to the Annulus Problem and their refutation of the Hauptvermutung, the latter works rely on Sullivan's theory mentioned above, so that their complete proof is thus also not yet contained in the literature.

In the mid-1970s Sullivan proposed an interesting idea: he proposed a construction of Lipschitz structures on manifolds, together with a sketch proof of their existence and uniqueness on purely continuous manifolds (using the solution of the Annulus Problem, since this fact can in no way be considered elementary). However, the proof of these theorems has never been written up completely. Subsequently Sullivan together with Teleman, and also Weinberger in the 1990s, started the construction of analysis and operator theory on Lipschitz manifolds. As a spin-off of their theory, leading to a proof of the topological invariance of Pontryagin classes, Novikov conjectured in 1970 a classification of the homotopy invariant expressions in the Pontryagin classes; these are the integrals of the Pontryagin–Hirzebruch polynomials over cycles dual to the cohomology classes lying over the fundamental group (the Novikov conjecture, or the higher signature conjecture). From Lusztig's work (1971), a whole series of authors obtained deep results here by the methods of geometric and functional analysis; among these A. Mishchenko, Kasparov, Gromov, Connes and others. This conjecture is not completely resolved to this day. In 1974 Chapman proved the topological invariance of Reidemeister–Whitehead torsion by a beautiful method in elementary geometry. In the 1980s and 1990s Lawson and Gromov discovered extremely interesting properties for simply connected manifolds of positive scalar curvature: the property that such a metric should exist is an invariant of spin cobordism. There is an analogue of the Novikov conjecture in the non-simply connected case, the Lawson–Gromov–Rosenberg conjecture, in which the Hirzebruch polynomial is replaced by the A-genus appearing in the theory of the Dirac operator. Kreck and Stolz made successful use of cobordism theory for problems of geometry during the 1990s.

Returning to problems of algebraic topology and homotopy theory, one should mention the beautiful general categorical construction of "p-adic localisation" of homotopy types (not just stable homotopy types, but of all simply connected complexes, and even a bit more). The proposal was due to Sullivan and Quillen around 1970, and allowed them to prove the Adams conjecture mentioned above bounding the order of the group of fibrewise homotopy equivalent classes of stable vector bundles – which are important for the classification problems of topology – in terms of the Adams operations in K-theory. The fact that the Grassmannian varieties are algebraically defined over the integers plays a decisive role in this construction. A very curious feature of this method is the fact that its general categorical nature allows us to solve hard problems in concrete applications. More recent years have seen a series of valuable results in homotopy theory, including the non-trivial finite-dimensional H-spaces discovered in the 1970s that do not reduce to Lie groups or the 7-sphere (Hilton and Mislin), Nishida's proof that any element of the stable homotopy ring of spheres is nilpotent, and a whole series of very valuable constructions and computations based on the methods of cobordism carried out by many authors, including Wilson, Ravenel, Buchstaber, Morava and Miller.

A useful and purely ring-theoretic theory of "minimal models" of rational and real homotopy types was constructed in the 1970s; Sullivan succeeded in formulating this theory starting from the ring of Whitney–Thom simplicial differential forms on complexes (in other words, collections of forms on simplexes that agree when restricted to common faces). Sullivan indicated a subring with rational coefficients that is functorially determined. This leads to a model of rational homotopy type. This turns out to be an effective tool in a series of cases to study the topology of concrete varieties: Kähler manifolds (where the weight of rational types is determined by the cohomology ring, as proved in joint work of Deligne, Griffiths, Morgan and Sullivan), homogeneous spaces, and others.

Some years before this, Chen initiated interesting analytic ideas – to use "iterated integration" in the ring of differential forms in order to write homotopy invariant expressions on maps of the circle (already in the 1950s), and for maps from spheres into manifolds (later in the 1970s). These questions received a completely natural treatment within the framework of the ring-theoretic theory of homotopy types.

There are interesting algebraic constructions and problems here, related to the question of how to characterise rational integrals starting from the analytically given ring of smooth differential forms, started by Novikov in 1984 for the purposes of field theory. We note that integrals of this type were first written down by Whitehead in the 1950s in the study of Hopf invariants of homotopy groups of spheres, based on Kelvin's integral in the fluid mechanics of vortices in the 19th century.

To summarise our report on the period of highest flowering of classical algebraic topology (that is, the 1950s and 60s), we see that a considerable number of deep results were also obtained later in the 1970s and 80s. However, the community of topologists pays less and less attention to the significance of their ideas for the rest of mathematics, restricting their interests and their horizon more and more, and making their language more and more isolated and abstract. Moreover, as we pointed out above, the community has also lost control of the extent to which even the best results of the subject are rigorously proved. The last aspect undoubtedly bears witness to a lowering of the level of the subject, when the central theorems are not proved, and the following generation is not even aware of this, and "trusts the classics".

We should say that classical algebraic topology of the period 1950–70, with the exception of isolated pieces, is not described in any cycle of generally accessible textbooks: the existing books either only skim its initial stages, or are mindbogglingly abstract. The subject is best studied based on the original works, which at the time were clear and written out in detail (a list of these can be found in [1] and [2]). At the end of this article we list some books [3]–[11] that are more convenient to read than the overwhelming majority of the others.

As we will see below, in the period 1985–95 topology lived through another period of uplift based on entirely new ideas, with the leading role taken by ideas (and often also people) coming from the world of quantum theory.

Low-dimensional and hyperbolic topology

Low-dimensional topology in the 1950s started with Papakyriakopoulos' proof of the so-called "Dehn lemma" in 1957. The first attempt to prove this was due to Dehn himself in the early 20th century, but this contained a deep gap. The theorem states that a knot in 3-dimensional space is trivial if the fundamental group of its complement is Abelian (cyclic). At the same time he also proved the "sphere theorem": if a 3-dimensional manifold has non-zero second homotopy group, it contains an embedded 2-sphere with non-trivial homotopy class. Milnor pointed out a series of beautiful corollaries of this result: all knot complements in the 3-sphere are aspherical. Any closed 3-manifold can be represented as a connected sum of "elementary" ones. Among orientable 3-manifolds, the elementary ones are the product of a 2-sphere by a circle, the manifolds with finite fundamental group and manifolds with contractible universal cover. In the 1960s Haken and Waldhausen constructed a deep structure theory of 3-manifolds and special "incompressible" surfaces in them. It was proved that the topological type of a knot is completely determined by its fundamental group with a marked Abelian subgroup (Waldhausen in the late 1960s). Haken constructed an algorithm that determines whether a knot is trivial, although for a long time no one succeeded in implementing it in practice. There was a start on developing a program of studies aiming to prove that the property of two knots (or links) to be isomorphic is algorithmically decidable. A series of followers of Haken were involved in this, but up until the present there are remaining gaps. It is possible that Matveev has succeeded, at the start of the 21st century, in completing this program, using his theory of complexity of 3-manifolds, but his work has not so far been checked. In the 1990s Rubinstein and Thompson constructed an algorithm that distinguishes a 3-sphere by the methods of Haken's theory.

At the end of the 1970s M. Freedman succeeded in completing a program, the main idea of which was first advanced by Casson, but which turned out to be very difficult: using a modification of Smale's type of arguments for the higher-dimensional case, but carrying them out as an infinite convergent series, it turned out to be possible to prove that two simply connected closed 4-manifolds that are homotopy equivalent are continuously homeomorphic – in particular, this proves the purely continuous analogue of the Poincaré conjecture for the 4-sphere. We will return later to the diffeomorphism problem for 4-manifolds; at present one only succeeds in establishing negative theorems, that manifolds are not diffeomorphic (see below). No progress is in sight in the positive direction.

Among results adjacent to topology, there is the famous theorem of Appel and Haken of the late 1980s, establishing the truth of the classical conjecture that a finite plane map can be coloured using 4 colours. An unusual feature of this work was the use of computer calculations inside mathematics, which were grandiose by the standards of the time.

In the 1970s Thurston discovered that very many 3-dimensional manifolds admit a metric of constant negative curvature. This applies in particular to many knot complements, that can be obtained from discrete groups of motions of Lobachevsky 3-space ("hyperbolic space") for which a fundamental domain has finite volume. Thurston worked out deep methods of studying and constructing hyperbolic manifolds. He advanced the "geometrisation conjecture", which asserts, roughly speaking, that in the absence of any obvious obstruction, a 3-manifold admits a metric of constant negative curvature. For closed manifolds this means that the fundamental group is infinite, cannot be decomposed as a free product, and all its Abelian subgroups are cyclic; fearing a refutation of the Poincaré conjecture, we say that in this case the manifold must be homotopy equivalent to a manifold of negative curvature, but Thurston's conjecture is much more general. Realising the ideas of Thurston in the special class of "Haken manifolds" was completed by Otal, who gave a complete proof in this case in the 1990s; the conjecture has also been proved in a number of other special cases. Incidentally, a knot complement is a Haken manifold. A considerable number of specialists developed these ideas.

Goldman discovered a curious Lie algebra for the homotopy classes of closed paths on a surface, as a spin-off of investigations related to moduli spaces of Riemann surfaces. Studying the topology of moduli spaces of Riemann surfaces and vector bundles over them is a deep theme of current work that does not come into the circle of ideas under discussion.

Returning to the theory of knots and links, we note that very recently, already into the 21st century, Dynnikov developed a completely new algebraic approach to the problem of classifying knots and links by representing them in a "many-page book" (the pages are half-spaces glued along a "common axis") with $k \geq 3$ pages. This representation was already introduced 100 years ago, but Dynnikov discovered its remarkable algebraic properties: the problem of classifying knots and links reduces to computing the centre of a finitely generated semigroup with unit. In the framework of the theory of groups nothing like this can be done. The algorithm for recognising the trivial knot is obtained here as amazingly simple and accessible, and one deduces a series of corollaries for the braid groups. This seems to be an extremely successful approach.

Recently Perel'man has begun putting out a sequence of preprints leading (it is asserted) to a proof of the Poincaré conjecture. Using the analytic method of "Ricci flow" with some corrections, one introduces on a homotopy 3-sphere a metric of positive curvature. A number of the top experts confirm that these preprints contain extraordinarily interesting and hopeful ideas. At present we do not know whether these ideas will lead to the desired results.

We should note that in the 1980s and early 1990s there was a far-reaching conceptual development in the theory of knots and 3-dimensional manifolds; remarkable polynomial invariants of knots were discovered around 1984–85, such as the Jones polynomial and its generalisations (the HOMFLY polynomial), which sheds new light on the classical Alexander polynomial. It soon became clear that polynomials such as the Jones polynomial are more naturally constructed, as Turaev showed, in the context of the Alexander–Markov model, in terms of an original series of representations of the braid groups: they are constructed starting from particular cases of solutions of the so-called "Yang–Baxter" equation, used in exact solutions of models in statistical physics and quantum field theory; it turns out that these are representations of the braid groups in the required cases. Geometric properties of the Jones polynomial allowed a proof of the conjectures of Tait from the end of the 19th century: except for trivial cases, the number of points of intersection in the alternating diagram of a knot is a topological invariant (Kauffman and Murasugi, 1987). This could not be proved by the methods of classical algebraic topology. In the late 1980s, the theory of the so-called "Vassiliev invariants" of finite order attracted the attention of many topologists. Their complete classification over a field of characteristic zero was given by Kontsevich at the end of 1992, by a method using ideas from quantum theory. Over finite fields the problem remains open. The property of a topological invariant to "have finite order" in the sense of Vassiliev is a very useful and general idea. The coefficients of the Alexander or Jones polynomials and their generalisations turn out to be of this kind, as proved by Bar-Natan, Birman and Lin in the early 1990s.

In the late 1980s, Witten noticed that the Jones polynomial has a treatment in quantum field theory as a well-known correlation function of "Wilson loops" related to a closed curve in the theory of Yang–Mills fields. In the given case we have a 3-dimensional Yang–Mills theory with a very special "topological" action functional – the so-called Chern–Simons functional. The ideas of topological quantum field theory have a curious history. In the early 1970s Singer and Ray started a theory of the determinant of elliptic operators and developed an analytic approach to the Reidemeister invariant, which was perfected by Atiyah. A lot later in the 1980s, Müller and Cheeger proved that the analytic Ray–Singer torsion does indeed coincide with the combinatorial Reidemeister torsion using simplicial approximation of the Laplace–Beltrami operator on differential forms.

At the end of the 1970s A. Schwarz posed the question: can one construct topological invariants of manifolds as a functional integral over all fields, in the style of quantum field theory (whatever that might mean) – in such a way that the action does not depend on the Riemannian metric? He considered the simplest Abelian gauge theory on 1-forms on 3-manifolds, given by the Kelvin–Whitehead action (the density of the Hopf invariant), where gauge transformations consist of adding an exact summand to the 1-form, that does not change the action. Systematically carrying through the theory of this gauge field (including the construction of the Faddeev–Popov "ghosts"). Schwarz showed that one can make sense of the quantity that arises: it equals the Reidemeister–Ray–Singer torsion. In general the theory of determinants became very popular in gauge invariant field theory in connection with the Faddeev–Popov "ghosts".

At the end of the 1980s Witten started to develop actively topological quantum field theories on different manifolds, especially those of dimension 2, 3 and 4. In the 3-fold case for the simplest non-Abelian theory, the Yang–Mills theory with action the Chern–Simons functional, Witten discovered that the so-called "Wilson loop", when it can be defined, coincides with the Jones polynomial of a knot. Without going deeply into this theory, we indicate its purely topological axiomatisation by Atiyah and Segal: any such "theory", called a TQFT, assigns to any $(n-1)$-dimensional closed oriented manifold a finite-dimensional Euclidean space, and takes a union of two manifolds to their tensor product. To every n-dimensional oriented manifold with boundary one assigns a distinguished vector in the Euclidean space corresponding to this boundary, together with its orientation. If the boundary consists of several components, then the corresponding space is a vector in the tensor product, together with tensorial indices. On gluing two manifolds with boundary along a whole component of the boundary we obtain a new distinguished vector from the tensor product of the two distinguished vectors by convolution in all indices corresponding to the common component. In classical topology, only two examples were known: in both examples all the spaces are 1-dimensional and the distinguished element is the exponential either of the Euler characteristic of a manifold with boundary (for $n = 2k$), or of its signature (in the case $n = 4k$). Justifying the behaviour under gluing gives the lemma on additivity under gluing along a component of the boundary for the Euler characteristic and the Novikov–Rokhlin lemma (1965–66) on the additivity of the signature under gluing, which according to Jänich (1967) is an axiom that determines these two quantities. TQFTs with spaces of dimension more that one have only arisen from modern quantum theory. New topological invariants, for example, those of 3-dimensional manifolds, are obtained from TQFT; if we represent the 3-manifold by gluing two manifolds with boundary along a common boundary, then this invariant equals the scalar product of the distinguished vectors. An effective combinatorial construction of a TQFT is not an easy problem. It was solved in some cases by Turaev–Reshetikhin and Viro–Turaev in the 1990s, who constructed the best-known "quantum" topological invariants of 3-manifolds, that can be considered exact definitions of Witten's continuum integral. A series of results were also obtained by Axelrod–Singer in the 1990s. Topological applications of these quantities could become extremely significant if one could succeed in understanding more deeply their connections with the topological properties of manifolds and with classical algebraic topology.

New ideas in Morse theory

The interaction with modern mathematical physics and theoretical physics introduced a series of new ideas in topological aspects of the calculus of variations. Starting out from the Hamiltonian formalism of a classical mechanical system (for example, a top in a gravitational field or in an ideal fluid), and also from the theory of motions of charged particles in a topologically non-trivial magnetic field, Novikov in 1981 initiated the construction of an analogue of Morse theory with single-valued functions replaced by closed 1-forms. The action functional is not well defined in these systems, but its variation is a closed 1-form on the space of paths. He gave a classification of this kind of local Lagrangians in higher-dimensional field theory (the Wess–Zumino–Novikov–Witten action) and formulated the condition of "topological quantisation of the coupling constants" that arises from the requirement that the Feynman amplitude (the exponential of the action times the imaginary unit) is a single-valued functional. Thus the variation of the action should be an integral 1-dimensional cohomology class on the space of fields. This is an alternative formulation of the condition of Dirac quantisation for a particle in the field of a magnetic monopole, but formulated in the context of the now generally accepted Feynman quantisation, rather than Schrödinger quantisation; this is the form that is convenient for generalisation to higher dimension. Deser–Jackiw–Templeton (1982) and Witten (1983) came to similar ideas in the case of several particles. An analogue of Morse theory ("Morse–Novikov theory") was developed by Novikov for 1-forms on finite-dimensional manifolds. It turns out that in this case the analogue of the Morse inequalities for the number of critical points ("Morse–Novikov inequalities") arises on using cohomology with coefficients in the so-called Novikov rings, the simplest of which coincides with the ring of Laurent series of finite order with integer coefficients in one variable. This theory was developed by Farber (1984), who proved that these inequalities are sharp under certain conditions (an analogue of Smale's theorem). The construction of Morse–Novikov theory on loop spaces turned out to be very difficult for a particle in a magnetic field: Novikov and Taimanov formulated and applied the "principle of deleting cycles" for multivalued functionals, but the justification was only rigorously proved in certain special cases.

Significantly later, starting with Floer's work (1988) on the Arnol'd conjecture about the number of periodic orbits in non-autonomous Hamiltonian systems with periodic time-dependence on compact symplectic manifolds, multivalued functionals also actually appeared in symplectic topology, although this was not immediately realised. Hamilton's variational principle on compact symplectic varieties always leads to the situation as in Novikov's work: the action functional is always multivalued. This theory was developed extensively in later works of MacDuff, Salamon and Hofer (early 1990s). We will not discuss any further the extremely deep aspects of symplectic and contact topology, such as the Gromov–Witten invariants, the theory of Hofer–Eliashberg–Givental, mirror symmetry and so on.

In 1982 Witten proposed an original analytic method for proving the usual classical Morse inequalities. His idea turned out to be extremely fruitful, but we do not go into details here. Witten's method was applied by Pazhitnov (1987) to the Novikov problem discussed above of bounding the number of critical points of closed 1-forms. Novikov attempted to apply this method also to non-closed forms (vector fields). Atiyah and Bott successfully applied Witten's method to holomorphic vector fields, but this subject would take us beyond the framework of this article.

A series of investigations were carried out for non-simply connected manifolds in the 1980s, using manifolds of representations of the fundamental group – for the purposes of Morse theory (Novikov, 1986), and also for knot theory (Le Thang, early 1990s). This subject has significant outlook, and a series of topologists in recent years have given a lot of attention to the study of manifolds of representations.

New ideas in topology of the 1980s and 1990s: four-manifolds

The theoretical physics community began to show an ever-widening interest in the ideas of algebraic topology already from the 1970s. It became clear that many real phenomena of nature are topological in nature, and nowhere more so than in quantum physics. Everywhere that non-trivial phenomena appear, related to the magnetic field (say) or its analogues, such as Yang–Mills fields (differential geometric connections on vector bundles), topology is bound to appear. Everywhere fields of a complicated mathematical nature appear, with non-trivial singularities, topology will appear. Similar situations appear in the low-temperature phases of many substances, in liquid crystals, and in other cases. We only mention here the ideas of physicists that had a large influence in return on the ideas of topology itself. In the theory of Yang–Mills fields topological phenomena were discovered such as the Polyakov–'t Hooft magnetic monopole (1973), still not observed experimentally, and the instanton discovered in the work of Polyakov–Schwarz–Belavin–Tyupkin (1974) and also the remarkable self-duality equation that this instanton satisfies, which was also discovered in the same paper.

Without discussing the fate of these ideas in physics, we mention that at the start of the 1980s, Donaldson discovered a remarkable application of instantons to the self-duality equation in 4-dimensional topology: for example, he succeeded in proving that every (closed, simply connected, smooth) 4-manifold with a positive-definite intersection form on 2-cycles is homotopy equivalent to a connected sum of complex projective planes and their complex conjugates – that is, only a trivial subset of the positive-definite forms is actually realised geometrically. Donaldson discovered that there are different smooth structures on certain 4-manifolds. This remarkable fact is based on deep results of the qualitative theory of non-linear elliptic PDEs – as the outcome of a large and very difficult theory. It also depends on Freedman's theorem on homeomorphism of 4-manifolds discussed above. Already from the time of Kodaira there have been lists of algebraic surfaces for which one can without special trouble find two simply connected 4-manifolds, given by different constructions, but having isomorphic intersection forms. By Freedman, they are actually homeomorphic. However, these manifolds have different manifolds of instantons on them (the solutions of the self-duality equation). This decides the question – they are not diffeomorphic.

Later arguments of this type were substantially improved, and the Seiberg–Witten equation was discovered, which is much simpler to study, making the foundational theorems a lot easier, and allowing us to obtain many similar results. Thom's conjecture on the minimal genus of a surface realising a 2-cycle was proved by Kronheimer and Mrowka in the late 1990s. In the mid-1990s Fintushel discovered a beautiful connection between the Seiberg–Witten equations and the Alexander polynomial of knots that holds for manifolds constructed in a certain way. He obtained effective methods of computing these invariants in a series of examples, leading to strong results. We leave the details of this new period of development of topology to more specialised articles.

We do not discuss in this paper areas such as the topology and geometry of foliations on manifolds, singularity theory, contact and symplectic topology and geometry, although these ideas are closely related to the new methods of topology just discussed.

A whole series of deep homological investigations were carried out for the needs of the theory of algebraic manifolds with singularities; for example, the intersection cohomology of Goresky and MacPherson, and its computation in a number of cases. Many results were obtained in the study and computation of different types of equivariant cohomology for the action of groups, especially Abelian groups (Kirwan, Jeffrey and others). Especially deep discoveries were made in the geometry and topology of moduli spaces of Riemann surfaces, with the ideas arising from string theory (Polyakov, Takhtajan and Zograf) and from the theory of matrix models (Witten's conjecture on Chern numbers). We do not have space to discuss these ideas here.

A whole series of deep aspects of the topology of non-simply connected manifolds also remains outside the framework of our article; this includes the theory of "von Neumann invariants" – an analogue of Betti numbers, the index formula, the Morse inequalities and Reidemeister torsion, developed by Atiyah, Singer, Novikov, Shubin, Lott, Gromov and Farber in the 1980s.

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Steklov Institute of Mathematics,
Russian Academy of Sciences;
University of Maryland at College Park
E-mail address: novikov@ipst.umd.edu

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