Infinite Cyclic Coverings

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The first two sections of this manuscript are expository in nature, and describe the homology of an infinite cyclic covering of a finite complex. Section 3 is a digression concerning torsion and the zeta function. Section 4 proves a Poincaré duality theorem for infinite cyclic coverings of manifolds, and Section 5 applies this duality theorem to describe the Trotter–Murasugi signature of a knot. An appendix computes the Reidemeister, Franz, de Rham torsion associated with an infinite cyclic covering.

I want to thank J. G. Hocking for his help in the preparation of this manuscript.

1. AN ALGEBRAIC PRELIMINARY

We will need one algebraic definition before proceeding. Let M be a finitely generated module over a principal ideal domain P. Then M

is isomorphic to a direct sum of cyclic modules, say

$$M \cong \frac{P}{(p_1)} \oplus \cdots \oplus \frac{P}{(p_k)}$$

where (p_i) denotes the principal ideal spanned by an element $p_i \in P$.

DEFINITION. The product ideal $(p_1 p_2 \cdots p_k)$ is called the *order* of M.

This order function is well defined, and is multiplicative:

ASSERTION 1. *If* $M_1 \subset M_2$ *then* order M_2 = (order M_1)(order M_2/M_1).

The proof, which is not difficult, can be based either on the Jordan–Hölder theorem, or on the interpretation of order in terms of the determinants† of a relation matrix.

Here are two immediate consequences of the definition:

ASSERTION 2. *The order of M is* (1) *if and only if M is the zero module.*

ASSERTION 3. *The order of M equals* (0) *if and only if M possesses a P-free cyclic direct summand; and* order $M \neq 0$ *if and only if M is a torsion module over P.* (That is, if and only if for each $m \in M$ there exists $p \in P$ with $pm = 0$, $p \neq 0$.)

2. THE HOMOLOGY MODULE OF A COVERING

Now consider the following geometric situation. Let X be a finite connected simplicial complex or CW-complex and $\tilde{X}$ the infinite cyclic covering of X, determined by some homomorphism of the fundamental group $\pi_1 X$ onto an infinite cyclic group Π. Thus Π acts freely as the group of covering transformations of the infinite complex $\tilde{X}$, and the quotient complex $\tilde{X}/\Pi$ can be identified with X.

Choosing some coefficient field F, consider the chain complex $C_*(\tilde{X}; F)$ and its homology groups $H_i(\tilde{X}; F)$. The infinite cyclic group

† See for example Zassenhaus [26].

Π of covering transformations operates on these groups. Hence we can think of $C_i(\tilde{X}; F)$ and $H_i(\tilde{X}; F)$ either as vector spaces over F, or alternatively as modules over the group algebra $F\Pi$.

Note that $C_i(\tilde{X}; F)$ is free and finitely generated† over $F\Pi$, with one generator for each i-cell of X. Since the ring $F\Pi$ is Noetherian, it follows immediately that the homology $H_i(\tilde{X}; F)$ is also finitely generated over $F\Pi$.

In fact $F\Pi$ is a principal ideal domain. [*Proof:* If t generates the infinite cyclic group Π then every ideal $\mathfrak{a}$ of $F\Pi$ certainly intersects the polynomial ring $F[t]$ in a principal ideal. But every element of the ideal $\mathfrak{a}$ can be expressed as the product of a polynomial in t and a unit of $F\Pi$.]

Hence the order of the module $H_i(\tilde{X}; F)$ is a well-defined ideal in $F\Pi$.

Now let us forget the $F\Pi$-module structure for a moment, and think of $H_i(\tilde{X}; F)$ as a vector space over F.

ASSERTION 4. *If $H_i(\tilde{X}; F)$ is a finite dimensional vector space over F then the ideal* order $H_i(\tilde{X}; F)$ *in $F\Pi$ is non-zero, and is spanned by the characteristic polynomial of the F-linear transformation $t_*: H_i(\tilde{X}; F) \to H_i(\tilde{X}; F)$ (where $t_*(\xi) = t\xi$). On the other hand if $H_i(\tilde{X}; F)$ is infinite dimensional over F, then* order $H_i(\tilde{X}; F) = (0)$.

The proof is not difficult. (In the finite dimensional case note that the cohomology module $H^i(\tilde{X}; F)$ is also finitely generated, and that order $H^i(\tilde{X}; F) =$ order $H_i(\tilde{X}; F)$.)

Here are a few examples:

EXAMPLE 1. If $X = S^1 \vee S^2$, then $\tilde{X}$ can be visualized as an infinite string with infinitely many balloons attached (Figure 1). The covering transformation $t \in \Pi$ carries each balloon onto the next one.

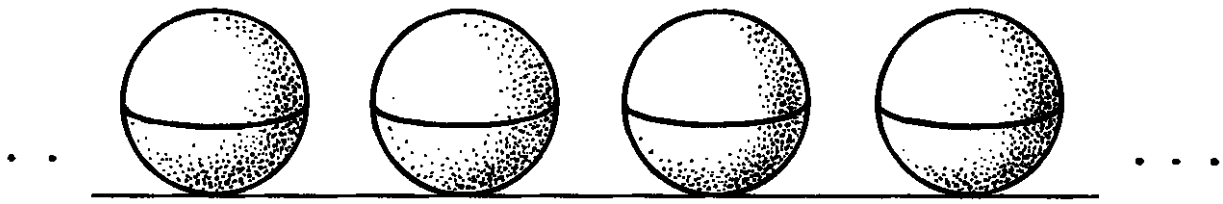

Figure 1.

† In fact $C_*(\tilde{X}; F)$ can be identified with the chains of the original complex X, taking $F\Pi$ suitably twisted as coefficient group.

Clearly $H_2(\tilde{X}; F)$ is not finitely generated over F. In fact $H_2(\tilde{X}; F)$ is free cyclic over $F\Pi$, so that order $H_2(\tilde{X}; F) = (0)$. On the other hand, $H_0(\tilde{X}; F)$ is a one-dimensional vector space over F, with Π operating trivially; so that order $H_0(\tilde{X}; F) = (t - 1)$. Finally, it is clear that order $H_i(\tilde{X}; F) = (1)$ for $i \neq 0, 2$.

EXAMPLE 2. If $X = S^1 \times S^2$ then $\tilde{X}$ has the homotopy type of S^2, and the group Π operates trivially. Hence order $H_0 =$ order $H_2 = (t - 1)$ and the other homology groups are trivial.

EXAMPLE 3. If X is a Klein bottle then $\tilde{X}$ has the homotopy type of a circle, and order $H_0 = (t - 1)$, order $H_1 = (t + 1)$; the other homology groups being trivial.

These examples suggest the general problem of deciding, for any given complex X, whether the ideals order $H_i(\tilde{X}; F)$ are zero or non-zero. Here are two partial results.

ASSERTION 5. *If X has the homology of a circle $[H_*(X; F) \cong H_*(S^1; F)]$, then the ideals* order $H_i(\tilde{X}; F)$ *are all non-zero; so that $H_*(\tilde{X}; F)$ is finitely generated over F.*

ASSERTION 6. *If $H_*(\tilde{X}; F)$ is finitely generated over F, then the euler characteristic $\chi(X)$ must be zero.*

Proof of Assertion 5: The short exact sequence

$$0 \longrightarrow C_*\tilde{X} \xrightarrow{t-1} C_*\tilde{X} \longrightarrow C_*X \longrightarrow 0$$

of chain complexes gives rise to a long exact sequence

$$\cdots \xrightarrow{\partial} H_i\tilde{X} \xrightarrow{t-1} H_i\tilde{X} \longrightarrow H_iX \xrightarrow{\partial} H_{i-1}\tilde{X} \longrightarrow \cdots$$

of homology; the coefficient group F (or any other fixed coefficient domain) being understood throughout.

For $i \geq 2$, since $H_iX = 0$, the sequence

$$H_i\tilde{X} \xrightarrow{t-1} H_i\tilde{X} \longrightarrow 0$$

asserts that every element of $H_i\tilde{X}$ is divisible by $t - 1$. Hence $H_i\tilde{X}$ cannot admit any $F\Pi$-free cyclic direct summand. And, since $H_0X \cong H_1X \cong H_0\tilde{X} \cong F$, our homology sequence evidently terminates with

$$\xrightarrow{0} H_1X \xrightarrow{\cong} H_0\tilde{X} \xrightarrow{0} H_0\tilde{X} \xrightarrow{\cong} H_0X \longrightarrow 0.$$

Therefore a similar argument works for $i = 1$.

Proof of Assertion 6: If $H_*\tilde{X}$ is finitely generated over F then the sequence

$$\cdots \xrightarrow{\partial} H_i\tilde{X} \longrightarrow H_i\tilde{X} \longrightarrow H_iX \xrightarrow{\partial} H_{i-1}\tilde{X} \longrightarrow$$

implies by a standard argument that $\chi(\tilde{X}) = \chi(\tilde{X}) + \chi(X)$, so that $\chi(X) = 0$.

[*Remark:* Our exact sequence can also be formulated as a "universal coefficient theorem"

$$0 \to H_i\tilde{X} \underset{F\Pi}{\otimes} F \to H_iX \to \mathrm{Tor}_{F\Pi}(H_{i-1}(\tilde{X}, F)) \to 0$$

which is the analogue, for the prime element $t - 1 \in F\Pi$, of the usual theory associated with a prime element $p \in Z$. Similarly one could construct a "Bockstein spectral sequence" relating H_*X to the $F\Pi$-free part of $H_*\tilde{X}$. Compare [3].]

EXAMPLE 4. Let $f: S^n \to S^{n+2}$ be a nice (say a differentiable) embedding of the sphere S^n into S^{n+2}. Remove from S^{n+2} an open tubular neighborhood of $f(S^n)$ to obtain the space X. Then X can be triangulated as a finite complex, and by Alexander duality X has the homology of S^1. Thus Assertion 5 applies; so $H_*\tilde{X}$ is finitely generated over F.

As usual, the module $H_0\tilde{X} \cong F$ has order ideal $(t - 1)$.

The module $H_1\tilde{X}$ depends only on the fundamental group of X. In fact note that the corresponding integral homology group $H_1(\tilde{X}; Z)$ can be identified with the abelianized commutator subgroup of π_1X.

DEFINITION. Any generator of the ideal order $H_1(\tilde{X}; F)$ is called the *Alexander polynomial* of π_1X. (See for example Crowell [5].)

In the case $n = 1$ of classical knot theory the modules $H_i\tilde{X}$ are trivial for $i \geq 2$; but for $n > 1$ we obtain further invariants $H_2\tilde{X}, \ldots, H_n\tilde{X}$ of our knotted n-sphere. These have been studied by Levine [13].

To what extent do the statements above depend on the use of field coefficients? If we use the integers Z as coefficient domain, then the group ring $Z\Pi$ is still Noetherian. Hence $H_i(\tilde{X}; Z)$ is still finitely generated (and a torsion module) over $Z\Pi$. However there is no really satisfactory theory of finitely generated $Z\Pi$-modules. And, even if $H_i(\tilde{X}; Z)$ happens to be well behaved over $Z\Pi$, it may not be finitely generated over Z.

To illustrate this point consider the two knots of Figure 2. Following Alexander and Briggs these knots are called 3_1 and 5_2 respectively. (Compare [20].)

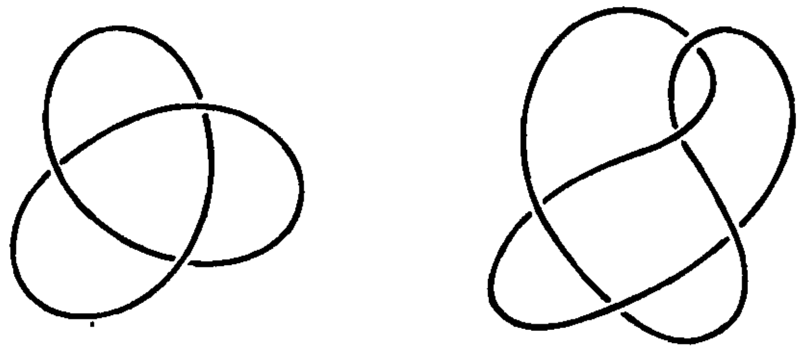

Figure 2.

In the case of the trefoil knot 3_1, Stallings and Neuwirth [18] have shown that the complementary complex X is a fibre bundle over the circle, each fibre M^2 being a surface of genus 1 bounded by a circle (i.e., M^2 is a torus with an open disk removed). Thus the covering $\tilde{X}$ is homeomorphic to the product $M^2 \times R$, and $H_1(\tilde{X}; Z) \cong H_1(M^2; Z) \cong Z \oplus Z$. This group certainly is finitely generated over Z. (Over the group ring $Z\Pi$, the homology $H_1(\tilde{X}; Z)$ can be described as the $Z\Pi$-cyclic module of order $(t^2 - t + 1)$.)

But for the knot 5_2 there is no such fibration, and the homology $H_1(\tilde{X}; Z)$ is not finitely generated over Z. In fact $H_1(\tilde{X}; Z)$ can be described over Z as a torsion-free group of rank 2 in which every element is divisible by 2. (Compare [19], [6].)

Over $Z\Pi$, however, there is still a tidy description: the module $H_1(\tilde{X}; Z)$ is $Z\Pi$-cyclic† of order $(2t^2 - 3t + 2)$.

Presumably, for a suitably chosen knot, the module $H_1(\tilde{X}; Z)$ cannot even be described as a direct sum of $Z\Pi$-cyclic modules?

I am grateful to H. Trotter for pointing out that an example of such a knot has been described by R. H. Fox and N. Smythe, "An ideal class invariant of knots," Proc. Amer. Math. Soc., 15 (1964), 707–709. In fact Fox and Smythe define an invariant called the row class of a moldule, which is clearly trivial in the case of a direct sum of cyclic modules. But in the case of a "pretzel knot" with crossing numbers 25, −3, 13, thinking of $H_1(\tilde{X}; Z)$ as a module over the

† More generally, whenever the group $\pi_1 X$ is generated by two elements one can verify that $H_1(\tilde{X}; Z)$ is $Z\Pi$-cyclic.

quotient ring $Z\Pi/[A(t)]$, they show that the row class of $H_1(\tilde{X}; Z)$ is nontrivial. (Here $[A(t)]$ denotes the prime ideal spanned by the Alexander polynomial $A(t) = 53t^2 - 105t + 53$.) It follows that $H_1(\tilde{X}; Z)$ is not a direct sum of $Z\Pi$-cyclic modules.

3. A DIVERSION

I want to spend a few minutes to describe a remarkably useless theorem: namely, I will show that the "zeta function" of André Weil is almost the same thing as the "torsion" invariant of Reidemeister, Franz, and de Rham. Presumably this result is of interest only as an example of the way in which an algebraic formalism can recur in widely separated areas of mathematics.

First recall the concept of torsion. (For a list of references, see [16].) Given a finite complex X of dimension n and a homomorphism h from $\pi_1 X$ to the group of units of a commutative ring P, one can consider the associated system of local coefficients P_h over X and the twisted homology $H_*(X, P_h)$. If $H_*(X, P_h) = 0$ then the *torsion* $\tau(X, h)$ is a unit of P, well-defined up to multiplication by $\pm h(\pi_1 X)$.

In particular suppose that we are given an infinite cyclic covering $\tilde{X}$ of X with $H_*(\tilde{X}; F)$ finitely generated over F. As ring P we choose the quotient field $F(t)$ consisting of all rational functions $f(t)/g(t)$ in the indeterminate t. Let $h\colon \pi_1 X \to F(t)$ be the composition of the homomorphism $\pi_1 X \to \Pi$ associated with our covering and the inclusion $\Pi \subset \text{Units}\,[F(t)]$. Then clearly the chain complex

$$C_*[X; F(t)_h] \cong F(t) \otimes_{F\pi} C_*(\tilde{X}; F)$$

has trivial homology. Hence the Reidemeister, Franz, de Rham torsion $\tau(X, h) \in F(t)$ is well-defined up to multiplication by units $\pm t^i$ of $Z\Pi$.

ASSERTION 7. *The torsion invariant $\tau(X, h)$ associated with our covering is equal to a unit of $F\Pi$ times the alternating product*[†]

$$f_0(t) f_1(t)^{-1} f_2(t) f_3(t)^{-1} \cdots f_n(t)^{\pm 1},$$

where $f_j(t)$ denotes the characteristic polynomial of the F-linear transformation

$$t_*\colon H_j(\tilde{X}; F) \to H_j(\tilde{X}; F).$$

[†] Special cases of this formula have been proved by Kervaire [12] and myself [15]. (Note that our τ corresponds to Δ^{-1} in [15].)

In more concise language this formula can be expressed as follows: The fractional ideal $(\tau) = (F\Pi)_T$ in $F(t)$ is equal to the quotient order $H_{\text{even}}\tilde{X}/\text{order } H_{\text{odd}}\tilde{X}$.

A proof will be given in the appendix.

Now let me recall the Weil zeta function (in a topological context rather than the number theoretic context in which Weil introduced it). Let K be a finite complex and $g: K \rightarrow K$ a continuous mapping. We would like to count the number of fixed points of g and of its various iterates:

$$g^2(x) = g[g(x)], g^3(x) = g(g[g(x)]), \ldots .$$

But this is usually too difficult,† so instead we look at the Lefschetz numbers $L(g\ )$ of the iterates. (If g has only finitely many fixed points, then $L(g)$ can be interpreted as the "algebraic number" of fixed points, each being counted with a suitable multiplicity.) We consider the sequence $L(g), L(g^2), L(g^3), \ldots$ of integers and try to build some pattern out of it.

According to Weil [25] the useful way to do this is by means of the zeta function

$$\zeta(s) = \exp\left(\sum_{\nu \geq 1} L(g^\nu) \frac{s^\nu}{\nu} \right).$$

This expression defines a formal power series in an indeterminate s which is concocted in such a way that

(1) knowing $\zeta(s)$ is equivalent to knowing all of the Lefschetz numbers $L(g^\nu)$,

(2) $\zeta(s)$ can be described in a simple way in terms of the action of f_* on the homology of H_*K in each dimension.

In fact Weil showed that $\zeta(s)$ is a rational function of the form

$$\zeta(a) = p_0(s)^{-1}p_1(s)p_2(s)^{-1}p_3(s)\cdots p_n(s)^{\pm 1},$$

where each $p_i(s)$ is a polynomial closely related to the characteristic polynomial of the linear transformation $g_*: H_i(K; Q) \rightarrow H_i(K; Q)$. More precisely, if the characteristic polynomial is

$$f_i(s) = s^k + a_1 s^{k-1} + \cdots + a_k$$

then $p_i(s) = 1 + a_1 s + \cdots + a_k s^k = s^k f_i(s^{-1})$ where k denotes the ith Betti number of K.

† Compare Artin and Mazur [1].

Introducing the abbreviation $\tau(s) = f_0(s)f_1(s)^{-1}\cdots f_n(s)^{\pm 1}$ in analogy with Assertion 7, the relation between τ and ζ can be expressed by the concise formula $\zeta(s^{-1})\tau(s) = s^{\chi(Y)}$ where $\chi(Y)$ denotes the euler characteristic.

(For example, if we map an even-dimensional sphere to itself with degree d, so that $L(g^\nu) = 1 + d$, then

$$\tau(s) = (s - 1)(s - d),$$
$$\zeta(s) = (1 - s)^{-1}(1 - ds)^{-1},$$

and $\zeta(s^{-1})\tau(s) = s^2$.)

4. POINCARÉ DUALITY

Next I want to describe a Poincaré duality theorem for infinite cyclic coverings. Suppose that M is a compact, connected, triangulated† n-manifold without boundary, and let $\tilde{M}$ be an infinite cyclic covering of M. We will assume that $\tilde{M}$ is orientable.

Taking coefficients in a field F, there are two possibilities. The homology groups $H_i(\tilde{M}; F)$ may be finite dimensional vector spaces over F, or some $H_i(\tilde{M}; F)$ may be an infinite dimensional vector space. We will be interested in the first possibility only.

DUALITY THEOREM. *If $H_*(\tilde{M}; F)$ is finitely generated over F, then $H^{n-1}(\tilde{M}; F)$ is one-dimensional over F, and the vector spaces $H^i(\tilde{M}; F)$ and $H^{n-1-i}(\tilde{M}; F)$ are dual to each other, being orthogonally paired to $H^{n-1}(\tilde{M}; F) \cong F$ by the cup product pairing.*

In other words $\tilde{M}$ has the homology properties of a compact manifold of dimension $n - 1$.

Remark: In many cases $\tilde{M}$ actually splits as the cartesian product of an $(n - 1)$-dimensional manifold and the real line. For example, Siebenmann and Novikov have proved that such a splitting occurs whenever $\tilde{M}$ is dominated by a finite complex and has free abelian fundamental group, providing that $n \geq 6$. (Compare [9], [22], as well as [4].) The present Duality Theorem has a weaker hypothesis and a correspondingly weaker conclusion.

† The triangulation will be used in the proof, although I conjecture that it shouldn't be needed.

The proof will depend on H. Hopf's assertion that any infinite cyclic covering of a finite complex must have exactly two "ends." (See [7], [11].)

DEFINITION. An *end* ϵ of a locally compact space Y is a function which assigns to each compact subset $K \subset Y$ precisely one component $\epsilon(K)$ of the complement $Y - K$, subject to the requirement that $\epsilon(K) \supset \epsilon(L)$ whenever $K \subset L$. Any set $N \subset Y$ which contains some $\epsilon(K)$ is called a *neighborhood* of the end ϵ.

The *cohomology*† *group* $H^i(Y, \epsilon)$ of Y modulo an end is defined to be the direct limit of the groups $H^i(Y, N)$ as N ranges over all neighborhoods of ϵ, using for example singular cohomology theory.

Taking coefficients in a field F, we will prove:

ASSERTION 8. *If $\tilde{X}$ is the infinite cyclic covering of a finite complex and if $H_*(\tilde{X}; F)$ is finitely generated over F, then $H^*(\tilde{X}, \epsilon; F) = 0$, $H^*(\tilde{X}, \epsilon'; F) = 0$, where ϵ and ϵ' denote the two ends of $\tilde{X}$.*

Proof: Choose a finite subcomplex $K \subset \tilde{X}$ so that $\tilde{X}$ is covered by the translates $\ldots t^{-1}K, K, tK, t^2K, \ldots$ under the group Π of covering transformations (Figure 3). Note that $t^iK \cap t^jK = \varnothing$ for

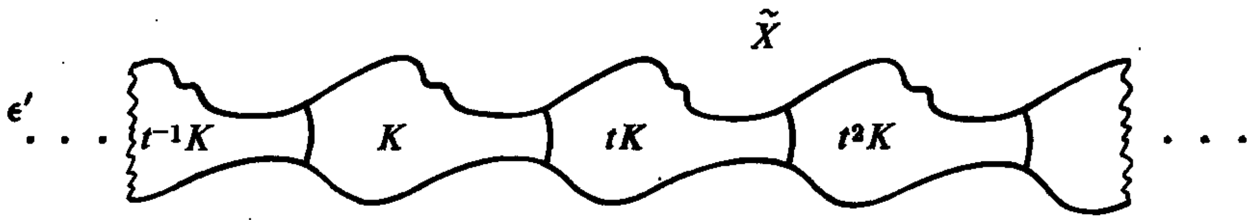

Figure 3.

$|i - j| \geq$ constant. It will be convenient to assume that K is connected, and intersects tK. It is then easy to verify that the set $N_p = t^pK \cup t^{p+1}K \cup t^{p+2}K \cup \cdots$ is a neighborhood of one end, say ϵ, of K. In fact the sets $N_0 \supset N_1 \supset N_2 \supset \cdots$ form a cofinal sequence of neighborhoods: in other words, every neighborhood of ϵ contains some N_p.

† The corresponding homology groups $H_i(Y, \epsilon)$ can also be defined, but more care is needed. One must first construct $C_*(Y, \epsilon) = \lim C_*(Y, N)$ and then take the homology of this chain complex.

Similarly the sets $N'_q = t^{-q}K \cup t^{-q-1}K \cup \cdots$ form a cofinal sequence of neighborhoods for the other end ϵ' of $\tilde{X}$.

We must prove that

$$\varinjlim H^*(\tilde{X}, N_p) = 0.$$

First note that $H_* N_0$ is finitely generated over F. This follows immediately from the Meyer–Vietoris sequence

$$\cdots \xrightarrow{\partial} H_i(N_0 \cap N'_0) \longrightarrow H_i N_0 \otimes H_i N'_0 \longrightarrow H_i \tilde{X} \xrightarrow{\partial} \cdots,$$

where $H_* \tilde{X}$ is finitely generated by hypothesis and $N_0 \cap N'_0$ is a finite complex.†

It follows that the relative group $H_*(\tilde{X}, N_0)$ is also finitely generated (using the exact sequence of the pair $(\tilde{X}, N_0)$). Hence there must exist some finite subcomplex $L \subset \tilde{X}$ so that every homology class of $H_*(\tilde{X}, N_0)$ is represented by a cycle lying within L.

If we choose the neighborhood

$$N_{-s} = t^{-s}K \cup t^{-s+1}K \cup \cdots \cup K \cup tK \cup \cdots$$

large enough so that $N_{-s} \supset L$, then it follows that every homology class in $H_*(\tilde{X}, N_0)$ is represented by a cycle lying within N_{-s}. In other words the natural homomorphism $H_*(N_{-s}, N_0) \to H_*(\tilde{X}, N_0)$ is surjective. Hence, from the exact sequence of the triple $\tilde{X} \supset N_{-s} \supset N_0$, it follows that the natural homomorphism $H_*(\tilde{X}, N_0) \to H_*(\tilde{X}, N_{-s})$ is zero.

Now apply the automorphism t^{p+s} which carries the triple $(\tilde{X}, N_{-s}, N_0)$ onto the triple $(\tilde{X}, N_p, N_{p+s})$. Thus we have proved:

LEMMA. *There exists an integer $s > 0$ so that, for all p, the natural homomorphism $H_*(\tilde{X}, N_{p+s}) \to H_*(\tilde{X}, N_p)$ is zero.*

Using field coefficients, it follows immediately that the dual cohomology homomorphism $H^*(\tilde{X}, N_p) \to H^*(\tilde{X}, N_{p+s})$ is also zero. Hence $H^*(\tilde{X}, \epsilon) = \varinjlim H^*(\tilde{X}, N_p) = 0$. This proves Assertion 8.

Proof of the Duality Theorem: Note that the direct limit of the groups $H^*(\tilde{M}, N_p \cup N'_q)$ as $p, q \to \infty$ is just the cohomology of $\tilde{M}$

† The triangulation of $\tilde{X}$ really seems to be needed at this point.

with compact support, denoted by $H_{\text{compact}}\tilde{M}$. Thus if we consider the Meyer–Vietoris sequence

$$\cdots \longrightarrow H^{i-1}\tilde{M} \xrightarrow{\delta} H^i(\tilde{M}, N_p \cup N'_q) \longrightarrow$$
$$H^i(\tilde{M}, N_p) \oplus H^i(\tilde{M}, N'_q) \longrightarrow \cdots$$

and pass to the direct limit, we obtain the exact sequence

$$\cdots \xrightarrow{0} H^{i-1}\tilde{M} \xrightarrow{\delta} H^i_{\text{compact}}\tilde{M} \longrightarrow 0 \oplus 0 \longrightarrow \cdots,$$

which shows that the homomorphism δ is an isomorphism.

But the oriented n-manifold $\tilde{M}$ is known to satisfy a Poincaré duality theorem of the form

$$H^i_{\text{compact}}\tilde{M} \cong H_{n-i}\tilde{M}.$$

This yields a duality isomorphism $H^{i-1}\tilde{M} \cong H_{n-i}\tilde{M}$ which is close to the statement of our theorem.

To obtain the required statement we need only to note that, with field coefficients, and with $H_*\tilde{M}$ finitely generated, the cup product

$$H^i_{\text{compact}}\tilde{M} \otimes H^{n-i}\tilde{M} \to H^n_{\text{compact}}\tilde{M} \cong F$$

provides a dual pairing. Using the identity $(\delta x) \cup y = \delta(x \cup y)$ for $x \in H^{i-1}\tilde{M}, y \in H^{n-i}\tilde{M}$, this completes the proof.

Remark 1: With somewhat more care, the analogous theorem

$$\cap\, u: H^{i-1}\tilde{M} \xrightarrow{\cong} H_{n-i}\tilde{M}$$

with integer coefficients can also be proved. It is easiest to base the proof on the statement that the homology groups

$$H_i(\tilde{M}, \epsilon) = H_i[\varprojlim C_*(\tilde{M}, N_p)]$$

are zero, in place of Assertion 8.

Remark 2: For any $x \in H^{i-1}\tilde{M}, y \in H^{n-i}\tilde{M}$ note the identity $(t^*x) \cup (t^*y) = t^*(x \cup y) = \pm x \cup y$ where the plus sign holds if and only if the base manifold M is orientable. From this, one easily derives the duality formula

$$\text{order } H^{i-1}\tilde{M} = \overline{\text{order}}\ H^{n-i}\tilde{M},$$

where the bar stands for the conjugation automorphism $f(t) \mapsto f(\pm t^{-1})$ of $F\Pi$ (again using the plus sign if M is orientable). This duality formula is due to Blanchfield [2, § 4 8].

There is also a relative form of the duality theorem, proved in the same way:

ASSERTION 9. *Let M be a compact triangulated n-manifold with boundary, and $\tilde{M}$ an orientable infinite cyclic covering of M. If $H_*\tilde{M}$ is finitely generated over the coefficient field F, then the groups $H^{i-1}\tilde{M}$ and $H^{n-i}(\tilde{M}, \partial\tilde{M})$ are orthogonally paired to $H^{n-1}(\tilde{M}, \partial\tilde{M}) \cong F$ by the cup product operation.*

5. THE SIGNATURE OF A KNOT

As an example let X be the complex which is obtained from the sphere S^3 by removing an open tubular neighborhood of a knotted 1-sphere. Thus X is a 3-manifold bounded by a torus, and $H_*(\tilde{X}; F)$ is finite over F by Assertion 5. Hence the pair $(\tilde{X}, \partial\tilde{X})$ has the cohomology properties of a 2-manifold bounded by a circle† (Assertion 9). In particular, studying the cohomology exact sequence of this pair, one easily verifies that the natural homomorphism $H^1(\tilde{X}, \partial\tilde{X}) \to H^1\tilde{X}$ is an isomorphism. Hence, using Assertion 9 again, the skew-symmetric cup product pairing $H^1(\tilde{X}, \partial\tilde{X}) \otimes H^1(\tilde{X}, \partial\tilde{X}) \to H^2(X^2, \partial X^2) \cong F$ is nonsingular.

One cannot hope to extract much information from a skew-symmetric pairing alone. But we have some additional structure which arises from the fact that $\tilde{X}$ is an infinite cyclic covering. Namely we have the covering automorphism t^* of $H^1(\tilde{X}, \partial\tilde{X}) \cong H^1\tilde{X}$. *Out of the skew cup product pairing together with the automorphism t^* we construct a symmetric bilinear pairing from $H^1(\tilde{X}, \partial\tilde{X}) \otimes H^1(\tilde{X}, \partial\tilde{X})$ to F by the formula $\langle x, y\rangle = (t^*x) \cup y + (t^*y) \cup x$.*

DEFINITION. This symmetric pairing is called the *quadratic form* of the knot.

This concept (in somewhat different form) is due to Trotter [24]. (Compare [10].)

ASSERTION 10. *Using real or rational coefficients, the quadratic form of a knot is non-singular.*

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† In fact for a Neuwirth-Stallings knot such as 3_1 the pair $(\tilde{X}, \partial\tilde{X})$ actually splits as a product $(M^2, S^1) \times R$. (See Section 2.)

Proof: Using the identity $(ty) \cup (tz) = y \cup z$ we see that $(ty) \cup x = y \cup t^{-1}x = -(t^{-1}x) \cup y$, and hence that $\langle x, y \rangle = [(t - t^{-1})x] \cup y$. Thus, for fixed $\bar{x}$, the equation $\langle \bar{x}, y \rangle = 0$ holds for all y if and only if $(t - t^{-1})\bar{x} = 0$. Since $t - t^{-1} = (t - 1)(t + 1)t^{-1}$, this equation has a solution $\bar{x} \neq 0$ if and only if the characteristic polynomial $A(t)$ has ± 1 as a root.

But according to Alexander the polynomial $A(t)$ (with integer coefficients) satisfies $A(1) = 1$ and hence $A(-1) \equiv 1 \bmod 2$. (Compare [21], [13].) Hence $A(1) \neq 0$, $A(-1) \neq 0$ in F; which proves Assertion 10.

DEFINITION (Murasugi [17]). The ***signature*** σ of a knot is the signature of its quadratic form (again using real or rational coefficients).

This signature is an even integer whose sign depends only on the choice of orientation for the containing space S^3. (In other words the sign does not depend on the choice of generator for Π, or for the homology group of the knot.)

According to Murasugi:

(1) The signature $\sigma(k_1 \# k_2)$ of the composition of two knots is the sum $\sigma(k_1) \# \sigma(k_2)$. In fact the quadratic form of $k_1 \# k_2$ is the direct sum of the corresponding quadratic forms.

(2) If a knot k is "cobordant" to k' (that is if $k \times 0$ and $k' \times 1$ together bound a locally flat annulus within $S^3 \times [0, 1]$) then $\sigma(k) = \sigma(k')$.

Hence σ gives rise to a homomorphism from the group of knot cobordism classes† onto an infinite cyclic group. (Note the analogy between σ and the Thom signature of a compact oriented $4n$-manifold.)

Here is an example in which σ is easy to compute:

ASSERTION 11. *If the Alexander polynomial $A(t)$ of k has degree 2, then the signature $\sigma(k)$ is either 0 or ± 2 according as the roots of $A(t)$ are real or lie on the complex unit circle.*

These possibilities are mutually exclusive since $A(\pm 1) \neq 0$. For example, since the polynomials $t^2 - t + 1$ and $2t^2 - 3t + 2$ have no real roots, this shows that the knots 3_1 and 5_2 both have signature ± 2

† Compare [8], [12].

(where the sign depends on the choice of orientation). It follows, for example, that the "granny knot" $3_1 \# 3_1$ has signature ± 4 and hence is not cobordant to the circle O.

Proof: Clearly the automorphism t^* of $H^1(\tilde{X}, \partial\tilde{X})$ leaves the quadratic form invariant. But every isometry of an indefinite quadratic form of rank 2 has real eigenvalues, and every isometry of a definite quadratic form of rank 2 has eigenvalues lying on the unit circle.

Now consider the direct sum decomposition of the $F\Pi$ module $H^1(\tilde{X}, \partial\tilde{X})$ into $[p(t)]$-primary summands corresponding to the various prime ideals $[p(t)]$ in $F\Pi$. Using the identity $\langle f(t)x, y\rangle = \langle x, f(t^{-1})y\rangle$ we see that the $[p(t)]$-primary summand is orthogonal to the $[q(t)]$-primary summand unless $[p(t)] = [q(t^{-1})]$. It follows that the only contributions to the signature arise from those primary summands for which $[p(t)] = [p(t^{-1})]$. Taking real coefficients this will happen only if the irreducible polynomial $p(t)$ is a quadratic polynomial of the form $p_\theta(t) = t^2 - 2t\cos\theta + 1$, $0 < \theta < \pi$, with roots $\cos\theta \pm i\sin\theta$ lying on the unit circle.

DEFINITION. For each such θ, let $\sigma_\theta(k)$ denote the contribution of the $[p_\theta(t)]$-primary component of $H^1(\tilde{X}, \partial\tilde{X})$ to the signature $\sigma(k)$. (Thus $\sigma(k)$ is the sum, over all θ such that $p_\theta(t)$ divides the Alexander polynomial, of $\sigma_\theta(k)$.)

THEOREM. *Each of these signatures $\sigma_\theta(k)$ is a cobordism invariant of the knot k. In particular the signatures σ_θ corresponding to the values*

$$\cos\theta = \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{5}{6}, \tfrac{7}{8}, \ldots$$

are independent and give rise to a homomorphism of the knot cobordism group onto a free abelian group with infinitely many generators.

Similar assertions hold for knotted $(2q-1)$-spheres in S^{2q+1} for all odd values of q, making use of Levine [13]. (Compare Kervaire [12].)

The proof that σ_θ is a cobordism invariant can be sketched as follows. If $k \subset S^3$ is cobordant to 0 then k bounds a locally flat 2-cell within the disk D^4. (Compare [8].) Removing a tubular neighborhood of this 2-cell, we obtain a manifold M^4 where ∂M^4 can be obtained from S^3 by surgery along k. (See for example [14].) Using the excision isomorphism $H^*(\tilde{X}, \partial\tilde{X}) \cong H^*(\partial\tilde{M}^4, \tilde{S}^1 \times D^2)$ we see that the signatures σ and σ_θ can just as well be defined using the manifold $\partial\tilde{M}^4$ in place of $(\tilde{X}, \partial\tilde{X})$.

Now consider the exact cohomology sequence of the pair $(\tilde{M}^4, \partial\tilde{M}^4)$. Just as in Thom [23, Theorem V.II], the natural homomorphism $i^*: H^1(\tilde{M}^4) \to H^1(\partial\tilde{M}^4)$ is dual to $\delta: H^1(\partial\tilde{M}^4) \to H^2(\tilde{M}^4, \partial\tilde{M}^4)$. Hence the image of i^* is a subvector space of half the dimension of $H^1(\partial\tilde{M}^4)$ on which the quadratic form is identically zero. Therefore $\sigma(k) = 0$. Splitting this exact sequence up as a direct sum of its various $[p(t)]$-primary components, we see similarly that each $\sigma_\theta(k)$ is zero.

The statement that $\sigma_\theta(k \# k') = \sigma_\theta(k) + \sigma_\theta(k')$ follows immediately from the fact that the quadratic form of $k \# k'$ splits as a direct sum.

Now if k_1 is cobordant to k_2 then it follows as in [8] that the composition $k_1 \# (-k_2)$ is cobordant to 0, and hence that $\sigma_\theta(k_1) - \sigma_\theta(k_2) = 0$.

Finally we must show that infinitely many of the invariants σ_θ are independent. It follows from Seifert [21, p. 589] that, for each integer $m \geq 1$, there exists a knot k_m with polynomial $A(t)$ equal to $mt^2 - (2m - 1)t + m$. (For example we can take 3_1 and 5_2 as k_1 and k_2 respectively.) Dividing by m, this polynomial corresponds to $p_\theta(t)$ with $\cos\theta = (2m - 1)/2m$. Hence

$$\sigma_\theta(k_m) = \pm 2 \text{ if } \cos\theta = \frac{2m-1}{2m},$$

and

$$\sigma_\theta(k_m) = 0 \text{ if } \cos\theta \neq \frac{2m-1}{2m}.$$

This completes the outlined proof.

APPENDIX: A COMPUTATION OF TORSION

Let C_* be a free finitely generated chain complex over a principal ideal domain P. If each C_i is assigned a preferred basis, and if the homology groups $H_i(C_*)$ are torsion modules over P, then the torsion τ of the corresponding complex over the quotient field of P is an element of the quotient field $Q(P)$, well-defined up to sign. It can be constructed as follows.

Let c be the preferred basis for

$$C_{\text{even}} = C_0 \oplus C_2 \oplus C_4 \oplus \cdots$$

and c' the preferred basis for C_{odd}. We will think of each "basis" as a column vector of module elements.

Choose a basis b for the (necessarily free) submodule B_{even} of boundaries in C_{even}, and choose a column vector x' of elements of C_{odd} so that $\partial x' = b$. Similarly choose a basis $\partial x = b'$ for B_{odd}. Then the entries of x and b can be expressed as linear combinations of the basis elements of C_{even}, yielding matrix equations $x = Xc$, $b = Yc$, and hence

$$\begin{pmatrix} x \\ b \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} c.$$

Choosing X' and Y' similarly, the torsion of $Q(P) \otimes_P C_*$ can now be defined by

$$\tau = \pm \frac{\det \begin{pmatrix} X \\ Y \end{pmatrix}}{\det \begin{pmatrix} X' \\ Y' \end{pmatrix}}.$$

(Compare [16, §3 and §1.2].)

We will prove that the element $\det \begin{pmatrix} X \\ Y \end{pmatrix}$ of P spans the ideal order H_{even} = (order H_0)(order H_2)(order H_4)$\cdots$ and similarly that $\det \begin{pmatrix} X' \\ Y' \end{pmatrix}$ spans order H_{odd}. Dividing these two ideals, this implies:

THEOREM. *The fractional ideal* (τ) *is equal to*

$$\frac{\text{order } H_{even}}{\text{order } H_{odd}}.$$

Clearly this implies Assertion 7 of Section 3.

To compute $\det \begin{pmatrix} X \\ Y \end{pmatrix}$ choose a basis z for the module Z_{even} of cycles, and note that $\begin{pmatrix} x \\ z \end{pmatrix}$ is then a basis for C_{even}. Setting $z = Mc$, it follows that

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} X \\ M \end{pmatrix} c,$$

so that the matrix $\begin{pmatrix} X \\ M \end{pmatrix}$ must be invertible. Hence its determinant must be a unit u of our ring.

Now let $b = Rz$ so that $RM = Y$. Then the identity

$$\begin{pmatrix} I & O \\ O & R \end{pmatrix}\begin{pmatrix} X \\ M \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

shows that det R times the unit u is equal to det $\begin{pmatrix} X \\ Y \end{pmatrix}$.

But clearly R is a relation matrix for the quotient module

$$\frac{Z_{\text{even}}}{B_{\text{even}}} = H_{\text{even}}.$$

Therefore the determinant of R spans the order ideal of H_{even}. (Compare Zassenhaus [26].) This completes the proof.

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