# *On the Pontrjagin square and the signature*

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## §1. Statement of results.

Let $V$ be a $Z_2$-vector space and let

$$\mu : V \otimes V \to Z_2$$

be a non-singular symmetric pairing. A function $\eta : V \to Z_4$ is said to be quadratic with respect to $\mu$ if

$$\eta(x+y)=\eta(x)+\eta(y)+j\mu(x\otimes y) \quad \text{for all} \quad x, y \in V ,$$

where $j : Z_2 \to Z_4$ is the non-trivial homomorphism. Then according to E. H. Brown [1], we define the Arf invariant $\sigma(V, \eta) \in Z_8$ as follows.

If we put $\alpha(\eta)=\sum_{x\in V} i^{\eta(x)} \in C$, (where $i^2=-1$, and $Z_4$ acts naturally on $\{1, i, -1, -i\}$), then it can be shown that (see the proof of Prop. 2-3)

$$\alpha(\eta)^8=(\sqrt{2}^{\dim V})^8 ;$$

therefore

$$\alpha(\eta)=\sqrt{2}^{\dim V}\cdot\left(\frac{1+i}{\sqrt{2}}\right)^m$$

for some $m \in Z_8$. We put $\sigma(V, \eta)=m$.

Now let $M^{4n}$ be an oriented Poincaré complex of formal dimension $4n$, then the Pontrjagin square

$$P_2 : H^{2n}(M ; Z_2) \to H^{4n}(M ; Z_4)=Z_4$$

is quadratic with respect to the cup-product

$$\mu : H^{2n}(M ; Z_2)\otimes H^{2n}(M ; Z_2) \to H^{4n}(M ; Z_2)=Z_2 .$$

Hence we can define

$$\sigma(M, P_2)=\sigma(H^{2n}(M ; Z_2), P_2) \in Z_8 .$$

Our results are

THEOREM 1-1. *If $M^{4n}$ is an oriented Poincaré complex of formal dimension $4n$, then*

$$\sigma(M, P_2)= \text{signature } M \bmod 8 .$$

This theorem was conjectured by E. H. Brown in [2].

COROLLARY 1-2. *Let $M^{4n}$ be an oriented Poincaré complex, then signature* $M\equiv 0 \pmod 4$ *if and only if*

$$P_2(v_{2n})=0 .$$

*Here $v_{2n}$ is the 2n-th Wu class of M.*

Theorem 1-1 and Corollary 1-2 will be proved in §4.

I would like to express my gratitude to Professor A. Hattori for many advices and encouragement.

*Added in proof.* I have heard from G. Brumfiel that he and E. H. Brown, E. Thomas also have proved Theorem 1-1.

## §2. Some remarks on the Arf invariant.

The following proposition is due to E. H. Brown.

PROPOSITION 2-1. (i). *Let $\eta_i: V_i \to Z_4$ $(i=1, 2)$ be two quadratic functions with respect to $\mu_i$. If we define*

$$\eta_1+\eta_2 : V_1\oplus V_2 \to Z_4$$

*by $(\eta_1+\eta_2)(x_1, x_2)=\eta_1(x_1)+\eta_2(x_2)$, then $\eta_1+\eta_2$ is quadratic with respect to $\mu_1+\mu_2$ and*

$$\sigma(V_1+V_2, \eta_1+\eta_2)=\sigma(V_1, \eta_1)+\sigma(V_2, \eta_2) .$$

(ii). *If $L: V\to Z_4$ is linear, then*

$$\alpha(L)=2^{\dim V} \quad \text{if} \quad L=0,$$

$$\alpha(L)=0 \quad \text{if} \quad L\neq 0 .$$

(iii). *If $\eta: U\to Z$ is a unimodular quadratic form over Z, $\eta: U/2U\to Z_4$ is well defined and quadratic and*

$$\sigma(U/2U, \eta)= \text{signature}\, \eta \bmod 8 .$$

COROLLARY 2-2. *If $V=A\oplus B$,* $\dim A=\dim B$ *and $\mu$ is zero on A, then for any quadratic function $\eta: V\to Z_4$ with respect to $\mu$ such that $\eta(A)=0$,*

$$\sigma(V, \eta)=0 .$$

PROOF. Let $A_b=\{a+b\,; a\in A\}$ for any $b\in B$, and we give a $Z_2$-vector space structure on $A_b$ by

$$(a_1+b)\dot{+}(a_2+b)=a_1+a_2+b .$$

Consider the function $\eta_b: A_b\to Z_4$ defined by

$$\eta_b(a+b)=\eta(a+b)-\eta(b)=j\mu(a\otimes b) ,$$

then $\eta_b$ is linear with respect to $\dot{+}$, hence by Proposition 2-1 if $b\neq 0$, then $\alpha(\eta_b)=0$, and if $b=0$, then $\alpha(\eta_0)=2^{\dim A}$. Therefore

$$\alpha(\eta)=\sum_{b\in B}\alpha(\eta|A_b)=\sum_{b\in B}\alpha(\eta_b)\cdot i^{\eta(b)}=\alpha(\eta_0)=2^{\dim A} .$$

Hence

$$\sigma(V, \eta)=0 . \qquad \text{Q.E.D.}$$

Now let $M^{2n}$ be an oriented Poincaré complex and let $\eta: H^n(M; Z_2)\to Z_4$ be a quadratic function with respect to the cup-product, then we prove a formula relating $\sigma(M, \eta)$ with $\eta(v_n)$, where $v_n$ is the $n$-th Wu class of $M$.

PROPOSITION 2-3. *In the above situation,* $\sigma(M, \eta)=\eta(v_n) \pmod 4$.

PROOF. We write $V$ for $H^n(M; Z_2)$. Consider

$$\eta+\eta: V\oplus V\to Z_4 .$$

Let $V_v=\{(u+v, u); u\in V\}$ and we give $V_v$ a $Z_2$-vector space structure by

$$(u_1+v, u_1)\dot{+}(u_2+v, u_2)=(u_1+u_2+v, u_1+u_2) .$$

Consider

$$2\eta|V_v: V_v\to Z_4 ,$$

we have

$$2\eta(u+v, u)=\eta(u+v)+\eta(u)=\eta(v)+2\cdot\eta(u)+j\mu(u\otimes v)=\eta(v)+j\mu(u\otimes u)+j\mu(u\otimes v) .$$

Hence if we define $\eta_v: V_v\to Z_4$ by

$$\eta_v(u+v, u)=2\eta(u+v, u)-\eta(v) ,$$

then $\eta_v$ is linear with respect to $\dot{+}$. Now if

$$u^2+uv=0 \quad \text{for all} \quad u\in V ,$$

then $v=v_n$, the $n$-th Wu class. Hence if $v\neq v_n$, then $\alpha(\eta_v)=0$, and $\alpha(\eta_{v_n})=2^{\dim V}$ (by Prop. 2-1). Therefore

$$\alpha(2\eta)=\sum_{u\in V}\alpha(2\eta|V_v)=\sum_{u\in V}\alpha(\eta_v)\cdot i^{\eta(v)}=\alpha(\eta_{v_n})\cdot i^{\eta(v_n)}=2^{\dim V}\cdot i^{\eta(v_n)}$$
$$=2^{\dim V}\cdot\left(\frac{1+i}{\sqrt{2}}\right)^{2\eta(v_n)} .$$

Hence

$$2\sigma(M, \eta)=2\sigma(V, \eta)=\sigma(V\oplus V, \eta+\eta)=2\eta(v_n) ,$$

hence

$$\sigma(M, \eta) \equiv \eta(v_n) \pmod 4 . \qquad \text{Q.E.D.}$$

COROLLARY 2-4. *Let $M^{2n}$ be an oriented Poincaré complex, $n$; odd, and let $\eta : H^n(M; Z_2) \to Z_4$ be a quadratic function, then*

$$\sigma(M, \eta) \equiv 0 \pmod 4$$

i.e.,

$$\sigma(M, \eta) = 0 \textit{ or } 4.$$

PROOF. By Proposition 2-3,

$$\sigma(M, \eta) \equiv \eta(v_n) \pmod 4,$$

but since $n$ is odd and $M$ is orientable, $v_n = 0$, hence

$$\sigma(M, \eta) \equiv 0 \pmod 4. \qquad \text{Q.E.D.}$$

## § 3. The Bockstein Spectral Sequence.

Let $M^{4n}$ be an oriented Poincaré complex and let $\{E_r^*, d_r\}$ and $\{E_*^r, d^r\}$ be the mod 2 Bockstein spectral sequence in cohomology and homology respectively. Then by Browder [1], we have

PROPOSITION 3-1. (i). *$\{E_r^*, d_r\}$ and $\{E_*^r, d^r\}$ are dual each other by the Kronecker index.*

(ii) $E_\infty^* = H^*(M)/\mathrm{Tor} \otimes Z_2$, $E_*^\infty = H_*(M)/\mathrm{Tor} \otimes Z_2$.

Now since $M^{4n}$ is orientable, we have

$$E_1^{4n} = E_2^{4n} = \cdots = E_\infty^{4n} = Z_2$$

and

$$E_{4n}^1 = E_{4n}^2 = \cdots = E_{4n}^\infty = Z_2 .$$

Let $\mu_2 \in E_{4n}^1 = \cdots = E_{4n}^\infty$ be the mod 2 fundamental class. Then we can prove

PROPOSITION 3-2. *The Poincaré duality holds for $\{E_r^*, d_r\}$ and $\{E_*^r, d^r\}$;*

(i) *The cup-product*

$$\mu : E_r^a \otimes E_r^b \to E_r^{a+b}$$

*is well defined and similarly for the cap-product.*

(ii) *$\cap \mu_2 : E_r^k \to E_{4n-k}^r$ is an isomorphism for all $k$ and $r$, and*

$$d^r(x \cap \mu_2) = d^r x \cap \mu_2 \quad \textit{for all} \quad x \in E_r^k.$$

(iii) *The cup-product*

$$\mu : E_r^k \otimes E_r^{4n-k} \to E_r^{4n} = Z_2$$

*is non singular.*

PROOF. (i) If $r=1$, then it is clear. Assume the statement holds for $r \leqq m$, then we define

$$\mu : E_{m+1}^a \otimes E_{m+1}^b \to E_{m+1}^{a+b}$$

by

$$\mu([x] \otimes [y]) = [\mu(x \otimes y)] ,$$

where $x \in E_m^a$, $d_m x = 0$ and $y \in E_m^b$, $d_m y = 0$ . Since

$$d_m(x \cdot y) = d_m x \cdot y + x \cdot d_m y = 0 ,$$

and

$$d_m x' \cdot y = d_m(x' \cdot y) \quad \text{for} \quad x' \in E_m^{a-1} ,$$

$\mu$ is well defined.

The proof for the cap-product is similar.

(ii) Clearly $\cap \mu_2 : E_1^k \to E_{4n-k}^1$ is an isomorphism. And

$$d^1(x \cap \mu_2) = d_1 x \cap \mu_2 + x \cap d^1 \mu_2 ,$$

but since $M$ is orientable, $d^1 \mu_2 = 0$ and

$$d^1(x \cap \mu_2) = d_1 x \cap \mu_2 .$$

Now suppose

$$\cap \mu_2 : E_r^k \simeq E_{4n-k}^r, \quad \text{and} \quad d^r(x \cap \mu_2) = d_r x \cap \mu_2$$
$$\text{for all} \quad r \leqq m \quad \text{and} \quad x \in E_r^k.$$

Then we define

$$\cap \mu_2 : E_{m+1}^k \to E_{4n-k}^{m+1}$$

as follows.

Let $x \in E_m^k$ and suppose $d_m x = 0$, then $[x] \in E_{m+1}^k$ and $d^m(x \cap \mu_2) = d_m x \cap \mu_2 = 0$. Hence

$$[x \cap \mu_2] \in E_{4n-k}^{m+1}$$

and we define

$$[x] \cap \mu_2 = [x \cap \mu_2] .$$

Now since

$$[x+d_m x']\cap\mu_2=[(x+d_m x')\cap\mu_2]=[x\cap\mu_2]+[d_m x'\cap\mu_2]$$
$$=[x\cap\mu_2]+[d^m(x'\cap\mu_2)]=[x\cap\mu_2]$$

$\cap\mu_2$ is well defined.

(a) $\cap\mu_2$ is epimorphic.

Take any $[y]\in E^{m+1}_{4n-k}$, $y\in E^m_{4n-k}$, $d^m y=0$, then there is an element $x\in E^k_m$ such that

$$x\cap\mu_2=y\ .$$

Then

$$0=d^m y=d^m(x\cap\mu_2)=d_m x\cap\mu_2\ ,$$

hence $d_m x=0$ and $[x]\in E^k_{m+1}$.

Therefore

$$[x]\cap\mu_2=[y]$$

and $\cap\mu_2$ is epimorphic.

(b) $\cap\mu_2$ is monomorphic.

Suppose

$$[x]\cap\mu_2=0\ ,$$

then $x\cap\mu_2=d^m y$ for some $y\in E^m_{4n-k-1}$. Let $x'\in E^{k+1}_m$ be such that

$$x'\cap\mu_2=y\ ,$$

then

$$d_m x'\cap\mu_2=d^m(x'\cap\mu_2)=d^m y\ .$$

Hence

$$d_m x'\cap\mu_2=x\cap\mu_2,\quad x=d_m x',\quad [x]=0.$$

(c) $d^{m+1}([x]\cap\mu_2)=d_{m+1}[x]\cap\mu_2$, for

$$d^{m+1}([x]\cap\mu_2)=d_{m+1}[x]\cap\mu_2+[x]\cap d^{m+1}\mu_2=d_{m+1}[x]\cap\mu_2\ .$$

(iii) Suppose

$$x\cdot y=0\quad\text{for all } x,$$

then

$$\langle x\cdot y,\mu_2\rangle=0,\quad \langle x, y\cap\mu_2\rangle=0$$

hence $y=0$.

Q.E.D.

## §4. Proof of Theorem 1-1 and Corollary 1-2.

Let $\{E_r^*, d_r\}$ and $\{E_*^r, d^r\}$ be the mod 2 Bockstein spectral sequence for $H^*(M)$ and $H_*(M)$, where $M^{4n}$ is an oriented Poincaré complex. We prove the following statement by induction on $r$.

$(Q_r)$ ; $P_2^{(r)} : E_r^{2n} \to Z_4$ can be defined so as to be quadratic with respect to the cup-product

$$\mu : E_r^{2n} \otimes E_r^{2n} \to E_r^{4n} = Z_2$$

and $\sigma(E_r^{2n}, P_2^{(r)}) = \sigma(E_{r-1}^{2n}, P_2^{(r-1)})$.

If $r=1$, then $(Q_1)$ is trivial and

$$\sigma(E_1^{2n}, P_2^{(1)}) = \sigma(M, P_2) .$$

Now assume $(Q_r)$ holds for $r \leqq m$, then we define

$$P_2^{(m+1)} : E_{m+1}^{2n} \to Z_4$$

as follows. If $[x] \in E_{m+1}^{2n}$, $x \in E_m^{2n}$, $d_m x=0$, then we set

$$P_2^{(m+1)}([x]) = P_2^{(m)}(x) .$$

To show that $P_2^{(m+1)}$ is well defined, it suffices to show that

$$P_2^{(m)}(\mathrm{im}\, d_m) = 0 ,$$

for

$$\begin{aligned} P_2^{(m)}(x+d_m y) &= P_2^{(m)}(x) + P_2^{(m)}(d_m y) + j(x \cdot d_m y) \\ &= P_2^{(m)}(x) + P_2^{(m)}(d_m y) + j d_m(x \cdot y) \\ &= P_2^{(m)}(x) + P_2^{(m)}(d_m y) , \end{aligned}$$

where $x \in E_m^{2n}$, $d_m x=0$, $y \in E_m^{2n-1}$. Here we have used the fact that $d_m | E_m^{2n-1} = 0$ which follows from the orientability of $M$.

Now let $d_m x \in E_m^{2n}$, $x \in E_m^{2n-1}$, then if $x$ is represented by an integral cochain $u$,

$$\delta u = 2^m \cdot a$$

and $d_m x$ is represented by $1/2^m \cdot \delta u = a$, $[a] \in H^{2n}(M; Z)$ and

$$P_2^{(m)}(d_m x) = P_2[a] .$$

But clearly $2^m[a]=0$, hence $[a]^2$ is a torsion element in $H^{4n}(M; Z)=Z$, therefore

$$[a]^2=0, \quad P_2[a] = [a]^2 \bmod 4 = 0 .$$

Clearly $P_2^{(m+1)}$ is quadratic with respect to the cup-product.

Next we prove $\sigma(E^{2n}_{m+1}, P_2^{(m+1)})=\sigma(E^{2n}_m, P_2^{(m)})$. Let $[x_1], [x_2], \cdots, [x_p]$ be a basis for $E^{2n}_{m+1}$ and let

$V$=the subspace of $E^{2n}_m$, spanned by $x_1, x_2, \cdots, x_p$, $x_i \in E^{2n}_m$, $d_m x_i=0$.
Let

$$\bar{V}=\{y \in E^{2n}_m ; x\cdot y=0 \quad \text{for all} \quad x \in V\} ,$$

then

LEMMA 4-1. $E^{2n}_m = V \oplus \bar{V}$.

PROOF. If $x \in V \cap \bar{V}$, then since $x \in V$, we have $d_m x=0$ and $[x] \in E^{2n}_{m+1}$. Let $[y] \in E^{2n}_{m+1}$ be any element, then we may take $y$ in $V$ and

$$[x]\cdot[y]=[x\cdot y]=0 .$$

Hence $[x]=0$ and since $V\cap \operatorname{im} d_m=\{0\}$, we have

$$x=0 .$$

By counting dimension, we obtain the lemma. Q.E.D.

Now we have

$$\sigma(E^{2n}_m, P_2^{(m)})=\sigma(V, P_2^{(m)} \mid V)+\sigma(\bar{V}, P_2^{(m)} \mid \bar{V}) ,$$

but clearly

$$\sigma(V, P_2^{(m)} \mid V)=\sigma(E^{2n}_{m+1}, P_2^{(m+1)}) ,$$

hence we have only to prove

$$\sigma(\bar{V}, P_2^{(m)} \mid \bar{V})=0 .$$

To show this, it suffices to show that there is a subspace $A\subset \bar{V}$ such that

(i) $P_2^{(m)}(A)=0$,

(ii) The cup-product is trivial on $A$,

(iii) $\dim A=1/2\cdot\dim \bar{V}$. (See Cor. 2-2).

Now we claim that the subspace

$$\operatorname{im}(d_m : E^{2n-1}_m \to E^{2n}_m)\subset \bar{V}$$

satisfies the conditions above.

(i) has already been proved (well definedness of $P_2^{(m+1)}$).

(ii) If $d_m x$, $d_m y \in \operatorname{im}(d_m : E^{2n-1}_m \to E^{2n}_m)$, then

$$d_m x\cdot d_m y=d_m(x\cdot d_m y)=0 ,$$

since $M$ is orientable.

(iii) First we prove

LEMMA 4-2. $\dim E_m^{2n} = \dim \ker (d_m ; E_m^{2n} \to E_m^{2n+1}) + \dim \mathrm{im} (d_m : E_m^{2n-1} \to E_m^{2n})$.

PROOF. We have

(1) $\dim E_m^{2n} - \dim \ker (d_m : E_m^{2n} \to E_m^{2n+1}) = \dim \mathrm{im} (d_m : E_m^{2n} \to E_m^{2n+1})$.

Now we prove that

$$(\mathrm{im} (d_m : E_m^{2n} \to E_m^{2n+1}))^{\perp} = \ker (d^m : E^m_{2n+1} \to E^m_{2n}) .$$

In fact if $\alpha \in (\mathrm{im}(d_m : E_m^{2n} \to E_m^{2n+1}))^{\perp}$, then

$$\langle d_m x, \alpha \rangle = 0 \quad \text{for all} \quad x \in E_m^{2n} .$$

Hence

$$\langle x, d^m \alpha \rangle = 0 \quad \text{for all } x.$$

Therefore

$$d^m \alpha = 0 \quad \text{i.e.,} \quad \alpha \in \ker (d^m : E^m_{2n+1} \to E^m_{2n}) .$$

Conversely, assume $d^m \alpha = 0$, then

$$\langle d_m x, \alpha \rangle = \langle x, d^m \alpha \rangle = 0 \quad \text{for all } x.$$

Hence $\alpha \in (\mathrm{im} (d_m : E_m^{2n} \to E_m^{2n+1}))^{\perp}$.

Therefore

(2) $\dim \mathrm{im} (d_m : E_m^{2n} \to E_m^{2n+1}) = \dim E^m_{2n+1} - \dim \ker (d^m : E^m_{2n+1} \to E^m_{2n})$.

Consider the following diagram, which is commutative by Proposition 3-2.

$$\begin{array}{ccc} E^m_{2n+1} & \xrightarrow{d^m} & E^m_{2n} \\ \wr \uparrow \cap \mu_2 & & \wr \uparrow \cap \mu_2 \\ E_m^{2n-1} & \xrightarrow[d_m]{} & E_m^{2n} \end{array}$$

We have

(3) $\dim E^m_{2n+1} - \dim \ker (d^m : E^m_{2n+1} \to E^m_{2n}) = \dim \mathrm{im} (d_m : E_m^{2n-1} \to E_m^{2n})$.

Combining (1), (2) and (3), we obtain the lemma. Q.E.D.

Proof of (iii);

$$\dim \bar{V} = \dim E_m^{2n} - \dim E_{m+1}^{2n} = \dim \ker d_m + \dim \mathrm{im}\, d_m$$
$$- (\dim \ker d_m - \dim \mathrm{im}\, d_m) = 2 \cdot \dim \mathrm{im} (d_m : E_m^{2n-1} \to E_m^{2n}) .$$

This proves (iii).

Thus we have proved

$$\sigma(E_{m+1}^{2n}, P_2^{(m+1)}) = \sigma(E_m^{2n}, P_2^{(m)})$$

for all $m$. Therefore

$$\sigma(M, P_2) = \sigma(E_1^{2n}, P_2^{(1)}) = \sigma(E_\infty^{2n}, P_2^{(\infty)}) = \text{signature } M \bmod 8 \text{ (by Prop. 2-1)}.$$

This proves Theorem 1-1.

PROOF OF COROLLARY 1-2. By Theorem 1-1

$$\text{signature } M \bmod 8 = \sigma(M, P_2)$$

and by Proposition 2-3,

$$\sigma(M, P_2) \equiv P_2(v_{2n}) \pmod 4,$$

hence

$$\text{signature } M \equiv P_2(v_{2n}) \pmod 4. \qquad \text{Q.E.D.}$$

## Bibliography

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