The definite quadratic forms in eight variables with determinant unity;

By L.-J. MORDELL,

Let

$$(1) \qquad f(x) = \sum_{r,s=1}^{n} a_{rs} x_r x_s \qquad (a_{rs} = a_{sr})$$

be a positive definite quadratic form with integer coefficients and determinant

$$(2) \qquad A = \| a_{rs} \|.$$

We consider the special case when $A = 1$. It is well known that then there is for each value of $n \leqq 7$ exactly one class of nonequivalent forms, namely

$$(3) \qquad \sum_{r=1}^{n} x_r^2.$$

This result was given by Hermite (¹) for $3 \leqq n \leqq 8$ but his proof for $n = 7, 8$ was vitiated by a numerical error. Stouff, however, has verified the result for $n = 7$. Minkowski (²) proved in 1882 that the result

(¹) *Œuvres de Charles Hermite*, I, 1905, p. 122-130. *See* in particular, the footnote, p. 129.

See also BACHMANN, *Die Arithmetik der quadratischen Formen*, 2, 1923, p. 350-358. On page 356, he reproduces Hermite's mistake and $C < 7,50$ should be $C \leqq 8,76$ and so $C = 8$.

(²) *Gesammelte Abhandlungen von Hermann Minkowski*, I, 1909, p. 77, or *Mémoires présentés par divers savants à l'Académie des Sciences de l'Institut national de France*, 29, 1884.

was false for $n=8$ by giving an improperly primitive form of determinant unity. Dickson (¹) credits him with having corrected the error of including $n=8$. They have both apparently overlooked a simpler form of determinant unity given by Korkine (²) and Zolotareff in 1873, in connection with their theory of *extreme* forms, namely,

$$\sum_1^8 x_r^2 + \left(\sum_1^8 x_r\right)^2 - 2x_1x_2 - 2x_2x_8. \tag{4}$$

Some recent arithmetical work on the representation of quadratic forms as sums of squares of linear forms with integer coefficients, in which Mr. Chao Ko and myself have been interested, suggested the desirability of investigating the class number for forms in eight variables with determinant unity. There is no theoretical difficulty attached to finding it by the method of Hermite, but a great deal of arithmetical work is involved. This can now be avoided by making use of two deep theorems in the theory of quadratic forms recently published. I prove the following

THEOREM. — *There are exactly two classes of forms in eight variables of determinant unity, namely the properly primitive class* $\sum_1^8 x_r^2$, *and the improperly primitive class* $\sum_1^8 x_r^2 + \left(\sum_1^8 x_r\right)^2 - 2x_1x_2 - 2x_3x_8$.

The first theorem used is that $f(x)$ is equivalent to a form in which

$$a_{11} \leq \sqrt[n]{\lambda_n \Delta}, \tag{5}$$

where $\lambda_6 = 64/3$, $\lambda_7 = 64$, $\lambda_8 = 256$. This is given by Blichfeldt (³), but Hofreiter (⁴), also gave the value of λ_8. The second theorem is due to the latter and states that if the equality sign in (5) holds for $n=6$,

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(¹) *History of the Theory of Numbers*, 3, 1923, p. 235.
(²) *Mathematische Annalen*, 6, 1873, p. 366-389.
(³) *Mathematische Zeitschrift*, 39, 1934, p. 1-15.
(⁴) *Monatsheft für Mathematik und Physik*, 40, 1933, p. 129-152.

then $f(x)$ is equivalent to the form

$$(6)\qquad \sqrt[8]{\frac{A}{3}}\left[\sum_1^8 x_r^2+\left(\sum_1^6 x_r\right)^2-2x_1x_2-2x_7x_8\right].$$

For $A=3$, this can be written as

$$(7)\qquad 2\left(x_1+\frac{1}{3}(x_3+x_4+x_5+x_6)\right)^2+2\left(x_2+\frac{1}{3}(x_3+x_4+x_5)\right)^2$$
$$+\left(x_3+\frac{1}{2}x_6\right)^2+\left(x_4+\frac{1}{2}x_6\right)^2+\left(x_5+\frac{1}{2}x_6\right)^2+\frac{3}{4}x_6^2.$$

I may note that by using this result and proceeding rather differently than herein, Mr Ko has at the same time as myself proved that there is one class of properly primitive forms in eight variables with determinant unity.

Let

$$(8)\qquad F(x)=\sum A_{rs}x_rx_s,$$

where

$$(9)\qquad A_{11}=\|a_{rs}\|\qquad (r, s=2, 3, \dots, n).$$

etc.,
be the adjoint form of $f(x)$.

It is easy to see that if any a_{rr} or A_{rr} is equal to unity, then both $f(x)$, $F(x)$ are equivalent to $\sum_1^8 x_r^2$, since the class number for the definite form of seven variables with determinant unity is one. We seek now the definite forms $f(x)$ with determinant $A=1$. From (5), we may assume $a_{11}\leq\sqrt[8]{256}$, i. e. $a_{11}=1$, 2, and need only consider $a_{11}=2$. Hence

$$2f(x)=(2x_1+a_{12}x_2+a_{13}x_3+\dots)^2+g(x),$$

where

$$g(x)=b_{22}x_2^2+2b_{23}x_2x_3+\dots.$$

Now the determinant of $g(x)$ is $2^8A/2^2=64$, and hence we may suppose $g(x)$ equivalent to a form in which $b_{22}<\sqrt[7]{2^6 2^6}<4$.

Then on replacing x_1 by x_1+bx_2, we may suppose $a_{12}=0, 1$. But if $b_{22} = 0, 1, 2,$

$$a_{22} = \frac{1}{2}(a_{12}^2 + b_{22}) \leqq \frac{3}{2},$$

and so $a_{22} = 1$. Hence we need only consider $b_{22} = 3$, and then $a_{12} = 1$, and

$$f(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_1(a_{13}x_3 + \ldots) + 2x_2(a_{23}x_3 + \ldots) + \ldots.$$

Take now the part $h(x)$, not involving terms x_1, x_2, of the adjoint form of $f(x)$. Its determinant is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \Delta^5 = 3.$$

Hence $h(x)$ is equivalent to a form in which $a_{33} \leqq \sqrt[6]{64} = 2$, and we need only consider the case $a_{33} = 2$. Then $h(x)$ is equivalent to the extreme form (7). We now apply a linear transformation in only the six variables $x_3, \ldots, x_8$, transforming $h(x)$ into (7), with, however, variables $x_3, x_4, \ldots,$ and this leaves unaltered the three terms $2x_1^2 + 2x_1x_2 + 2x_2^2$ in the new $f(x)$. Let the adjoint form of the new $f(x)$ have $k(x)$, say, for the part independent of x_1. The determinant of $k(x)$ is

$$2 = a_{11}\Delta^6 = \| A_{rs} \| \qquad (r, s = 2, 3, \ldots, 8).$$

We can now construct $k(x)$ knowing its determinant 2 and (7) the part independent of x_1, say. From (7), on permuting the variables it must take the form

$$2\left(x_1 + \frac{1}{2}(x_3 + x_4 + x_5 + x_6) + c_1x_7\right)^2 + 2\left(x_2 + \frac{1}{2}(x_3 + x_4 + x_5) + c_2x_7\right)^2$$
$$+ \sum_{r=3,4,5} \left(x_r + \frac{1}{2}x_6 + c_rx_7\right)^2 + \frac{3}{4}(x_6 + c_6x_7)^2 + c_7x_7^2,$$

where the c's are constant. From its determinant, $c_7 = \frac{2}{3}$. Also the coefficients of x_1x_7 etc., must be even integers. Hence (all mod 1),

$$2c_1 \equiv 0, \qquad 2c_2 \equiv 0, \qquad c_1 + c_2 + c_r \equiv 0 \qquad (r = 3, 4, 5),$$

$$c_1 + \frac{1}{2}(c_3 + c_4 + c_5) + \frac{3}{4}c_6 \equiv 0,$$

$$2c_1^2 + 2c_2^2 + c_3^2 + c_4^2 + c_5^2 + \frac{3}{4}c_6^2 + \frac{2}{3} \equiv 0.$$

Hence $3c_6 \equiv 0$ and since $2c_r \equiv 0$ $(r=1, 2, 3, 4, 5)$, the last two equations show that $c_6 = l/3$, where l is an integer $\not\equiv 0 \bmod 3$. By replacing x_6 by $x_6 - \lambda_6 x_7$, we may take $l = \pm 2$.

Since $c_3 \equiv c_4 \equiv c_5 \equiv 0$ or $\frac{1}{2}$, we may, on replacing x_3, x_4, x_5 by $x_3 - \lambda_3 x_7$ etc., suppose that

$$c_3 = c_4 = c_5 = c,$$

where $c = 0$ or $\frac{1}{2}$.

The first gives $c_1 \pm \frac{1}{3} \equiv 0$. Then $c_2 \equiv \pm \frac{1}{3}$ and on replacing x_1, x_2 by $x_1 - \lambda_1 x_7$ etc., we have $c_1 = \frac{1}{3}$, $c_2 = \frac{1}{3}$ and

$$(10)\qquad \begin{aligned} k(x) = {} & 2\left(x_1 + \frac{1}{3}(x_3 + x_4 + x_5 + x_6 + x_7)\right)^2 \\ & + 2\left(x_2 + \frac{1}{3}(x_3 + x_4 + x_5 + x_6)\right)^2 \\ & + \sum_{r=3,4,5}\left(x_r + \frac{1}{2}x_6\right)^2 + \frac{3}{4}\left(x_6 \pm \frac{2}{3}x_7\right)^2 + \frac{2}{3}x_7^2. \end{aligned}$$

We can take $+\frac{2}{3}$ as otherwise on replacing x_7 by $-x_7$, x_1 by $-x_1 - (x_3 + x_4 + x_5 + x_6 - x_7)$, we get $h(x)$ again. The last terms of $k(x)$ can be written also as $\left(x_7 + \frac{1}{2}x_6\right)^2 + \frac{1}{2}x_6^2$.

Next the case $c_3 = c_4 = c_5 = \frac{1}{2}$ is impossible since

$$\frac{1}{2}(c_3 + c_4 + c_5) = \frac{3}{4} \qquad \text{and} \qquad \frac{3c_6}{4} = \pm\frac{1}{2}.$$

We must now construct $F(x)$ of determinant unity knowing $k(x)$ the part given by taking $x_8 = 0$. Interchange the role of x_6, x_7, and so we must have

$$\begin{aligned} F(x) = {} & 2\left(x_1 + \frac{1}{3}(x_3 + x_4 + x_5 + x_6 + x_7) + d_1 x_8\right)^2 \\ & + 2\left(x_2 + \frac{1}{3}(x_3 + x_4 + x_5 + x_6) + d_2 x_8\right)^2 \\ & + \sum_{r=3}^{5}\left(x_r + \frac{1}{2}x_7 + d_r x_8\right)^2 + \frac{1}{2}(x_7 + d_7 x_8)^2 + d_8 x_8^2. \end{aligned}$$

where the d's are constants.

From the determinant of $F(x)$, $d_8 = \frac{1}{2}$.

From the coefficients of $x_1 x_8$ etc., we have (all mod 1)

$$2d_1 \equiv 0, \qquad 2d_2 \equiv 0, \qquad d_1 + d_2 + d_r \equiv 0 \qquad (r = 3, 4, 5, 6),$$

$$d_1 + \frac{1}{2}(d_3 + d_4 + d_5 + d_6 + d_7) \equiv 0,$$

$$2d_1^2 + 2d_2^2 + \sum_{r=3}^{6} d_r^2 + \frac{1}{2} d_7^2 + \frac{1}{2} \equiv 0.$$

From the latter, on multiplying by 2, since $d_3^2 \equiv (d_1 + d_2)^2$ etc., d_7 is an integer which on putting $x_7 = x_7 - \lambda_7 x_8$ can be taken as 0. Also $2d_1^2 + 2d_2^2 + \frac{1}{2} \equiv 0$, i. e. we can take $d_1 = \frac{1}{2}$, $d_2 = 0$ or $d_1 = 0$, $d_2 = \frac{1}{2}$.

The first leads to $d_3 = d_4 = d_5 = d_6 = \frac{1}{2}$ which does not satisfy the last equation but one above. The second leads to $d_3 = d_4 = d_5 = d_6 = \frac{1}{2}$, and gives the self-adjoint form

$$(11) \qquad F(x) = 2\left(x_1 + \frac{1}{2}(x_3 + x_4 + x_5 + x_6 + x_7)\right)^2 + 2\left(x_2 + \frac{1}{2}(x_3 + x_4 + x_5 + x_6 + x_8)\right)^2 + \sum_{r=3,4,5,6}\left(x_r + \frac{1}{2}x_7 + \frac{1}{2}x_8\right)^2 + \frac{1}{2}x_7^2 + \frac{1}{2}x_8^2$$

$$= \sum_1^8 x_r^2 + \left(\sum_1^8 x_r\right)^2 - 2x_1x_2 - 2x_1x_8.$$

This proves the result on interchanging the role x_1, x_2 and noting that this and the previous interchange of x_6, x_7 give a transformation of determinant unity. It is of interest to note that all the forms (11) or (4), (10) and (7) have been given by Korkine (4) and Zolatareff as *extreme* forms in 8, 7, 6 variables respectively.