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Noncommutative localisation in algebraic K-theory II

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Abstract

In [Amnon Neeman, Andrew Ranicki, Noncommutative localisation in algebraic K-theory I, Geom. Topol. 8 (2004) 1385–1425] we proved a localisation theorem in the algebraic K-theory of noncommutative rings. The main purpose of the current article is to express the general theorem of the previous paper in a more user-friendly fashion, in a way more suitable for applications. In the process we compare our result to the existing theorems in the literature, showing how the previous paper improves all the existing results.

It should be pointed out that there have been two very interesting recent preprints on related topics. The reader is referred to the beautiful papers of Krause [Henning Krause, Cohomological quotients and smashing localizations, http://wwwmath.upb.de/~hkrause/publications.html. [8]] and Dwyer [William G. Dwyer, Noncommutative localization in homotopy theory, preprint, http://www.nd.edu/~wgd/. [4]]. Krause studies the lifting of chain complexes and the relation with the telescope conjecture, and Dwyer generalises to the homotopy theoretic framework.
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0. Introduction

This article is a sequel to [10]. We begin by briefly recalling the main result of [10]. Let A be an associative ring with unit. Let $\sigma = \{s_i : P_i \to Q_i\}$ be a set of homomorphisms of finitely generated, projective (left) A-modules. Let $\sigma^{-1}A$ be the Cohn localisation. The main theorem of [10] is a localisation theorem; it identifies the homotopy fiber of the natural map $K(A) \to K(\sigma^{-1}A)$. Actually, only up to nonsense in degree -1. The homotopy fiber is a spectrum which, in general, has a nonvanishing (-1)th homotopy group. The main theorem of [10] says nothing about π_{-1} of the fiber. What it really gives is the (-1)-connected cover of the homotopy fiber. In order to state the theorem precisely, we first remind the reader of our notation.

Let $A \to \sigma^{-1}A$ be as above. Let $C^{\mathrm{perf}}(A)$ be the Waldhausen category of perfect complexes of A-modules. The objects are the perfect complexes, the morphisms are the chain maps, the cofibrations are the chain maps which are split monomorphisms in each degree, and the weak equivalences are the homotopy equivalences. In $C^{\mathrm{perf}}(A)$, let $\mathbf{R}$ be generated by σ. That is, $\mathbf{R}$ is the smallest Waldhausen subcategory of $C^{\mathrm{perf}}(A)$ containing σ and all acyclic complexes, and closed under mapping cones and direct summands. The main K-theoretic result from [10] asserts:

Theorem 0.1. *Suppose that the localisation $\sigma^{-1}A$ is stably flat over A. We remind the reader: The localisation $A \to \sigma^{-1}A$ is stably flat if, for all $n > 0$,*

$$\operatorname{Tor}_n^A\left(\sigma^{-1}A, \sigma^{-1}A\right) = 0.$$

Then the (-1)-connected cover of the homotopy fiber of $K(A) \to K(\sigma^{-1}A)$ is naturally identified with $K(\mathbf{R})$.

This theorem gives, in great generality, a description of the (-1)-connected cover of the homotopy fiber of the map $K(A) \to K(\sigma^{-1}A)$. In practise one is often willing to sacrifice some generality to obtain a smaller, more manageable model for the homotopy fiber. The usual theorems have generally made simplifying assumptions about the set $\sigma = \{s_i : P_i \to Q_i\}$ of morphisms that become inverted. The usual assumption is that all the morphisms in σ are monomorphisms. Both to compare our results with older work, and more importantly in order to see what the theorem says in this very useful special case, we will study the situation that arises. If A is commutative and the maps in σ are all monomorphisms then Quillen showed (see Grayson [7, p. 229]) that the homotopy fiber of $K(A) \to K(\sigma^{-1}A)$ can be obtained as the Quillen K-theory of a relatively small and comprehensible exact category $\mathcal{E}$. We generalise this to the noncommutative case. This is probably the most important, useful result of the current article.

But it seems sensible to be through about comparing our result to older work. The article does this. It should be noted that the article is purely K-theoretic. The applications to L-theory will come separately, in [11]. In this article we compare the K-theory results that can be deduced from [10] with what exists in the literature. We show quite explicitly how the methods can be used to sharply improve three existing theorems in the K-theory literature.

The first is Weibel–Yao [14]. Weibel and Yao study the case where all the elements of σ are maps

$$s_i : A \to A.$$

In other words, in $\sigma = \{s_i : P_i \to Q_i\}$ we have $P_i = Q_i = A$ for every i. The elements of σ are 1×1 matrices, that is ordinary elements of A. Weibel and Yao [14, p. 220] prove a localisation theorem, under some extra hypotheses. The hypotheses of their theorem only hold if $\sigma^{-1}A$ is flat over A.

Let us note that if $\sigma^{-1}A$ is flat, either as a left or as a right A-module, then for all $n > 0$ the groups $\mathrm{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A)$ must vanish. In other words, flatness implies stable flatness. In Theorem 0.1 we see that stable flatness suffices for a localisation theorem. The first thing the reader might wish to see is examples of stably flat localisations which are not flat; this will establish that Theorem 0.1 is genuinely more powerful than Weibel and Yao [14].

We see such an example in Section 1. In the example, σ is even a set of 1×1 matrices. Explicitly, Section 1 will establish the easy

Example 0.2. Let k be a field, and let $A = k\langle x, y\rangle$ be the free associative algebra in two noncommuting variables x and y. Let σ consist of the single 1×1 matrix

$$x : A \longrightarrow A.$$

We assert that

(i) The ring $\sigma^{-1}A$ is not flat over A, either as a right or as a left module.
(ii) The ring $\sigma^{-1}A$ is stably flat over A.

In other words, even in the "classical" case where we only invert elements of the ring A (as opposed to matrices of them), Theorem 0.1 applies in situations not covered by Weibel and Yao [14]. For further examples, including localisations which are not even stably flat, see [12].

Next we come to the main result of the article, generalising the theorem of Quillen's. We begin by reminding the reader of Quillen's theorem; see Grayson [7, p. 229]. Grayson states the theorem in the language of algebraic geometry. If we assume that his scheme X is affine, with $X = \mathrm{Spec}(A)$, the theorem simplifies to

Theorem 0.3. *Let A be a commutative ring, and let $\sigma = \{a\}$ be a set containing one nonzero-divisor in A. Let T be the exact category of all finitely presented A-modules M such that*

(i) *There exists an integer $n > 0$ with $a^n M = 0$.*
(ii) *The module M has projective dimension $\leqslant 1$.*

Then the (-1)-connected cover of the homotopy fiber of the map $K(A) \to K(\sigma^{-1}A)$ is the Quillen K-theory $K(T)$.

The immediate question that springs to mind is what happens in the noncommutative situation. Suppose we are willing to assume that all the maps in σ are monomorphisms. We would like to define an exact category $\mathcal{E}$ of torsion modules for the pair (A, σ). And we would like the Quillen K-theory $K(\mathcal{E})$ to agree with the (-1)-connected cover of the homotopy fiber of the map $K(A) \to K(\sigma^{-1}A)$.

Definition 0.4. Assume all the maps in σ are monomorphisms. We define an exact category $\mathcal{E}$. It is a full subcategory of the category of all A-modules. All the objects in $\mathcal{E}$ are finitely presented A-modules, of projective dimension $\leqslant 1$. The category $\mathcal{E}$ is completely determined by

(i) For every $s_i : P_i \to Q_i$ in σ, the cokernel $M_i = Q_i/P_i$ lies in $\mathcal{E}$.
(ii) In any short exact sequence of finitely presented A-modules of projective dimension $\leqslant 1$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0,$$

if two of the objects M', M and M'' lie in $\mathcal{E}$ then so does the third.
(iii) $\mathcal{E}$ contains all direct summands of its objects.
(iv) $\mathcal{E}$ is minimal, subject to (i)–(iii).

If we want to remind ourselves that $\mathcal{E}$ depends on A and σ, we will write it as $\mathcal{E}(A, \sigma)$. If we wish to refer to it in words (as opposed to symbols), we will call it the category of (A, σ)-torsion modules. An object of $\mathcal{E}$ will be an (A, σ)-torsion module.

This defined the exact category $\mathcal{E} = \mathcal{E}(A, \sigma)$. To state the next theorem, we remind the reader of the notation of [10]. The Waldhausen category $\mathbf{R}$ has an associated derived category, which in [10] we denoted $D(\mathbf{R}) \cong \mathcal{R}^c$. Our next result is

Theorem 0.5. *Assume all the maps in σ are monomorphisms. With $\mathcal{E}$ as in Definition* 0.4 *and $\mathcal{R}^c \cong D(\mathbf{R})$ as in* [10]*, there is a natural equivalence of categories $D^b(\mathcal{E}) \cong \mathcal{R}^c$. The equivalence is compatible with choices of models, inducing a homotopy equivalence in K-theory*

$$K(\mathcal{E}) \cong K\big(C^b(\mathcal{E})\big) \cong K(\mathbf{R}).$$

(Proof in Section 2.)

Remark 0.6. It should be noted that in Theorem 0.5 we *do not* assume that $A \to \sigma^{-1}A$ is a stably flat localisation. The hypotheses are as stated in the theorem.

If $A \to \sigma^{-1}A$ happens to be stably flat, and all the maps in σ are monomorphisms, then Theorems 0.1 and 0.5 both apply. We combine our results: Theorem 0.1 tells us that $K(\mathbf{R})$ is the (-1)-connected cover of the homotopy fiber of the map $K(A) \to K(\sigma^{-1}A)$. And Theorem 0.5 says that $K(\mathbf{R})$ can be identified with $K(\mathcal{E})$. So far, we have identified the (-1)-connected cover of the homotopy fiber with the Quillen K-theory of some exact category $\mathcal{E}(A, \sigma)$ of torsion modules. All that remains is to show that, in the special case where A is commutative and $\sigma = \{a\} \subset A$, our more general category $\mathcal{E}$ specialises to Quillen's exact category T of Theorem 0.3. To this end, we note

Proposition 0.7. *An A-module M belongs to $\mathcal{E}$ if and only if*

(i) *M is finitely presented, and of projective dimension $\leqslant 1$.*
(ii) $\{\sigma^{-1}A\} \otimes_A M = 0 = \mathrm{Tor}_1^A(\sigma^{-1}A, M)$.

(Proof in Corollary 3.3.)

In the special case where A is commutative and $\sigma = \{a\}$, $\sigma^{-1}A$ is flat over A. Hence $\mathrm{Tor}_1^A(\sigma^{-1}A, M) = 0$. Since M is finitely generated, the module

$$\{\sigma^{-1}A\} \otimes_A M = A\left[\frac{1}{a}\right] \otimes_A M$$

vanishes if and only if there exists an integer $n > 0$ with $a^n M = 0$. Proposition 0.7 now tells us that $\mathcal{E}$ precisely agrees with Quillen's T. Hence Theorem 0.3 is a formal consequence of Theorems 0.5 and 0.1.

If $A \to \sigma^{-1}A$ is a stably flat localisation, then Theorem 0.1 gives a long exact sequence

$$\cdots \to K_2(\sigma^{-1}A) \to K_1(\mathbf{R}) \to K_1(A) \to K_1(\sigma^{-1}A) \to K_0(\mathbf{R}) \to K_0(A) \to K_0(\sigma^{-1}A).$$

If the localisation is not stably flat, the best result to date is Schofield's [13, Theorem 4.12, p. 60]. We remind the reader:

Theorem 0.8. *(Schofield). Suppose the morphism $A \to \sigma^{-1}A$ is injective. Then, with the exact category $\mathcal{E}$ of Definition* 0.4*, there is an exact sequence*

$$K_1(A) \longrightarrow K_1(\sigma^{-1}A) \longrightarrow K_0(\mathcal{E}) \longrightarrow K_0(A) \longrightarrow K_0(\sigma^{-1}A).$$

There is no hypothesis that $A \to \sigma^{-1}A$ is stably flat.

Remark 0.9. It should be noted that if $A \to \sigma^{-1}A$ is injective then all the maps in σ must be monomorphisms. This is well known, but we include a brief proof in Proposition 2.2. If $A \to \sigma^{-1}A$ is injective, then the category $\mathcal{E}$ of Definition 0.4 is well-defined. The assertion of Theorem 0.8 makes sense.

It is natural to ask if we can prove Schofield's theorem by our techniques, or maybe even improve on it. It turns out that we can. We will prove that, with no injectivity hypothesis on maps in σ, there is always an exact sequence

$$K_1(\mathbf{R}) \to K_1(A) \to K_1(\sigma^{-1}A) \to K_0(\mathbf{R}) \to K_0(A) \to K_0(\sigma^{-1}A).$$

If the maps in σ all happen to be monomorphisms then Theorem 0.5 tells us $K(\mathcal{E}) = K(\mathbf{R})$. We recover a longer version of Schofield's exact sequence; we have six terms, Schofield only had five. Even better: our theorem generalises to arbitrary σ.

The way our proof works is the following. In [10] we produced a diagram

$$\begin{array}{ccccccccccc}
K_1(\mathbf{R}) & \longrightarrow & K_1(A) & \longrightarrow & K_1(\mathbf{T}) & \longrightarrow & K_0(\mathbf{R}) & \longrightarrow & K_0(A) & \longrightarrow & K_0(\mathbf{T}) \\
 & & & & \Big\downarrow{\scriptstyle K_1(T)} & & & & & & \Big\downarrow{\scriptstyle K_0(T)} \\
 & & & & K_1(\sigma^{-1}A) & & & & & & K_0(\sigma^{-1}A).
\end{array}$$

By [10, Corollary 4.9] we know that the top row is exact. In fact, this exact sequence can be continued arbitrarily far to the left. What we prove here is:

Theorem 0.10. *The maps*

$$\begin{aligned} &K_1(T): K_1(\mathbf{T}) \longrightarrow K_1(\sigma^{-1}A) \\ \textit{and}\quad &K_0(T): K_0(\mathbf{T}) \longrightarrow K_0(\sigma^{-1}A) \end{aligned}$$

are both isomorphisms.

(Proof for K_0 in Section 5, for K_1 in Section 6.)

Up until now we have discussed how the results of [10] can be used to improve and generalise known localisation theorems in algebraic K-theory. Our next theorem says that the results are best possible by the techniques.

We first remind ourselves a little of the proof of the main theorem of [10]. In glorious generality we constructed a diagram of triangulated categories

$$\begin{array}{ccccccc}
\mathcal{R}^c & \longrightarrow & D^{\mathrm{perf}}(A) & \xrightarrow{\quad\pi\quad} & \mathcal{T}^c & \xrightarrow{\quad T\quad} & D^{\mathrm{perf}}(\sigma^{-1}A). \\
 & & & \searrow \qquad \nearrow{\scriptstyle i} & & & \\
 & & & \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} & & &
\end{array}$$

The category $\mathcal{T}^c$ in this diagram was defined to be the idempotent completion of $\frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c}$. The functor i is by definition fully faithful, and every object of $\mathcal{T}^c$ is a direct summand of an object in the image of i. The key result [10, Theorem 0.7], was that T is an equivalence of categories if and only if the localisation is stably flat. Compacting the diagram to

$$\begin{array}{ccccc}
\mathcal{R}^c & \longrightarrow & D^{\mathrm{perf}}(A) & \xrightarrow{\quad T\pi\quad} & D^{\mathrm{perf}}(\sigma^{-1}A) \\
 & & & \searrow \qquad \nearrow{\scriptstyle Ti} & \\
 & & & \frac{D^{\mathrm{perf}}(A)}{\mathcal{R}^c} &
\end{array}$$

the key result can be restated to say that the localisation is stably flat if and only if the functor Ti is fully faithful, and every object of $D^{\mathrm{perf}}(\sigma^{-1}A)$ is a direct summand of an object in the image of Ti. The K-theoretic consequences follow formally from results of Waldhausen, Grayson and

Gillet, after we make a reasonable choice of Waldhausen models. Part of the work in [10] was to establish that a suitable choice of models can be made.

Still at the level of triangulated categories, it is not *a priori* clear that $\mathcal{R}^c$ is the best possible choice. We could consider other subcategories $\mathcal{K} \subset D^{\text{perf}}(A)$. It is conceivable that, for some other choice of $\mathcal{K} \subset D^{\text{perf}}(A)$, the natural map

$$\frac{D^{\text{perf}}(A)}{\mathcal{K}} \xrightarrow{\ \nu\ } D^{\text{perf}}\big(\sigma^{-1}A\big)$$

will be fully faithful, and that every object in $D^{\text{perf}}(\sigma^{-1}A)$ will be isomorphic to a direct summand of something in the image of ν. Our next theorem asserts that the only choice for $\mathcal{K}$ which has any chance of working is $\mathcal{K} = \mathcal{R}^c$. As we already know, $\mathcal{K} = \mathcal{R}^c$ only works when the localisation is stably flat.

If the functor

$$\frac{D^{\text{perf}}(A)}{\mathcal{K}} \xrightarrow{\ \nu\ } D^{\text{perf}}\big(\sigma^{-1}A\big)$$

is to be fully faithful, then the objects that map to zero by the composite

$$D^{\text{perf}}(A) \xrightarrow{\ \mu\ } \frac{D^{\text{perf}}(A)}{\mathcal{K}} \xrightarrow{\ \nu\ } D^{\text{perf}}\big(\sigma^{-1}A\big)$$

are precisely those in $\mathcal{K} \subset D^{\text{perf}}(A)$. This is only of interest if the composite $\nu\mu$ induces the natural map $K(A) \to K(\sigma^{-1}A)$. The natural functor that does this is $T\pi : D^{\text{perf}}(A) \to D^{\text{perf}}(\sigma^{-1}A)$. We remind the reader that $T\pi$ is the functor taking X to $\{\sigma^{-1}A\}^L \otimes_A X$. To show that $\mathcal{K} = \mathcal{R}^c$ is the only possibility, we prove

Theorem 0.11. *Let X be an object of $D^{\text{perf}}(A)$. Then $T\pi(X) \cong 0$ if and only if $X \in \mathcal{R}^c$. In the notation above $\mathcal{K} = \mathcal{R}^c$ is the only possible choice. Our K-theoretic results are optimal by the methods.*

(Proof in Proposition 3.2.)

This ends the summary of the theorems in the article. The results are mostly independent of each other, and the order in which the proofs are presented is a little arbitrary. We did try to put near the beginning the results which are relevant to the next article in this series, about L-theory. The results of Sections 5 and 6 play no role in the L-theory work. For this reason they are at the end. The technical lemmas of Section 4 are irrelevant to Section 5 but are needed in Section 6. From a logical point of view it might have been better to put them between the two sections. But, since these technical lemmas also play a role in L-theory, we placed them before the last two sections. This way an L-theorist can safely skip the last two sections of the article.

1. The comparison with Weibel–Yao

In this section we will establish the very easy Example 0.2. We remind the reader. Let k be any field. The ring A is the free associative algebra $k\langle x, y\rangle$ in two variables x and y. The set σ contains the singleton $\{x : A \to A\}$. We wish to prove that

(i) $\sigma^{-1}A$ is not flat over A, either as a right or as a left A-module.
(ii) $\sigma^{-1}A$ is stably flat over A.

Let us prove (ii) first. We know from [3, Corollary 2, p. 68] that the ring $A = k\langle x, y\rangle$ is hereditary; that is, every module has projective dimension $\leqslant 1$. It follows that $\operatorname{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A)$ must vanish whenever $n > 1$. The fact that $\operatorname{Tor}_1^A(\sigma^{-1}A, \sigma^{-1}A)$ vanishes may be found in Schofield [13, p. 58], or in [10, Lemma 8.6(ii)]. Putting this together we have that $\operatorname{Tor}_n^A(\sigma^{-1}A, \sigma^{-1}A)$ vanishes for all $n \geqslant 1$, that is the localisation is stably flat.

Next we prove (i). By symmetry it suffices to show that $\sigma^{-1}A$ is not flat as a right A-module. Put $k = A/(Ax + Ay)$. Since the intersection of the left ideals Ax and Ay is trivial, we have a free resolution for k

$$0 \longrightarrow A \oplus A \xrightarrow{(x,y)} A \longrightarrow k \longrightarrow 0.$$

Tensoring this with $\sigma^{-1}A$ we deduce an exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^A\big(\sigma^{-1}A, k\big) \longrightarrow \sigma^{-1}A \oplus \sigma^{-1}A \xrightarrow{(x,y)} \sigma^{-1}A \longrightarrow \sigma^{-1}A \otimes_A k \longrightarrow 0.$$

By the choice of σ, the map $x : \sigma^{-1}A \to \sigma^{-1}A$ is an isomorphism. The kernel of the map $(x, y) : \sigma^{-1}A \oplus \sigma^{-1}A \to \sigma^{-1}A$ therefore identifies as $\sigma^{-1}A$. Thus

$$\operatorname{Tor}_1^A\big(\sigma^{-1}A, k\big) = \sigma^{-1}A,$$

and $\sigma^{-1}A$ is not flat as a right A-module.

2. Torsion modules

Hypothesis 2.1. *In this section, we assume that all the morphisms in σ are injections.*

The following result is well known; since the proof is so easy we include it.

Proposition 2.2. *If $A \to \sigma^{-1}A$ is an injection then every $s_i : P_i \to Q_i$ in σ is an injection, i.e. Hypothesis* 2.1 *is satisfied.*

Proof. Since $A \to \sigma^{-1}A$ is a monomorphism and P_i is projective and therefore flat, we deduce that

$$A \otimes_A P_i \longrightarrow \big\{\sigma^{-1}A\big\} \otimes_A P_i$$

is a monomorphism. Abbreviate $\{\sigma^{-1}A\} \otimes_A P_i$ as $\sigma^{-1}P_i$. Then the above says that the map $P_i \to \sigma^{-1}P_i$ is mono. Consider the commutative diagram

$$\begin{array}{ccc} P_i & \xrightarrow{s_i} & Q_i \\ \downarrow & & \downarrow \\ \sigma^{-1}P_i & \xrightarrow{\sigma^{-1}s_i} & \sigma^{-1}Q_i. \end{array}$$

By the above, $P_i \to \sigma^{-1}P_i$ is an injection. Now $\sigma^{-1}s_i : \sigma^{-1}P_i \to \sigma^{-1}Q_i$ is an isomorphism. From the commutativity of the square we deduce that $s_i : P_i \to Q_i$ is an injection. $\square$

Example 2.3. The converse of Proposition 2.2 does not hold in general. The set $\sigma = \{0 \to A\}$ satisfies Hypothesis 2.1, but $A \to \sigma^{-1}A = 0$ is not injective.

In the rest of this section we will always assume that Hypothesis 2.1 holds. That is, all the maps $s_i : P_i \to Q_i$ are injective. Then all the cokernels $M_i = Q_i/P_i$ fit in short exact sequences

$$0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow M_i \longrightarrow 0.$$

The modules M_i are all finitely presented, and all have projective dimension $\leqslant 1$. In Definition 0.4 we let $\mathcal{E} = \mathcal{E}(A, \sigma)$ be the smallest exact subcategory of the category of all finitely presented A-modules of projective dimension $\leqslant 1$, which contains the M_i above and is closed under short exact sequences and direct summands. We remind the reader

Definition 2.4. The bounded derived category of the exact category $\mathcal{E}$, denoted $D^b(\mathcal{E})$, is defined as follows. The objects are bounded chain complexes of objects of $\mathcal{E}$. The morphisms are obtained from the chain maps by formally inverting the maps whose mapping cones are acyclic (as complexes of A-modules). There is an obvious functor $i : D^b(\mathcal{E}) \to D(A)$.

Lemma 2.5. *The functor $i : D^b(\mathcal{E}) \to D(A)$ is fully faithful.*

Proof. Let us begin by showing that, for any objects $M, N \in \mathcal{E}$ and any $n \in \mathbb{Z}$,

$$\{D^b(\mathcal{E})\}(M, \Sigma^n N) = \{D(A)\}(M, \Sigma^n N).$$

Take a map in $D^b(\mathcal{E})$ of the form $M \to \Sigma^n N$. There exists a bounded complex X of objects in $\mathcal{E}$, a quasi-isomorphism $g : X \to M$, and a map of complexes $f : X \to \Sigma^n N$ so that our map is fg^{-1}. That is, we have a complex

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \xrightarrow{\partial_0} X^1 \longrightarrow \cdots.$$

There is a quasi-isomorphism $X \to M$; in particular $H^0(X) = M$. We have an exact sequence

$$\longrightarrow X^{-1} \longrightarrow \ker(\partial_0) \longrightarrow M \longrightarrow 0.$$

But $M \in \mathcal{E}$ means that M is of projective dimension $\leqslant 1$. There is an exact sequence

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow M \longrightarrow 0$$

with P and Q (finitely generated) and projective A-modules. Since P and Q are projective A-modules, there exists a map

$$\begin{array}{ccccccc} P & \longrightarrow & Q & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \\ X^{-1} & \longrightarrow & \ker(\partial_0) & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

Let Z be given by the pushout square

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \downarrow & & \downarrow \\ X^{-1} & \longrightarrow & Z. \end{array}$$

In the short exact sequence

$$0 \longrightarrow X^{-1} \longrightarrow Z \longrightarrow M \longrightarrow 0$$

we are given that X^{-1} and M lie in $\mathcal{E}$. It immediately follows that Z is finitely presented and of projective dimension $\leqslant 1$. Definition 0.4(ii) now establishes that $Z \in \mathcal{E}$. The short exact sequence also tells us that the complex $0 \to X^{-1} \to Z \to 0$ is quasi-isomorphic to M. We deduce a quasi-isomorphism $h : X' \to X$ of complexes, given below:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & X^{-1} & \longrightarrow & Z & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & X^{-2} & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & X^1 & \longrightarrow & \cdots. \end{array}$$

It follows that the map $fg^{-1} : M \to \Sigma^n N$ is equal to the map $\{fh\}\{gh\}^{-1}$. Since X' is concentrated in degrees -1 and 0, it follows that fh vanishes unless $n = 0$ or 1. Unless $n = 0$ or 1, we have proved that $\{D^b(\mathcal{E})\}(M, \Sigma^n N)$ vanishes. As for

$$\{D(A)\}\big(M, \Sigma^n N\big) = \operatorname{Ext}^n_A(M, N),$$

it must vanish since the projective dimension of M is $\leqslant 1$. In other words, for $n \neq 0, 1$ the equality

$$\{D^b(\mathcal{E})\}\big(M, \Sigma^n N\big) = \{D(A)\}\big(M, \Sigma^n N\big)$$

is just because both sides vanish.

We leave to the reader to check that the two sides are equal also when $n = 0$ or 1. For $n = 0$ both sides identify as $\mathcal{E}(M, N)$, while for $n = 1$ both sides identify as $\mathrm{Ext}^1_A(M, N)$.

Let M be an object of $\mathcal{E}$. Consider next the full subcategory $\mathcal{B} \subset D^b(\mathcal{E})$ defined by

$$\mathrm{Ob}(\mathcal{B}) = \bigl\{Y \in \mathrm{Ob}\bigl(D^b(\mathcal{E})\bigr) \bigm| \forall n \in \mathbb{Z},\ \bigl\{D^b(\mathcal{E})\bigr\}\bigl(M, \Sigma^n Y\bigr) \to \bigl\{D(A)\bigr\}\bigl(M, \Sigma^n Y\bigr) \text{ is an isomorphism}\bigr\}.$$

By the above $\mathcal{B}$ contains $\mathcal{E}$, and clearly $\mathcal{B}$ is triangulated. Hence $\mathcal{B}$ contains all of $D^b(\mathcal{E})$. Next, take any Y in $D^b(\mathcal{E})$, and consider the full subcategory $\mathcal{C} \subset D^b(\mathcal{E})$ given by

$$\mathrm{Ob}(\mathcal{C}) = \bigl\{X \in \mathrm{Ob}\bigl(D^b(\mathcal{E})\bigr) \bigm| \forall n \in \mathbb{Z},\ \bigl\{D^b(\mathcal{E})\bigr\}\bigl(X, \Sigma^n Y\bigr) \to \bigl\{D(A)\bigr\}\bigl(X, \Sigma^n Y\bigr) \text{ is an isomorphism}\bigr\}.$$

By the above $\mathcal{E} \subset \mathcal{C}$, and $\mathcal{C}$ is clearly triangulated. Hence $\mathcal{C}$ contains $D^b(\mathcal{E})$. □

Lemma 2.6. *Assume that maps in σ are all injections. The natural map $D^b(\mathcal{E}) \to D(A)$ factors through $\mathcal{R}^c \subset D(A)$, and the induced map $D^b(\mathcal{E}) \to \mathcal{R}^c$ is an equivalence of categories.*

Proof. Let E be the full subcategory of $\mathcal{S} = D(A)$ containing all objects isomorphic to objects in the image of $i : D^b(\mathcal{E}) \to D(A)$. By Lemma 2.5, E is a full, triangulated subcategory of $\mathcal{S} = D(A)$. If M_i fits in an exact sequence

$$0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow M_i \longrightarrow 0$$

with $s_i \in \sigma$, then $M_i \in \mathcal{E} \subset D^b(\mathcal{E})$. Thus M_i, which is quasi-isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow 0 \longrightarrow \cdots$$

lies in $D^b(\mathcal{E}) \subset E \subset D(A)$. Since E contains any isomorph of any of its objects, σ is contained in $E \subset D(A)$.

From [9, Lemma 2.2] we know that $\mathcal{R}^c \subset D(A)$ is the smallest thick subcategory containing σ. That is, $\mathcal{R}^c$ is the smallest triangulated subcategory of $D(A)$, containing σ and closed under direct summands. We know that $D^b(\mathcal{E})$ is equivalent to E, and E contains σ. We want to show that $E = \mathcal{R}^c$. All we need for this is to prove that E is closed under direct summands and minimal. The minimality is clear, from the minimality of $\mathcal{E}$. We need to show that E is closed under direct summands.

But Definition 0.4(iii) tells us that every idempotent in $\mathcal{E}$ splits. Theorem 2.8 of Balmer and Schlichting's [1] allows us to deduce that $D^b(\mathcal{E})$ is idempotent complete; all direct summands of objects in $D^b(\mathcal{E})$ lie in $D^b(\mathcal{E})$ (up to isomorphism). Hence the equivalent category E is also closed under direct summands. □

Theorem 2.7. *Suppose every morphism in σ is injective. Then the Waldhausen K-theory of the Waldhausen category* **R** *is isomorphic to the Quillen K-theory of the exact category $\mathcal{E}$.*

Proof. By Lemma 2.6, the natural map $D^b(\mathcal{E}) \to D(A)$ induces a triangulated equivalence of $D^b(\mathcal{E})$ with $\mathcal{R}^c$. In order to turn this into a statement in K-theory we need to choose models wisely.

Let $C'(A)$ be the following Waldhausen model category. The objects are bounded chain complexes of A-modules, where every module in the chain complex is finitely presented and of projective dimension $\leqslant 1$. The morphisms are the chain maps. The cofibrations are the degreewise split monomorphisms, and the weak equivalences are the homology isomorphisms. There is an inclusion functor of Waldhausen categories

$$C^{\mathrm{perf}}(A) \longrightarrow C'(A).$$

On the level of derived categories it induces an equivalence $D^{\mathrm{perf}}(A) \to D'(A)$. The inclusion $\mathbf{R} \hookrightarrow C^{\mathrm{perf}}(A)$ induces a fully faithful triangulated functor $D(\mathbf{R}) \to D^{\mathrm{perf}}(A) \simeq D'(A)$. We define $\mathbf{R}' \subset C'(A)$ to be the full Waldhausen subcategory of all objects which become isomorphic in $D'(A)$ to objects in the image of $D(\mathbf{R}) \simeq \mathcal{R}^c$. By [10, Lemma 2.5], the natural map $D(\mathbf{R}) \to D(\mathbf{R}')$ is an equivalence.

Let $C^b(\mathcal{E})$ be the following Waldhausen model category. The objects are bounded chain complexes of objects of $\mathcal{E}$. The morphisms are the chain maps. The cofibrations are the degreewise split monomorphisms, and the weak equivalences are the homology isomorphisms. The natural inclusions give maps of Waldhausen model categories

$$\begin{array}{ccc} & & C^b(\mathcal{E}) \\ & & \downarrow \\ \mathbf{R} & \longrightarrow & \mathbf{R}'. \end{array}$$

At the level of derived categories we get equivalences

$$\begin{array}{ccc} & & D^b(\mathcal{E}) \\ & & \downarrow \wr \\ D(\mathbf{R}) & \xrightarrow{\sim} & D(\mathbf{R}'). \end{array}$$

By [10, Theorem 2.2] we deduce homotopy equivalences of K-theory spectra

$$\begin{array}{ccc} & & K\big(C^b(\mathcal{E})\big) \\ & & \downarrow \wr \\ K(\mathbf{R}) & \xrightarrow{\sim} & K(\mathbf{R}'). \end{array}$$

Now Gillet's [5, 6.2] tells us that the natural map $K(\mathcal{E}) \to K(C^b(\mathcal{E}))$ is a homotopy equivalence, completing the construction of a homotopy equivalence $K(\mathcal{E}) \simeq K(\mathbf{R})$. $\square$

3. The kernel of the tensor product

The main aim of this section is to prove Theorem 0.11, which we will do in Proposition 3.2. As a corollary we will also deduce Proposition 0.7. But first it might help to remind the reader of our notation.

Recall [10, Lemma 5.3]. The tensor product defines a functor $D(A) \to D(\sigma^{-1}A)$. It is the functor

$$X \mapsto \{\sigma^{-1}A\}^L \otimes_A X.$$

In [10, Lemma 5.3] we proved that this functor has a natural factorisation

$$D(A) \xrightarrow{\pi} \mathcal{T} \xrightarrow{T} D(\sigma^{-1}A).$$

Furthermore, we proved that the functor T takes compacts to compacts. We have induced functors

$$D^{\mathrm{perf}}(A) \xrightarrow{\pi} \mathcal{T}^c \xrightarrow{T} D^{\mathrm{perf}}(\sigma^{-1}A).$$

The main results of this section are the following statements, which are slight variants of each other.

(i) Let X be an object of $D^{\mathrm{perf}}(A)$. We have $T\pi(X) \cong 0$ if and only if $X \in \mathcal{R}^c$. (See Proposition 3.2.)
(ii) Let t be an object of $\mathcal{T}^c$. If $T(t) \cong 0$ then $t \cong 0$. (See Proposition 3.4.)

For the sake of compactness of notation, we will adopt the abbreviation

$$\sigma^{-1}P = \{\sigma^{-1}A\} \otimes_A P$$

whenever P is a projective A-module. As in [10], we abbreviate $D^{\mathrm{perf}}(A) = \mathcal{S}^c$. And we introduce the new shorthand, for this section,

$$\mathcal{D} = D(\sigma^{-1}A).$$

Proposition 3.1. *For any two finitely generated projective A-modules P and Q one has*

$$\mathcal{T}^c(\pi P, \pi Q) = \mathcal{D}(T\pi P, T\pi Q) = \mathrm{Hom}_{\sigma^{-1}A}(\sigma^{-1}P, \sigma^{-1}Q).$$

Proof. The identity $\mathcal{D}(T\pi P, T\pi Q) = \mathrm{Hom}_{\sigma^{-1}A}(\sigma^{-1}P, \sigma^{-1}Q)$ is almost by definition. After all, $T\pi$ is the functor taking P to $\sigma^{-1}P = \{\sigma^{-1}A\} \otimes_A P$. And the inclusion of the category of $\sigma^{-1}A$-modules into its derived category $\mathcal{D} = D(\sigma^{-1}A)$ is fully faithful;

$$\mathcal{D}(T\pi P, T\pi Q) = \mathcal{D}(\sigma^{-1}P, \sigma^{-1}Q) = \mathrm{Hom}_{\sigma^{-1}A}(\sigma^{-1}P, \sigma^{-1}Q).$$

There is a natural map, induced by the functor T,

$$\mathcal{T}^c(\pi P, \pi Q) \longrightarrow \mathcal{D}(T\pi P, T\pi Q).$$

We need to prove it an isomorphism. The case where $P = Q = A$ is easy; we have

$$\begin{aligned}\mathcal{T}^c(\pi A, \pi A) &= \bigl\{\sigma^{-1}A\bigr\}^{\text{op}} \qquad \text{by [10, Theorem 7.4]}\\ &= \mathcal{D}\bigl(\sigma^{-1}A, \sigma^{-1}A\bigr)\\ &= \mathcal{D}(T\pi A, T\pi A).\end{aligned}$$

But the collection of all P and Q for which the map $T : \mathcal{T}^c(\pi P, \pi Q) \to \mathcal{D}(T\pi P, T\pi Q)$ is an isomorphism is clearly closed under finite direct sums and direct summands, and hence contains all the finitely generated projective modules. □

Proposition 3.2. *For any object X in $D^{\text{perf}}(A)$, we have the implication*

$$\Bigl\{\bigl\{\sigma^{-1}A\bigr\}^L \otimes_R X = 0\Bigr\} \quad \Leftrightarrow \quad \bigl\{X \in \mathcal{R}^c\bigr\}.$$

Proof. If $X \in \mathcal{R}^c$ then $\pi X = 0$, hence $\{\sigma^{-1}A\}^L \otimes_A X = T\pi X = 0$. The nontrivial statement is the converse. We need to prove that if $T\pi X = 0$ then $X \in \mathcal{R}^c$.

Take any $X \in D^{\text{perf}}(A)$. That is, X is a bounded complex of finitely generated projective A-modules. Up to suspension, X may be written as a complex

$$\to 0 \to X^0 \to X^1 \to \cdots \to X^{n-1} \to X^n \to 0 \to$$

and all the X^i are finitely generated and projective. Assume $\{\sigma^{-1}A\}^L \otimes_R X = 0$. We need to prove that $X \in \mathcal{R}^c$. But $\{\sigma^{-1}A\}^L \otimes_R X = 0$ means that the complex

$$\to 0 \to \sigma^{-1}X^0 \to \sigma^{-1}X^1 \to \cdots \to \sigma^{-1}X^{n-1} \to \sigma^{-1}X^n \to 0 \to$$

must be contractible. There are maps $\sigma^{-1}X^i \to \sigma^{-1}X^{i-1}$ so that, for each i, the sum of the two composites

$$\begin{array}{ccc}\sigma^{-1}X^i & \longrightarrow & \sigma^{-1}X^{i+1}\\ \downarrow & & \downarrow\\ \sigma^{-1}X^{i-1} & \longrightarrow & \sigma^{-1}X^i\end{array}$$

is the identity on $\sigma^{-1}X^i$. By Proposition 3.1, the contracting homotopy may be lifted to the complex

$$\to 0 \to \pi X^0 \xrightarrow{\partial} \pi X^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} \pi X^{n-1} \xrightarrow{\partial} \pi X^n \to 0 \to .$$

For each i there are maps $D : \pi X^i \to \pi X^{i-1}$, so that the two composites

$$\begin{array}{ccc} \pi X^i & \xrightarrow{\partial} & \pi X^{i+1} \\ {\scriptstyle D}\downarrow & & \downarrow{\scriptstyle D} \\ \pi X^{i-1} & \xrightarrow{\partial} & \pi X^i \end{array}$$

add to the identity on πX^i.

Now let $Y^i \in D^{\mathrm{perf}}(A)$ be the complex

$$\to 0 \to X^0 \to X^1 \to \cdots \to X^{i-1} \to X^i \to 0 \to .$$

For each i, there is a triangle

$$\Sigma^{-i-1} X^{i+1} \longrightarrow Y^{i+1} \longrightarrow Y^i \longrightarrow \Sigma^{-i} X^{i+1}.$$

The functor π is triangulated, and hence for each i we deduce a triangle

$$\Sigma^{-i-1}\pi X^{i+1} \longrightarrow \pi Y^{i+1} \longrightarrow \pi Y^i \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1}.$$

We shall prove, by induction on i, that

(i) The map

$$\pi Y^i \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1}$$

is a split monomorphism in $\mathcal{T}^c$.

(ii) For each i we shall produce an explicit splitting; that is, we shall produce a map

$$\Sigma^{-i}\pi X^{i+1} \xrightarrow{\theta_i} \pi Y^i$$

so that $\theta_i \rho_i$ is the identity on πY^i.

(iii) $1 - \rho_i\theta_i$ is an endomorphism of $\Sigma^{-i}\pi X^{i+1}$. We shall show it to be the composite

$$\Sigma^{-i}\pi X^{i+1} \xrightarrow{\Sigma^{-i}\partial} \Sigma^{-i}\pi X^{i+2} \xrightarrow{\Sigma^{-i}D} \Sigma^{-i}\pi X^{i+1}$$

with ∂ and D as above, satisfying $1 = D\partial + \partial D$.

Note that for $i < -1$, $X^{i+1} = Y^i = 0$, and there is nothing to do. We may assume that (i)–(iii) hold for some i. We only need to show the induction step; that is, if it holds for i then it holds also for $i + 1$.

It is easy to compute, in the derived category $D(A)$, the composite $\alpha\beta$, with α and β the morphisms in the triangles below

$$\Sigma^{-i-1}X^{i+1} \xrightarrow{\beta} Y^{i+1} \longrightarrow Y^{i} \longrightarrow \Sigma^{-i}X^{i+1},$$

$$\Sigma^{-i-2}X^{i+2} \longrightarrow Y^{i+2} \longrightarrow Y^{i+1} \xrightarrow{\alpha} \Sigma^{-i-1}X^{i+2}.$$

The morphism $\alpha\beta$ is just Σ^{-i-1} applied to the differential $\partial : X^{i+1} \to X^{i+2}$. Applying the functor π we conclude the following. By the part (iii) of the induction hypothesis, the composite

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\Sigma^{-i-1}\partial} \Sigma^{-i-1}\pi X^{i+2} \xrightarrow{\Sigma^{-i-1}D} \Sigma^{-i-1}\pi X^{i+1}$$

is equal to $1 - \Sigma^{-1}(\rho_i\theta_i)$. By the above, it factors further as

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \xrightarrow{\pi\alpha} \Sigma^{-i-1}\pi X^{i+2} \xrightarrow{\Sigma^{-i-1}D} \Sigma^{-i-1}\pi X^{i+1}.$$

Now look at the longer composite

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \xrightarrow{\{\Sigma^{-i-1}D\}\circ(\pi\alpha)} \Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1}.$$

It is equal to $(\pi\beta)[1 - \Sigma^{-1}(\rho_i\theta_i)]$. The distinguished triangle

$$\Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \longrightarrow \pi Y^{i} \xrightarrow{\rho_i} \Sigma^{-i}\pi X^{i+1},$$

coupled with the fact that ρ_i is a split monomorphism, guarantees that the triangle is really a split exact sequence in $\mathcal{T}$

$$0 \longrightarrow \Sigma^{-1}\pi Y^{i} \xrightarrow{\Sigma^{-1}\rho_i} \Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi\Sigma Y^{i+1} \longrightarrow 0.$$

But then $\pi\beta$ is a split epimorphism, and its composite with $\Sigma^{-1}\rho_i$ vanishes. From the vanishing of $\{\pi\beta\}\{\Sigma^{-1}\rho_i\}$ it follows that

$$(\pi\beta)\big\{\Sigma^{-i-1}D\big\}(\pi\alpha)(\pi\beta) = (\pi\beta)\big[1 - \Sigma^{-1}(\rho_i\theta_i)\big] = \pi\beta.$$

Hence

$$\big[1 - (\pi\beta)\circ\big\{\Sigma^{-i-1}D\big\}\circ(\pi\alpha)\big](\pi\beta) = 0.$$

Since $\pi\beta$ is a split epimorphism, we conclude that

$$(\pi\beta)\circ\big\{\Sigma^{-i-1}D\big\}\circ(\pi\alpha) = 1.$$

But $\pi\alpha : \pi Y^{i+1} \to \Sigma^{-i-1}\pi X^{i+2}$ is nothing other than the map ρ_{i+1}, and if we put $\theta_{i+1} = (\pi\beta) \circ \{\Sigma^{-i-1}D\}$, then we have proved parts (i) and (ii) for $i+1$.

It only remains to establish (iii). By construction, $\rho_{i+1}\theta_{i+1}$ is given by the composite

$$\Sigma^{-i-1}\pi X^{i+2} \xrightarrow{\Sigma^{-i-1}D} \Sigma^{-i-1}\pi X^{i+1} \xrightarrow{\pi\beta} \pi Y^{i+1} \xrightarrow{\pi\alpha} \Sigma^{-i-1}\pi X^{i+2},$$

which is nothing other than $\Sigma^{-i-1}(\partial D)$. Hence this equals $1 - \Sigma^{-i-1}(D\partial)$.

This completes the induction. Now choose $i > n$. The complex Y^i is nothing other than $X \in D^{\mathrm{perf}}(A)$, and by (i) we conclude that πX is a direct summand of $\pi X^{i+1} = 0$. It follows that $\pi X = 0$, as an object of $\mathcal{S}^c/\mathcal{R}^c \subset \mathcal{T}^c$. This forces $X \in \mathcal{R}^c$. $\square$

Corollary 3.3. *Suppose all the morphisms in σ are monomorphisms, so that the exact category $\mathcal{E}$ of Definition* 0.4 *makes sense. If M is an A-module, then M lies in $\mathcal{E}$ if and only if*

(i) *M is finitely presented, and of projective dimension $\leqslant 1$.*
(ii) $\{\sigma^{-1}A\} \otimes_A M = 0 = \mathrm{Tor}_1^A(\sigma^{-1}A, M)$.

Proof. By Proposition 3.2, $\{\sigma^{-1}A\}^L \otimes_A X = 0$ if and only if $X \in \mathcal{R}^c$. Since all the morphisms in σ are monomorphisms, Lemma 2.6 asserts that $\mathcal{R}^c \simeq D^b(\mathcal{E})$. Combining the results, $\{\sigma^{-1}A\}^L \otimes_A X = 0$ if and only if X is isomorphic to an object in $D^b(\mathcal{E})$.

If M is an object of $\mathcal{E} \subset D^b(\mathcal{E})$, then by Definition 0.4 M is finitely presented and of projective dimension $\leqslant 1$, and by the above $\{\sigma^{-1}A\}^L \otimes_A M = 0$. That is, $\{\sigma^{-1}A\} \otimes_A M = 0 = \mathrm{Tor}_1^A(\sigma^{-1}A, M)$. We need to prove the converse.

Suppose therefore that M is a finitely presented A-module of projective dimension $\leqslant 1$, and that $\{\sigma^{-1}A\} \otimes_A M = 0 = \mathrm{Tor}_1^A(\sigma^{-1}A, M)$. We need to show that $M \in \mathcal{E}$. Because M is of projective dimension $\leqslant 1$, we have $\mathrm{Tor}_n^A(\sigma^{-1}A, M) = 0$ for all $n > 1$. By hypothesis $\{\sigma^{-1}A\} \otimes_A M = 0 = \mathrm{Tor}_1^A(\sigma^{-1}A, M)$. It follows that $\{\sigma^{-1}A\}^L \otimes_A M = 0$. By the first paragraph of the proof we deduce that M is quasi-isomorphic to an object in $D^b(\mathcal{E})$. There is a chain complex in $\mathcal{E}$

$$0 \to X^{-m} \to \cdots \to X^{-1} \to X^0 \to X^1 \to \cdots \to X^n \to 0$$

whose only cohomology is M in degree 0. For $i \geqslant 0$ define K^i to be the kernel of $X^i \to X^{i+1}$. We know that $K^n = X^n \in \mathcal{E}$. For each $i \geqslant 0$ we have short exact sequences

$$0 \longrightarrow K^i \longrightarrow X^i \longrightarrow K^{i+1} \longrightarrow 0.$$

If $K^{i+1} \in \mathcal{E}$, then we reason that both K^{i+1} and X^i lie in $\mathcal{E}$, hence are finitely presented and of projective dimension $\leqslant 1$. The exact sequence says first that K^i is finitely presented and of projective dimension $\leqslant 1$. But then Definition 0.4(ii) tells us that $K^i \in \mathcal{E}$. By descending induction we conclude $K^i \in \mathcal{E}$ for all $i \geqslant 0$. In particular $K^0 \in \mathcal{E}$.

For all $i \geqslant 0$ define I^{-i} to be the image of $X^{-i-1} \to X^{-i}$. We have a short exact sequence

$$0 \longrightarrow I^0 \longrightarrow K^0 \longrightarrow M \longrightarrow 0.$$

We know that M is finitely presented and of projective dimension $\leqslant 1$, while $K^0 \in \mathcal{E}$ (hence also finitely presented and of projective dimension $\leqslant 1$). It follows that I^0 is finitely presented and of projective dimension $\leqslant 1$. The short exact sequences

$$0 \longrightarrow I^{-i} \longrightarrow X^{-i} \longrightarrow I^{-i+1} \longrightarrow 0$$

tell us that if I^{-i+1} is finitely presented and of projective dimension $\leqslant 1$, then so is I^{-i} (because $X^{-i} \in \mathcal{E}$ must have the property). By descending induction all the modules I^{-i} must be finitely presented and of projective dimension $\leqslant 1$. But $I^{-m} = 0 \in \mathcal{E}$. The exact sequences

$$0 \longrightarrow I^{-i} \longrightarrow X^{-i} \longrightarrow I^{-i+1} \longrightarrow 0,$$

coupled with Definition 0.4(ii), tell us that if $I^{-i} \in \mathcal{E}$ then $I^{-i+1} \in \mathcal{E}$. Ascending induction allows us to conclude that all the $I^{-i} \in \mathcal{E}$. In particular $I^0 \in \mathcal{E}$.

But now in the exact sequence

$$0 \longrightarrow I^0 \longrightarrow K^0 \longrightarrow M \longrightarrow 0$$

we have I^0 and K^0 both lying in $\mathcal{E}$. Hence $M \in \mathcal{E}$. □

The following is a technical improvement of Proposition 3.2 which we will need in Section 5.

Proposition 3.4. *Suppose t is an object in $\mathcal{T}^c$. If $Tt = 0$ then $t = 0$.*

Proof. By Proposition 3.2 we know that if $x \in D^{\mathrm{perf}}(A)$, and if

$$T\pi x = \left\{\sigma^{-1}A\right\}^L \otimes_R x = 0,$$

then $x \in \mathcal{R}^c$, in other words $\pi x = 0$. The proposition is therefore true for all objects $\pi x \in \mathcal{T}^c$, with $x \in D^{\mathrm{perf}}(A)$.

Now we turn to the general case. Suppose t is an object in $\mathcal{T}^c$ with $Tt = 0$. We wish to show $t = 0$. By the last sentences of [9, Theorem 2.1], t is a direct summand of an object πx, with $x \in D^{\mathrm{perf}}(A)$. That is, there exists an object $t' \in \mathcal{T}^c$ with

$$\pi x \cong t \oplus t'.$$

Consider the distinguished triangle

$$t \oplus t' \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} t \oplus t' \longrightarrow t \oplus \Sigma t \longrightarrow \Sigma(t \oplus t').$$

It is a triangle of the form

$$\pi x \longrightarrow \pi x \longrightarrow t \oplus \Sigma t \longrightarrow \Sigma \pi x.$$

Since the image of the functor π, that is the subcategory $\mathcal{S}^c/\mathcal{R}^c \subset \mathcal{T}^c$, is full and triangulated (see [9, Theorem 2.1]), the entire distinguished triangle above must lie in $\mathcal{S}^c/\mathcal{R}^c$. That means $t \oplus \Sigma t$ is isomorphic to an object in $\mathcal{S}^c/\mathcal{R}^c$. It follows there exists an object $y \in D^{\mathrm{perf}}(A) \cong \mathcal{S}^c$ with

$$\pi y \cong t \oplus \Sigma t.$$

This makes

$$T\pi y \cong Tt \oplus \Sigma Tt \cong 0.$$

By the above $\pi y = 0$. But t is a direct summand of πy; hence $t = 0$. □

4. A bound on the length of complexes in $\mathcal{R}^c$

The category $\mathcal{R}^c$ is the smallest thick subcategory of $D^{\mathrm{perf}}(A)$ containing σ. Every object in $\mathcal{R}^c$ is a direct summand of an object made up of iterated mapping cones on objects in σ. For technical reasons we want bounds on how long the iterated mapping cones need to be. This section is devoted to proving such bounds.

Definition 4.1. The full subcategory of all objects in $\mathcal{S} = D(A)$ which vanish outside the range $[m, n]$ will be denoted $\mathcal{S}[m, n]$. We allow m or n to be infinite; the categories $\mathcal{S}[m, \infty)$ and $\mathcal{S}(-\infty, n]$ have the obvious definitions.

Remark 4.2. The reader should note that the categories $\mathcal{S}[n, \infty)$ and $\mathcal{S}(-\infty, n]$ should not be confused with $\mathcal{S}^{\geqslant n}$ and $\mathcal{S}^{\leqslant n}$. It is true that every object in $\mathcal{S}^{\leqslant n}$ is *isomorphic in* $\mathcal{S}$ to a chain complex

$$\cdots \to X^m \to X^{m+1} \to \cdots \to X^{n-1} \to X^n \to 0 \to 0 \to \cdots.$$

An isomorphism in $\mathcal{S} = D(A)$ is after all just a homology isomorphism. For any object in $\mathcal{S}^{\leqslant n}$, there is an object in $\mathcal{S}(-\infty, n]$ homology isomorphic to it. But for once we want to have a name for the complexes which are actually supported on the interval $[m, n]$, not just isomorphic in $\mathcal{S}$ to such objects.

Definition 4.3. The category $\mathcal{K}$ will be the smallest full subcategory of $\mathcal{S}$ such that

4.3.1. *Every suspension of every object in σ lies in $\mathcal{K}$. That is, $\mathcal{K}$ contains all the complexes*

$$\cdots \longrightarrow 0 \longrightarrow P_i \xrightarrow{s_i} Q_i \longrightarrow 0 \longrightarrow \cdots.$$

4.3.2. *Given any chain map of objects in $\mathcal{K}$*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial} & X^{i-1} & \xrightarrow{\partial} & X^i & \xrightarrow{\partial} & X^{i+1} & \xrightarrow{\partial} & \cdots \\
 & & \downarrow{\scriptstyle f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} & & \\
\cdots & \xrightarrow{\partial} & Y^{i-1} & \xrightarrow{\partial} & Y^i & \xrightarrow{\partial} & Y^{i+1} & \xrightarrow{\partial} & \cdots
\end{array}$$

then the mapping cone

$$\cdots \to X^i \oplus Y^{i-1} \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_i & \partial \end{pmatrix}} X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -\partial & 0 \\ f_{i+1} & \partial \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \to \cdots$$

also lies in $\mathcal{K}$.

As in Remark 4.2 we mean equality of chain complexes, not homotopy equivalence or homology isomorphism.

We note the obvious lemmas.

Lemma 4.4. *Let* $\widetilde{\mathcal{K}}$ *be the full subcategory of all objects in* $\mathcal{S}$ *isomorphic to objects in* $\mathcal{K}$. *That is, any object of* $\mathcal{S} = D(A)$ *isomorphic to a chain complex in* $\mathcal{K}$ *lies in* $\widetilde{\mathcal{K}}$. *The subcategory* $\widetilde{\mathcal{K}} \subset \mathcal{S}$ *is triangulated.*

Proof. The point is that the objects of $\mathcal{K}$ are bounded chain complexes of projectives. Let $f : X \to Y$ be a morphism in $D(A)$ between objects in $\mathcal{K}$. Because X is a bounded-above complex of projectives, there is a chain map representing the morphism. The mapping cone on this chain map completes $f : X \to Y$ to a triangle, and lies in $\mathcal{K}$. Up to isomorphism in $\mathcal{S} = D(A)$, all triangles on morphisms in $\mathcal{K}$ are contained in $\mathcal{K}$. □

Lemma 4.5. *The category* $\mathcal{K}$ *is contained in* $\mathcal{R}^c$. *Furthermore, every object in* $\mathcal{R}^c$ *is a direct summand of an object isomorphic in* $D(A)$ *to an object in* $\mathcal{K}$.

Proof. The inclusion $\mathcal{K} \subset \mathcal{R}^c$ is easy. The category $\mathcal{R}^c$ contains σ and is closed under mapping cones, and $\mathcal{K}$ is the smallest such.

Next observe that, by [9, Lemma 2.2], the category $\mathcal{R}^c$ is the smallest thick subcategory of $\mathcal{S}$ containing σ, and hence $\mathcal{R}^c$ is the smallest thick subcategory containing the triangulated subcategory $\widetilde{\mathcal{K}}$ of Lemma 4.4. Therefore every object of $\mathcal{R}^c$ is a direct summand of an object in $\widetilde{\mathcal{K}}$. □

The point of the exercise is that any object in $\mathcal{K}$ can be expressed as a mapping cone on a map of shorter objects. We need a definition, and then we are ready to state our main lemma.

Definition 4.6. The subcategories $\mathcal{K}[m,n]$ are defined as the intersection

$$\mathcal{K}[m,n] = \mathcal{K} \cap \mathcal{S}[m,n].$$

As in Definition 4.1, we allow m and n to be infinite.

Lemma 4.7. *Suppose $n \in \mathbb{Z}$ is an integer. Then every object $Z \in \mathcal{K}$ can be expressed as a mapping cone on a chain map $Z_1 \to Z_2$, as below*

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\partial} & Z_1^{n-1} & \xrightarrow{\partial} & Z_1^n & \xrightarrow{\partial} & Z_1^{n+1} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & Z_2^n & \xrightarrow{\partial} & Z_2^{n+1} & \xrightarrow{\partial} & Z_2^{n+2} & \xrightarrow{\partial} & \cdots
\end{array}$$

that is, $Z_1 \in \mathcal{K}(-\infty, n+1]$ and $Z_2 \in \mathcal{K}[n, \infty)$.

Proof. Let $\mathcal{B}$ be the full subcategory of $\mathcal{K}$ containing the objects for which the assertion of the lemma holds. That is, an object $Z \in \mathcal{K}$ belongs to $\mathcal{B}$ if and only if, for every $n \in \mathbb{Z}$, there exist $Z_1 \in \mathcal{K}(-\infty, n+1]$ and $Z_2 \in \mathcal{K}[n, \infty)$ and a chain map $Z_1 \to Z_2$ so that Z is equal to the mapping cone. It suffices to prove that $\mathcal{B} = \mathcal{K}$, for which we need only show that any suspension of an object of σ lies in $\mathcal{B}$, and that mapping cones on maps in $\mathcal{B}$ lie in $\mathcal{B}$.

Assume therefore that we are given a complex s below

$$\cdots \longrightarrow 0 \longrightarrow c^\ell \longrightarrow c^{\ell+1} \longrightarrow 0 \longrightarrow \cdots$$

which is some suspension of an object in σ. Choose any $n \in \mathbb{Z}$. If $n \leqslant \ell$, then $s \in \mathcal{K}[n, \infty)$, and s is the mapping cone of the chain map $0 \to s$. If $n \geqslant \ell + 1$, then $\Sigma^{-1}s \in \mathcal{K}(-\infty, n+1]$ and s is isomorphic to the mapping cone on the chain map $\Sigma^{-1}s \to 0$. Either way, $s \in \mathcal{B}$.

Next suppose we are given two object X and Y in $\mathcal{B}$, and a chain map $f : X \to Y$. Let Z be the mapping cone of f. We need to show that Z is in $\mathcal{B}$. For every integer $n \in \mathbb{Z}$, we need to express Z as a mapping cone on a map of objects $Z_1 \to Z_2$, with $Z_1 \in \mathcal{K}(-\infty, n+1]$ and $Z_2 \in \mathcal{K}[n, \infty)$. Without loss of generality assume $n = 0$.

Because $X \in \mathcal{B}$ we may express it as a mapping cone on a map $X_1 \to X_2$, with $X_1 \in \mathcal{K}(-\infty, 2]$ and $X_2 \in \mathcal{K}[1, \infty)$. Because $Y \in \mathcal{B}$ we may express it as the mapping cone on a map $Y_1 \to Y_2$, with $Y_1 \in \mathcal{K}(-\infty, 1]$ and $Y_2 \in \mathcal{K}[0, \infty)$. We have a diagram, where the rows are short exact sequences of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & \Sigma X_1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle f} & & & & \\
0 & \longrightarrow & Y_2 & \longrightarrow & Y & \longrightarrow & \Sigma Y_1 & \longrightarrow & 0.
\end{array}$$

The composite

$$\begin{array}{ccccc}
X_2 & \longrightarrow & X & & \\
 & & \downarrow{\scriptstyle f} & & \\
 & & Y & \longrightarrow & \Sigma Y_1
\end{array}$$

is a chain map from $X_2 \in \mathcal{K}[1,\infty)$ to $\Sigma Y_1 \in \mathcal{K}(-\infty,0]$, and therefore must vanish. It follows that we may complete to a commutative diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_2 & \longrightarrow & X & \longrightarrow & \Sigma X_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Sigma f_2} & & \\ 0 & \longrightarrow & Y_2 & \longrightarrow & Y & \longrightarrow & \Sigma Y_1 & \longrightarrow & 0. \end{array}$$

Let Z_1 be the mapping cone on $f_1 : X_1 \to Y_1$, and let Z_2 be the mapping cone on $f_2 : X_2 \to Y_2$. Then $Z_1 \in \mathcal{K}(-\infty,1]$ while $Z_2 \in \mathcal{K}[0,\infty)$. Furthermore Z, which is the mapping cone on $f : X \to Y$, can also be expressed as a mapping cone on a map $Z_1 \to Z_2$. □

5. T induces a K_0-isomorphism

In this section we shall prove that the functor $T : \mathbf{T} \to \mathbf{U}$ of [10, Summary 5.4] induces an isomorphism $K_0(\mathbf{T}) \to K_0(\mathbf{U})$. Note that in higher K-theory there is a need to worry about models. But for any Waldhausen category $\mathbf{C}$, the Grothendieck group $K_0(\mathbf{C})$ is an invariant of the triangulated category $D(\mathbf{C})$. As long as we confine ourselves to K_0 computations we can quite safely work directly with the triangulated categories. This is what we will do in the current section.

Thus we must show that the functor of triangulated categories $T : \mathcal{T}^c \to D^{\mathrm{perf}}(\sigma^{-1}A)$ induces an isomorphism in K_0. We shall do it through a sequence of lemmas. We resume the shorthands of Section 3. For every projective A-module we let $\sigma^{-1}P = \{\sigma^{-1}A\} \otimes_A P$, and we let $\mathcal{S} = D(A)$, $\mathcal{S}^c \cong D^{\mathrm{perf}}(A)$ and $\mathcal{D} = D^{\mathrm{perf}}(\sigma^{-1}A)$. For the first time in this article we will need to occasionally mention unbounded derived categories. We remind the reader that the functor $\pi : D(A) \to \mathcal{T}$ has a right adjoint $G : \mathcal{T} \to D(A)$.

Lemma 5.1. *Let n be an integer. Let $X \in \mathcal{S}^c$ be an object of $\mathcal{S}^{\leqslant n}$, and let P be a finitely generated projective A-module. Then the functor $T : \mathcal{T}^c \to \mathcal{D}$ gives a homomorphism*

$$\mathcal{T}^c\big(\pi\Sigma^{-n}P, \pi X\big) \longrightarrow \mathcal{D}\big(T\pi\Sigma^{-n}P, T\pi X\big).$$

We assert that this map is an isomorphism.

Proof. By translation we may assume $n = 0$. We need to prove the map injective and surjective. Let us prove surjectivity first. Recall that $X \in \mathcal{S}^c \cap \mathcal{S}^{\leqslant 0}$ is isomorphic to a chain complex of finitely generated projective A-modules

$$\to X^m \to X^{m+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to 0 \to .$$

This makes $T\pi X$ the chain complex

$$\to \sigma^{-1}X^m \to \sigma^{-1}X^{m+1} \to \cdots \to \sigma^{-1}X^{-1} \to \sigma^{-1}X^0 \to 0 \to 0 \to .$$

Let P be a finitely generated projective A-module, concentrated in degree 0. Now the complex of $\sigma^{-1}A$-modules $T\pi P$ is a single projective module $\sigma^{-1}P$, concentrated in degree 0. Any map in the derived category, from the bounded above complex of projectives $\sigma^{-1}P = \{\sigma^{-1}A\} \otimes_A P$ to the complex $\{\sigma^{-1}A\}L \otimes_A X$, can be represented by a chain map. There is a map $\sigma^{-1}P \to \sigma^{-1}X_0$ inducing it. By Proposition 3.1, this comes from a map $\pi P \to \pi X^0$. But then the composite

$$\pi P \longrightarrow \pi X^0 \longrightarrow \pi X$$

gives a map $\pi P \to \pi X$ in $\mathcal{T}^c$, inducing $T\pi P \to T\pi X$.

This proved the surjectivity. For the injectivity, note that there is a short exact sequence of chain complexes

$$\begin{array}{ccccccccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{-1} & \longrightarrow & X^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow .
\end{array}$$

Write the corresponding triangle as

$$X^0 \longrightarrow X \longrightarrow Y \longrightarrow \Sigma X^0.$$

The functor π takes this to the triangle

$$\pi X^0 \longrightarrow \pi X \longrightarrow \pi Y \longrightarrow \pi \Sigma X^0.$$

Let P be a finitely generated projective A-module, concentrated in degree 0. Suppose we are given a map $\pi P \to \pi X$. Composing to Y, we deduce a map

$$\pi P \longrightarrow \pi X \longrightarrow \pi Y.$$

By adjunction, this corresponds to a map

$$P \longrightarrow G\pi Y,$$

which must vanish. After all $Y \in \mathcal{S}^{\leqslant -1}$, and by [10, Lemma 6.4] it follows that $G\pi Y$ is also in $\mathcal{S}^{\leqslant -1}$. The map from a projective object P in degree 0 to the complex $G\pi Y \in \mathcal{S}^{\leqslant -1}$ has to vanish.

It follows that the map $\pi P \to \pi X$ must factor as

$$\pi P \longrightarrow \pi X^0 \longrightarrow \pi X.$$

Now assume that the map vanishes in $\mathcal{D} = D^c(\sigma^{-1}A)$. That is, the composite

$$\sigma^{-1}P \longrightarrow \sigma^{-1}X^0 \longrightarrow \left\{\sigma^{-1}A\right\}^L \otimes_A X$$

vanishes in $\mathcal{D}$. Then it must be null homotopic. The map $\sigma^{-1}P \to \sigma^{-1}X^0$ must factor as

$$\sigma^{-1}P \longrightarrow \sigma^{-1}X^{-1} \longrightarrow \sigma^{-1}X^0.$$

By Proposition 3.1, this tells us that the map $\pi P \to \pi X^0$ must factor as

$$\pi P \longrightarrow \pi X^{-1} \longrightarrow \pi X^0$$

and hence the map

$$\pi P \longrightarrow \pi X^{-1} \longrightarrow \pi X^0 \longrightarrow \pi X$$

must vanish. □

Lemma 5.2. *Let n be an integer. Let Z be an object of $\mathcal{T}^c$, and suppose for all $r > n$, $H^r(TZ) = 0$. Then there is an object $X \in D^{\mathrm{perf}}(A)$, that is a bounded complex of projective A-modules*

$$\to 0 \to X^m \to X^{m+1} \to \cdots \to X^{\ell-1} \to X^\ell \to 0 \to$$

so that Z is a direct summand of an isomorph of πX, and $\ell \leqslant n$.

Proof. By suspending we may assume $n = 0$. Because every object of $\mathcal{T}^c$ is a direct summand of an isomorph of an object in $\mathcal{S}^c/\mathcal{R}^c$, we may certainly find an $X \in D^{\mathrm{perf}}(A)$ and a $Z' \in \mathcal{T}^c$ with $\pi X \cong Z \oplus Z'$. What is not clear is that we may choose X to be a complex

$$\to 0 \to X^m \to X^{m+1} \to \cdots \to X^{\ell-1} \to X^\ell \to 0 \to$$

with $\ell \leqslant 0$. Assume therefore that $\ell > 0$, and we shall show that we may reduce ℓ by 1.

We recall the short exact sequence of chain complexes

$$\begin{array}{ccccccccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & X^\ell & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{\ell-1} & \longrightarrow & X^\ell & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & 0 & \longrightarrow & X^m & \longrightarrow & X^{m+1} & \longrightarrow & \cdots & \longrightarrow & X^{\ell-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \ .
\end{array}$$

It gives a triangle which we write as

$$\Sigma^{-\ell}X^\ell \longrightarrow X \xrightarrow{\ a\ } Y \longrightarrow \Sigma^{-\ell+1}X^\ell.$$

We also have that Z is a direct summand of πX. That is, there are maps

$$\pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

so that $bc = 1_Z$. Now we wish to consider the composite

$$\pi\Sigma^{-\ell}X^\ell \longrightarrow \pi X \xrightarrow{b} Z \xrightarrow{c} \pi X.$$

We know that X^ℓ is a finitely generated projective A-module, and $X \in \mathcal{S}$ lies in $\mathcal{S}^{\leqslant \ell}$. The conditions are as in Lemma 5.1. In order to prove that the composite vanishes, it suffices to prove that T of it vanishes, in $\mathcal{D} = D^{\mathrm{perf}}(\sigma^{-1}A)$.

But in $\mathcal{D}$ the map becomes the composite

$$T\pi\Sigma^{-\ell}X^\ell \longrightarrow T\pi X \longrightarrow TZ \longrightarrow T\pi X.$$

We assert that already the shorter composite, $T\pi\Sigma^{-\ell}X^\ell \to T\pi X \to TZ$ must vanish. After all, it is a map

$$T\pi\Sigma^{-\ell}X^\ell \longrightarrow TZ.$$

By hypothesis, TZ vanishes above degree 0. It is quasi-isomorphic to a complex of $\sigma^{-1}A$-modules in degree $\leqslant 0$. And $T\pi\Sigma^{-\ell}X^\ell = \Sigma^{-\ell}\sigma^{-1}X^\ell$ is a single projective $\sigma^{-1}A$-module, concentrated in degree $\ell > 0$. Hence the vanishing. The composite

$$\pi\Sigma^{-\ell}X^\ell \longrightarrow \pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

must therefore vanish. Since c is a split monomorphism, we deduce that the composite

$$\pi\Sigma^{-\ell}X^\ell \longrightarrow \pi X \xrightarrow{b} Z$$

also vanishes.

But now the triangle

$$\pi\Sigma^{-\ell}X^\ell \longrightarrow \pi X \xrightarrow{a} \pi Y \longrightarrow \pi\Sigma^{-\ell+1}X^\ell$$

tells us that the map $b : \pi X \to Z$ must factor as

$$\pi X \xrightarrow{a} \pi Y \xrightarrow{\beta} Z.$$

The composite

$$Z \xrightarrow{c} \pi X \xrightarrow{a} \pi Y \xrightarrow{\beta} Z$$

is the identity, and hence Z is a direct summand of an isomorph of πY, with Y the complex

$$\longrightarrow 0 \longrightarrow X^m \longrightarrow X^{m+1} \longrightarrow \cdots \longrightarrow X^{\ell-1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdot \qquad \square$$

Lemma 5.3. *Let n be an integer. Let Z be an object of $\mathfrak{T}^c$, and suppose for all $r > n$, $H^r(TZ) = 0$. Given any finitely generated projective A-module P, and any map*

$$T\pi P = \sigma^{-1}P \xrightarrow{a} H^n(TZ),$$

there is a map in $\mathfrak{T}^c$

$$\pi\Sigma^{-n}P \xrightarrow{\mu} Z$$

so that $H^n(T\mu) = a$.

Proof. By translating we may assume $n = 0$. Let Z be an object of $\mathfrak{T}^c$ and suppose that, for all $r > 0$, $H^r(TZ) = 0$. By Lemma 5.2 there exists a complex $X \in D^{\mathrm{perf}}(A)$

$$\to 0 \to X^m \to X^{m+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to$$

so that Z is a direct summand of an isomorph of πX. We have two maps

$$\pi X \xrightarrow{b} Z \xrightarrow{c} \pi X$$

so that $bc = 1_Z$. This gives us two maps

$$T\pi X \xrightarrow{Tb} TZ \xrightarrow{Tc} T\pi X$$

with $(Tb)(Tc) = 1$. Given any map

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ),$$

we can form the composite

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ) \xrightarrow{H^0(Tc)} H^0(T\pi X).$$

Of course, $T\pi X$ is just the chain complex

$$\cdots \to \sigma^{-1}X^{-1} \to \sigma^{-1}X^0 \to 0 \to$$

and any map from a projective $\sigma^{-1}P$ to $H^0(T\pi X)$ lifts to a map

$$\sigma^{-1}P \longrightarrow T\pi X.$$

By Lemma 5.1 the above map is $T\gamma$, for a (unique) map

$$\pi P \xrightarrow{\gamma} \pi X.$$

Now let μ be the composite

$$\pi P \xrightarrow{\gamma} \pi X \xrightarrow{b} Z.$$

Applying the functor $H^0 \circ T$, we compute $H^0(T\mu)$ to be the composite

$$\sigma^{-1}P \xrightarrow{a} H^0(TZ) \xrightarrow{H^0(Tc)} H^0(T\pi X) \xrightarrow{H^0(Tb)} H^0(TZ),$$

which is nothing other than the map a. $\square$

Lemma 5.4. *For any finitely generated projective $\sigma^{-1}A$-module M, there is a canonically unique object $\widetilde{M} \in \mathcal{T}^c$ so that*

5.4.1. $H^n(T\widetilde{M}) = 0$ *for* $n \neq 0$.

5.4.2. $H^0(T\widetilde{M}) = M$.

The functor $H^0(T-)$ is an equivalence of categories between the full subcategory of objects $\widetilde{M} \in \mathcal{T}^c$ and finitely generated projective $\sigma^{-1}A$-modules.

Proof. Let us first prove existence. Let M be a finitely generated projective $\sigma^{-1}A$-module. There exists a $\sigma^{-1}A$-module N, so that $M \oplus N \cong \{\sigma^{-1}A\}^r$. There is an idempotent $\{\sigma^{-1}A\}^r \to \{\sigma^{-1}A\}^r$ which is the map

$$M \oplus N \xrightarrow{1_M \oplus 0_N} M \oplus N.$$

Write this map as $1_M \oplus 0_N : T\pi A^r \to T\pi A^r$. By Proposition 3.1 there is a unique lifting $e : \pi A^r \to \pi A^r$. The uniqueness of the lifting allows us to easily show that $e^2 = e$. But idempotents split in $\mathcal{T}$, by [2, Proposition 3.2 or Remark 3.3]. Define $\widetilde{M}$ by splitting the idempotent e.

Then $H^n(T\widetilde{M})$ is computed by splitting the idempotent $H^n(Te)$ on $H^n(\sigma^{-1}A^r)$; this gives us zero when $n \neq 0$, and M when $n = 0$. We have proved the existence of an $\widetilde{M}$ satisfying 5.4.1 and 5.4.2.

Now suppose X is an object of $\mathcal{T}^c$, and that

(i) $H^n(TX) = 0$ for $n \neq 0$,
(ii) $H^0(TX) = M$.

We wish to produce an isomorphism $\widetilde{M} \to X$. In any case we have a map

$$\{\sigma^{-1}A\}^r \longrightarrow M = H^0(TX),$$

namely the projection to the direct summand. By Lemma 5.3 there is a map

$$\pi A^r \longrightarrow X$$

which induces the projection. We may form the composite

$$\widetilde{M} \longrightarrow \pi A^r \longrightarrow X,$$

and it is very easy to check that the map

$$T\widetilde{M} \longrightarrow TX$$

is a homology isomorphism, hence an isomorphism in $D^{\mathrm{perf}}(\sigma^{-1}A)$. If we complete $\widetilde{M} \to X$ to a triangle in $\mathcal{T}^c$

$$\widetilde{M} \longrightarrow X \longrightarrow Y \longrightarrow \Sigma\widetilde{M},$$

then $TY = 0$. But by Proposition 3.4 it then follows that $Y = 0$, and $\widetilde{M} \to X$ is an isomorphism.

Finally it remains to check that $\mathcal{T}^c(\widetilde{M}, \widetilde{N}) = \mathrm{Hom}_{\sigma^{-1}A}(M, N)$. By the construction of $\widetilde{M}$ and $\widetilde{N}$ as direct summands of πA^r and πA^s, this reduces to knowing that

$$\mathcal{T}^c(\pi A^r, \pi A^s) = \mathcal{D}(T\pi A^r, T\pi A^s)$$

is an isomorphism. But we know this from Proposition 3.1. □

Theorem 5.5. *The map $T : \mathcal{T}^c \to \mathcal{D}$ induces a K_0-isomorphism.*

Proof. We define maps of categories

$$\mathcal{P}(\sigma^{-1}A) \xrightarrow{a} \mathcal{T}^c \xrightarrow{T} \mathcal{D}$$

with $\mathcal{P}(\sigma^{-1}A)$ the category of finitely generated projective $\sigma^{-1}A$-modules. The map T we already know. The map a takes a finitely generated projective $\sigma^{-1}A$-module M to $a(M) = \widetilde{M}$. In K-theory, the composite

$$K_0(\sigma^{-1}A) \to K_0(\mathcal{T}^c) \to K_0(\mathcal{D})$$

is clearly an isomorphism. To prove that both maps are isomorphisms, it suffices to show that the map $K_0(a) : K_0(\sigma^{-1}A) \to K_0(\mathcal{T}^c)$ is onto. This is what we shall do.

Let Z be an object of $\mathcal{T}^c$. We want to show that its class $[Z] \in K_0(\mathcal{T}^c)$ lies in the image of $K_0(a)$. We shall prove this by induction on the length of TZ. For the purpose of this proof, the length of $TZ \in D^{\mathrm{perf}}(\sigma^{-1}A)$ is defined to be the smallest integer n for which there exists an integer m with

$$H^i(TZ) = 0 \quad \text{unless } m \leqslant i \leqslant m + n.$$

Suppose the length of TZ is zero. Replacing Z by a suspension, this means that $H^n(TZ)=0$ unless $n=0$. Since $TZ \in D^{\mathrm{perf}}(\sigma^{-1}A)$, we have that $H^0(TZ)=M$ must be a finitely generated projective $\sigma^{-1}A$-module. By Lemma 5.4 we know that Z is (canonically) isomorphic to $\widetilde{M}$. Thus Z is in the image of a.

Suppose now that we know the induction hypothesis. We are given $n \geqslant 0$. We know that if Z is an object of $\mathcal{T}^c$ so that the length of TZ is $\leqslant n$, then the class $[Z] \in K_0(\mathcal{T}^c)$ lies in the image of $K_0(a)$. Let Z be a complex of length $n+1 \geqslant 1$. Replacing Z by a suspension, this means that $H^r(TZ)=0$ unless $-n-1 \leqslant r \leqslant 0$. Now $H^0(TZ)$ is a finitely presented $\sigma^{-1}A$-module; we may choose a finitely generated free A-module F, and a surjection $\sigma^{-1}F \to H^0(TZ)$. By Lemma 5.3 there is a map

$$\pi F \longrightarrow Z$$

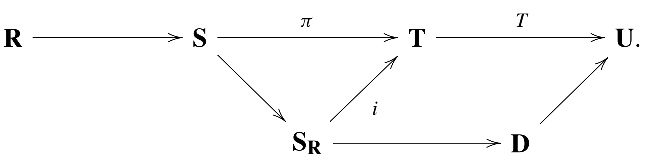
lifting this surjection. Form the triangle in $\mathcal{T}^c$

$$\pi F \longrightarrow Z \longrightarrow Y \longrightarrow \Sigma\pi F.$$

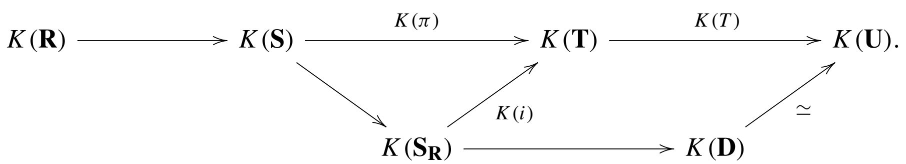
It is easily computed that the length of TY is $\leqslant n$, so by induction $[Y]$ lies in the image of $K_0(a): K_0(\sigma^{-1}A) \to K_0(\mathcal{T}^c)$. Clearly $[\pi F]=[\sigma^{-1}F]$ also lies in the image of $K_0(a)$, and the triangle tells us that $[Z]=[Y]+[\pi F]$. □

6. T induces a K_1-isomorphism

In [10, Summary 5.4] we constructed a diagram of Waldhausen models



On applying the K-theory functor, we deduce a diagram



What we want to show is that $K_1(\mathbf{T}) \simeq K_1(\mathbf{D})$. As the diagram indicates, we already know that the natural map $K(\mathbf{D}) \to K(\mathbf{U})$ is a homotopy equivalence. I want to remind the reader that Grayson's cofinality theorem [10, Theorem 2.4] tells us that the map $K_1(i): K_1(\mathbf{S_R}) \to K_1(\mathbf{T})$ is also an isomorphism. It is only at the level of K_0 that the map $K(i)$ fails to be an isomorphism.

The diagrams above therefore reduce us to showing that the natural map $K_1(\mathbf{S_R}) \to K_1(\mathbf{D})$ is an isomorphism. The advantage is that we do not have to work with the models $\mathbf{T}$ and $\mathbf{U}$, which involve unbounded complexes.

In the proof of Theorem 5.5 we introduced a functor $\mathcal{P}(\sigma^{-1}A) \to \mathcal{T}^c$. We do not know a Waldhausen model for this map. But we shall now show that there is an induced map on K_1. The group $K_1(\sigma^{-1}A)$ is generated by determinants of automorphisms of free (or projective) modules. This means: Given any projective A-module P, and an automorphism $\phi : \sigma^{-1}P \to \sigma^{-1}P$, the determinant of ϕ is an element of $K_1(\sigma^{-1}A)$. The collection of all determinants of all ϕ's generates $K_1(\sigma^{-1}A)$. We want to produce a map $K_1(\sigma^{-1}A) \to K_1(\mathbf{S_R})$; to define the map, it suffices to say what it does on all ϕ's as above.

To define what the map does to ϕ, let us remind ourselves that the zero-space of the spectrum $K(\mathbf{S_R})$ has a Gillet–Grayson model (see [6]), which we denote $GG(\mathbf{S_R})$. That is, there is a homotopy equivalence

$$GG(\mathbf{S_R}) \simeq \Omega^\infty K(\mathbf{S_R}).$$

The space $GG(\mathbf{S_R})$ is an H-space, and hence

$$K_1(\mathbf{S_R}) = \pi_1 K(\mathbf{S_R}) = \pi_1 GG(\mathbf{S_R}) = H_1 GG(\mathbf{S_R}).$$

Starting with an automorphism $\phi : \sigma^{-1}P \to \sigma^{-1}P$, we need to produce a class in the first homology group $H_1 GG(\mathbf{S_R})$.

We note that, by Proposition 3.1, $\phi : \sigma^{-1}P \to \sigma^{-1}P$ corresponds to a unique automorphism

$$\varphi : \pi P \to \pi P.$$

This is an automorphism defined in $\mathcal{S}^c/\mathcal{R}^c$, and $\mathbf{S_R}$ is a Waldhausen model for $\mathcal{S}^c/\mathcal{R}^c$. It follows that there exist weak equivalences $a : Q \to P$ and $b : Q \to P$, with $\varphi = ab^{-1}$. But then $(P,0)$ and $(Q,0)$ are 0-cells in the Gillet–Grayson model $GG(\mathbf{S_R})$. The weak equivalences $(a,0) : (Q,0) \to (P,0)$ and $(b,0) : (Q,0) \to (P,0)$ are 1-cells. The difference, which we denote $[a] - [b]$, is a cycle. It is an element in $H_1(GG(\mathbf{S_R})) = \pi_1(GG(\mathbf{S_R})) = K_1(\mathbf{S_R})$. We leave it to the reader to check that the map sending ϕ to $[a] - [b]$ extends to a well-defined homomorphism $\psi : K_1(\sigma^{-1}A) \to K_1(\mathbf{S_R})$.

Recall that $\mathbf{D} = C^{\mathrm{perf}}(\sigma^{-1}A)$, and that the map $\mathbf{S_R} \to \mathbf{D}$ is just tensor product with $\sigma^{-1}A$. The composite

$$K_1\big(\sigma^{-1}A\big) \xrightarrow{\ \psi\ } K_1(\mathbf{S_R}) \xrightarrow{\ \theta\ } K_1(\mathbf{D})$$

is easily seen to be the K_1 part of the natural map $K(\sigma^{-1}A) \to K(\mathbf{D}) = K(C^{\mathrm{perf}}(\sigma^{-1}A))$, and by Gillet's [5, 6.2] we know it to be an isomorphism. To prove that both ψ and θ are isomorphisms it suffices to check that $\psi : K_1(\sigma^{-1}A) \to K_1(\mathbf{S_R})$ is epi. We have a localisation exact sequence

$$K_1(\mathbf{S}) \longrightarrow K_1(\mathbf{S_R}) \longrightarrow K_0(\mathbf{R}) \longrightarrow K_0(\mathbf{S}).$$

Note that $K(\mathbf{S}) = K(C^{\mathrm{perf}}(A)) = K(A)$. The composite $K_1(A) \to K_1(\sigma^{-1}A) \to K_1(\mathbf{S_R})$ is easily computed to agree with the natural $K_1(A) = K_1(\mathbf{S}) \to K_1(\mathbf{S_R})$; we deduce a commutative diagram where the bottom row is exact

$$\begin{array}{ccccccc} K_1(A) & \longrightarrow & K_1(\sigma^{-1}A) & & & & \\ \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \psi} & & & & \\ K_1(A) & \longrightarrow & K_1(\mathbf{S_R}) & \longrightarrow & K_0(\mathbf{R}) & \longrightarrow & K_0(A). \end{array}$$

To prove ψ epi, it suffices to show that the composite

$$\begin{array}{ccc} K_1\big(\sigma^{-1}A\big) & & \\ \Big\downarrow{\scriptstyle \psi} & & \\ K_1(\mathbf{S_R}) & \longrightarrow & K_0(\mathbf{R}) \end{array}$$

surjects to the kernel of $K_0(\mathbf{R}) \to K_0(A)$. But the composite is easy to compute. Take an automorphism $\phi : \sigma^{-1}A \otimes P \to \sigma^{-1}A \otimes P$ as above, which corresponds as above to an automorphism

$$\varphi : \pi P \to \pi P.$$

Choose weak equivalences $a : Q \to P$ and $b : Q \to P$, with $\varphi = ab^{-1}$. Then ϕ gets sent to $[A] - [B]$, where

$$A: \quad \cdots \to 0 \to Q \xrightarrow{a} P \to 0 \to \cdots,$$
$$B: \quad \cdots \to 0 \to Q \xrightarrow{b} P \to 0 \to \cdots.$$

It will therefore suffice to show that every element in the kernel of the map $K_0(\mathbf{R}) \to K_0(A)$ can be expressed as a difference $[A] - [B]$ as above. Note that this is a statement about K_0, and K_0 is an invariant of the derived category, independent of choices of models. From now on we can forget all about models. We shall prove something a little stronger than we need.

Theorem 6.1. *Every element in $K_0(\mathbf{R}) = K_0(\mathcal{R}^c)$ is a linear combination of complexes of length $\leqslant 1$. That is, it may be written as $\sum \pm [A_i]$, with $A_i \in \mathcal{R}^c$ being complexes of finitely generated projective A-modules of the form*

$$\cdots \to 0 \to X \to Y \to 0 \to \cdots.$$

Before the proof of Theorem 6.1, let us state the main corollary.

Corollary 6.2. *Every object in the kernel of the map $K_0(\mathcal{R}^c) \to K_0(\mathcal{S}^c) = K_0(A)$ is of the form $[A] - [B]$, where A is a complex*

$$\cdots \to 0 \to Q \xrightarrow{a} P \to 0 \to \cdots$$

and B is a complex

$$\cdots \to 0 \to Q \xrightarrow{b} P \to 0 \to \cdots.$$

By the discussion preceding Theorem 6.1 *this means that the map* $\mathbf{S_R} \to \mathbf{D}$ *induces a* K_1*-isomorphism.*

Proof that Corollary 6.2 follows from Theorem 6.1. Suppose we have an element of the kernel of the map $K_0(\mathcal{R}^c) \to K_0(\mathcal{S}^c) = K_0(A)$. By Theorem 6.1, just by virtue of being an element of $K_0(\mathcal{R}^c)$, it has an expression as $\sum \pm [A_i]$, with $A_i \in \mathcal{R}^c$ being complexes of finitely generated projective A-modules of the form

$$\cdots \to 0 \to X_i \to Y_i \to 0 \to \cdots.$$

Recalling that $[\Sigma A_i] = -[A_i]$, up to changing signs in the sum we may assume that all the X_i are in degree -1, all the Y_i in degree 0. Collecting together all the terms of equal sign, we may rewrite the sum as

$$\Big[\bigoplus A_i\Big] - \Big[\bigoplus B_j\Big].$$

That is, we have an element $[A] - [B]$ in the kernel of $K_0(\mathcal{R}^c) \to K_0(\mathcal{S}^c) = K_0(A)$, where A, B are complexes of the form

$$A\colon \quad \cdots \to 0 \to A^{-1} \xrightarrow{a} A^0 \to 0 \to \cdots,$$

$$B\colon \quad \cdots \to 0 \to B^{-1} \xrightarrow{b} B^0 \to 0 \to \cdots.$$

The fact that $[A] - [B]$ lies in the kernel of the map $K_0(\mathcal{R}^c) \to K_0(A)$ tells us that, in $K_0(A)$, there is an identity

$$[A^{-1}] + [B^0] = [B^{-1}] + [A^0].$$

This in turn says that there is a projective A-module X, and an isomorphism

$$A^{-1} \oplus B^0 \oplus X \cong B^{-1} \oplus A^0 \oplus X.$$

The object $[A] \in \mathcal{R}^c$ is isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow A^{-1} \oplus B^0 \oplus X \xrightarrow{a \oplus 1_{B^0} \oplus 1_X} A^0 \oplus B^0 \oplus X \longrightarrow 0 \longrightarrow \cdots$$

while the object $[B] \in \mathcal{R}^c$ is isomorphic to the complex

$$\cdots \longrightarrow 0 \longrightarrow B^{-1} \oplus A^0 \oplus X \xrightarrow{b \oplus 1_{A^0} \oplus 1_X} B^0 \oplus A^0 \oplus X \longrightarrow 0 \longrightarrow \cdots.$$

Put $Q = A^{-1} \oplus B^0 \oplus X \cong B^{-1} \oplus A^0 \oplus X$, and $P = A^0 \oplus B^0 \oplus X$. Then A is isomorphic in $\mathcal{R}^c$ to a complex

$$\cdots \to 0 \to Q \xrightarrow{\alpha} P \to 0 \to \cdots$$

and B is isomorphic in $\mathcal{R}^c$ to a complex

$$\cdots \to 0 \to Q \xrightarrow{\beta} P \to 0 \to \cdots$$

as required. □

Proof of Theorem 6.1. It remains to prove Theorem 6.1. Let X be an object of $\mathcal{R}^c$. We need to show that the class $[X]$ in $K_0(\mathcal{R}^c)$ can be written as a linear combination of classes of objects of length $\leqslant 1$.

Because $X \in \mathcal{R}^c \subset \mathcal{S}^c$, we have that X is isomorphic to a bounded complex of finitely generated projective A-modules. Suspending suitably and replacing by an isomorph, we may assume X has the form

$$\to 0 \to X^{-m} \to X^{-m+1} \to \cdots \to X^{-1} \to X^0 \to 0 \to .$$

If $m \leqslant 1$ we are done; the complex has length $\leqslant 1$. The proof is by induction on m. Assume we are given an integer $n \geqslant 1$. Assume further that, for every $X \in \mathcal{R}^c$ of length $m \leqslant n$, $[X]$ is equal in $K_0(\mathcal{R}^c)$ to a linear combination of complexes of length $\leqslant 1$. Take a complex X as above, with $m = n + 1 \geqslant 2$. We need to show that it can also be expressed as a linear combination of complexes of length $\leqslant 1$.

By Lemma 4.5 we have that every object of $\mathcal{R}^c$ is a direct summand of an isomorph of an object in $\mathcal{K}$, with $\mathcal{K}$ as in Definition 4.3. Choose a chain complex $Y \in \mathcal{K}$ and maps $X \to Y \to X$ composing to the identity on X. Clearly the map $H^0(Y) \to H^0(X)$ must be surjective.

By Lemma 4.7 there exist exists a triangle

$$U \longrightarrow Y \longrightarrow V \longrightarrow \Sigma U$$

with $U \in \mathcal{K}[1, \infty)$ and $V \in \mathcal{K}(-\infty, 1]$. The composite

$$U \longrightarrow Y \longrightarrow X$$

is a map from $U \in \mathcal{K}[1, \infty)$ to $X \in \mathcal{S}^{\leqslant 0}$, which must vanish. It follow that $Y \to X$ factors as $Y \to V \to X$. And since $H^0(Y) \to H^0(X)$ is epi and factors through $H^0(V)$, we deduce that $H^0(V) \to H^0(X)$ must be epi. Replacing Y by V we may assume $Y \in \mathcal{K}(-\infty, 1]$.

Next we apply Lemma 4.7 again, this time to deduce that $Y \in \mathcal{K}(-\infty, 1]$ can be expressed as the mapping cone on a map $U \to V$, with $U \in \mathcal{K}(-\infty, 0]$ and $V \in \mathcal{K}[-1, 1]$. There is a triangle

$$U \longrightarrow V \longrightarrow Y \longrightarrow \Sigma U,$$

hence an exact sequence

$$H^0(V) \longrightarrow H^0(Y) \longrightarrow H^1(U) = 0.$$

The map $H^0(V) \to H^0(Y) \to H^0(X)$ is the composite of two epis, hence is epi. Replacing $Y \to X$ by the composite $V \to Y \to X$, we may assume $Y \in \mathcal{K}[-1, 1]$.

The last time we apply Lemma 4.7 is to express Y as the mapping cone of a map $U \to Z$, with $U \in \mathcal{K}[0, 1]$ and $Z \in \mathcal{K}[0, 1]$. The only observations we wish to make is that $H^1(Z) \to H^1(Y)$ is epi, and that in $K_0(\mathcal{R}^c)$, $[Z]$ and $[Y] = [Z] - [U]$ are both linear combinations of objects in $\mathcal{R}^c$ of length $\leqslant 1$. Let us summarise: we have constructed maps $Z \to Y \to X$, with

(i) $Z \in \mathcal{K}[0, 1]$, $Y \in \mathcal{K}[-1, 1]$,
(ii) $H^0(Y) \to H^0(X)$ epi,
(iii) $H^1(Z) \to H^1(Y)$ epi,
(iv) both $[Y]$ and $[Z]$ are linear combinations of objects in $\mathcal{R}^c$ of length $\leqslant 1$.

Form the mapping cone on the map $Y \to X$ to obtain a triangle

$$Y \longrightarrow X \longrightarrow X' \longrightarrow \Sigma Y.$$

Since $Y \in \mathcal{K}[-1, 1]$ while $X \in \mathcal{R}^c$ is supported on the interval $[-m, 0]$ with $m \geqslant 2$, the mapping cone X' is an object of $\mathcal{R}^c$ supported in $[-m, 0]$. The long exact sequence in homology gives

$$H^0(Y) \longrightarrow H^0(X) \longrightarrow H^0(X') \longrightarrow H^1(Y) \longrightarrow H^1(X).$$

We have $H^1(X) = 0$, while $H^0(Y) \to H^0(X)$ is an epimorphism. Hence $H^0(X') = H^1(Y)$. But we know that the map $H^1(Z) \to H^1(Y)$ is an epimorphism, by (iii). And Z is a complex of the form

$$\cdots \to 0 \to Z^0 \to Z^1 \to 0 \to \cdots$$

that is a complex of length $\leqslant 1$. It follows that we can extend the epimorphism $\beta : H^1(Z) \to H^0(X')$ to a map from the presentation Z; there is a map $\Sigma Z \to X'$, inducing β in H^0. We may form the mapping cone, obtaining a triangle

$$\Sigma Z \longrightarrow X' \longrightarrow X'' \longrightarrow \Sigma^2 Z.$$

Since $\Sigma Z \in \mathcal{K}[-1, 0]$ and X' is supported on $[-m, 0]$ with $m \geqslant 2$, we conclude that X'' is supported on $[-m, 0]$. But now the long exact homology sequence

$$H^0(\Sigma Z) \xrightarrow{\ \alpha\ } H^0(X') \longrightarrow H^0(X'') \longrightarrow H^1(\Sigma Z)$$

has α surjective, while $H^1(\Sigma Z) = 0$. We conclude that $H^0(X'') = 0$. The complex X'' may be written as

$$\to 0 \to \widetilde{X}^{-m} \to \widetilde{X}^{-m+1} \to \cdots \to \widetilde{X}^{-1} \to \widetilde{X}^0 \to 0 \to$$

and since $H^0(X'') = 0$ the map $\widetilde{X}^{-1} \to \widetilde{X}^0$ is an epimorphism. Since $\widetilde{X}^0$ is projective, this epimorphism must be split. The complex X'' is homotopy equivalent to a complex

$$\to 0 \to \widetilde{X}^{-m} \to \widetilde{X}^{-m+1} \to \cdots \to \widetilde{Y}^{-1} \to 0 \to 0 \to$$

supported in the interval $[-m, -1]$. By induction, its class in $K_0(\mathcal{R}^c)$ is a linear combination of complexes of length $\leqslant 1$.

But now the triangles above give the identities

$$[X] = [X'] + [Y], \qquad [X'] = [X''] - [Z]$$

and hence $[X] = [X''] + [Y] - [Z]$, and all the terms on the right may be expressed as linear combinations of complexes of length $\leqslant 1$. □

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