

# The algebraic construction of the Novikov complex of a circle-valued Morse function

Andrew Ranicki

Received: 23 November 2000 / Revised version: 3 May 2001 /

Published online: 28 February 2002 – © Springer-Verlag 2002

**Abstract.** The Novikov complex of a circle-valued Morse function  $f : M \rightarrow S^1$  is constructed algebraically from the Morse-Smale complex of the restriction of the real-valued Morse function  $\bar{f} : \bar{M} \rightarrow \mathbb{R}$  to a fundamental domain of the pullback infinite cyclic cover  $\bar{M} = f^*\mathbb{R}$  of  $M$ .

## 1. Introduction

The relationship between real-valued Morse functions and chain complexes is well understood. The Morse-Smale complex of a Morse function  $f : M \rightarrow \mathbb{R}$  on a compact  $m$ -dimensional manifold  $M$  is defined using a choice of gradient-like vector field  $v$  satisfying the transversality condition, to be a based f.g. free  $\mathbb{Z}[\pi_1(M)]$ -module chain complex  $C^{MS}(M, f, v)$  with  $\text{rank}_{\mathbb{Z}[\pi_1(M)]} C_i^{MS}(M, f, v) = c_i(f)$  the number of critical points of  $f$  with index  $i$ , and the differentials defined by counting the downward  $v$ -gradient flow lines in the universal cover  $\tilde{M}$  of  $M$ . The pair  $(f, v)$  determines a handlebody decomposition  $M = \bigcup_{i=0}^m \bigcup_{c_i(f)} D^i \times D^{m-i}$  with one  $i$ -handle for each critical point of index  $i$ , and the Morse-Smale complex is the cellular chain complex of  $\tilde{M}$   $C^{MS}(M, f, v) = C(\tilde{M})$ .

The relationship between circle-valued Morse functions  $f : M \rightarrow S^1$  and chain complexes is more complicated, and not so well understood. A lift of  $f$  to the pullback infinite cyclic cover  $\bar{M} = f^*\mathbb{R}$  is a real-valued Morse function  $\bar{f} : \bar{M} \rightarrow \mathbb{R}$  on a non-compact manifold, so traditional Morse theory does not apply directly. The methods developed (by Novikov, Farber, Pajitnov, the author and others) to count the critical points of  $f$  use the structure of the fundamental group ring  $\mathbb{Z}[\pi_1(M)]$  as a Laurent polynomial extension of  $\mathbb{Z}[\pi_1(\bar{M})]$ , as well as a completion and a localization of  $\mathbb{Z}[\pi_1(M)]$ . For any map  $f : M \rightarrow S^1$  with  $M$  and  $\bar{M}$  connected the ring  $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(\bar{M})]_\alpha[z, z^{-1}] = \mathbb{Z}[\pi_1(\bar{M})]_\alpha[z][z^{-1}]$  is the  $\alpha$ -twisted Laurent polynomial extension with  $\alpha : \pi_1(\bar{M}) \rightarrow \pi_1(\bar{M})$  the monodromy automorphism and  $az = z\alpha(a)$  ( $a \in \mathbb{Z}[\pi_1(\bar{M})]$ ). The completion

A. RANICKI

Department of Mathematics and Statistics, University of Edinburgh, Edinburgh EH9 3JZ, Scotland, UK (e-mail: aar@maths.ed.ac.uk)

$\mathbb{Z}[\widehat{\pi_1(M)}] = \mathbb{Z}[\pi_1(\overline{M})]_\alpha((z)) = \mathbb{Z}[\pi_1(\overline{M})]_\alpha[[z]][z^{-1}]$  is called the *Novikov ring* of  $\mathbb{Z}[\pi_1(M)]$ . Let  $\Sigma$  be the set of square matrices in  $\mathbb{Z}[\pi_1(\overline{M})]_\alpha[[z]] \subset \mathbb{Z}[\pi_1(M)]$  which become invertible in  $\mathbb{Z}[\pi_1(\overline{M})]$  under the augmentation  $z \mapsto 0$ . The non-commutative localization  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$  of  $\mathbb{Z}[\pi_1(M)]$  (in the sense of Cohn [C]) is a ring with a morphism  $\mathbb{Z}[\pi_1(M)] \rightarrow \Sigma^{-1}\mathbb{Z}[\pi_1(M)]$  such that any ring morphism  $\mathbb{Z}[\pi_1(M)] \rightarrow R$  which sends  $\Sigma$  to invertible matrices in  $R$  has a unique factorization  $\mathbb{Z}[\pi_1(M)] \rightarrow \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \rightarrow R$ . The inclusion  $\mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{Z}[\widehat{\pi_1(M)}]$  sends  $\Sigma$  to invertible matrices in  $\mathbb{Z}[\widehat{\pi_1(M)}]$ , so there is a natural ring morphism  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{Z}[\widehat{\pi_1(M)}]$ .

A vector field on a manifold  $M$  is a section  $v : M \rightarrow \tau_M$  of the tangent bundle  $\tau_M$ . The gradient of a Morse function  $f : M \rightarrow \mathbb{R}$  is a section  $\nabla f = (\partial f / \partial x_i) : M \rightarrow \tau_M^*$  of the cotangent bundle  $\tau_M^*$ , with zeros the critical points of  $f$ . A vector field  $v : M \rightarrow \tau_M$  is 'gradient-like' for  $f$  if there exists a Riemannian metric  $\beta : \tau_M \cong \tau_M^*$  on  $M$  such that  $\beta \circ v = \nabla f : M \rightarrow \tau_M^*$ . A  $v$ -gradient flow line  $\gamma : \mathbb{R} \rightarrow M$  satisfies  $\gamma'(t) = -v(\gamma(t)) \in \tau_M(\gamma(t))$ . The limits  $\lim_{t \rightarrow -\infty} \gamma(t) = p$ ,  $\lim_{t \rightarrow \infty} \gamma(t) = q \in M$  are critical points of  $f$ . For every non-critical point  $x \in M$  there is a  $v$ -gradient flow line  $\gamma_x : \mathbb{R} \rightarrow M$  (which is unique up to scaling) such that  $\gamma_x(0) = x \in M$ . The unstable and stable manifolds of a critical point  $p \in M$  of index  $i$  are the open manifolds  $W^u(p, v) = \{x \in M \mid \lim_{t \rightarrow -\infty} \gamma_x(t) = p\}$ ,  $W^s(p, v) = \{x \in M \mid \lim_{t \rightarrow \infty} \gamma_x(t) = p\}$  which are diffeomorphic to  $\mathbb{R}^i, \mathbb{R}^{m-i}$  respectively. Let  $\mathcal{GT}(f)$  denote the space of all gradient-like vector fields  $v$  for  $f$  which satisfy the Morse-Smale transversality condition that for any distinct critical points  $p, q \in M$  the submanifolds  $W^u(p, v), W^s(q, v) \subset M$  intersect transversely. For  $v \in \mathcal{GT}(f)$  and critical points  $p, q \in M$  of index  $i, i - 1$  there is only a finite number of  $v$ -gradient flow lines  $\gamma : \mathbb{R} \rightarrow M$  which start at  $p$  and terminate at  $q$ , and choosing orientations there is obtained an algebraic number  $n(p, q) \in \mathbb{Z}$ .

A circle-valued Morse function  $f : M \rightarrow S^1$  lifts to a real-valued Morse function  $\bar{f} : \overline{M} \rightarrow \mathbb{R}$  on the infinite cyclic cover  $\overline{M} = f^*\mathbb{R}$ . Let  $\mathcal{GT}(f)$  be the space of all vector fields  $v$  on  $M$  which lift to a gradient-like vector field  $\bar{v}$  on  $\overline{M}$  satisfying the transversality condition, so that  $\bar{v} \in \mathcal{GT}(\bar{f})$ .

Novikov [N] and Pajitnov [P1] used the completion  $\mathbb{Z}[\widehat{\pi_1(M)}]$  to construct geometrically for any circle-valued Morse function  $f : M \rightarrow S^1$  and  $v \in \mathcal{GT}(f)$  a based f. g. free  $\mathbb{Z}[\widehat{\pi_1(M)}]$ -module chain complex  $C^{Nov}(M, f, v)$  such that  $\text{rank}_{\mathbb{Z}[\widehat{\pi_1(M)}]} C_i^{Nov}(M, f, v) = c_i(f)$ , with  $c_i(f)$  the number of critical points of  $f$  with index  $i$ . As in the real-valued case the differentials are defined by counting the  $\bar{v}$ -gradient flow lines in the universal cover  $\tilde{M}$ . Moreover, there is a chain equivalence  $C^{Nov}(M, f, v) \simeq C(\tilde{M}; \mathbb{Z}[\widehat{\pi_1(M)}])$  with  $C(\tilde{M}; \mathbb{Z}[\widehat{\pi_1(M)}]) = \mathbb{Z}[\widehat{\pi_1(M)}] \otimes_{\mathbb{Z}[\pi_1(M)]} C(\tilde{M})$  the  $\mathbb{Z}[\widehat{\pi_1(M)}]$ -coefficient cellular chain complex of  $\tilde{M}$ , for any CW structure on  $M$ .

Pajitnov [P2,P3,P4] constructed for any Morse function  $f : M \rightarrow S^1$  a  $C^0$ -dense subspace  $\mathcal{G}\mathcal{C}\mathcal{T}(f) \subset \mathcal{G}\mathcal{T}(f)$ , such that for  $v \in \mathcal{G}\mathcal{C}\mathcal{T}(f)$  the coefficients in the corresponding Novikov complex  $C^{Nov}(M, f, v)$  are rational, in the sense that  $C^{Nov}(M, f, v) = \widehat{\mathbb{Z}[\pi_1(M)]} \otimes_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C^{Paj}(M, f, v)$  for a based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complex  $C^{Paj}(M, f, v)$ , with a chain equivalence  $C^{Paj}(M, f, v) \simeq C(\tilde{M}; \Sigma^{-1}\mathbb{Z}[\pi_1(M)])$ . (Strictly speaking,  $C^{Paj}(M, f, v)$  was only defined for abelian  $\pi_1(M)$ , when the natural ring morphism  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)] \rightarrow \widehat{\mathbb{Z}[\pi_1(M)]}$  is injective, but this was for algebraic convenience rather than out of geometric necessity.)

For a Morse function  $f : M \rightarrow S^1$  which is transverse regular at  $0 \in S^1$  the lift  $\bar{f} : \bar{M} \rightarrow \mathbb{R}$  is transverse regular at  $\mathbb{Z} \subset \mathbb{R}$ , and the restriction to a fundamental domain is a Morse function  $f_N = \bar{f}| : (M_N; N, N_1) = \bar{f}^{-1}(I; \{0\}, \{1\}) \rightarrow (I; \{0\}, \{1\})$  with  $c_i(f_N) = c_i(f)$  critical points of index  $i$ , and  $N_1$  a copy of  $N$ . Every  $v \in \mathcal{G}\mathcal{T}(f)$  lifts to  $\bar{v} \in \mathcal{G}\mathcal{T}(\bar{f})$ , and  $v_N = \bar{v}| \in \mathcal{G}\mathcal{T}(f_N)$  determines a handlebody decomposition  $M_N = N \times I \cup \bigcup_{i=0}^m \bigcup_{c_i(f)} D^i \times D^{m-i}$  with one  $i$ -handle

for each index  $i$  critical point of  $f$ . Given a  $CW$  structure on  $N$  with  $c_i(N)$   $i$ -cells use the handlebody structure on  $M_N$  to define a  $CW$  structure on  $M_N$  with  $c_i(N) + c_i(f)$   $i$ -cells. (In practice, the  $CW$  structure will be the one determined by a Morse function  $N \rightarrow \mathbb{R}$ ). The inclusion  $g : N \rightarrow M_N$  induces an inclusion of based f.g. free  $\mathbb{Z}[\pi_1(\bar{M})]$ -module chain complexes  $g : C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  with  $\tilde{N}, \tilde{M}_N$  the covers of  $N, M_N$  induced from the universal cover  $\tilde{M}$  of  $\bar{M}$ . Write the inclusion of  $N_1$  in  $M_N$  as  $h : N \rightarrow M_N$ , and note that in general  $N_1 = h(N)$  is not a  $CW$  subcomplex of  $M_N$ . A *chain approximation* for  $h$  is a  $\mathbb{Z}[\pi_1(\bar{M})]$ -module chain map  $h : C(\tilde{N}_1) = \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  in the chain homotopy class induced by the map  $h : N \rightarrow M_N$  as given by the cellular approximation theorem. For any choice of chain approximation  $h$  Farber and Ranicki [FR] defined algebraically a based f. g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complex  $C^{FR}(M, f, v, h)$  such that  $\text{rank}_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C_i^{FR}(M, f, v, h) = c_i(f)$  as a deformation of the  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -coefficient Morse-Smale complex

$$\begin{aligned} & C^{MS}(M_N, f_N, v_N; \Sigma^{-1}\mathbb{Z}[\pi_1(M)]) \\ &= \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(\bar{M})]} C^{MS}(M_N, f_N, v_N) \end{aligned}$$

with a chain equivalence  $C^{FR}(M, f, v, h) \simeq \Sigma^{-1}\mathbb{Z}[\pi_1(M)] \otimes_{\mathbb{Z}[\pi_1(\bar{M})]} C(\tilde{M})$ . There is also a  $\widehat{\mathbb{Z}[\pi_1(M)]}$ -coefficient version

$$C^{FR}(M, f, v, h; \widehat{\mathbb{Z}[\pi_1(M)]}) = \widehat{\mathbb{Z}[\pi_1(M)]} \otimes_{\Sigma^{-1}\mathbb{Z}[\pi_1(M)]} C^{FR}(M, f, v, h) .$$

**Cokernel Theorem 6.6.** *The chain complex of [FR] is isomorphic to the cokernel of the morphism of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes  $\phi = g - zh : \Sigma^{-1}C(\tilde{N})_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}C(\tilde{M}_N)_\alpha[z, z^{-1}]$  which is a split injection in each degree,  $C^{FR}(M, f, v, h) \cong \text{coker}(\phi)$ .  $\square$*

The expression  $C^{FR}(M, f, v, h)$  as a cokernel makes it possible to prove invariance results such as :

**Invariance Theorem 6.7.** *Let  $f : M \rightarrow S^1$  be a Morse function, and let  $v \in \mathcal{GT}(f)$ . For any regular values  $0, 0' \in S^1$ , CW structures on  $N = f^{-1}(0)$ ,  $N' = f^{-1}(0')$  and chain approximations  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$ ,  $h' : \alpha C(\tilde{N}') \rightarrow C(\tilde{M}_{N'})$  there is defined a simple isomorphism of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes  $C^{FR}(M, f, v, h) \cong C^{FR}(M, f, v, h')$ .  $\square$*

Here, simple means that the torsion of the isomorphism is in the image of  $\{\pm\pi_1(M)\} \subseteq K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)])$ .

Given  $f : M \rightarrow S^1$ ,  $v \in \mathcal{GT}(f)$  and a CW structure on  $N = f^{-1}(0) \subset M$  we shall say that a chain approximation  $h^{gra} : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  is *gradient-like* if it counts the  $v$ -gradient flow lines in the universal cover  $\tilde{M}$ . (The precise definition is given in 6.8).

**Isomorphism Theorem 6.10.** *For  $v \in \mathcal{GT}(f)$  with a gradient-like chain approximation  $h^{gra}$  there are basis-preserving isomorphisms  $C^{Nov}(M, f, v) \cong C^{FR}(M, f, v, h^{gra}; \mathbb{Z}[\widehat{\pi_1(M)}])$  and  $C^{Paj}(M, f, v) \cong C^{FR}(M, f, v, h^{gra})$ .  $\square$*

Pajitnov [P3] (Theorem 7.2) showed that every  $v \in \mathcal{GT}(f)$  is  $C^0$ -close to  $v^{gra} \in \mathcal{GCCT}(f)$  with a gradient-like chain approximation  $h^{gra} : C(\tilde{N}) \rightarrow C(\tilde{M}_N)$ . A similar construction was obtained by Hutchings and Lee [HL]. In fact, Cornea and Ranicki [CR] prove that every  $v \in \mathcal{GT}(f)$  admits a gradient-like chain approximation  $h^{gra}$ .

The plan of the paper is as follows. Section 1 is purely algebraic in nature, concerning the chain homotopy properties of the algebraic mapping cones and cokernels of chain maps. The glueing properties of the Morse-Smale complex  $C^{MS}(M, f, v)$  are described in Sect. 2 for finite unions, and in Sect. 3 for infinite unions. Section 4 gives a brief account of the Cohn localization. Section 5 deals with the cokernel and infinite union construction of chain complexes over a twisted polynomial extension  $A_\alpha[z, z^{-1}]$  for any ring  $A$  with automorphism  $\alpha : A \rightarrow A$ , the localization  $\Sigma^{-1}A_\alpha[z, z^{-1}]$  and the Novikov ring  $A_\alpha((z)) = A_\alpha[[z]][z^{-1}]$ . The Cokernel, Invariance and Isomorphism Theorems are proved in Sect. 6.

The remainder of the Introduction is an outline of the proof of the Cokernel, Invariance and Isomorphism Theorems in the special case when  $f_* : \pi_1(M) \rightarrow \pi_1(S^1)$  is an isomorphism, so that  $\pi_1(\overline{M}) = \{1\}$ ,  $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[z, z^{-1}]$ ,  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)] = (1 + z\mathbb{Z}[z])^{-1}\mathbb{Z}[z, z^{-1}]$  and  $\mathbb{Z}[\widehat{\pi_1(M)}] = \mathbb{Z}((z))$ . The Novikov complex of a Morse function  $f : M \rightarrow S^1$  with respect to  $v \in \mathcal{GT}(f)$  is the based f.g. free  $\mathbb{Z}((z))$ -module chain complex  $C = C^{Nov}(M, f, v)$  with

$$d_C : C_i = \mathbb{Z}((z))^{c_i(f)} \rightarrow C_{i-1} = \mathbb{Z}((z))^{c_{i-1}(f)} ; \bar{p} \mapsto \sum_{\bar{q}} n(\bar{p}, \bar{q})\bar{q}$$

where  $n(\bar{p}, z^j \bar{q})$  is the algebraic number of  $\bar{v}$ -gradient flow lines in  $\bar{M}$  from a critical point  $\bar{p} \in \bar{M}$  of index  $i$  to a critical point  $\bar{q} \in \bar{M}$  of index  $i - 1$ , using the transversality property of  $v$  to ensure that these numbers are finite. Here  $z : \bar{M} \rightarrow \bar{M}$  is the generating covering translation parallel to the  $v$ -gradient flow of  $f$ . If  $0 \in S^1$  is a regular value of  $f$  the Morse function  $f_N = \bar{f}| : (M_N; N, z^{-1}N) = \bar{f}^{-1}(I; \{0\}, \{1\}) \rightarrow (I; \{0\}, \{1\})$  has  $c_i(f_N) = c_i(f)$  critical points of index  $i$ . Each critical point  $p \in M$  of  $f$  can be regarded as a critical point  $\bar{p} \in M_N$  of  $f_N$ . For  $v \in \mathcal{GT}(f)$  the Morse-Smale complex of  $f_N$  with respect to  $v_N \in \mathcal{GT}(f_N)$  is the based f.g. free  $\mathbb{Z}$ -module chain complex  $C^{MS}(M_N, f_N, v_N) = F$  with

$$d_F : F_i = \mathbb{Z}^{c_i(f)} \rightarrow F_{i-1} = \mathbb{Z}^{c_{i-1}(f)} ; \bar{p} \mapsto \sum_{\bar{q}} n(\bar{p}, \bar{q})$$

where  $n(\bar{p}, \bar{q})$  is the algebraic number of  $v_N$ -gradient flow lines in  $M_N$  from a critical point  $\bar{p} \in M_N$  of index  $i$  to a critical point  $\bar{q} \in M_N$  of index  $i - 1$ .

The Novikov complex  $C^{Nov}(M, f, v) = C$  counts the  $\bar{v}$ -gradient flow lines which start at a critical point  $\bar{p} \in z^j M_N \subset \bar{M}$  and terminate at a critical point  $\bar{q} \in z^k M_N \subset \bar{M}$  with  $k \leq j$ . The Morse-Smale complex  $C^{MS}(M_N, f_N, v_N) = F$  only counts such flow lines with  $j = k = 0$ . In order to construct  $C$  from  $F$  we glue together an infinite number of copies of an 'algebraic fundamental domain' which gives an algebraic picture of the way the  $v_N$ -flow lines enter  $M_N$  at  $z^{-1}N$  and either die at a critical point of  $f_N$  in  $M_N$  or exit at  $N$ .

As above, given an arbitrary  $CW$  structure on  $N = f^{-1}(0)$  with  $c_i(N)$   $i$ -cells use the handlebody decomposition  $M_N = N \times I \cup \bigcup_{i=0}^m \bigcup_{c_i(f)} D^i \times D^{m-i}$ , to give  $M_N$  a  $CW$  structure with  $c_i(N) + c_i(f)$   $i$ -cells, one for each  $i$ -cell of  $N$  and one for each  $i$ -handle of  $(M_N; N, z^{-1}N)$ . The cellular chain complex of  $M_N$  is of the form  $C(M_N) = E$  with

$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1},$$

where  $D = C(N)$ ,  $F = C(\bar{M}_N, N \times I) = C^{MS}(M_N, f_N, v_N)$ ,  $D_i = \mathbb{Z}^{c_i(N)}$ ,  $E_i = \mathbb{Z}^{c_i(N)+c_i(f)}$ ,  $F_i = \mathbb{Z}^{c_i(f)}$ ,  $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i$ . Let  $h : D \rightarrow E$  be a chain approximation for the inclusion  $h : z^{-1}N \rightarrow M$ , with components  $h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i$ . The cellular  $\mathbb{Z}[z, z^{-1}]$ -module chain complex of  $\bar{M}$  is the algebraic mapping cone  $C(\bar{M}) = C(g-zh : D[z, z^{-1}] \rightarrow E[z, z^{-1}])$ . The chain complex of Farber and Ranicki [FR] was defined to be the based f.g. free  $\Sigma^{-1}\mathbb{Z}[z, z^{-1}]$ -module chain complex  $C^{FR}(M, f, v, h) = \widehat{F}$  with

$$d_{\widehat{F}} = d_F + zh_F(1 - zh_D)^{-1}c : \widehat{F}_i = \Sigma^{-1}F_i[z, z^{-1}] \rightarrow \widehat{F}_{i-1} = \Sigma^{-1}F_{i-1}[z, z^{-1}]$$

a deformation of  $\Sigma^{-1}F[z, z^{-1}]$ . The inverse of  $1 - zh_D$  is defined over the Novikov ring  $\mathbb{Z}((z))$  by  $(1 - zh_D)^{-1} = 1 + zh_D + z^2(h_D)^2 + z^3(h_D)^3 + \dots$ , so the induced  $\mathbb{Z}((z))$ -module chain complex is  $C^{FR}(M, f, v, h; \mathbb{Z}((z))) = \mathbb{Z}((z)) \otimes_{\Sigma^{-1}\mathbb{Z}[z, z^{-1}]} C^{FR}(M, f, v, h) = \overline{F}$  with

$$d_{\overline{F}} = d_F + \sum_{j=1}^{\infty} z^j h_F(h_D)^{j-1} c : \overline{F}_i = F_i((z)) \rightarrow \overline{F}_{i-1} = F_{i-1}((z)).$$

The chain approximation  $h$  is an algebraic model for the  $\bar{v}$ -gradient flow across a fundamental domain  $(M_N; N, z^{-1}N)$  of  $\overline{M}$ . The formula for  $d_{\overline{F}}$  is interpreted in Sect. 5 as the generating function for the number of flow lines in  $\overline{M}$  of prescribed length, with  $h_F(h_D)^{j-1}c$  counting the flow lines which start in  $M_N$  and terminate in  $z^j M_N$ , crossing the  $j$  walls  $N, zN, \dots, z^{j-1}N$  between the adjacent fundamental domains  $M_N, zM_N, \dots, z^j M_N$ .

The algebraic treatment in Sect. 1 of cokernels of chain maps will be used in Sect. 6 to prove that the inclusions  $\widehat{F}_i \rightarrow \Sigma^{-1}E_i[z, z^{-1}]$  induce the isomorphism  $C^{FR}(M, f, v, h) \cong \text{coker}(\phi)$  of the Cokernel Theorem 6.6, with  $\phi = g - zh : \Sigma^{-1}D[z, z^{-1}] \rightarrow \Sigma^{-1}E[z, z^{-1}]$ . A chain homotopy  $k : h \simeq h' : \alpha D \rightarrow E$  determines an isomorphism  $\text{coker}(\phi) \cong \text{coker}(\phi')$  of  $\Sigma^{-1}\mathbb{Z}[z, z^{-1}]$ -module chain complexes (Proposition 5.3), giving the isomorphism of the Invariance Theorem 6.7

$$C^{FR}(M, f, v, h) \cong \text{coker}(\phi) \cong \text{coker}(\phi') \cong C^{FR}(M, f, v, h')$$

in the special case  $N = N'$ . The general case is proved by an algebraic treatment of handle exchanges (Proposition 5.4).

For a gradient-like chain approximation  $h^{gra} = \begin{pmatrix} h_D^{gra} \\ h_F^{gra} \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i$  the algebraic numbers of  $\bar{v}$ -gradient flow lines between critical points of  $\bar{f} : \overline{M} \rightarrow \mathbb{R}$  are given by

$$n(\overline{p}, z^j \overline{q}) = \begin{cases} (\overline{p}, \overline{q})\text{-coefficient of } d_F : F_i \rightarrow F_{i-1} & \text{if } j = 0 \\ (\overline{p}, \overline{q})\text{-coefficient of } h_F^{gra} (h_D^{gra})^{j-1} c : F_i \rightarrow F_{i-1} & \text{if } j > 0 \\ 0 & \text{if } j < 0 \end{cases}$$

for any critical points  $\overline{p}, \overline{q} \in M_N$  of  $f_N$  with index  $i, i - 1$  respectively. It follows that for any  $-\infty < j < k < \infty$  the Morse-Smale complex of the real-valued Morse function  $f_N[j, k] = \bar{f}| : (M_N[j, k]; z^{-j}N, z^{-k}N) =$

$\bar{f}^{-1}([j, k]; \{j\}, \{k\}) \rightarrow ([j, k]; \{j\}, \{k\})$  is

$$\begin{aligned} & C^{MS}(M_N[j, k], f_N[j, k], v_N[j, k]) \\ &= \left( \sum_{\ell=-k}^{-j} z^\ell F_i, d_F + \sum_{m=1}^{k-j} z^m h_F^{gra} (h_D^{gra})^{m-1} c \right) \\ &\cong \text{coker} \left( g - zh^{gra} : \sum_{\ell=-k+1}^{-j} z^\ell D \rightarrow \sum_{\ell=-k}^{-j} z^\ell E \right). \end{aligned}$$

Passing to the direct limit as  $k \rightarrow \infty$  gives the Morse-Smale complex of the proper real-valued Morse function  $f_N[j, \infty) = \bar{f} | : (M_N[j, \infty), \partial M_N[j, \infty)) = \bar{f}^{-1}([j, \infty), \{j\}) \rightarrow ([j, \infty), \{j\})$  to be

$$\begin{aligned} & C^{MS}(M_N[j, \infty), f_N[j, \infty), v_N[j, \infty)) \\ &= \varinjlim_k C^{MS}(M_N[j, k], f_N[j, k], v_N[j, k]) \\ &\cong \text{coker} \left( g - zh^{gra} : \sum_{\ell=-\infty}^{-j} z^\ell D \rightarrow \sum_{\ell=-\infty}^{-j} z^\ell E \right). \end{aligned}$$

Passing to the inverse limit as  $j \rightarrow -\infty$  gives the Isomorphism Theorem 6.10 for the Novikov complex, with a basis-preserving  $\mathbb{Z}((z))$ -module isomorphism

$$\begin{aligned} C^{Nov}(M, f, v) &= \varprojlim_j C^{MS}(M_N[j, \infty), f_N[j, \infty), v_N[j, \infty)) \\ &\cong \text{coker}(g - zh^{gra} : D((z)) \rightarrow E((z))) \\ &\cong C^{FR}(M, f, v, h^{gra}; \mathbb{Z}((z))). \end{aligned}$$

The geometric differential in  $C = C^{Nov}(M, f, v)$  is just the algebraic differential in  $\bar{F} = C^{FR}(M, f, v, h^{gra}; \mathbb{Z}((z)))$ , with

$$\begin{aligned} d_C(\bar{p}) &= \sum_{\bar{q}} \sum_{j \in \mathbb{Z}} n(\bar{p}, z^j \bar{q}) z^j \bar{q} = (d_F + \sum_{j=1}^{\infty} z^j h_F^{gra} (h_D^{gra})^{j-1} c)(\bar{p}) \\ &= (d_F + zh_F^{gra} (1 - zh_D^{gra})^{-1} c)(\bar{p}) = d_{\bar{F}}(\bar{p}) \in \bar{F}_{i-1} = \mathbb{Z}^{c_{i-1}(f)}, \end{aligned}$$

so that there is also a basis-preserving  $\Sigma^{-1}\mathbb{Z}[z, z^{-1}]$ -module isomorphism  $C^{Paj}(M, f, v) \cong C^{FR}(M, f, v, h^{gra})$ . The projection

$$\begin{aligned} p : C(\bar{M}; \mathbb{Z}((z))) &= \mathcal{C}(g - zh^{gra} : D((z)) \rightarrow E((z))) \\ &\rightarrow \text{coker}(g - zh^{gra} : D((z)) \rightarrow E((z))) = C^{Nov}(M, f, v) \end{aligned}$$

pieces together  $\bar{v}$ -gradient flow lines in  $\bar{M}$  from their intersections with the translates  $z^j M_N \subset \bar{M}$  ( $j \in \mathbb{Z}$ ) of the fundamental domain  $M_N$ ;  $p$  is a chain equivalence with torsion

$$\begin{aligned} \tau(p) &= \sum_{i=0}^{\infty} (-)^{i+1} \tau(1 - zh_D^{gra} : D_i((z)) \rightarrow D_i((z))) \\ &= \prod_{i=0}^{\infty} \det(1 - zh_D^{gra} : D_i((z)) \rightarrow D_i((z)))^{(-)^{i+1}} \\ &\in \widehat{W}(\mathbb{Z}) \subset K_1(\mathbb{Z}((z))) = K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z}) \oplus \widehat{W}(\mathbb{Z}) \end{aligned}$$

with  $\widehat{W}(\mathbb{Z}) = 1 + z\mathbb{Z}[[z]]$  under multiplication. The kernel of  $p$  corresponds to the closed orbits of the  $v$ -gradient flow lines in  $M$ , which avoid the critical points of  $f$  and so do not contribute to the Novikov complex, and which are counted by the torsion of  $p$ .

I am grateful to Andrei Pajitnov for valuable suggestions for improving the preprint version of the paper.

### 1. Cones and cokernels

The Novikov complex of a circle-valued Morse function  $f : M \rightarrow S^1$  will be shown in Sect. 6 to be isomorphic to the cokernel of a chain map constructed from the Morse-Smale complex of a fundamental domain for the infinite cyclic cover  $\bar{M} = f^*\mathbb{R}$  of  $M$ ; the algebraic mapping cone of the chain map is a cellular chain complex of  $\bar{M}$ . This section is accordingly devoted to the relationship between the algebraic mapping cone and the cokernel of a chain map. The algebraic mapping cone is a chain homotopy invariant. The cokernel is not a chain homotopy invariant, although it is a homology invariant. The main novelty of this section is the introduction of an equivalence relation on chain maps called ‘chain isotopy’, which is stronger than chain homotopy, and is such that the cokernels of chain isotopic maps are isomorphic. The chain map with cokernel the Novikov complex depends on a choice of chain map in a chain homotopy class; a different choice will give a chain isotopic chain map, with isomorphic cokernel.

Let  $A$  be a ring. The *algebraic mapping cone* of an  $A$ -module chain map  $\phi : D \rightarrow E$  is the  $A$ -module chain complex  $\mathcal{C}(\phi)$  defined by

$$d_{\mathcal{C}(\phi)} = \begin{pmatrix} d_E & (-)^{i-1} \phi \\ 0 & d_D \end{pmatrix} : \mathcal{C}(\phi)_i = E_i \oplus D_{i-1} \rightarrow \mathcal{C}(\phi)_{i-1} = E_{i-1} \oplus D_{i-2} .$$

The natural projections

$$p : \mathcal{C}(\phi)_i = E_i \oplus D_{i-1} \rightarrow \text{coker}(\phi : D_i \rightarrow E_i) ; (x, y) \mapsto [x]$$

define a chain map  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi)$ .

**Proposition 1.1.** *For an injective chain map  $\phi : D \rightarrow E$  the natural projection  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi)$  induces isomorphisms in homology  $p_* : H_*(\mathcal{C}(\phi)) \cong H_*(\text{coker}(\phi))$ .*

*Proof.* The short exact sequences of  $A$ -module chain complexes

$$0 \rightarrow E \rightarrow \mathcal{C}(\phi) \rightarrow D_{*-1} \rightarrow 0, \quad 0 \rightarrow D \xrightarrow{\phi} E \rightarrow \text{coker}(\phi) \rightarrow 0$$

induce long exact sequences in homology, which are related by a natural transformation

$$\begin{array}{ccccccc} \dots & \rightarrow & H_i(D) & \xrightarrow{\phi_*} & H_i(E) & \longrightarrow & H_i(\mathcal{C}(\phi)) & \longrightarrow & H_{i-1}(D) & \rightarrow \dots \\ & & \downarrow 1 & & \downarrow 1 & & \downarrow p_* & & \downarrow 1 & \\ \dots & \rightarrow & H_i(D) & \xrightarrow{\phi_*} & H_i(E) & \longrightarrow & H_i(\text{coker}(\phi)) & \longrightarrow & H_{i-1}(D) & \rightarrow \dots \end{array}$$

It now follows from the 5-lemma that the induced morphisms  $p_*$  are isomorphisms.  $\square$

As usual, a *chain homotopy*  $\theta : \phi \simeq \phi' : D \rightarrow E$  between chain maps  $\phi, \phi' : D \rightarrow E$  is a collection of  $A$ -module morphisms  $\theta : D_i \rightarrow E_{i+1}$  such that

$$\phi' - \phi = d_E \theta + \theta d_D : D_i \rightarrow E_i .$$

Chain homotopic chain maps have isomorphic algebraic mapping cones :

**Proposition 1.2.** *A chain homotopy  $\theta : \phi \simeq \phi' : D \rightarrow E$  determines an isomorphism  $I = \begin{pmatrix} 1 & \pm \theta \\ 0 & 1 \end{pmatrix} : \mathcal{C}(\phi) \rightarrow \mathcal{C}(\phi')$  of the algebraic mapping cones. If  $D, E$  are based f.g. free then  $I$  is a simple isomorphism.*

*Proof.* By construction.  $\square$

In general, the cokernels of chain homotopic chain maps are not isomorphic (or even chain equivalent). The following relation will be convenient in dealing with cokernels of chain maps, in order to avoid this problem.

**Definition 1.3.** A *chain isotopy*  $\psi : \phi \sim \phi' : D \rightarrow E$  between chain maps  $\phi, \phi' : D \rightarrow E$  is a collection of  $A$ -module morphisms  $\psi : E_i \rightarrow E_{i+1}$  such that

(i) for each  $i$

$$\phi' = (1 + d_E \psi + \psi d_E) \phi : D_i \rightarrow E_i ,$$

defining a chain homotopy  $\psi \phi : \phi \simeq \phi' : D \rightarrow E$ ,

(ii) each  $1 + d_E\psi + \psi d_E : E_i \rightarrow E_i$  is an automorphism. □

**Proposition 1.4.** *Chain isotopy is an equivalence relation on chain maps.*

*Proof.* Reflexivity: every chain map  $\phi : D \rightarrow E$  is isotopic to itself by  $0 : \phi \sim \phi$ .

Symmetry: for any chain isotopy  $\psi : \phi \sim \phi' : D \rightarrow E$  define a chain isotopy  $\psi^- : \phi' \sim \phi$  by  $\psi^- = -(1 + d_E\psi + \psi d_E)^{-1}\psi : E_i \rightarrow E_{i+1}$ , with  $1 + d_E\psi^- + \psi^- d_E = (1 + d_E\psi + \psi d_E)^{-1} : E_i \rightarrow E_i$ .

Transitivity: for any chain isotopies  $\psi : \phi \sim \phi', \psi' : \phi' \sim \phi'' : D \rightarrow E$  define a chain isotopy  $\psi'' : \phi \sim \phi''$  by

$$\psi'' = \psi + \psi'(1 + d_E\psi + \psi d_E) : E_i \rightarrow E_{i+1}$$

with  $1 + d_E\psi'' + \psi'' d_E = (1 + d_E\psi' + \psi' d_E)(1 + d_E\psi + \psi d_E) : E_i \rightarrow E_i$ . □

Isotopic chain maps have isomorphic cokernels :

**Proposition 1.5.** *A chain isotopy  $\psi : \phi \sim \phi' : D \rightarrow E$  determines isomorphisms of chain complexes*

$$q = \begin{pmatrix} 1 \pm \psi\phi \\ 0 & 1 \end{pmatrix} : \mathcal{C}(\phi) \rightarrow \mathcal{C}(\phi'),$$

$$r = [1 + d_E\psi + \psi d_E] : \text{coker}(\phi) \rightarrow \text{coker}(\phi')$$

and a chain homotopy  $s : rp \simeq p'q : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi')$  with  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi), p' : \mathcal{C}(\phi') \rightarrow \text{coker}(\phi')$  the projections.

*Proof.* The isomorphism  $q$  is a special case of 1.2.

The isomorphism  $r$  is given by the morphism of exact sequences

$$\begin{array}{ccccccc} D & \xrightarrow{\phi} & E & \longrightarrow & \text{coker}(\phi) & \longrightarrow & 0 \\ \parallel & & \cong \downarrow & & \cong \downarrow & & \\ D & \xrightarrow{\phi'} & E & \longrightarrow & \text{coker}(\phi') & \longrightarrow & 0 \end{array}$$

The  $A$ -module morphisms

$$s : \mathcal{C}(\phi)_i = E_i \oplus D_{i-1} \rightarrow \text{coker}(\phi' : D_{i+1} \rightarrow E_{i+1}) ; (x, y) \mapsto [\psi(x)]$$

define a chain homotopy  $s : rp \simeq p'q$  in the diagram

$$\begin{array}{ccc} \mathcal{C}(\phi) & \xrightarrow{p} & \text{coker}(\phi) \\ q \downarrow & \swarrow s & \downarrow r \\ \mathcal{C}(\phi') & \xrightarrow{p'} & \text{coker}(\phi') \end{array}$$

□

**Definition 1.6.** An embedding of chain complexes is a chain map  $\phi : D \rightarrow E$  such that each  $\phi : D_i \rightarrow E_i$  is a split injection.  $\square$

**Proposition 1.7.** Let  $\phi : D \rightarrow E$  be an embedding of  $A$ -module chain complexes.

- (i) The natural projection  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi)$  is a chain equivalence.
- (ii) If each  $D_i, E_i, \text{coker}(\phi : D_i \rightarrow E_i)$  is a based f.g. free  $A$ -module, then  $p$  is a chain equivalence with torsion  $\tau(p) = \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}_i) \in K_1(A)$ , where  $\tau(\mathcal{E}_i) \in K_1(A)$  is the torsion of the short exact sequence of based f.g. free  $A$ -modules

$$\mathcal{E}_i : 0 \longrightarrow D_i \xrightarrow{\phi} E_i \longrightarrow \text{coker}(\phi : D_i \rightarrow E_i) \longrightarrow 0 .$$

- (iii) If  $\psi : \phi \sim \phi' : D \rightarrow E$  is a chain isotopy then  $\phi' : D \rightarrow E$  is also an embedding. With bases as in (ii), and the isomorphism given by 1.5  $r = [1 + d_E \psi + \psi d_E] : \text{coker}(\phi) \cong \text{coker}(\phi')$  has torsion

$$\tau(r) = \sum_{i=0}^{\infty} (-)^i (\tau(\mathcal{E}'_i) - \tau(\mathcal{E}_i)) \in K_1(A) .$$

*Proof.* (i) Extend each  $\phi : D_i \rightarrow E_i$  to a direct sum system

$$D_i \begin{array}{c} \xrightarrow{\phi} \\ \xleftarrow[e]{\phantom{\phi}} \end{array} E_i \begin{array}{c} \xrightarrow{j} \\ \xleftarrow[k]{\phantom{j}} \end{array} F_i$$

with  $F_i = \text{coker}(\phi : D_i \rightarrow E_i)$ . Let

$$c = ed_{E_k} : F_i \rightarrow D_{i-1}, d_F = jd_{E_k} : F_i \rightarrow F_{i-1},$$

so that there is defined an isomorphism of chain complexes  $(\phi k) : E' \rightarrow E$  with

$$d_{E'} = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E'_i = D_i \oplus F_i \rightarrow E'_{i-1} = D_{i-1} \oplus F_{i-1} .$$

The chain map  $p : \mathcal{C}(\phi) \rightarrow F = \text{coker}(\phi)$  is given by

$$p = (j \ 0) : \mathcal{C}(\phi)_i = E_i \oplus D_{i-1} \rightarrow F_i .$$

The chain map  $g : F \rightarrow \mathcal{C}(\phi)$  defined by

$$g = \begin{pmatrix} k \\ -c \end{pmatrix} : F_i \rightarrow \mathcal{C}(\phi)_i = E_i \oplus D_{i-1}$$

is such that  $pg = 1 : F \rightarrow F, h : gp \simeq 1 : \mathcal{C}(\phi) \rightarrow \mathcal{C}(\phi)$  with  $h$  the chain homotopy

$$h = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} : \mathcal{C}(\phi)_i = E_i \oplus D_{i-1} \rightarrow \mathcal{C}(\phi)_{i+1} = E_{i+1} \oplus D_i .$$

Thus  $p : \mathcal{C}(\phi) \rightarrow F, g : F \rightarrow \mathcal{C}(\phi)$  are inverse chain equivalences.

(ii) Immediate from the 0-dimensional case, which follows from the sum formula  $\tau(fg) = \tau(f) + \tau(g)$ .

(iii) By the chain homotopy invariance of torsion, the sum formula, the identity  $\tau(q) = 0$  with  $q$  as in 1.5 and (ii)

$$\begin{aligned} \tau(r) &= \tau(p') - \tau(p) + \tau(q) = \tau(p') - \tau(p) & (9) \\ &= \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}'_i) - \sum_{i=0}^{\infty} (-)^i \tau(\mathcal{E}_i) \in K_1(A). & \square \end{aligned}$$

**Proposition 1.8.** *Let  $E$  be an  $A$ -module chain complex of the form*

$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_D \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1} .$$

Let  $\phi : D \rightarrow E$  be a chain map, with  $\phi = \begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i$ .

(i) *If each  $\phi_D : D_i \rightarrow D_i$  is an automorphism then*

(a)  *$\phi$  is an embedding of chain complexes.*

(b) *The chain complex  $\widehat{F}$  defined by*

$$d_{\widehat{F}} = d_F - \phi_F(\phi_D)^{-1}c : \widehat{F}_i = F_i \rightarrow \widehat{F}_{i-1} = F_{i-1}$$

*is such that the inclusions  $F_i \rightarrow E_i$  induce an isomorphism of chain complexes  $\widehat{F} \cong \text{coker}(\phi)$ .*

(c) *The natural projection  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi)$  is a chain equivalence. The chain complex  $K$  defined by*

$$d_K = \begin{pmatrix} d_D & 0 \\ (-1)^i \phi_D & \phi_D d_D (\phi_D)^{-1} \end{pmatrix} : K_i = D_{i-1} \oplus D_i \rightarrow K_{i-1} = D_{i-2} \oplus D_{i-1}$$

*(i.e. the algebraic mapping cone of the isomorphism of chain complexes  $\phi_D : D = (D_i, d_D) \rightarrow (D_i, \phi_D d_D (\phi_D)^{-1})$  is contractible, and fits into an exact sequence*

$$0 \rightarrow K \rightarrow \mathcal{C}(\phi) \xrightarrow{p} \text{coker}(\phi) \rightarrow 0$$

*with*

$$K_i = D_{i-1} \oplus D_i \rightarrow \mathcal{C}(\phi)_i = D_{i-1} \oplus D_i \oplus E_i; (x, y) \mapsto (x, y, \phi_F(\phi_D)^{-1}(y)).$$

(d) If  $D, F$  are based f. g. free the natural projection  $p : \mathcal{C}(\phi) \rightarrow \text{coker}(\phi) \cong \widehat{F}$  is a chain equivalence of based f. g. free  $A$ -module chain complexes with torsion

$$\tau(p : \mathcal{C}(\phi) \rightarrow \widehat{F}) = -\tau(K) = -\sum_{i=0}^{\infty} (-)^i \tau(\phi_D : D_i \rightarrow D_i) \in K_1(A).$$

(ii) Given a chain homotopy  $\theta : \phi \simeq \phi' : D \rightarrow E$  with  $\phi_D, \phi'_D : D_i \rightarrow D_i$  automorphisms write  $\theta = \begin{pmatrix} \theta_D \\ \theta_F \end{pmatrix} : D_i \rightarrow E_{i+1} = D_{i+1} \oplus F_{i+1}$ . The morphisms defined by

$$\psi = \begin{pmatrix} \theta_D(\phi_D)^{-1} 0 \\ \theta_F(\phi_D)^{-1} 0 \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_i = D_i \oplus F_i$$

are such that

$$\phi' = (1 + d_E \psi + \psi d_E) \phi : D_i \rightarrow E_i.$$

Thus if each  $1 + d_E \psi + \psi d_E : E_i \rightarrow E_i$  is an automorphism there is defined a chain isotopy of embeddings  $\psi : \phi \sim \phi' : D \rightarrow E$  and as in 1.5 there is defined an isomorphism  $r = [1 + d_E \psi + \psi d_E] : \text{coker}(\phi) \cong \text{coker}(\phi')$ . Moreover, if  $D, F$  are based f. g. free the isomorphism

$$r : \widehat{F} \cong \text{coker}(\phi) \cong \text{coker}(\phi') \cong \widehat{F}'$$

is simple, with

$$\tau(r) = \sum_{i=0}^{\infty} (-)^i (\tau(\phi_D : D_i \rightarrow D_i) - \tau(\phi'_D : D'_i \rightarrow D'_i)) = 0 \in K_1(A).$$

*Proof.* (i) (a) Each  $\phi : D_i \rightarrow E_i$  is a split injection.

(b) It is clear that the inclusion  $F \rightarrow E$  induces a chain map  $\widehat{F} \rightarrow \text{coker}(\phi)$ . The  $A$ -module morphisms

$$(-\phi_F(\phi_D)^{-1} 1) : E_i = D_i \oplus F_i \rightarrow \widehat{F}_i = F_i$$

induce the inverse chain isomorphism  $\text{coker}(\phi) \rightarrow \widehat{F}$ .

(c) Immediate from 1.7 (i).

(d) Apply 1.7 (ii), noting that the short exact sequence

$$\mathcal{E}_i : 0 \longrightarrow D_i \xrightarrow{\begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix}} D_i \oplus F_i \xrightarrow{(-\phi_F(\phi_D)^{-1} 1)} F_i \longrightarrow 0$$

has torsion

$$\tau(\mathcal{E}_i) = -\tau(\phi_D : D_i \rightarrow D_i) \in K_1(A).$$

(ii) It follows from

$$\begin{aligned} \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi_E \end{pmatrix} &= \begin{pmatrix} \phi_D \\ \phi_E \end{pmatrix} d_D : D_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1}, \\ \begin{pmatrix} \phi'_D \\ \phi'_F \end{pmatrix} - \begin{pmatrix} \phi_D \\ \phi_E \end{pmatrix} &= \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} \theta_D \\ \theta_E \end{pmatrix} + \begin{pmatrix} \theta_D \\ \theta_E \end{pmatrix} d_D : D_i \rightarrow E_i = D_i \oplus F_i \end{aligned}$$

that

$$\begin{aligned} (1 + d_E \psi + \psi d_E) \phi &= \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} \theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0 \end{pmatrix} \right. \\ &\quad \left. + \begin{pmatrix} \theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0 \end{pmatrix} \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \right) \begin{pmatrix} \phi_D \\ \phi_E \end{pmatrix} \\ &= \begin{pmatrix} \phi'_D \\ \phi'_F \end{pmatrix} = \phi' : D_i \rightarrow E_i = D_i \oplus F_i. \end{aligned}$$

If  $1 + d_E \psi + \psi d_E : E_i \rightarrow E_i$  is an automorphism then  $\psi : \phi \simeq \phi' : D \rightarrow E$  is a chain isotopy, and by 1.7 (iii)

$$\begin{aligned} \tau(r : \widehat{F} \cong \widehat{F}') &= \sum_{i=0}^{\infty} (-)^i (\tau(\mathcal{E}'_i) - \tau(\mathcal{E}_i)) \\ &= \sum_{i=0}^{\infty} (-)^i (\tau(\phi_D : D_i \rightarrow D_i) - \tau(\phi'_D : D'_i \rightarrow D'_i)) \in K_1(A). \end{aligned}$$

Now there is defined an isomorphism of short exact sequences

$$\begin{array}{ccccccc} \mathcal{E}_i : 0 & \longrightarrow & D_i & \xrightarrow{\begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix}} & D_i \oplus F_i & \xrightarrow{(-\phi_F(\phi_D)^{-1} \ 1)} & \widehat{F}_i \longrightarrow 0 \\ & & \parallel & & \downarrow 1 + d_E \psi + \psi d_E & & \downarrow r \\ \mathcal{E}'_i : 0 & \longrightarrow & D_i & \xrightarrow{\begin{pmatrix} \phi'_D \\ \phi'_F \end{pmatrix}} & D_i \oplus F_i & \xrightarrow{(-\phi'_F(\phi'_D)^{-1} \ 1)} & \widehat{F}'_i \longrightarrow 0 \end{array}$$

so that

$$\tau(r : \widehat{F}_i \rightarrow \widehat{F}'_i) = \tau(1 + d_E \psi + \psi d_E : E_i \rightarrow E_i) \in K_1(A).$$

The chain complex automorphism  $1 + d_E \psi + \psi d_E : E \rightarrow E$  is chain homotopic to  $1 : E \rightarrow E$ , so

$$\begin{aligned} \tau(r : \widehat{F} \rightarrow \widehat{F}') &= \sum_{i=0}^{\infty} (-1)^i \tau(r : \widehat{F}_i \rightarrow \widehat{F}'_i) \\ &= \sum_{i=0}^{\infty} (-1)^i \tau(1 + d_E \psi + \psi d_E : E_i \rightarrow E_i) = \tau(1 + d_E \psi + \psi d_E : E \rightarrow E) \\ &= \tau(1 : E \rightarrow E) = 0 \in K_1(A). \end{aligned}$$

□

The formula  $d_{\widehat{F}} = d_F - \phi_F(\phi_D)^{-1}c : F_i \rightarrow F_{i-1}$  was first obtained in [FR,2.3].

### 2. The Morse-Smale complex $C^{MS}(M, f, v)$

This section recalls the properties of the Morse-Smale complex  $C^{MS}(M, f, v)$  of a real-valued Morse function on an  $m$ -dimensional cobordism  $f : (M; N, N') \rightarrow (I; \{0\}, \{1\})$  with respect to any  $v \in \mathcal{GT}(f)$ .

**Definition 2.1.** The Morse-Smale complex of  $f : M \rightarrow \mathbb{R}$  with respect to  $v \in \mathcal{GT}(f)$  is the based f.g. free  $\mathbb{Z}[\pi_1(M)]$ -module chain complex  $C^{MS}(M, f, v)$  with

- (i)  $\text{rank}_{\mathbb{Z}[\pi_1(M)]} C_i^{MS}(M, f, v) = c_i(f)$ , with one basis element  $\tilde{p}$  for each critical point  $p \in M$  of index  $i$ , corresponding to a choice of lift  $\tilde{p} \in \tilde{M}$ ,
- (ii) the boundary  $\mathbb{Z}[\pi_1(M)]$ -module morphisms

$$\begin{aligned} d : C_i^{MS}(M, f, v) = \mathbb{Z}[\pi_1(M)]^{c_i(f)} &\rightarrow C_{i-1}^{MS}(M, f, v) = \mathbb{Z}[\pi_1(M)]^{c_{i-1}(f)} ; \\ \tilde{p} &\mapsto \sum_{\tilde{q}} n(\tilde{p}, \tilde{q}) \tilde{q} \end{aligned}$$

with  $n(\tilde{p}, \tilde{q}) \in \mathbb{Z}$  the algebraic number of  $\tilde{v}$ -gradient flow lines in  $\tilde{M}$  joining  $\tilde{p}$  to  $\tilde{q}$ . □

A Morse function on a cobordism  $f : (M; N, N') \rightarrow (I; \{0\}, \{1\})$  and  $v \in \mathcal{GT}(f)$  determine a handlebody decomposition  $M = N \times I \cup \bigcup_{i=0}^m \bigcup_{c_i(f)} h^i$  with  $c_i(f)$   $i$ -handles  $h^i = D^i \times D^{m-i}$ . Given a CW structure on  $N$  with  $c_i(N)$   $i$ -cells  $e^i \subset N$  let  $M$  have the CW structure with  $c_i(M) = c_i(N) + c_i(f)$   $i$ -cells: there is one  $i$ -cell  $e^i \times I \subset M$  for each  $i$ -cell  $e^i \subset N$ , and one  $i$ -cell  $h^i \subset M$  for each

critical point of index  $i$ . Let  $\tilde{M}$  be the universal cover of  $M$ , and let  $\tilde{N}, \tilde{N}'$  be the corresponding covers of  $N, N'$ .

**Proposition 2.2.** *The Morse-Smale complex  $C^{MS}(M, f, v)$  is the relative cellular chain complex of  $(\tilde{M}, \tilde{N} \times I)$*

$$C^{MS}(M, f, v) = C(\tilde{M}, \tilde{N} \times I).$$

*Proof.* See Franks [Fr]. □

**Definition 2.3.** Given a Morse function  $f : (M; N, N') \rightarrow (I; \{0\}, \{1\})$ ,  $v \in \mathcal{GT}(f)$  and  $CW$  structures on  $N$  and  $N'$  write

$$D = C(\tilde{N}), D' = C(\tilde{N}'), E = C(\tilde{M}), F = C^{MS}(M, f, v) = C(\tilde{M}, \tilde{N} \times I).$$

(i) The cellular chain complex of  $\tilde{M}$  is of the form

$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1}$$

for a *birth* chain map  $c : F_{*+1} \rightarrow D$ , that is

$$E = C(\tilde{M}) = C(c).$$

(ii) The natural map  $g : N \rightarrow M$  is the inclusion of a  $CW$  subcomplex, inducing

the embedding  $g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : C(\tilde{N}) = D \rightarrow C(\tilde{M}) = E$ . A *crossing chain*

*approximation*  $h : C(\tilde{N}') = D' \rightarrow C(\tilde{M}) = E$  is a chain map induced by the inclusion  $h : N' \rightarrow M$  – in general this is not the inclusion of a  $CW$  subcomplex, so the construction requires the cellular approximation theorem, and the  $CW$  structures only determine a chain map  $h$  up to chain homotopy. The

components of  $h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : D'_i \rightarrow E_i = D_i \oplus F_i$  are such that

$$d_D h_D + c h_F = h_D d_{D'} : D'_i \rightarrow D_{i-1}, d_F h_F = h_F d_{D'} : D'_i \rightarrow F_{i-1}$$

defining a *death* chain map  $h_F : D' \rightarrow F$  and a *survival* chain homotopy  $h_D : c h_F \simeq 0 : D' \rightarrow D_{*-1}$ . □

*Remark 2.4.* For a Morse function  $f : (M; N, N') \rightarrow (I; \{0\}, \{1\})$  with  $v \in \mathcal{GT}(f)$  there are 4 types of  $v$ -gradient flow lines, which we shall call  $\alpha, \beta, \gamma, \delta$ , corresponding to the 4 morphisms

$$d_F : F_i \rightarrow F_{i-1}, c : F_i \rightarrow D_{i-1}, h_F : D'_i \rightarrow F_i, h_D : D'_i \rightarrow D_i,$$

as follows :

- (a) complete:  $\alpha$  starts at an index  $i$  critical point  $p \in M$  and terminates at an index  $i - 1$  critical point  $q \in M$ .
- (b) birth:  $\beta$  starts at an index  $i$  critical point  $p \in M$  and terminates in  $N$ .
- (c) death:  $\gamma$  starts in  $N'$  and terminates at an index  $i$  critical point  $p \in M$ .
- (d) survival:  $\delta$  starts in  $N'$  and terminates in  $N$ .

A crossing chain approximation  $h : D' \rightarrow E$  corresponds to the flow lines which start in  $N'$ , i.e. those of death and survival type. Pajitnov [P4,§4] obtained an analogue of the cellular approximation theorem for the gradient flow : for any Morse function  $f : M \rightarrow S^1$  with regular value  $0 \in S^1$  it is possible to choose  $v \in \mathcal{GT}(f)$  and handlebody structures on  $N$  and  $N'$  such that

- (i) every survival  $v$ -gradient flow line in  $M$  which starts in an  $i'$ -handle of  $N'$  ends in an  $i$ -handle of  $N$  with  $i \leq i'$ ,
- (ii) there is a finite number of rel  $\partial$  homology classes of survival flow lines as in (i) with  $i = i' - 1$ ,
- (iii) there exist  $d_F, c, h_F$  as in 2.3 which actually count the flow lines of type  $\alpha, \beta, \gamma$ , and  $h_D$  counts the rel  $\partial$  homology classes of survival flow lines  $\delta$  as in (ii),
- (iv) the function which sends  $y \in M \setminus \text{Crit}(f)$  to the endpoint  $\Phi(y) \in N \cup \text{Crit}(f)$  of the flow line of  $f$  which starts at  $y$

$$\Phi : M \setminus \text{Crit}(f) \rightarrow N \cup \text{Crit}(f) ; y \mapsto \Phi(y)$$

restricts to a function  $N' \cap \Phi^{-1}(N) \rightarrow N \cap \Phi(N')$ ;  $x' \mapsto \Phi(x')$  which is a partially defined map  $N' \rightarrow N$  inducing the 'partial chain map'  $h_D : D' = C(\tilde{N}') \rightarrow D = C(\tilde{N})$ .

(See also Hutchings and Lee [HL]). □

We shall now express the Morse-Smale complex  $C^{MS}(M, f, v)$  of a Morse function

$$f : (M; N, N'') \rightarrow ([0, 2]; 0, 2) (v \in \mathcal{GT}(f))$$

which is transverse regular at  $1 \in [0, 2]$  in terms of the Morse-Smale complexes  $C^{MS}(M', f', v'), C^{MS}(M'', f'', v'')$  of the restrictions

$$f' = f| : (M'; N, N') = f^{-1}([0, 1]; \{0\}, \{1\}) \rightarrow ([0, 1]; \{0\}, \{1\}) ,$$

$$f'' = f| : (M''; N', N'') = f^{-1}([1, 2]; \{1\}, \{2\}) \rightarrow ([1, 2]; \{1\}, \{2\})$$

using choices of  $CW$  structures for  $N, N', N''$  and crossing chain approximations  $h' : C(\tilde{N}') \rightarrow C(\tilde{M}')$ ,  $h'' : C(\tilde{N}'') \rightarrow C(\tilde{M}'')$ . If  $f'$  (resp.  $f''$ ) has  $c_i(f')$  (resp.  $c_i(f'')$ ) critical points of index  $i$  then  $f$  has  $c_i(f) = c_i(f') + c_i(f'')$  critical points of index  $i$ .

**Terminology 2.5.** Write the various chain complexes, birth, death, survival and crossing chain approximations for  $(f', v')$  and  $(f'', v'')$  as

$$D = C(\tilde{N}), D' = C(\tilde{N}'), D'' = C(\tilde{N}''), E' = C(\tilde{M}'), E'' = C(\tilde{M}''),$$

$$\begin{aligned}
 F' &= C^{MS}(M', f', v'), F'' = C^{MS}(M'', f'', v''), \\
 d_{E'} &= \begin{pmatrix} d_D & c' \\ 0 & d_{F'} \end{pmatrix} : E'_i = D_i \oplus F'_i \rightarrow E'_{i-1} = D_{i-1} \oplus F'_{i-1}, \\
 d_{E''} &= \begin{pmatrix} d_{D'} & c'' \\ 0 & d_{F''} \end{pmatrix} : E''_i = D'_i \oplus F''_i \rightarrow E''_{i-1} = D'_{i-1} \oplus F''_{i-1}, \\
 h' &= \begin{pmatrix} h'_D \\ h'_{F'} \end{pmatrix} : D'_i \rightarrow E'_i = D_i \oplus F'_i, h'' = \begin{pmatrix} h''_{D'} \\ h''_{F''} \end{pmatrix} : D''_i \rightarrow E''_i = D'_i \oplus F''_i.
 \end{aligned}$$

Define the chain complexes  $F, E$  by

$$\begin{aligned}
 d_F &= \begin{pmatrix} d_{F'} & h'_{F'}c'' \\ 0 & d_{F''} \end{pmatrix} : F_i = F'_i \oplus F''_i \rightarrow F_{i-1} = F'_{i-1} \oplus F''_{i-1}, \\
 d_E &= \begin{pmatrix} d_D & c' & h'_Dc'' \\ 0 & d_{F'} & h'_{F'}c'' \\ 0 & 0 & d_{F''} \end{pmatrix} : E_i = D_i \oplus F'_i \oplus F''_i \rightarrow E_{i-1} = D_{i-1} \oplus F'_{i-1} \oplus F''_{i-1},
 \end{aligned}$$

and let  $c : F \rightarrow D_{*-1}, h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : D'' \rightarrow E$  be the chain maps defined by

$$\begin{aligned}
 c &= (c' \ h'_Dc'') : F_i = F'_i \oplus F''_i \rightarrow D_{i-1}, \\
 h_D &= h'_Dh''_{D'} : D''_i \rightarrow D_i, h_F = \begin{pmatrix} h'_{F'}h''_{D'} \\ h''_{F''} \end{pmatrix} : D''_i \rightarrow F_i = F'_i \oplus F''_i. \quad \square
 \end{aligned}$$

**Proposition 2.6.** (i) *The Morse-Smale complex of*

$$f = f' \cup f'' : (M; N, N'') = (M'; N, N') \cup (M''; N', N'') \rightarrow ([0, 2]; \{0\}, \{2\})$$

with respect to  $v = v' \cup v'' \in \mathcal{GT}(f)$  is the algebraic mapping cone  $C^{MS}(M, f, v) = C(a : F''_{*+1} \rightarrow F')$  of the chain map  $a : F''_{*+1} \rightarrow F'$  defined by

$$\begin{aligned}
 a : F''_{i+1} &= \mathbb{Z}[\pi_1(M)]^{c_{i+1}(f'')} \rightarrow \\
 F'_i &= \mathbb{Z}[\pi_1(M)]^{c_i(f')} ; \tilde{p} \mapsto \sum_{\tilde{q}} \sum_{u \in \pi_1(M)} n(\tilde{p}, u\tilde{q})u\tilde{q}
 \end{aligned}$$

with  $n(\tilde{p}, u\tilde{q}) \in \mathbb{Z}$  the algebraic number of  $\tilde{v}$ -gradient flow lines in  $\tilde{M}$  joining  $\tilde{p} \in \tilde{M}''$  to  $u\tilde{q} \in \tilde{M}'$ .

(ii) For any death chain map  $h'_{F'} : D' \rightarrow F'$  of  $(f', v')$  and any birth chain map  $c'' : F''_{*+1} \rightarrow D'$  of  $(f'', v'')$  there exists a chain homotopy  $b : a \simeq h'_{F'}c'' : F''_{*+1} \rightarrow F'$  such that

(a)  $b$  determines a simple isomorphism

$$I = \begin{pmatrix} 1 & \pm b \\ 0 & 1 \end{pmatrix} : C(a) = C^{MS}(M, f, v) \rightarrow F = C(h'_{F'}c''),$$

- (b)  $cI : C^{MS}(M, f, v)_{*+1} \rightarrow D = C(\tilde{N})$  is a birth chain map of  $(f, v)$ ,
- (c) the cellular chain complex of  $\tilde{M}$  is  $C(\tilde{M}) = C(cI)$  and there is defined a simple isomorphism  $1 \oplus I : E \rightarrow C(\tilde{M})$ ,
- (d)  $(1 \oplus I)h : D'' = C(\tilde{N}) \rightarrow C(\tilde{M})$  is a crossing chain approximation for  $(f, v)$ , with components a death chain map

$$((1 \oplus I)h)_F = Ih_F = \begin{pmatrix} h'_{F'}h''_{D'} \pm bh''_{F''} \\ h''_{F''} \end{pmatrix} : D''_i \rightarrow C^{MS}(M, f, v)_i = F'_i \oplus F''_i$$

and a survival chain map  $((1 \oplus I)h'' )_D = h_D = h'_D h''_{D'} : D''_i \rightarrow D_i$ .

*Proof.* (i) This is a direct consequence of the construction of the Morse-Smale complex (2.1).

(ii) Use the handlebody decomposition of  $M$  (resp.  $M', M''$ ) determined by  $(f, v)$  (resp.  $(f', v'), (f'', v'')$ ) to extend the CW structure on  $N$  (resp.  $N, N'$ ) to a CW structure on  $M$  (resp.  $M', M''$ ). The existence of a chain homotopy  $b : a \simeq h_F c'$  is immediate from the observation that  $a$  and  $h'_{F'} c''$  are both connecting chain maps  $\partial : F''_{*+1} = C(\tilde{M}, \tilde{M}')_{*+1} \rightarrow C(\tilde{M}, \tilde{N}) = F'$  for the triad of CW subcomplexes  $M \supset M' \supset N$ , which is unique up to chain homotopy. Property (a) is just an application of 1.2. The composite of cellular approximations to the inclusions  $N'' \rightarrow M'', M'' \rightarrow M$  is a cellular approximation to the inclusion  $N'' \rightarrow M$ , giving (b),(c) and (d). □

*Remark 2.7.* Cornea and Ranicki [CR] obtain a sharper version of 2.6 : for any Morse map

$$f = f' \cup f'' : (M; N, N'') = (M; N, N') \cup (M'; N', N'') \rightarrow ([0, 2]; \{0\}, \{2\})$$

and  $v = v' \cup v'' \in \mathcal{GT}(f)$  there exist Morse maps  $\hat{f} : M \rightarrow \mathbb{R}, g : N' = f^{-1}(1) \rightarrow \mathbb{R}$  and  $\hat{v} \in \mathcal{GT}(\hat{f}), w \in \mathcal{GT}(g)$  such that

- (a)  $(\hat{f}, \hat{v})$  agrees with  $(f, v)$  outside a tubular neighbourhood of  $N' f^{-1}[1 - \epsilon, 1 + \epsilon] = N' \times [1 - \epsilon, 1 + \epsilon] \subset M$  for some small  $\epsilon > 0$ .
- (b)  $(\hat{f}, \hat{v})$  restricts to translates of  $(g, w)$

$$\begin{aligned} (\hat{f}, \hat{v})|_{(g_+, w_+)} &= (g_+, w_+) : N' \times \{1 + \epsilon/2\} \rightarrow \mathbb{R}, \\ (\hat{f}, \hat{v})|_{(g_-, w_-)} &= (g_-, w_-) : N' \times \{1 - \epsilon/2\} \rightarrow \mathbb{R} \end{aligned}$$

with

$$\begin{aligned} \text{Crit}_i(g_+) &= \text{Crit}_i(g) \times \{1 + \epsilon/2\}, \text{Crit}_i(g_-) = \text{Crit}_i(g) \times \{-\epsilon/2\}, \\ \text{Crit}_i(\hat{f}) &= \text{Crit}_{i-1}(g_+) \cup \text{Crit}_i(g_-) \cup \text{Crit}_i(f). \end{aligned}$$

- (c) The  $v$ -gradient flow lines are in one-one correspondence with the broken  $\widehat{v}$ -gradient flow lines i.e. joined up sequences of  $\widehat{v}$ -gradient flow lines which start at critical points of  $f''$  and terminate at critical points of  $f'$ .
- (d) The Morse-Smale complex of  $(\widehat{f}, \widehat{v})$  is of the form

$$d_{C^{MS}(M, \widehat{f}, \widehat{v})} = \begin{pmatrix} d_{F'} & h'_{F'} & 0 & 0 \\ 0 & -d_{D'} & 0 & 0 \\ 0 & 1 & d_{D'} & c'' \\ 0 & 0 & 0 & d_{F''} \end{pmatrix} :$$

$$C_i^{MS}(M, \widehat{f}, \widehat{v}) = F'_i \oplus D'_{i-1} \oplus D'_i \oplus F''_i \rightarrow C_i^{MS}(M, \widehat{f}, \widehat{v}) = F'_{i-1} \oplus D'_{i-2} \oplus D'_{i-1} \oplus F''_{i-1}$$

with  $D' = C(N', g, w)$ , giving choices of 'gradient-like' crossing chain approximations  $h', h''$  such that the chain homotopy in 2.6 (ii) is

$$b = 0 : a \simeq h'_{F'} c'' : F'' = C^{MS}(M'', f'', v'') \rightarrow F'_{*-1} = C^{MS}(M', f', v')_{*-1}$$

(i.e.  $a = h'_{F'} c''$ ) and the simple isomorphism

$$I : C^{MS}(M, f, v) = \mathcal{C}(a) \rightarrow F = \mathcal{C}(h'_{F'} c'')$$

is the identity. □

### 3. The proper Morse-Smale complex

The construction of  $C^{MS}(M, f, v)$  applies just as well to a proper real-valued Morse function on a non-compact manifold :

**Definition 3.1.** Let  $(M, \partial M)$  be a non-compact manifold with compact boundary and a proper real-valued Morse function  $f : (M, \partial M) \rightarrow ([0, \infty), \{0\})$ , and let  $v \in \mathcal{GT}(f)$ . The *proper Morse-Smale complex*  $C^{MS}(M, f, v)$  is defined exactly as in the compact case, with

- (i)  $C_i^{MS}(M, f, v)$  the based free  $\mathbb{Z}[\pi_1(M)]$ -module generated by  $\gamma_i(f)$ , the set of critical points of  $f$  with index  $i$ ,
- (ii) the boundary  $\mathbb{Z}[\pi_1(M)]$ -module morphisms are given by

$$d : C_i^{MS}(M, f, v) = \mathbb{Z}[\pi_1(M)]^{\gamma_i(f)} \rightarrow C_{i-1}^{MS}(M, f, v) = \mathbb{Z}[\pi_1(M)]^{\gamma_{i-1}(f)} ;$$

$$\tilde{p} \mapsto \sum_{\tilde{q}} \sum_{u \in \pi_1(M)} n(\tilde{p}, u\tilde{q}) u\tilde{q}$$

with  $n(\tilde{p}, u\tilde{q}) \in \mathbb{Z}$  the algebraic number of  $v$ -gradient flow lines in  $\tilde{M}$  joining  $\tilde{p}$  to  $u\tilde{q}$ . □

Given a proper Morse function  $f : (M, \partial M) \rightarrow ([0, \infty), \{0\})$  and a gradient-like vector field for  $f$  and a CW structure for  $\partial M$  let  $M$  have the CW structure given by the handle decomposition  $M = \partial M \times I \cup \bigcup_{i=0}^m \bigcup_{\gamma_i(f)} h^i$ . The expression of 2.6 for the Morse-Smale complex of the union of two Morse functions on adjoining compact cobordisms will now be applied to obtain an isomorphism  $I : C^{MS}(M, f, v) \cong C(\tilde{M}, \partial\tilde{M})$  between the Morse-Smale complex  $C^{MS}(M, f, v)$  and the relative cellular chain complex of  $(\tilde{M}, \partial\tilde{M})$ .

**Terminology 3.2.** For  $j = 0, 1, 2, \dots$  let

$$f[j] = f| : (M[j]; N[j], N[j + 1]) = f^{-1}([j, j + 1]; \{j\}, \{j + 1\}) \rightarrow ([j, j + 1]; \{j\}, \{j + 1\})$$

be the Morse functions on compact cobordisms given by the restriction of  $f$ . The inclusions  $g[j] : N[j] \rightarrow M[j], h[j] : N[j + 1] \rightarrow M[j]$  induce embeddings of based f. g. free  $\mathbb{Z}[\pi_1(M)]$ -module chain complexes  $g[j] : C(\tilde{N}[j]) \rightarrow C(\tilde{M}[j])$  and chain maps  $h[j] : C(\tilde{N}[j + 1]) \rightarrow C(\tilde{M}[j])$ . Write

$$\begin{aligned} D[j] &= C(\tilde{N}[j]), \quad E[j] = C(\tilde{M}[j]), \quad F[j] = C(\tilde{M}[j], \tilde{N}[j]), \\ d_{E[j]} &= \begin{pmatrix} d_{D[j]} & c[j] \\ 0 & d_{F[j]} \end{pmatrix} : \\ &E_i[j] = D_i[j] \oplus F_i[j] \rightarrow E_{i-1}[j] = D_{i-1}[j] \oplus F_{i-1}[j], \\ g[j] &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i[j] \rightarrow E_i[j] = D_i[j] \oplus F_i[j], \\ h[j + 1] &= \begin{pmatrix} h_{D[j+1]} \\ h_{F[j+1]} \end{pmatrix} : D_i[j + 1] \rightarrow E_i[j] = D_i[j] \oplus F_i[j]. \quad \square \end{aligned}$$

**Proposition 3.3.** Let  $f : (M, \partial M) \rightarrow ([0, \infty), \{0\})$  be a proper Morse function, and let  $v \in \mathcal{GT}(f)$ .

(i) The cellular chain complex of  $(M, \partial M)$  is of the form

$$C(\tilde{M}, \partial\tilde{M}) = \text{coker}(g - h : \sum_{j=0}^{\infty} C(\tilde{N}[j]) \rightarrow \sum_{j=0}^{\infty} C(\tilde{M}[j])),$$

a based free  $\mathbb{Z}[\pi_1(M)]$ -module complex with basis the images of the basis elements in  $\sum_{j=0}^{\infty} C(\tilde{M}[j], \tilde{N}[j])$ , and may be expressed as

$$d_{C(\tilde{M}, \partial\tilde{M})} = \begin{pmatrix} d_{F[0]} & h_{F[0]}c[1] & h_{F[0]}h_{D[1]}c[2] & h_{F[0]}h_{D[1]}h_{D[2]}c[3] & \dots \\ 0 & d_{F[1]} & h_{F[1]}c[2] & h_{F[1]}h_{D[2]}c[3] & \dots \\ 0 & 0 & d_{F[2]} & h_{F[2]}c[3] & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$: C_i(\tilde{M}, \partial\tilde{M}) = \sum_{k=0}^{\infty} F_i[k] \rightarrow C_{i-1}(\tilde{M}, \partial\tilde{M}) = \sum_{j=0}^{\infty} F_{i-1}[j].$$

(ii) The Morse-Smale complex of  $(f, v)$  is of the form

$$d_{C^{MS}(M, f, v)} = \begin{pmatrix} d_{C^{MS}(M[0], f[0], v[0])} & c^{MS}[0, 1] & c^{MS}[0, 2] & c^{MS}[0, 3] \dots \\ 0 & d_{C^{MS}(M[1], f[1], v[1])} & c^{MS}[1, 2] & c^{MS}[1, 3] \dots \\ 0 & 0 & d_{C^{MS}(M[2], f[2], v[2])} & c^{MS}[2, 3] \dots \\ \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}$$

$$: C_i^{MS}(M, f, v) = \sum_{k=0}^{\infty} C_i^{MS}(M[k], f[k], v[k])$$

$$\rightarrow C_{i-1}^{MS}(M, f, v) = \sum_{j=0}^{\infty} C_{i-1}^{MS}(M[j], f[j], v[j])$$

with

$$c^{MS}[j, k] : C_i^{MS}(M[k], f[k], v[k]) \rightarrow C_{i-1}^{MS}(M[j], f[j], v[j]) \quad (j < k)$$

counting the  $\tilde{v}$ -gradient flow lines in the universal cover  $\tilde{M}$  which start at an index  $i$  critical point of  $\tilde{f}[k]$  and terminate at an index  $i - 1$  critical point of  $\tilde{f}[j]$ .

(iii) There exists an isomorphism of chain complexes  $I : C^{MS}(M, f, v) \cong C(\tilde{M}, \partial\tilde{M})$  of the form

$$I = 1 + \sum_{j' < j} b[j', j] : C_i^{MS}(M, f, v) = \sum_{j=0}^{\infty} F_i[j]$$

$$\rightarrow C_i(\tilde{M}, \partial\tilde{M}) = \sum_{j'=0}^{\infty} F_i[j'].$$

*Proof.* (i) The exact sequence

$$0 \rightarrow \sum_{j=0}^{\infty} C(\tilde{N}[j]) \xrightarrow{g-h} \sum_{j=0}^{\infty} C(\tilde{M}[j]) \rightarrow C(\tilde{M}, \partial\tilde{M}) \rightarrow 0$$

is just the chain level Mayer-Vietoris sequence for the union  $M = M[\text{even}] \cup M[\text{odd}]$ , with

$$M[\text{even}] = \bigcup_{j \text{ even}} M[j], \quad M[\text{odd}] = \bigcup_{j \text{ odd}} M[j].$$

The matrix formula for  $\text{coker}(g - h)$  is a direct application of 1.8 (i) (b) with

$$\phi = g - h = \begin{pmatrix} 1 - h_D & & \\ & -h_F & \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i ,$$

noting that each

$$1 - h_D = \begin{pmatrix} 1 - h_{D[0]} & 0 & \dots \\ 0 & 1 & -h_{D[1]} \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : E_i = \sum_{j=0}^{\infty} F_i[j] \rightarrow E_i = \sum_{j=0}^{\infty} F_i[j]$$

is an automorphism, with inverse

$$(1 - h_D)^{-1} = \begin{pmatrix} 1 & h_{D[0]} & h_{D[0]}h_{D[1]} & \dots \\ 0 & 1 & h_{D[1]} & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : E_i = \sum_{j=0}^{\infty} F_i[j] \rightarrow E_i = \sum_{j=0}^{\infty} F_i[j] .$$

(ii) By construction.

(iii) For  $k = 1, 2, \dots$  define the Morse function

$$f[0, k] = \bigcup_{j=1}^k f[j] = f : M[0, k] = \bigcup_{j=1}^k M[j] = f^{-1}[0, k] \rightarrow [0, k] ,$$

and assume inductively that there is given an isomorphism

$$I[0, k] : C^{MS}(M[0, k], f[0, k], v[0, k]) \cong C(\tilde{M}[0, k], \partial\tilde{M})$$

of the form

$$I[0, k] = 1 + \sum_{j' < j} b[j', j] :$$

$$C_i^{MS}(M[0, k], f[0, k], v[0, k]) = \sum_{j=0}^k F_i[j] \rightarrow C_i(\tilde{M}[0, k], \partial\tilde{M}) = \sum_{j'=0}^k F_i[j'] .$$

Now apply 2.6 (i) to the Morse function

$$\begin{aligned} f[0, k + 1] &= f[0, k] \cup f[k] : \\ M[0, k + 1] &= M[0, k] \cup M[k] \rightarrow [0, k + 1] = [0, k] \cup [k, k + 1], \end{aligned}$$

extending  $I[0, k]$  to an isomorphism  $I[0, k + 1]$  of the same form, and pass to the direct limit to obtain an isomorphism

$$I = \varinjlim_k I[0, k] : C^{MS}(M, f, v) = \varinjlim_k C^{MS}(M[0, k], f[0, k], v[0, k]) \rightarrow C(\tilde{M}, \partial\tilde{M}) = \varinjlim_k C(\tilde{M}[0, k], \partial\tilde{M}).$$

of the form  $I = 1 + \sum_{j' < j} b[j', j]$ . □

### 4. The Cohn noncommutative localization

We refer to Cohn [C] and Schofield [Scho] for general accounts of the localization  $\Sigma^{-1}R$  of a ring  $R$  inverting a set  $\Sigma$  of square matrices. The natural morphism  $R \rightarrow \Sigma^{-1}R$  has the universal property that a morphism of rings  $R \rightarrow A$  which sends  $\Sigma$  to invertible matrices in  $A$  has a unique factorization

$$R \rightarrow \Sigma^{-1}R \rightarrow A.$$

The Gerasimov-Malcolmson normal form expresses every morphism of f. g. free  $\Sigma^{-1}R$ -modules  $\phi : \Sigma^{-1}R^n \rightarrow \Sigma^{-1}R^p$  as a composite

$$\phi = f\sigma^{-1}g : \Sigma^{-1}R^n \rightarrow \Sigma^{-1}R^m \rightarrow \Sigma^{-1}R^m \rightarrow \Sigma^{-1}R^p$$

(nonuniquely) for some  $R$ -module morphisms  $f : R^m \rightarrow R^p, \sigma : R^m \rightarrow R^m, g : R^n \rightarrow R^m$  such that  $\sigma$  is  $\Sigma^{-1}R$ -invertible.

**Proposition 4.1.** (Sheiham [Sh, 3.1]) *Given a ring morphism  $\epsilon : R \rightarrow A$  let  $\Sigma$  be the set of all square matrices in  $R$  which become invertible in  $A$ .*

- (i) *The ring morphism  $\epsilon$  extends to a ring morphism  $\epsilon : \Sigma^{-1}R \rightarrow A$ .*
- (ii) *An endomorphism  $\phi : \Sigma^{-1}R^n \rightarrow \Sigma^{-1}R^n$  of a f. g. free  $\Sigma^{-1}R$ -module is an automorphism if and only if  $\epsilon(\phi) : A^n \rightarrow A^n$  is an  $A$ -module automorphism.*

*Proof.* (i) By the universal property of  $R \rightarrow \Sigma^{-1}R$ .

(ii) It is clear that if  $\phi$  is an automorphism then so is  $\epsilon(\phi)$ .

Conversely, suppose that  $\epsilon(\phi)$  is an automorphism. Express  $\phi$  in the Gerasimov-Malcolmson normal form

$$\phi = f\sigma^{-1}g : \Sigma^{-1}R^n \rightarrow \Sigma^{-1}R^n$$

for some  $R$ -module morphisms  $f : R^m \rightarrow R^n, \sigma : R^m \rightarrow R^m, g : R^n \rightarrow R^m$  such that  $\epsilon(\sigma) : A^m \rightarrow A^m$  is an automorphism. The  $R$ -module endomorphism defined by

$$\theta = \begin{pmatrix} 0 & -f \\ g & \sigma \end{pmatrix} : R^n \oplus R^m \rightarrow R^n \oplus R^m$$

is  $A$ -invertible, since the induced  $\Sigma^{-1}R$ -module endomorphism

$$\theta = \begin{pmatrix} \phi & -f \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sigma^{-1}g & 1 \end{pmatrix} : \Sigma^{-1}(R^n \oplus R^m) \rightarrow \Sigma^{-1}(R^n \oplus R^m)$$

is  $A$ -invertible. Thus  $\theta : \Sigma^{-1}(R^n \oplus R^m) \rightarrow \Sigma^{-1}(R^n \oplus R^m)$  is a  $\Sigma^{-1}R$ -module automorphism, and hence so is  $\phi$ . □

In the application of Cohn localization in Sect. 5 below

$$\epsilon : R = A_\alpha[z] \rightarrow A ; z \mapsto 0 .$$

### 5. Polynomial extensions

The relationship between the algebraic mapping cone and cokernel worked out in Sect. 1 will now be applied to the algebraic situation arising from a circle-valued Morse function  $f : M \rightarrow S^1$  with  $v \in \mathcal{GT}(f)$ . The actual application to the Novikov complex  $C^{Nov}(M, f, v)$  will be carried out in Sect. 6.

**Definition 5.1.** Let  $A$  be a ring with an automorphism  $\alpha : A \rightarrow A$ , and let  $z$  be an indeterminate over  $A$  such that

$$az = z\alpha(a) \quad (a \in A) .$$

(i) The  $\alpha$ -twisted Laurent polynomial extension of  $A$

$$A_\alpha[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j A = A_\alpha[z, z^{-1}]$$

is the ring of polynomials  $\sum_{j=-\infty}^{\infty} a_j z^j$  ( $a_j \in A$ ) such that  $\{j \in \mathbb{Z} \mid a_j \neq 0\}$  is finite.

(ii) The  $\alpha$ -twisted Novikov completion of  $A_\alpha[z, z^{-1}]$

$$A_\alpha((z)) = \varprojlim_k \sum_{j=-\infty}^k z^j A = A_\alpha[[z]][z^{-1}]$$

is the ring of formal power series  $\sum_{j=-\infty}^{\infty} a_j z^j$  ( $a_j \in A$ ) such that  $\{j \leq 0 \mid a_j \neq 0\}$  is finite. □

Given an  $A$ -module  $B$  and  $j \in \mathbb{Z}$  let  $z^j B$  be the  $A$ -module with elements  $z^j x$  ( $x \in B$ ) and

$$a(z^j x) = z^j \alpha^j(a)x , \quad z^j x + z^j x' = z^j(x + x') \quad (a \in A, x, x' \in B) .$$

The induced  $A_\alpha[z, z^{-1}]$ -module is then given by

$$A_\alpha[z, z^{-1}] \otimes_A B = B_\alpha[z, z^{-1}] = \sum_{j=-\infty}^{\infty} z^j B .$$

For any  $A$ -module  $C$  and  $k \in \mathbb{Z}$  the  $A$ -module morphisms  $z^j B \rightarrow z^k C$  are given by

$$z^{k-j}\theta : z^j B \rightarrow z^k C ; z^j x \mapsto z^k \theta(x)$$

with  $\theta : B \rightarrow C$  a morphism of the additive groups such that

$$\theta(ax) = \alpha^{k-j}(a)\theta(x) \in C \ (a \in A, x \in B) .$$

We shall write  $z^{-1}B$  as  $\alpha B$ .

For a f.g. free  $A$ -module  $B$  and any  $A$ -module  $C$  every  $A_\alpha[z, z^{-1}]$ -module morphism  $\psi : B_\alpha[z, z^{-1}] \rightarrow C_\alpha[z, z^{-1}]$  is given by

$$\psi = \sum_{j=-\infty}^{\infty} z^j \psi_j : \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} \psi_j(x_k)$$

with  $\psi_j : B \rightarrow z^j C$   $A$ -module morphisms such that  $\{j \in \mathbb{Z} \mid \psi_j \neq 0\}$  is finite. Similarly, every  $A_\alpha((z))$ -module morphism  $\widehat{\psi} : B_\alpha((z)) \rightarrow C_\alpha((z))$  is given by

$$\widehat{\psi} = \sum_{j=-\infty}^{\infty} z^j \widehat{\psi}_j : \sum_{k=-\infty}^{\infty} z^k x_k \mapsto \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} \widehat{\psi}_j(x_k)$$

with  $\widehat{\psi}_j : B \rightarrow z^j C$   $A$ -module morphisms such that  $\{j \leq 0 \mid \widehat{\psi}_j \neq 0\}$  is finite.

**Definition 5.2.** Let  $\Sigma^{-1}A_\alpha[z, z^{-1}]$  be the localization of  $A_\alpha[z, z^{-1}]$  inverting the set  $\Sigma$  of square matrices in  $A_\alpha[z] \subset A_\alpha[z, z^{-1}]$  which become invertible in  $A$  under the augmentation

$$\epsilon : A_\alpha[z] \rightarrow A ; z \mapsto 0 . \quad \square$$

**Proposition 5.3.** Let  $D, E$  be f.g. free  $A$ -module chain complexes, and let  $g : D \rightarrow E, h : \alpha D \rightarrow E$  be  $A$ -module chain maps such that

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i , \ h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : \alpha D_i \rightarrow E_i = D_i \oplus F_i ,$$

$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1} .$$

Given bases for  $D, F$  let  $E$  have the corresponding basis.

(i) The  $\Sigma^{-1}D_\alpha[z, z^{-1}]$ -module morphism

$$\phi = g - zh : \Sigma^{-1}D_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E_\alpha[z, z^{-1}]$$

is an embedding, the natural projection  $p : C = \mathcal{C}(\phi) \rightarrow \widehat{C} = \text{coker}(\phi)$  is a chain equivalence, and the inclusions  $F_i \rightarrow E_i$  induce an isomorphism  $\widehat{F} \cong \widehat{C}$  with  $\widehat{F}$  the  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complex given by

$$\begin{aligned} d_{\widehat{F}} &= d_F + zh_F(1 - zh_D)^{-1}c : \widehat{F}_i = \Sigma^{-1}(F_i)_\alpha[z, z^{-1}] \\ &\rightarrow \widehat{F}_{i-1} = \Sigma^{-1}(F_{i-1})_\alpha[z, z^{-1}]. \end{aligned}$$

If  $\widehat{C}$  is given the basis determined by the bases of  $D, F$  and the isomorphism  $\widehat{F} \cong \widehat{C}$  then

$$\begin{aligned} \tau(p : C \simeq \widehat{C}) &= - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \\ &\rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \in K_1(\Sigma^{-1}A_\alpha[z, z^{-1}]) . \end{aligned}$$

(ii) An  $A$ -module chain homotopy  $k : h \simeq h' : \alpha D \rightarrow E$  determines a  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain isotopy

$$\psi : \phi = g - zh \sim \phi' = g - zh' : \Sigma^{-1}D_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E_\alpha[z, z^{-1}]$$

and simple isomorphisms  $C \cong C', \widehat{F} \cong \widehat{F}', \widehat{C} \cong \widehat{C}'$  of  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complexes where  $C = \mathcal{C}(\phi), C' = \mathcal{C}(\phi')$  etc., with

$$\begin{aligned} \tau(C \cong C') &= 0 \in K_1(\Sigma^{-1}A_\alpha[z, z^{-1}]) , \\ \tau(\widehat{C} \cong \widehat{C}') &= \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &\quad - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh'_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &= 0 \in K_1(\Sigma^{-1}A_\alpha[z, z^{-1}]) . \end{aligned}$$

(iii) Let  $\{D(k)\}, \{E(k)\}, \{F(k)\}$  be the inverse systems of  $A$ -module chain complexes defined by

$$\begin{aligned} D(k) &= D_\alpha[z, z^{-1}] / \sum_{j=k+1}^{\infty} z^j D = \sum_{j=-\infty}^k z^j D , \\ E(k) &= E_\alpha[z, z^{-1}] / \sum_{j=k+1}^{\infty} z^j E = \sum_{j=-\infty}^k z^j E , \\ F(k) &= \text{coker}(g - zh : D(k) \rightarrow E(k)) \end{aligned}$$

with structure maps the natural projections

$$D(k) \rightarrow D(k - 1) , E(k) \rightarrow E(k - 1) , F(k) \rightarrow F(k - 1) .$$

The short exact sequence of inverse systems of  $A$ -module chain complexes

$$0 \rightarrow D(k) \xrightarrow{g-zh} E(k) \rightarrow F(k) \rightarrow 0$$

induces a short exact sequence of the inverse limit  $A_\alpha((z))$ -module chain complexes

$$0 \rightarrow \varprojlim_k D(k) \xrightarrow{g-zh} \varprojlim_k E(k) \rightarrow \varprojlim_k F(k) \rightarrow 0$$

with

$$\varprojlim_k D(k) = D_\alpha((z)), \varprojlim_k E(k) = E_\alpha((z)).$$

Moreover, the inclusions  $F_i \rightarrow E_i$  induce isomorphisms

$$A_\alpha((z)) \otimes_{\Sigma^{-1}A_\alpha[z, z^{-1}]} \widehat{F} \cong \varprojlim_k F(k).$$

*Proof.* (i) Immediate from 1.8 (i), since

$$\begin{aligned} \phi &= \begin{pmatrix} \phi_D \\ \phi_F \end{pmatrix} = \begin{pmatrix} 1 - zh_D \\ -zh_F \end{pmatrix} : \\ &\Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(E_i)_\alpha[z, z^{-1}] = \Sigma^{-1}(D_i \oplus F_i)_\alpha[z, z^{-1}] \end{aligned}$$

with  $\phi_D = 1 - zh_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]$  a  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module automorphism.

(ii) Write the  $A$ -module chain homotopy  $k : h \simeq h' : \alpha D \rightarrow E$  as

$$k = \begin{pmatrix} k_D \\ k_F \end{pmatrix} : \alpha D_i \rightarrow E_{i+1} = D_{i+1} \oplus F_{i+1},$$

so that

$$\begin{pmatrix} h'_D \\ h'_F \end{pmatrix} - \begin{pmatrix} h_D \\ h_F \end{pmatrix} = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} k_D \\ k_F \end{pmatrix} + \begin{pmatrix} k_D \\ k_F \end{pmatrix} d_D : \alpha D_i \rightarrow E_i = D_i \oplus F_i.$$

Define a  $\Sigma^{-1}A_\alpha[z, z^{-1}]$  chain homotopy

$$\theta : \phi \simeq \phi' : \Sigma^{-1}D_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E_\alpha[z, z^{-1}]$$

by

$$\begin{aligned} \theta &= \begin{pmatrix} \theta_D \\ \theta_F \end{pmatrix} = \begin{pmatrix} -zk_D \\ -zk_F \end{pmatrix} : \\ &\Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(E_{i+1})_\alpha[z, z^{-1}] = \Sigma^{-1}(D_{i+1} \oplus F_{i+1})_\alpha[z, z^{-1}]. \end{aligned}$$

As in 1.8 (ii) the  $\Sigma^{-1}A_\alpha[z]$ -module morphisms

$$\psi = \begin{pmatrix} \theta_D(\phi_D)^{-1} & 0 \\ \theta_F(\phi_D)^{-1} & 0 \end{pmatrix} = \begin{pmatrix} -zk_D(1 - zh_D)^{-1} & 0 \\ -zk_F(1 - zh_D)^{-1} & 0 \end{pmatrix} :$$

$$\Sigma^{-1}(E_i)_\alpha[z] = \Sigma^{-1}(D_i \oplus F_i)_\alpha[z] \rightarrow \Sigma^{-1}(E_i)_\alpha[z] = \Sigma^{-1}(D_i \oplus F_i)_\alpha[z]$$

are such that

$$\phi' = (1 + d_E\psi + \psi d_E)\phi : D \rightarrow E .$$

The  $\Sigma^{-1}A_\alpha[z]$ -module endomorphism

$$\begin{aligned} 1 + d_E\psi + \psi d_E &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \begin{pmatrix} -zk_D(1 - zh_D)^{-1} & 0 \\ -zk_F(1 - zh_D)^{-1} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} -zk_D(1 - zh_D)^{-1} & 0 \\ -zk_F(1 - zh_D)^{-1} & 0 \end{pmatrix} \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} \\ &: \Sigma^{-1}(E_i)_\alpha[z] \rightarrow \Sigma^{-1}(E_i)_\alpha[z] \end{aligned}$$

has augmentation an  $A$ -module automorphism  $\epsilon(1 + d_E\psi + \psi d_E) = 1 : E_i \rightarrow E_i$ , so that  $1 + d_E\psi + \psi d_E : \Sigma^{-1}(E_i)_\alpha[z] \rightarrow \Sigma^{-1}(E_i)_\alpha[z]$  is a  $\Sigma^{-1}A_\alpha[z]$ -module automorphism by 4.1, and  $\psi$  defines a chain isotopy  $\psi : \phi \sim \phi' : \Sigma^{-1}D_\alpha[z] \rightarrow \Sigma^{-1}E_\alpha[z]$ . Define the isomorphisms

$$\begin{aligned} q &= \begin{pmatrix} 1 \pm \psi\phi \\ 0 & 1 \end{pmatrix} : C = \mathcal{C}(\phi) \rightarrow C' = \mathcal{C}(\phi') , \\ r &= [1 + d_E\psi + \psi d_E] : \widehat{C} = \text{coker}(\phi) \rightarrow \widehat{C}' = \text{coker}(\phi') \end{aligned}$$

as in 1.5, with  $q$  simple. By 1.8 (ii) and (i)

$$\begin{aligned} \tau(r) &= \tau(p : C \simeq \widehat{C}) - \tau(p' : C' \simeq \widehat{C}') \\ &= \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &\quad - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh'_D : \Sigma^{-1}(D_i)_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}(D_i)_\alpha[z, z^{-1}]) \\ &= 0 \in K_1(\Sigma^{-1}A_\alpha[z, z^{-1}]) . \end{aligned}$$

(iii) The  $A$ -module chain map  $\phi(k) = g - zh : D(k) \rightarrow E(k)$  is of the type considered in 1.8, with the components of

$$\phi(k) = \begin{pmatrix} \phi_D(k) \\ \phi_F(k) \end{pmatrix} : D(k)_i \rightarrow E(k)_i = D(k)_i \oplus F(k)_i$$

given by

$$\phi_D(k) = \begin{pmatrix} 1 - zh_D & 0 & \dots \\ 0 & 1 & -zh_D & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : D(k)_i = \sum_{j=-\infty}^k z^j D \rightarrow D(k)_i = \sum_{j=-\infty}^k z^j D_i ,$$

$$\phi_F(k) = \begin{pmatrix} 0 & -zh_F & 0 & \dots \\ 0 & 0 & -zh_F & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : D(k)_i = \sum_{j=-\infty}^k z^j D \rightarrow E(k)_i = \sum_{j=-\infty}^k z^j E_i.$$

As in 1.8  $\phi_D(k)$  is an automorphism, with inverse

$$\phi_D(k)^{-1} = \begin{pmatrix} 1 & zh_D & z^2(h_D)^2 & \dots \\ 0 & 1 & zh_D & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} : D(k)_i = \sum_{j=-\infty}^k z^j D_i \rightarrow D(k)_i = \sum_{j=-\infty}^k z^j D_i,$$

and the chain complex  $\overline{F}(k)$  defined by

$$d_{\overline{F}(k)} = d_F - \phi_F(k)\phi_D(k)^{-1}c = \begin{pmatrix} d_F & zh_{FC} & z^2h_Fh_{DC} & \dots \\ 0 & d_F & zh_{FC} & \dots \\ 0 & 0 & d_F & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} :$$

$$\overline{F}(k)_i = \sum_{j=-\infty}^k z^j F_i \rightarrow \overline{F}(k)_i = \sum_{j=-\infty}^k z^j F_i$$

is such that the inclusions  $\overline{F}(k)_i \rightarrow E(k)_i$  induce isomorphisms  $\overline{F}(k) \cong F(k)$  and  $\varprojlim_k \overline{F}(k) \cong \varprojlim_k F(k)$ . The identification  $\varprojlim_k D(k) = D_\alpha((z))$  is immediate from the identifications

$$D(k) = \sum_{j=-\infty}^k z^j D \rightarrow D(k-1) = \sum_{j=-\infty}^{k-1} z^j D ; \sum_{j=-\infty}^k z^j x_j \mapsto \sum_{j=-\infty}^{k-1} z^j x_j .$$

Similarly for  $\varprojlim_k E(k) = E_\alpha((z))$ ,  $\varprojlim_k \overline{F}(k) = \overline{F}$ . The short exact sequence of inverse systems

$$0 \rightarrow D(k) \xrightarrow{\phi} E(k) \rightarrow F(k) \rightarrow 0$$

determines an exact sequence of the inverse and derived limits

$$0 \rightarrow \varprojlim_k D(k) \xrightarrow{g-zh} \varprojlim_k E(k) \rightarrow \varprojlim_k F(k) \rightarrow \varprojlim_k^1 D(k) \rightarrow \dots .$$

Since the structure maps  $D(k) \rightarrow D(k-1)$  are onto the derived limit is  $\varprojlim_k^1 D(k) = 0$ , and the inverse limits actually fit into a short exact sequence

$$0 \rightarrow \varprojlim_k D(k) \xrightarrow{\overline{\phi}} \varprojlim_k E(k) \rightarrow \varprojlim_k F(k) \rightarrow 0$$

as required, with  $\overline{\phi} = g - zh$ . Alternatively, identify  $\text{coker}(\overline{\phi}) = \overline{F}$  by a direct application of 1.8 (i). □

The following result on algebraic handle exchanges will be required in Sect. 6.

**Proposition 5.4.** *Let  $D, D', E^+, E^-$  be f.g. free  $A$ -module chain complexes, and let*

$$g^+ : D \rightarrow E^+, \quad g^- : D' \rightarrow E^-, \quad h^+ : D' \rightarrow E^+, \quad h^- : \alpha D \rightarrow E^-$$

be  $A$ -module chain maps such that

$$\begin{aligned} g^+ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \rightarrow E_i^+ = D_i \oplus F_i^+, \quad g^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D'_i \rightarrow E_i^- = D'_i \oplus F_i^-, \\ h^+ &= \begin{pmatrix} h_D^+ \\ h_F^+ \end{pmatrix} : D'_i \rightarrow E_i^+ = D_i \oplus F_i^+, \quad h^- = \begin{pmatrix} h_D^- \\ h_F^- \end{pmatrix} : \alpha D_i \rightarrow E_i^- = D'_i \oplus F_i^-, \\ d_{E^+} &= \begin{pmatrix} d_D & c^+ \\ 0 & d_{F^+} \end{pmatrix} : E_i^+ = D_i \oplus F_i^+ \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1}^+, \\ d_{E^-} &= \begin{pmatrix} d_{D'} & c^- \\ 0 & d_{F^-} \end{pmatrix} : E_i^- = D'_i \oplus F_i^- \rightarrow E_{i-1}^- = D'_{i-1} \oplus F_{i-1}^-. \end{aligned}$$

Given bases for  $D, D', F^+, F^-$  there are now defined two collections of data as in 5.3

$$(g : D \rightarrow E, h : \alpha D \rightarrow E, F), \quad (g' : D' \rightarrow E', h' : \alpha D' \rightarrow E', F')$$

with

$$\begin{aligned} d_E &= \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1}, \\ d_F &= \begin{pmatrix} d_{F^+} & h_F^+ c^- \\ 0 & d_{F^-} \end{pmatrix} : F_i = F_i^+ \oplus F_i^- \rightarrow F_{i-1} = F_{i-1}^+ \oplus F_{i-1}^-, \\ c &= (c^+ \ h_F^+ c^-) : F_i = F_i^+ \oplus F_i^- \rightarrow D_{i-1}, \quad g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i, \\ h_D &= h_D^+ h_D^- : \alpha D_i \rightarrow D_i, \quad h_F = \begin{pmatrix} h_F^+ h_D^- \\ h_F^- \end{pmatrix} : \alpha D_i \rightarrow F_i = F_i^+ \oplus F_i^- \end{aligned}$$

and

$$\begin{aligned} d_{E'} &= \begin{pmatrix} d_{D'} & c' \\ 0 & d_{F'} \end{pmatrix} : E'_i = D'_i \oplus F'_i \rightarrow E'_{i-1} = D'_{i-1} \oplus F'_{i-1}, \\ d_{F'} &= \begin{pmatrix} d_{F^+} & 0 \\ h_F^- c^+ & d_{F^-} \end{pmatrix} : F'_i = F_i^+ \oplus F_i^- \rightarrow F'_{i-1} = F_{i-1}^+ \oplus F_{i-1}^-, \\ c' &= (h_D^- c^+ \ c^-) : F'_i = F_i^+ \oplus F_i^- \rightarrow D'_{i-1}, \quad g' = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D'_i \rightarrow E'_i = D'_i \oplus F'_i, \\ h'_D &= h_D^- h_D^+ : \alpha D'_i \rightarrow D'_i, \quad h'_F = \begin{pmatrix} h_F^+ h_D^- \\ h_F^- h_D^+ \end{pmatrix} : \alpha D'_i \rightarrow F'_i = F_i^+ \oplus F_i^-. \end{aligned}$$

The cokernels of the corresponding embeddings of the based f.g. free  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complexes

$$\begin{aligned} \phi &= g - zh : \Sigma^{-1}D_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E_\alpha[z, z^{-1}], \\ \phi' &= g' - zh' : \Sigma^{-1}D'_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E'_\alpha[z, z^{-1}] \end{aligned}$$

are related by an isomorphism of the based f.g. free  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complexes  $I : \text{coker}(\phi) \cong \text{coker}(\phi')$  which sends basis elements to  $z^\delta$  (basis elements), with  $\delta = 0$  or  $1$ .

*Proof.* Use 5.3 to identify  $\text{coker}(\phi) = \widehat{F}$ ,  $\text{coker}(\phi') = \widehat{F}'$  with  $\widehat{F}$ ,  $\widehat{F}'$  the based f.g. free  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complexes defined by

$$\begin{aligned} d_{\widehat{F}} &= d_F + zh_F(1 - zh_D)^{-1}c : \widehat{F}_i = \Sigma^{-1}(F_i)_\alpha[z, z^{-1}] \rightarrow \widehat{F}_{i-1} \\ &= \Sigma^{-1}(F_{i-1})_\alpha[z, z^{-1}], \\ d_{\widehat{F}'} &= d_{F'} + zh'_F(1 - zh'_D)^{-1}c' : \widehat{F}'_i \\ &= \Sigma^{-1}(F'_i)_\alpha[z, z^{-1}] \rightarrow \widehat{F}'_{i-1} = \Sigma^{-1}(F'_{i-1})_\alpha[z, z^{-1}]. \end{aligned}$$

Define an isomorphism  $I : \widehat{F} \cong \widehat{F}'$  of based f.g. free  $\Sigma^{-1}A_\alpha[z, z^{-1}]$ -module chain complexes by

$$\begin{aligned} I : \widehat{F}_i &= \Sigma^{-1}(F_i^+ \oplus F_i^-)_\alpha[z, z^{-1}] \rightarrow \widehat{F}'_i \\ &= \Sigma^{-1}(F_i^+ \oplus F_i^-)_\alpha[z, z^{-1}]; (x^+, x^-) \rightarrow (x^+, zx^-). \quad \square \end{aligned}$$

### 6. The Novikov complexes $C^{Nov}(M, f, v)$ , $C^{Paj}(M, f, v)$ , $C^{FR}(M, f, v, h)$

This section starts with a review of the geometric constructions of the Novikov complex  $C^{Nov}(M, f, v)$  and the Pajitnov complex  $C^{Paj}(M, f, v)$  of a circle-valued Morse function  $f : M \rightarrow S^1$  with respect to  $v \in \mathcal{GT}(f)$ . Then the proper real-valued Morse function  $\bar{f} : \bar{M} = f^*\mathbb{R} \rightarrow \mathbb{R}$  is used to identify the Novikov complex with the inverse limit of the proper Morse-Smale complexes  $C^{Nov}(M, f, v) = \varprojlim_k C^{MS}(M(k), f(k), v(k))$  of the proper real-valued Morse functions  $f(k) = \bar{f}| : M(k) = \bar{f}^{-1}[-k, \infty) \rightarrow [-k, \infty)$  ( $k \geq 0$ ) with  $v(k) = \bar{v}$ . This is followed by a review of the algebraic construction of the chain complex  $C^{FR}(M, f, v, h)$  of Farber and Ranicki [FR]. Finally, all this is put together to prove the Cokernel, Invariance and Isomorphism Theorems already stated in the Introduction. In particular, for  $v \in \mathcal{GT}(f)$  with a gradient-like chain approximation  $h^{gra}$  there exist basis-preserving isomorphisms  $C^{Nov}(M, f, v) \cong C^{FR}(M, f, v, h^{gra}; \widehat{\mathbb{Z}[\pi_1(M)]})$  and  $C^{Paj}(M, f, v) \cong C^{FR}(M, f, v, h^{gra})$ .

The infinite cyclic cover of  $M$  determined by  $f : M \rightarrow S^1$  is

$$\bar{M} = f^*\mathbb{R} = \{(x, y) \in M \times \mathbb{R} \mid f(x) = [y] \in S^1\},$$

with  $\bar{f} : \bar{M} \rightarrow \mathbb{R}; (x, y) \mapsto y$ . The generating covering translation  $z : \bar{M} \rightarrow \bar{M}; (x, y) \mapsto (x, y - 1)$  is parallel to the downward  $v$ -gradient flow (and so acts from right to left, or rather from top to bottom). Assume that  $M$  and  $\bar{M} = f^*\mathbb{R}$  are connected, so that  $\pi_1(M) = \pi_1(\bar{M}) \times_\alpha \mathbb{Z}$  with  $\alpha = z_* : \pi_1(M) \rightarrow \pi_1(\bar{M})$  the monodromy automorphism. The group ring  $\mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(\bar{M})]_\alpha[z, z^{-1}]$  is the  $\alpha$ -twisted Laurent polynomial extension of  $\mathbb{Z}[\pi_1(\bar{M})]$ . Write the Novikov ring of  $\mathbb{Z}[\pi_1(M)]$  as  $\widehat{\mathbb{Z}[\pi_1(M)]} = \widehat{\mathbb{Z}[\pi_1(\bar{M})]}_\alpha((z)) = \widehat{\mathbb{Z}[\pi_1(\bar{M})]}_\alpha[[z]][[z^{-1}]]$ .

**Definition 6.1.** ([N], [P1]) The Novikov complex  $C^{Nov}(M, f, v)$  of a Morse function  $f : M \rightarrow S^1$  with respect to  $v \in \mathcal{GT}(f)$  is the based f.g. free  $\widehat{\mathbb{Z}[\pi_1(M)]}$ -module chain complex with

- (i)  $\text{rank}_{\widehat{\mathbb{Z}[\pi_1(M)]}} C_i^{Nov}(M, f, v) = c_i(f)$ , with one basis element  $\tilde{p}$  for each critical point  $p \in M$  of index  $i$ , corresponding to a choice of lift  $\tilde{p} \in \tilde{M}$
- (ii) the boundary  $\widehat{\mathbb{Z}[\pi_1(M)]}$ -module morphisms are given by

$$d : C_i^{Nov}(M, f, v) = \widehat{\mathbb{Z}[\pi_1(M)]}^{c_i(f)} \rightarrow C_{i-1}^{Nov}(M, f, v) = \widehat{\mathbb{Z}[\pi_1(M)]}^{c_{i-1}(f)} ;$$

$$\tilde{p} \mapsto \sum_{\tilde{q}} \sum_{u \in \pi_1(M)} n(\tilde{p}, u\tilde{q})u\tilde{q}$$

with  $n(\tilde{p}, u\tilde{q}) \in \mathbb{Z}$  the algebraic number of  $\tilde{v}$ -gradient flow lines in  $\tilde{M}$  from  $\tilde{p}$  to  $u\tilde{q}$ .  $\square$

Let  $\Sigma^{-1}\widehat{\mathbb{Z}[\pi_1(M)]}$  be the localization of  $\widehat{\mathbb{Z}[\pi_1(M)]}$  defined in 5.2.

**Definition 6.2.** (Pajitnov [P2,P3,P4])

For a Morse function  $f : M \rightarrow S^1$  and  $v \in \mathcal{GECT}(f)$  the Novikov complex is of the form  $C^{Nov}(M, f, v) = \widehat{\mathbb{Z}[\pi_1(M)]} \otimes_{\Sigma^{-1}\widehat{\mathbb{Z}[\pi_1(M)]}} C^{Paj}(M, f, v)$  with the Pajitnov complex  $C^{Paj}(M, f, v)$  a based f.g. free  $\Sigma^{-1}\widehat{\mathbb{Z}[\pi_1(M)]}$ -module chain complex such that  $\text{rank}_{\Sigma^{-1}\widehat{\mathbb{Z}[\pi_1(M)]}} C_i^{Paj}(M, f, v) = c_i(f)$ .  $\square$

From now on it will be assumed that  $0 \in S^1$  is a regular value of  $f$ , with inverse image a codimension 1 framed submanifold  $N^{m-1} = f^{-1}(0) \subset M^m$ . Thus  $\bar{f} : \bar{M} \rightarrow \mathbb{R}$  is transverse regular at  $\mathbb{Z} \subset \mathbb{R}$ , and cutting  $M$  along  $N$  gives a cobordism  $(M_N; N, z^{-1}N) = \bar{f}^{-1}(I; \{0\}, \{1\})$  which is a fundamental domain for  $\bar{M} = \bigcup_{j=-\infty}^{\infty} z^j M_N$  with a Morse function  $f_N = \bar{f}| : (M_N; N, z^{-1}N) \rightarrow (I; \{0\}, \{1\})$ .

**Proposition 6.3.** The Novikov complex of a Morse function  $f : M \rightarrow S^1$  with respect to any  $v \in \mathcal{GT}(f)$  is the inverse limit

$$C^{Nov}(M, f, v) = \varprojlim_k C^{MS}(M(k), f(k), v(k))$$

of the Morse-Smale complexes of the proper real-valued Morse functions  $f(k) = \bar{f}| : M(k) = \bar{f}^{-1}[-k, \infty) = \bigcup_{j=-\infty}^k z^j M_N \rightarrow [-k, \infty)$  on the non-compact

manifolds with boundary  $(M(k), \partial M(k)) = \bar{f}^{-1}([-k, \infty), \{-k\}) = (\bigcup_{j=-\infty}^k z^j M_N, z^k N)$  with respect to the projections

$$\begin{aligned} C^{MS}(M(k+1), f(k+1), v(k+1)) &= C\left(\bigcup_{j=-\infty}^{k+1} z^j M_N, z^{k+1} N\right) \\ &\rightarrow C\left(\bigcup_{j=-\infty}^{k+1} z^j M_N, z^{k+1} M_N\right) \\ &= C\left(\bigcup_{j=-\infty}^k z^j M_N, z^k N\right) = C^{MS}(M(k), f(k), v(k)). \end{aligned}$$

*Proof.* Identify  $C^{Nov}(M, f, v) = \varprojlim_k C^{MS}(M(k), f(k), v(k))$  using the expression of the Novikov ring as the inverse limit  $\mathbb{Z}[\widehat{\pi_1(\bar{M})}] = \varprojlim_k \mathbb{Z}[\pi_1(\bar{M})]_\alpha(k)$  of the  $\mathbb{Z}[\pi_1(\bar{M})]$ -modules

$$\mathbb{Z}[\pi_1(\bar{M})]_\alpha(k) = \sum_{j=-\infty}^k z^j \mathbb{Z}[\pi_1(\bar{M})] \subset \mathbb{Z}[\pi_1(M)] = \mathbb{Z}[\pi_1(\bar{M})]_\alpha[z, z^{-1}]$$

with respect to the natural projections

$$\mathbb{Z}[\pi_1(\bar{M})]_\alpha(k+1) \rightarrow \mathbb{Z}[\pi_1(\bar{M})]_\alpha(k); \quad \sum_{j=-\infty}^k a_j z^j \mapsto \sum_{j=-\infty}^{k-1} a_j z^j. \quad \square$$

As before, let  $f : M \rightarrow S^1$  have  $c_i(f)$  critical points of index  $i$ . The real-valued Morse function  $f_N = \bar{f}| : (M_N; N, z^{-1}N) \rightarrow (I; \{0\}, \{1\})$  has  $c_i(f_N) = c_i(f)$  critical points of index  $i$ , and as in Sect. 2 there is a handle decomposition  $M_N = N \times I \cup \bigcup_{i=0}^m \bigcup_{c_i(f)} h^i$ . Let  $N$  have a CW structure with  $c_i(N)$

$i$ -cells  $e^i \subset N$  and let  $M_N$  have the corresponding CW structure with  $c_i(N)$   $i$ -cells of type  $e^i \times I \subset M_N$  and  $c_i(f)$   $i$ -cells of type  $h^i \subset M_N$ . Let  $\tilde{M}$  be the universal cover of  $M$ , and let  $\tilde{M}_N, \tilde{N}$  be the corresponding covers of  $M_N, N$ . The inclusion  $g : N \rightarrow M_N$  induces an inclusion of the cellular  $\mathbb{Z}[\pi_1(\bar{M})]$ -module chain complexes  $g : C(\tilde{N}) \rightarrow C(\tilde{M}_N)$ . The inclusion  $h : z^{-1}N \rightarrow M_N$  induces a  $\mathbb{Z}[\pi_1(\bar{M})]$ -module chain map  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$ .

Now  $M = M_N / (N = z^{-1}N)$  has a CW structure with  $c_i(M) = c_i(f) + c_i(N) + c_{i-1}(N)$   $i$ -cells.

The following terminology will be used in dealing with the cellular chain complexes associated to the fundamental domain  $(M_N; N, z^{-1}N)$  of  $\bar{M}$ .

**Terminology 6.4.** As in 2.3 write

$$D = C(\tilde{N}), E = C(\tilde{M}_N), F = C^{MS}(M_N, f_N, v_N) = C(\tilde{M}_N, \tilde{N} \times I),$$

$$d_E = \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix} : E_i = D_i \oplus F_i \rightarrow E_{i-1} = D_{i-1} \oplus F_{i-1}.$$

The inclusions  $g : N \rightarrow M, h : z^{-1}N \rightarrow M$  induce chain maps

$$g = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : D_i \rightarrow E_i = D_i \oplus F_i, h = \begin{pmatrix} h_D \\ h_F \end{pmatrix} : \alpha D_i \rightarrow E_i = D_i \oplus F_i. \quad \square$$

**Definition 6.5.** ([FR]) Given a Morse function  $f : M \rightarrow S^1$  with regular value  $0 \in S^1, v \in \mathcal{GT}(f)$ , a choice of CW structure for  $N = f^{-1}(0) \subset M$ , and a choice of chain approximation  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  let  $C^{FR}(M, f, v, h) = \widehat{F}$  be the based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complex given by

$$\widehat{F}_i = \Sigma^{-1}C_i(\tilde{M}_N, \tilde{N})_\alpha[z, z^{-1}] = \Sigma^{-1}\mathbb{Z}[\pi_1(M)]^{c_i(f)},$$

$$d_{\widehat{F}} = d_{C(\tilde{M}_N, \tilde{N})} + zh_F(1 - zh_D)^{-1}c : \widehat{F}_i \rightarrow \widehat{F}_{i-1}. \quad \square$$

**Cokernel Theorem 6.6.** (i) *The inclusions  $\widehat{F}_i \rightarrow \Sigma^{-1}C_i(\tilde{M}_N)_\alpha[z, z^{-1}]$  induce a basis-preserving (and a fortiori simple) isomorphism of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes*

$$C^{FR}(M, f, v, h) \cong \text{coker}(g - zh : \Sigma^{-1}C(\tilde{N})_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}C(\tilde{M}_N)_\alpha[z, z^{-1}]).$$

(ii) *The natural projection*

$$p : C(\tilde{M}; \Sigma^{-1}\mathbb{Z}[\pi_1(M)]) = C(g - zh : \Sigma^{-1}C(\tilde{N})_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}C(\tilde{M}_N)_\alpha[z, z^{-1}]) \rightarrow C^{FR}(M, f, v, h)$$

*is a chain equivalence of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes with torsion*

$$\tau(p) = - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D : \Sigma^{-1}C_i(\tilde{N})_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}C_i(\tilde{N})_\alpha[z, z^{-1}]) \in K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)]).$$

*Proof.* This is a direct application of 5.3 (i). □

**Invariance Theorem 6.7.** *Let  $f : M \rightarrow S^1$  be a Morse function, and let  $v \in \mathcal{GT}(f)$ . For any regular values  $0, 0' \in S^1$ , CW structures on  $N = f^{-1}(0), N' = f^{-1}(0')$  and chain approximations  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N), h' :$*

$\alpha C(\tilde{N}') \rightarrow C(\tilde{M}_{N'})$  there is defined a simple isomorphism  $C^{FR}(M, f, v, h) \cong C^{FR}(M, f, v, h')$  of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes.

*Proof.* The case  $0 = 0', N = N'$  is given by 5.3 (ii). So assume  $0 \neq 0' \in S^1$ , and let  $I^+, I^- \subset S^1$  be the two arcs joining  $0$  and  $0'$ . The restrictions of  $(f, v)$

$$\begin{aligned} (f^+, v^+) &= (f, v)| : (M^+; N, N') = f^{-1}(I^+; \{0\}, \{0'\}) \rightarrow (I^+; \{0\}, \{0'\}) , \\ (f^-, v^-) &= (f, v)| : (M^-; N', N) = f^{-1}(I^-; \{0'\}, \{0\}) \rightarrow (I^-; \{0'\}, \{0\}) \end{aligned}$$

are Morse functions with gradient-like vector fields such that

$$\begin{aligned} (f_N, v_N) &= (f^+, v^+) \cup (f^-, v^-) : \\ (M_N; N, z^{-1}N) &= (M^+; N, N') \cup (M^-; N', N) \rightarrow ([0, 1]; \{0\}, \{1\}), \\ (f_{N'}, v_{N'}) &= (f^-, v^-) \cup (f^+, v^+) : \\ (M_{N'}; N', z^{-1}N') &= (M^-; N', N) \cup (M^+; N, N') \rightarrow ([0, 1]; \{0\}, \{1\}) . \end{aligned}$$

Use the handlebody structure on  $(M^+; N, N')$  (resp.  $(M^-; N', N)$ ) determined by  $(f^+, v^+)$  (resp.  $(f^-, v^-)$ ) to extend the CW structure on  $N$  (resp.  $N'$ ) to a CW structure on  $M^+$  (resp.  $M^-$ ). The inclusions of CW subcomplexes  $g^+ : N \rightarrow M^+, g^- : N' \rightarrow M^-$  induce inclusions of subcomplexes

$$g^+ : D = C(\tilde{N}) \rightarrow E^+ = C(\tilde{M}^+), \quad g^- : D' = C(\tilde{N}') \rightarrow E^+ = C(\tilde{M}^-).$$

Cellular approximations to the inclusions  $h^+ : N' \rightarrow M^+, h^- : z^{-1}N \rightarrow M^-$  induce chain maps

$$h^+ : D' = C(\tilde{N}') \rightarrow E^+ = C(\tilde{M}^+), \quad h^- : \alpha D = \alpha C(\tilde{N}) \rightarrow E^- = C(\tilde{M}^-).$$

A direct application of 5.4 gives a simple isomorphism of based f.g. free  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain complexes

$$C^{FR}(M, f, v, h) = \text{coker}(g-zh) \cong C^{FR}(M, f, v, h') = \text{coker}(g'-zh'). \quad \square$$

**Definition 6.8.** A crossing chain approximation  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  is *gradient-like* if for any critical points  $\tilde{p}, \tilde{q} \in \tilde{M}_N$  of  $\tilde{f}_N : \tilde{M}_N \rightarrow \mathbb{R}$  with index  $i, i - 1$  the algebraic number of  $\tilde{v}$ -gradient flow lines in  $\tilde{M}$  joining  $\tilde{p}$  to  $z^j \tilde{q}$  is

$$n(\tilde{p}, z^j \tilde{q}) = \begin{cases} (\tilde{p}, \tilde{q})\text{-coefficient of } d_F : F_i \rightarrow F_{i-1} & \text{if } j = 0 \\ (\tilde{p}, \tilde{q})\text{-coefficient of } h_F(h_D)^{j-1}c : F_i \rightarrow F_{i-1} & \text{if } j > 0 \\ 0 & \text{if } j < 0 . \end{cases}$$

A gradient-like chain approximation  $h$  will be denoted by  $h^{gra}$ . □

*Remark 6.9.* (i) Pajitnov [P4] proved that for any Morse function  $f : M \rightarrow S^1$  and  $v \in \mathcal{GCCT}(f)$  there exists a handlebody structure on  $N$  with a gradient-like

chain approximation  $h^{gra} : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  (cf. Remark 2.4).

(ii) Cornea and Ranicki [CR] prove that for any Morse function  $f : M \rightarrow S^1$  and  $v \in \mathcal{GT}(f)$  there exists a CW structure on  $N = f^{-1}(0)$  with a chain approximation  $h$  such that  $C^{Nov}(M, f, \alpha)$  is simple isomorphic to  $((F_i)_\alpha((z)), d_F + zh_F(1 - zh_D)^{-1}c)$  by showing that there exist Morse functions  $f' : M \rightarrow S^1$ ,  $g : N \rightarrow \mathbb{R}$  with  $v' \in \mathcal{GT}(f')$ ,  $w \in \mathcal{GT}(g)$  such that  $(f', v')$  agrees with  $(f, v)$  outside a tubular neighbourhood of  $N$   $f^{-1}[-\epsilon, \epsilon] = N \times [-\epsilon, \epsilon] \subset M$  for some small  $\epsilon > 0$ , and  $(f', v')$  restricts to translates of  $(g, w)$   $(f', v')| = (g_\pm, w_\pm) : N \times \{\pm\epsilon/2\} \rightarrow S^1 \setminus \{0\}$ , with  $\text{Crit}_i(g_\pm) = \text{Crit}_i(g) \times \{\pm\epsilon/2\}$  and  $\text{Crit}_i(f') = \text{Crit}_{i-1}(g_+) \cup \text{Crit}_i(g_-) \cup \text{Crit}_i(f)$ . The  $\bar{v}$ -gradient flow lines are in one-one correspondence with the broken  $\bar{v}'$ -gradient flow lines i.e. joined up sequences of  $\bar{v}'$ -gradient flow lines which start and terminate at critical points of  $\bar{f}$ . The Morse-Smale complex of  $(\bar{f}', \bar{v}')$  is of the form

$$C^{MS}(\bar{M}, \bar{f}', \bar{v}') = \left( (D_{i-1} \oplus D_i \oplus F_i)_\alpha[z, z^{-1}], \begin{pmatrix} -d_D & 0 & 0 \\ 1 - zh_D & d_D & c \\ -zh_F & 0 & d_F \end{pmatrix} \right)$$

with

$$D = C^{MS}(N, g, w), F = C^{MS}(M_N, f_N, v_N) \text{ etc.}$$

The cellular chain complex of the cover  $\tilde{M}_N$  of  $M_N$  induced from the universal cover  $\tilde{M}$  of  $M$  is  $E = C(\tilde{M}_N) = (D_i \oplus F_i, \begin{pmatrix} d_D & c \\ 0 & d_F \end{pmatrix})$ . (This is the circle-valued analogue of Remark 2.7). Thus the algebraic mapping cone of the  $\mathbb{Z}[\pi_1(M)]$ -module chain map  $\phi = \begin{pmatrix} 1 - zh_D \\ -zh_F \end{pmatrix} : D_\alpha[z, z^{-1}] \rightarrow E_\alpha[z, z^{-1}]$  is  $C(\phi) = C^{MS}(\bar{M}, \bar{f}', \bar{v}')$  and the cokernel of the induced  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -module chain map is  $\text{coker}(\Sigma^{-1}\phi : \Sigma^{-1}D_\alpha[z, z^{-1}] \rightarrow \Sigma^{-1}E_\alpha[z, z^{-1}]) = C^{FR}(M, f, v, h)$ . The kernel of the projection  $K = \ker(p : \Sigma^{-1}C(\phi) \rightarrow \text{coker}(\Sigma^{-1}\phi))$  is an algebraic model for the closed orbits of the  $v$ -gradient flow. As in 1.8 (i) (c) identify

$$d_K = \begin{pmatrix} d_D & 0 \\ (-1)^i(1 - zh_D) & (1 - zh_D)^{-1}d_D(1 - zh_D) \end{pmatrix} : K_i = \Sigma^{-1}(D_{i-1} \oplus D_i)_\alpha[z, z^{-1}] \rightarrow K_{i-1} = \Sigma^{-1}(D_{i-2} \oplus D_{i-1})_\alpha[z, z^{-1}].$$

(iii) In applying 2.6 and 3.3 to  $f : M \rightarrow S^1$ ,  $v \in \mathcal{GT}(f)$  with a gradient-like chain approximation  $h^{gra}$ , the Morse-Smale complexes of unions of copies of a Morse function  $f_N : (M_N; N; z^{-1}N) \rightarrow (I; \{0\}, \{1\})$  and  $v_N \in \mathcal{GT}(f_N)$ , the chain homotopy  $b$  in 2.6 and the higher chain homotopies  $b[j, j']$  in 3.3 are 0. In particular, the Morse-Smale complex of the proper real-valued Morse function

$$\bar{f}^+ = \bigcup_{k=-\infty}^0 z^k f_N = \bar{f}| : \bar{M}^+ = \bigcup_{k=-\infty}^0 z^k M_N = \bar{f}^{-1}[0, \infty) \rightarrow [0, \infty)$$

is given as a based free  $\mathbb{Z}[\pi_1(\overline{M})]$ -module chain complex by

$$C^{MS}(\overline{M}^+, \overline{f}^+, \overline{v}^+) = \left( \sum_{k=-\infty}^0 z^k F_i, d_F + \sum_{j=1}^{\infty} z^j h_F^{gra} (h_D^{gra})^{j-1} c \right).$$

The coefficients of  $h_F^{gra} (h_D^{gra})^{j-1} c$  count the  $\tilde{v}$ -gradient flow lines which start at an index  $i$  critical point of  $\tilde{M}$ , cross  $j$  translates of  $\tilde{N} \subset \tilde{M}$ , and terminate at an index  $i - 1$  critical point of  $\tilde{M}$ . □

**Isomorphism Theorem 6.10.** *Given a Morse function  $f : M \rightarrow S^1$  with regular value  $0 \in S^1$ ,  $v \in \mathcal{GT}(f)$ , a choice of CW structure for  $N = f^{-1}(0) \subset M$ , and a choice of chain approximation  $h : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  there is defined a simple isomorphism  $I_h : C^{Nov}(M, f, v) \cong C^{FR}(M, f, v, h; \mathbb{Z}[\widehat{\pi_1(M)}])$  (resp.  $I_h : C^{Paj}(M, f, v) \cong C^{FR}(M, f, v, h)$ ) of based f. g. free  $R$ -module chain complexes with  $R = \mathbb{Z}[\widehat{\pi_1(M)}]$  (resp.  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ ). For a gradient-like chain approximation  $h^{gra} : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  the simple isomorphisms  $I_{h^{gra}}$  are basis-preserving.*

*Proof.* By 6.3 the Novikov complex is the inverse limit  $C^{Nov}(M, f, v) = \varprojlim_k F(k)$  of the inverse system  $\{F(k) = C^{MS}(M(k), f(k), v(k))\}$  of the Morse-Smale  $\mathbb{Z}[\pi_1(\overline{M})]$ -module chain complexes of the proper real-valued Morse functions

$$f(k) = \overline{f}| : (M(k), \partial M(k)) = \left( \bigcup_{j=-\infty}^k z^j M_N, z^k N \right) \rightarrow ([-k, \infty), \{-k\}).$$

By 3.3 there is defined an isomorphism of inverse systems  $I_h(k) : F(k) \rightarrow \text{coker}(g - zh : D(k) \rightarrow E(k))$  with  $D(k) = \sum_{j=-\infty}^k z^j C(\tilde{N})$  and  $E(k) = \sum_{j=-\infty}^k z^j C(\tilde{M}_N)$ . The inverse limits are given by 5.3 (iii) and 6.3 to be  $\varprojlim_k D(k) = C(\tilde{N})_\alpha((z))$  and  $\varprojlim_k E(k) = C(\tilde{M}_N)_\alpha((z))$ , with

$$\begin{aligned} & \varprojlim_k \text{coker}(g - zh : D(k) \rightarrow E(k)) \\ &= \text{coker}(g - zh : \varprojlim_k D(k) \rightarrow \varprojlim_k E(k)) \\ &= \text{coker}(g - zh : C(\tilde{N})_\alpha((z)) \rightarrow C(\tilde{M}_N)_\alpha((z))) \\ &\cong C^{FR} \left( M, f, v, h; \mathbb{Z}[\widehat{\pi_1(M)}] \right). \end{aligned}$$

Define  $I_h$  to be the induced isomorphism of inverse limits

$$\begin{aligned} I_h &= \varprojlim_k I_h(k) : C^{Nov}(M, f, v) = \varprojlim_k F(k) \\ &\rightarrow \varprojlim_k \text{coker}(g - zh : D(k) \rightarrow E(k)) \cong C^{FR} \left( M, f, v, h; \mathbb{Z}[\widehat{\pi_1(M)}] \right). \end{aligned}$$

For a gradient-like chain approximation  $h^{gra}$  3.3 gives identifications

$$d_{F(k)} = d_{C(\tilde{M}_N, \tilde{N})} + \sum_{j=1}^{\infty} z^j h_F(h_D^{gra})^{j-1} c :$$

$$F(k)_i = \sum_{j=-\infty}^k z^j C_i(\tilde{M}_N, \tilde{N}) \rightarrow F(k)_{i-1} = \sum_{j=-\infty}^k z^j C_{i-1}(\tilde{M}_N, \tilde{N})$$

(as in 6.9 (iii)). Passing to the inverse limit as  $k \rightarrow \infty$  gives

$$d_{C^{Nov}(M, f, v)} = d_{C(\tilde{M}_N, \tilde{N})} + \sum_{j=1}^{\infty} z^j h_F^{gra}(h_D^{gra})^{j-1} c :$$

$$C_i^{Nov}(M, f, v) = C_i(\tilde{M}_N, \tilde{N})_{\alpha}((z)) \rightarrow C_{i-1}^{Nov}(M, f, v) = C_{i-1}(\tilde{M}_N, \tilde{N})_{\alpha}((z))$$

so that  $I_{h^{gra}} : C^{Nov}(M, f, v) \cong C^{FR}(M, f, v, h^{gra}; \mathbb{Z}[\widehat{\pi_1(M)}])$  is a basis-preserving isomorphism, with zero torsion. There exists a chain homotopy  $h \simeq h^{gra} : \alpha C(\tilde{N}) \rightarrow C(\tilde{M}_N)$  so that  $\tau(I_h) = \tau(I_{h^{gra}}) = 0 \in K_1(\mathbb{Z}[\widehat{\pi_1(M)}])$  by 5.3 (ii). Similarly for  $\Sigma^{-1}\mathbb{Z}[\pi_1(M)]$ -coefficients.  $\square$

*Remark 6.11.* (i) The formulae given by 6.7 and 6.10

$$\tau(p : C(\tilde{M}; \Sigma^{-1}\mathbb{Z}[\pi_1(M)]) \rightarrow C^{Paj}(M, f, v))$$

$$= - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D^{gra} : \Sigma^{-1}C_i(\tilde{N})_{\alpha}[z, z^{-1}] \rightarrow \Sigma^{-1}C_i(\tilde{N})_{\alpha}[z, z^{-1}])$$

$$\in K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)]) ,$$

$$\tau(\hat{p} : C(\tilde{M}; \mathbb{Z}[\widehat{\pi_1(M)}]) \rightarrow C^{Nov}(M, f, v))$$

$$= - \sum_{i=0}^{\infty} (-)^i \tau(1 - zh_D^{gra} : C_i(\tilde{N}; \mathbb{Z}[\widehat{\pi_1(M)}])$$

$$\rightarrow C_i(\tilde{N}; \mathbb{Z}[\widehat{\pi_1(M)}])) \in K_1(\mathbb{Z}[\widehat{\pi_1(M)}])$$

are generalizations of the formulae of Hutchings and Lee [HL] and Pajitnov [P5],[P6] counting the critical points of  $f : M \rightarrow S^1$ , the  $\zeta$ -function of the closed orbits of the gradient flow (corresponding to  $h_D^{gra}$ ) and the Reidemeister torsion of  $M$ . Schütz [Sch1],[Sch2] extended these formulae to the closed orbits of a generic gradient flow of a closed 1-form using Hochschild homology and a chain equivalence of the type  $\hat{p} : C(\tilde{M}; \mathbb{Z}[\widehat{\pi_1(M)}]) \rightarrow C^{Nov}(M, f, v)$ .

(ii) See Chapters 10,14,15 of [R] for the splitting theorems for the torsion groups  $K_1(\mathbb{Z}[\widehat{\pi_1(M)}])$ ,  $K_1(\Sigma^{-1}\mathbb{Z}[\pi_1(M)])$  in the case  $\alpha = 1$  (which extend to the case of arbitrary  $\alpha$ , Pajitnov and Ranicki [PR]) and for the expressions of the torsions

$\tau(1-zh_D)$  in terms of (noncommutative) characteristic polynomials. For any ring  $A$  the classical Bass-Heller-Swan splitting  $K_1(A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus \widetilde{\text{Nil}}_0(A) \oplus \widetilde{\text{Nil}}_0(A)$  generalizes to splittings  $K_1(\Sigma^{-1}A[z, z^{-1}]) = K_1(A) \oplus K_0(A) \oplus \widetilde{\text{Nil}}_0(A) \oplus V(A)$  and  $K_1(A((z))) = K_1(A) \oplus K_0(A) \oplus \widetilde{\text{Nil}}_0(A) \oplus \widehat{V}(A)$  with  $\widehat{V}(A) \subseteq K_1(A((z)))$  the image of the multiplicative group of noncommutative Witt vectors  $\widehat{W}(A) = 1+zA[[z]]$ , and  $V(A) \subseteq K_1(\Sigma^{-1}A[z, z^{-1}])$  the image of the subgroup of the noncommutative rational Witt vectors  $W(A) \subseteq \widehat{W}(A)$  generated by  $1+zA[z]$ . (In [R] it was claimed that the natural surjections  $W(A)^{ab} \rightarrow V(A)$ ,  $\widehat{W}(A)^{ab} \rightarrow \widehat{V}(A)$  are isomorphisms, but an explicit counterexample was constructed in [PR]). The torsions of the chain equivalences  $p, \widehat{p}$  are such that  $\tau(p) \in V(A) \subseteq K_1(\Sigma^{-1}A[z, \alpha])$ ,  $\tau(\widehat{p}) \in \widehat{V}(A) \subseteq K_1(A((z)))$  respectively, with  $A = \mathbb{Z}[\pi_1(\overline{M})]$ .

(iii) The natural map  $\Sigma^{-1}A[z, z^{-1}] \rightarrow A((z))$  is injective for a commutative ring  $A$ , since in this case  $\Sigma^{-1}A[z, z^{-1}]$  is just the localization of  $A[z, z^{-1}]$  inverting all the elements of type  $1+az \in A[z, z^{-1}]$  ( $a \in A$ ). Sheiham [Sh] has constructed an example of a noncommutative ring  $A$  such that  $\Sigma^{-1}A[z, z^{-1}] \rightarrow A((z))$  is not injective.

(iv) Farber [F] extended the construction of Farber and Ranicki [FR] to obtain an algebraic Novikov complex for a closed 1-form, but did not relate it to the geometric Novikov complex.  $\square$

## References

- [C] P. M. Cohn, Free rings and their relations, 2nd edition. Academic Press 1985
- [CR] O. Cornea, A. A. Ranicki, Rigidity and glueing for the Morse and Novikov complexes. to appear, e-print math. AT/0107221
- [F] M. Farber, Morse-Novikov critical point theory, Cohn localization and Dirichlet units. e-print math.dg-ga/9911157, Commun. Contemp. Math **1** (1999), 467–495
- [FR] M. Farber, A. A. Ranicki, The Morse-Novikov theory of circle-valued functions and noncommutative localization. e-print math.dg-ga/9812122, Proc. 1998 Moscow Conference for the 60th Birthday of S. P. Novikov, Tr. Mat. Inst. Steklova **225** (1999), 381–388
- [Fr] J. Franks, Morse-Smale flows and homotopy theory. Topology **18** (1979), 199–215
- [HL] M. Hutchings, Y.-J. Lee, Circle-valued Morse theory and Reidemeister torsion. e-print math.dg-ga/9706012, Geom. Top. **3** (1999), 369–396
- [N] S. P. Novikov, The hamiltonian formalism and a multi-valued analogue of Morse theory. Uspeki Mat. **37** (1982), 3–49. English tr. Russian Math. Surveys **37** (1982), 1–56
- [P1] A. V. Pajitnov, On the Novikov complex for rational Morse forms. Annales de la Faculté de Sciences de Toulouse **4** (1995), 297–338
- [P2] A. V. Pajitnov, The incidence coefficients in the Novikov complex are generically rational functions. Algebra i Analiz **9** (1997), 92–139. English tr. St. Petersburg Math. J. **9** (1998), 969–1006
- [P3] A. V. Pajitnov, Simple homotopy type of Novikov complex for closed 1-forms and Lefschetz  $\zeta$ -function of the gradient flow. e-print math.dg-ga/9706014, Russian Math. Surveys **54** (1999), 117–170
- [P4] A. V. Pajitnov,  $C^0$ -generic properties of boundary operators in Novikov complex. e-print math.dg-ga/9812157, A.M.S. Translations **197** (1999), 29–117

- [P5] A. V. Pajitnov, Closed orbits of gradient flows and logarithms of non-abelian Witt vectors. e-print math.DG/9908010, *K-theory* **21** (2000), 301–324
- [P6] A. V. Pajitnov, Counting closed orbits of gradient flows of circle-valued maps. e-print math.AT/0104273
- [PR] A. V. Pajitnov, A. A. Ranicki, The Whitehead group of the Novikov ring. e-print math.AT/0012031, *K-theory* **21** (2000), 325–365
- [R] A. A. Ranicki, High dimensional knot theory. Springer Mathematical Monograph, Springer 1998
- [Scho] A. H. Schofield, Representations of rings over skew fields. L.M.S. Lecture Notes 92, Cambridge 1985
- [Sch1] D. Schütz, Gradient flows of closed 1-forms and their closed orbits. e-print math.DG/0009055, to appear in Forum Math
- [Sch2] D. Schütz, One parameter fixed point theory and gradient flows of closed 1-forms. e-print math.DG/0104245, to appear in *K-theory*
- [Sh] D. Sheiham Noncommutative characteristic polynomials and Cohn localisation. e-print math.RA/0104158, *J. Lond. Math. Soc.* 64, 13–28 (2001)