

A NOTE ON VON NEUMANN RHO-INVARIANT OF SURFACE BUNDLES OVER THE CIRCLE

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Abstract. In this short note, we give a formula for the von Neumann rho-invariant of surface bundles over the circle S^1 . As a corollary, we describe a relation among the von Neumann rho-invariant, the first Morita-Mumford class and the Rochlin invariant in a framework of the bounded cohomology.

1. Introduction. Let M be an oriented closed Riemannian 3-manifold. Then we can define the η -invariant $\eta(M)$ of the signature operator. If we are given a surjective homomorphism from $\pi_1 M$ to a discrete group Γ , we have a Γ -covering $\bar{M} \rightarrow M$ and we can lift the metric and the signature operator to \bar{M} . In this situation, the von Neumann η -invariant $\eta^{(2)}(\bar{M})$ is defined for \bar{M} . Cheeger and Gromov showed in [3] that the difference $\eta^{(2)} - \eta$ is independent of a Riemannian metric. This topological invariant $\eta^{(2)} - \eta$ is called the *von Neumann rho-invariant* and is denoted by $\rho^{(2)}(\bar{M})$.

Recently, an approximation theorem of the η -invariants is shown by Vaillant [11] and Lück-Schick [7]. To be more precise, for a sequence of normal subgroups $\Gamma \triangleright \Gamma_1 \triangleright \Gamma_2 \triangleright \cdots$ such that $[\Gamma : \Gamma_k] < \infty$ and $\bigcap_k \Gamma_k = \{e\}$, and Γ/Γ_k -coverings $M^k = \bar{M}/\Gamma_k \rightarrow M$, it holds that

$$\eta^{(2)}(\bar{M}) = \lim_{k \rightarrow \infty} \frac{\eta(M^k)}{[\Gamma : \Gamma_k]}.$$

Applying this formula to surface bundles over the circle S^1 , we can describe the von Neumann rho-invariant by virtue of Meyer's signature cocycle [8] (see Proposition 2.1).

Let Σ_g be an oriented closed surface of genus g ($g \geq 2$) and \mathcal{M}_g its mapping class group. Namely, \mathcal{M}_g is the group of all isotopy classes of orientation preserving diffeomorphisms of Σ_g . Then the *first Morita-Mumford class* $e_1 \in H^2(\mathcal{M}_g, \mathbf{Z})$ (see [10]) is defined to be the Gysin image (integration along the fiber) of the square of the Euler class of the central extension

$$0 \rightarrow \mathbf{Z} \rightarrow \mathcal{M}_{g,1} \rightarrow \mathcal{M}_{g,*} \rightarrow 1.$$

Here $\mathcal{M}_{g,1}$ is the mapping class group of Σ_g relative to an embedded disc $D \subset \Sigma_g$ and $\mathcal{M}_{g,*}$ denotes the one relative to a base point $* \in D$. It is also known that e_1 is a bounded

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cohomology class. For a technical reason (see Section 3), we consider e_1 on $\mathcal{M}_{g,*}$ rather than on \mathcal{M}_g .

In general, the pull back of e_1 via a holonomy homomorphism $f : \pi_1 S^1 \rightarrow \mathcal{M}_{g,*}$ of a Σ_g -bundle over S^1 is automatically vanishing, because $H^2(S^1, \mathbf{Z})$ is trivial. However, Kitano showed in [6] that $e_1/48$ makes sense as a bounded cohomology class in $H_b^2(S^1, \mathbf{Z})$ and furthermore it is essentially given by the Rochlin invariant μ , if the image of f is contained in the Torelli subgroup $\mathcal{I}_{g,*} = \text{Ker}\{\mathcal{M}_{g,*} \rightarrow \text{Sp}(2g, \mathbf{Z})\}$. Combining our formula for $\rho^{(2)}$ with a result of Kitano, we have the following result on the level 2 subgroup $\mathcal{M}_{g,*}(2) = \text{Ker}\{\mathcal{M}_{g,*} \rightarrow \text{Sp}(2g, \mathbf{Z}/2)\} \supset \mathcal{I}_{g,*}$.

THEOREM 1.1. *Let $f : \mathbf{Z} \rightarrow \mathcal{M}_{g,*}(2)$ be a holonomy homomorphism. Then the first Morita-Mumford class $f^*e_1/48 \in H_b^2(\mathbf{Z}, \mathbf{Z})$ is represented by $\mu f - \rho^{(2)} f/16 \in H^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z})$.*

REMARK 1.2. It is known that $H_b^2(\mathbf{Z}, \mathbf{Z}) \cong H_b^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z}) \cong H^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z}) \cong \mathbf{R}/\mathbf{Z}$. See [4], [6] for the proof.

As was mentioned in [6], $e_1/48$ and μ depend on a fixed spin structure on the fiber Σ_g . In principle, our theorem implies that the difference between the first Morita-Mumford class and the Rochlin invariant does not depend on a spin structure, and it is given by the von Neumann rho-invariant. In particular, we see from Corollary 2.4 that vanishing of the von Neumann rho-invariant gives a description of the first Morita-Mumford class via the Rochlin invariant on the Torelli group $\mathcal{I}_{g,*}$.

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2. A formula of $\rho^{(2)}$. In this section, we give a formula for the von Neumann rho-invariant $\rho^{(2)}$ of surface bundles over S^1 . As for the precise definition, see Cheeger-Gromov [3]. We remark that it does not depend on a Riemannian metric on the manifold.

Let M_φ be the mapping torus $\Sigma_g \times \mathbf{R}/(x, t) \sim (\varphi(x), t + 1)$ corresponding to a holonomy $\varphi \in \mathcal{M}_{g,*}$. Let $\mathbf{Z} \rightarrow \bar{M}_\varphi \rightarrow M_\varphi$ be the \mathbf{Z} -covering associated to the surjective homomorphism $p : \pi_1 M_\varphi \rightarrow \pi_1 S^1 \cong \mathbf{Z}$.

PROPOSITION 2.1. *The von Neumann rho-invariant of \bar{M}_φ is given by*

$$\rho^{(2)}(\bar{M}_\varphi) = - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k-1} \text{sign}(\varphi, \varphi^i),$$

where sign is Meyer's signature 2-cocycle [8] of the mapping class group $\mathcal{M}_{g,*}$.

PROOF. Set $M_\varphi^k = \bar{M}_\varphi/k!\mathbf{Z}$. It is easy to see that $\mathbf{Z}/k! \rightarrow M_\varphi^k \rightarrow M_\varphi$ is the $\mathbf{Z}/k!$ -covering associated to a homomorphism $p_k : \pi_1 M_\varphi \rightarrow \mathbf{Z}/k!$. We then apply an approximation theorem of the η -invariants, due to Vaillant [11] and Lück-Schick [7], to the sequence $\mathbf{Z} \triangleright 2!\mathbf{Z} \triangleright 3!\mathbf{Z} \triangleright \dots$. It follows that

$$\begin{aligned} \rho^{(2)}(\bar{M}_\varphi) &= \eta^{(2)}(\bar{M}_\varphi) - \eta(M_\varphi) = \lim_{k \rightarrow \infty} \frac{\eta(M_\varphi^k)}{k!} - \eta(M_\varphi) \\ &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{k!} \eta(M_{\varphi^{k!}}) - \eta(M_\varphi) \right\}. \end{aligned}$$

Since $M_{\varphi^{k!}} \rightarrow M_\varphi$ is a $k!$ -fold cyclic covering, we can directly apply our previous results to this covering (see [9] Propositions 2.2 and 3.1). Hence we obtain

$$\rho^{(2)}(\bar{M}_\varphi) = \lim_{k \rightarrow \infty} \frac{1}{3k!} \sum_{i=1}^{k!-1} c(\varphi, \varphi^i),$$

where c denotes Atiyah's 2-cocycle [1] of the mapping class group defined by the canonical 2-framing of 3-manifolds. Furthermore it is known that the cocycle c coincides with $-3 \times$ Meyer's signature cocycle sign (see [10] for instance). Therefore we have

$$\rho^{(2)}(\bar{M}_\varphi) = - \lim_{k \rightarrow \infty} \frac{1}{k!} \sum_{i=1}^{k!-1} \text{sign}(\varphi, \varphi^i) = - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k-1} \text{sign}(\varphi, \varphi^i). \quad \square$$

REMARK 2.2. By definition, Meyer's cocycle is a bounded 2-cocycle, so that the above limit exists. Moreover this defines a class function on $\mathcal{M}_{g,*}$. We also remark that the above formula holds for torus bundles (that is, $g = 1$).

EXAMPLE 2.3. Let us consider the genus one case. As is well-known, in this case $\mathcal{M}_{1,*} \cong SL(2, \mathbf{Z})$ holds. An element $A \in SL(2, \mathbf{Z})$ is classified by its trace into the following three cases:

(i) Elliptic case (namely, $|\text{tr } A| < 2$). Let $A_n \in SL(2, \mathbf{Z})$ have the order n ($n = 3, 4, 6$). We can take

$$A_3 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

An easy calculation shows that $\text{sign}(A_n, A_n) = \dots = \text{sign}(A_n, A_n^{n-2}) = -2$ and $\text{sign}(A_n, A_n^{n-1}) = \text{sign}(A_n, A_n^n) = 0$ (see [8]). Hence we have

$$\rho^{(2)}(\bar{M}_{A_n}) = \begin{cases} 2/3, & n = 3, \\ 1, & n = 4, \\ 4/3, & n = 6. \end{cases}$$

It should be noted that $\rho^{(2)}(\bar{M}_\varphi) = 0$ for any involution $\varphi \in \mathcal{M}_{g,*}$ (see [9]).

(ii) Parabolic case (namely, $|\text{tr } A| = 2$). We can take

$$A_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad (b \in \mathbf{Z}).$$

Then we obtain $\rho^{(2)}(\bar{M}_{A_b}) = -\text{sgn}(b)$, where $\text{sgn}(b) = b/|b|$ if $b \neq 0$ and 0 if $b = 0$.

(iii) Hyperbolic case (namely, $|\text{tr } A| > 2$). Since Meyer's function of genus one, that is a class function $\phi : SL(2, \mathbf{Z}) \rightarrow (1/3)\mathbf{Z}$ such that $\delta\phi = \text{sign}$, satisfies $\phi(A^k) = k\phi(A)$ for

a hyperbolic element A (see [8]), we have

$$\rho^{(2)}(\bar{M}_A) = - \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k-1} \text{sign}(A, A^i) = - \lim_{k \rightarrow \infty} \left\{ \phi(A) - \frac{1}{k} \phi(A^k) \right\} = 0.$$

COROLLARY 2.4. *If φ is an element of the Torelli group $\mathcal{I}_{g,*}$, then $\rho^{(2)}(\bar{M}_\varphi) = 0$.*

PROOF. Meyer’s 2-cocycle sign is originally defined on the Siegel modular group $\text{Sp}(2g, \mathbf{Z})$, and $\mathcal{I}_{g,*}$ is the kernel of the homomorphism $\mathcal{M}_{g,*} \rightarrow \text{Sp}(2g, \mathbf{Z})$, so that $\text{sign}(\varphi, \varphi^i)$ vanishes for any i . \square

3. Morita-Mumford class and Rochlin invariant. In this section, we summarize a work of Kitano [6], which gives a description of the Rochlin invariant as a secondary characteristic class within a framework of the bounded cohomology H_b^* . As for the definition of H_b^* , see Gromov [5].

Let (M, α) be an oriented spin 3-manifold with a spin structure α . It is a classical result that there exists a compact oriented spin 4-manifold (W, β) such that $\partial W = M$ and $\beta|_M = \alpha$. The *Rochlin invariant* $\mu(M, \alpha) \in \mathbf{Q}/\mathbf{Z}$ is defined by

$$\mu(M, \alpha) = \frac{\text{Sign } W}{16} \pmod{\mathbf{Z}},$$

where $\text{Sign } W$ denotes the signature of a 4-manifold W . By Rochlin’s theorem, $\mu(M, \alpha)$ does not depend on the choice of W .

Let us fix a spin structure α of Σ_g ($g \geq 2$). For each $\varphi \in \mathcal{M}_{g,*}(2)$, there exists a spin structure $\tilde{\alpha}$ on M_φ such that the restriction on each fiber is α . If we require that the restriction of $\tilde{\alpha}$ to the S^1 -orbit of $*$ $\in \Sigma_g$ is the bounding spin structure (namely, not the Lie group spin structure), then $\tilde{\alpha}$ is uniquely determined. This is the reason why we consider $\mathcal{M}_{g,*}(2)$ rather than \mathcal{M}_g .

Now consider the set of pairs $\{(\varphi, W)\}$, where $\varphi \in \mathcal{M}_{g,*}(2)$ and W is an oriented spin 4-manifold. Of course, $\partial W = M_\varphi$ and the induced spin structure on M_φ is $\tilde{\alpha}$. Two pairs (φ, W) and (φ, W') are said to be equivalent if $\text{Sign } W = \text{Sign } W'$. The set of equivalence classes, which we denote by $\mathcal{M}_{g,*}^\alpha(2)$, has a group structure defined by the fiber connected sum (see [6] for details), and there is a natural surjective homomorphism

$$\mathcal{M}_{g,*}^\alpha(2) \rightarrow \mathcal{M}_{g,*}(2)$$

given by $(\varphi, W) \mapsto \varphi$. Moreover we introduce a map $\tau : \mathcal{M}_{g,*}^\alpha(2) \rightarrow \mathbf{Q}$ by

$$\tau(\varphi, W) = \frac{1}{16} \text{Sign } W \in \frac{1}{16} \mathbf{Z} \subset \mathbf{Q}.$$

PROPOSITION 3.1 (Kitano [6]). *Under the setting above, the following hold.*

(i) *The Euler class e_α of the extension $\mathcal{M}_{g,*}^\alpha(2) \rightarrow \mathcal{M}_{g,*}(2)$ is a bounded cohomology class and is given by $e_1/48$ on the level 2 subgroup $\mathcal{M}_{g,*}(2)$.*

(ii) *Let $f : \mathbf{Z} \rightarrow \mathcal{M}_{g,*}(2)$ be a homomorphism. The pull back $f^*e_\alpha \in H_b^2(\mathbf{Z}, \mathbf{Z})$ is described by $\bar{\tau}_\infty f \in H^1(\mathbf{Z}, \mathbf{R}/\mathbf{Z})$, where $\bar{\tau}_\infty : \mathcal{M}_{g,*}(2) \rightarrow \mathbf{R}/\mathbf{Z}$ is the reduction mod \mathbf{Z} of*

a map

$$\tau_\infty(\varphi, W) = \lim_{k \rightarrow \infty} \frac{\tau((\varphi, W)^k)}{k} \in \mathbf{R}.$$

From the additivity of the signature, we obtain

$$\tau_\infty(\varphi, W) = \tau(\varphi, W) + \frac{1}{16} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^{k-1} \text{sign}(\varphi, \varphi^i).$$

Taking the reduction mod \mathbf{Z} and using Proposition 2.1, we have

$$\bar{\tau}_\infty(\varphi) = \mu(M_\varphi, \tilde{\alpha}) - \frac{1}{16} \rho^{(2)}(\bar{M}_\varphi) \in \mathbf{R}/\mathbf{Z}$$

for $\varphi \in \mathcal{M}_{g,*}(2)$. Therefore Theorem 1.1 follows from Proposition 3.1 and the proof is completed.

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