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Surgery with coefficients

By R. JAMES MILGRAM*

Dedicated to the memory of A. N. Milgram

After the discovery of the exotic spheres, Kervaire and Milnor ([18]) explored the structure of the groups Γ_k of all diffeomorphism classes of exotic spheres homeomorphic to S^k. Their main technique was surgery on degree 1 normal maps

$$\rho\colon \widetilde{M}^{k+1} \longrightarrow M^{k+1},$$

where M^{k+1} is the disc D^{k+1} and ρ is a homotopy equivalence on $\partial\widetilde{M}$. Browder and Novikov ([4], [30]) extended these techniques to more general classes of spaces, and applied them to a broad class of problems. Sullivan ([35]) next redescribed the evolving theory in terms of maps into the classifying spaces G/O and G/PL in the smooth and PL-cases, respectively.

Sullivan then presented the program of completely describing the homotopy type of a map $f\colon M^n \to G/\mathrm{PL}$ in terms of invariants (signatures and Kervaire invariants) for a basic set of surgery problems associated to M and f, and he made great strides towards completing it. But one step was missing.

In this paper, we present a more general type of surgery invariant, one that is appropriate to surgery on manifolds with coefficients, and which restricts to the ordinary index obstruction on oriented manifolds. But it is considerably more complex on, for example, Z_2-manifolds (for the definition and properties of Z_2-manifolds, see e.g. [34], Chapter VIII, pages 150–168, or §1, (1.15) and (1.20)).

It turns out that this more general invariant completes Sullivan's original program except for a problem in dimension 4, where something more subtle must happen.

This work arose out of the need to study the natural map

$$r\colon SG \longrightarrow G/\mathrm{PL}$$

of the space SG, the set of degree 1 homotopy equivalences of spheres, onto G/PL, which is the classifying space for PL-sphere-bundles, together with a fiber homotopy trivialization. G/PL can also be thought of as the fiber of the map of classifying spaces,

$$B_{\mathrm{PL}} \longrightarrow B_G .$$

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Sullivan's original description of G/PL was not sufficiently delicate, though, to give $r^*: H^*(G/\mathrm{PL}, Z_2) \to H^*(SG, Z_2)$.

Once the results of the current paper were obtained, the main results of [9] and [10] became direct calculations, and my point of view, at least, is that this work should be regarded primarily as a companion piece to [10]. This is reflected, for example, in our choice of method to prove the product formulas (§ 7) for the new invariant. There is a geometric method available, developed by Morgan and Sullivan ([29]); however, we use a strongly homotopical method similar to that which Browder and Brown ([3], [7], [8]) used for their treatments of the Kervaire invariant. The main justification for this is that the *techniques* involved in §§ 6 and 7 play a vital role in the work of [10], and, if they had not appeared here, they would certainly have had to be fully developed there.

A second advantage of our technique has been pointed out by Ib Madsen. The surgery theory of Poincaré spaces has been developed by Lowell Jones in his paper "Patch spaces: a geometric representation for Poincaré duality spaces," Ann. of Math **97** (1973), 306-343, and also later by Quinn, and all our proofs go through without essential modification, even for spaces as general as these. Of course, the interpretation of the invariants in terms of linking numbers given in § 6 is no longer valid, but this is the only point where geometric properties, as distinguished from homotopical properties, of manifolds are required.

Finally, almost no examples of degreee 1 normal maps, to which the new invariants are relevant, are considered here. This is because they arise most naturally in the context of [10]. Indeed, in that paper an enormous number of examples are constructed, and they play a decisive role in the proofs of the main results.

The paper is arranged so that it should be possible to read it if the reader is familiar with [5] or [18], and the work of Browder and Brown on the Kervaire invariant (in particular, [7] or [8] suffice). Since Sullivan's thesis is not in general circulation, its main ideas with respect to our present project are concisely presented. However, the reader may find [32] most helpful.

In § 1, we review the theory of bordism (e.g. as in [13, Chapters 1, 2]), and introduce coefficients. In order to preserve balance, both the geometric (constructions with maps and manifolds) and homotopic (calculations in bordism by homotopy theory) viewpoints are presented.

In § 2, we recall Sullivan's description of G/PL in terms of the index and Kervaire invariants for degree 1 normal maps, and in § 3 we abstractly solve the problem of doing surgery on Z_n-manifolds.

The results described so far have been known, I understand, to various workers in the field. However, our projected use of the generalized index invariant demanded effective methods for evaluating it. These are supplied in § 4, where we give a new account of the invariant in terms of a quadratic form on the torsion subgroup of the kernel (for the definition of the surgery kernels, see [5, Chapters 1, 2]) and a certain Gaussian sum.

In §§ 5 and 6, we remedy a flaw in the presentation of § 4. There the invariant was defined only after the normal map ρ was made highly connected and used special properties of this situation. In §§ 5 and 6, we use functional cohomology operations to define the quadratic form directly for an arbitrary ρ.

In § 7, we apply these techniques to prove a product formula (7.3) for the invariants (for analogous formulas for the Kervaire invariant, see [5, Chapter 3], or [32]). One of the more interesting consequences of (7.3) is that the Kervaire invariant on oriented manifolds becomes a special case of the new index invariant. Specifically, let E_6 be the 6-dimensional Z_2-manifold described in [34, p. 167]. Let $\rho\colon \widetilde{M}^{4k+2} \rightarrow M^{4k+2}$ be a degree 1 normal map of closed oriented manifolds; then ρ has Kervaire invariant 1 if and only if

$$1 \times \rho\colon E_6 \times \widetilde{M}^{4k+2} \longrightarrow E^6 \times M^{4k+2}$$

has Z_2-index invariant 1!

In § 8, we indicate how the new invariant gives the homotopy class of a map

$$f\colon M \longrightarrow G/\mathrm{PL}$$

except for some problems in dimension 4.

For the reader's convenience, we include an appendix which reviews those results we need about Q/Z-cohomology and cohomology operations.

I would like to take this opportunity to thank Paul Cohen and Ralph Phillips for their invaluable aid, and, in particular, Gregory Brumfiel, without whose help this paper would not have been possible. I would also like to thank the referees, whose extraordinarily careful reading of the original manuscript, and detailed criticisms are responsible for whatever degree of clarity and precision of exposition the present version can claim.

1. Bordism and bordism with coefficients

In this section, we recall the definitions and main properties of the PL-bordism groups, and bordism with coefficients.

DEFINITION 1.1. *Let* $(M^n, \partial M)$, $(N^n, \partial N)$ *be oriented* PL-*manifolds with* *boundaries. Suppose*

$$f\colon (M^n, \partial M) \longrightarrow (X, Y)\,,$$
$$g\colon (N^n, \partial N) \longrightarrow (X, Y)$$

are continuous maps. The triple $(f, M, \partial M)$ *is called bordant to* $(g, N, \partial N)$ *if there are:* (1) *a manifold* $(W^{n+1}, \partial W)$ *so* $\partial W = (M + N) \cup Z$ *with* $Z \cap M = \partial M$, $Z \cap N = \partial N$, *and* (2) *a map* $H\colon (W^{n+1}, Z) \to (X, Y)$ *so* $H \mid M = f$ *and* $H \mid N = g$.

(1.2)

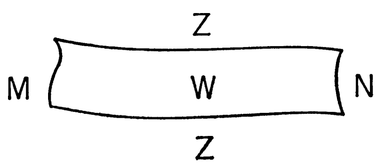

FIGURE 1.2.

The absolute PL-bordism group $\Omega_n^{\mathrm{PL}}(X)$ is defined as $\Omega_n^{\mathrm{PL}}(X, \phi)$, and so, for $(f, M, \partial M)$ to give a class in $\Omega_n^{\mathrm{PL}}(X)$, it is necessary that $\partial M = \phi$. Thus $\Omega_n^{\mathrm{PL}}(X)$ corresponds to maps of closed PL-manifolds into X.

The $\Omega_n^{\mathrm{PL}}(\ ,\ )$ satisfy the Eilenberg-Steenrod axioms except for the dimension axiom, and thus define a homology theory (see e.g. [13, p. 13]).

There is a second way in which homology theories arise: by taking homotopy groups of smash products of spectra and spaces, which we review now.

Recall the definitions of spectrum, smash products of spectra, and maps of smash products of spectra to spectra as given, for example, in [48, p. 637].

DEFINITION 1.2. $\mathcal{W}$ *is a ring spectrum if there is a map* $\mu\colon \mathcal{W} \wedge \mathcal{W} \to \mathcal{W}$.

We have the usual definition for associativity.

The $\mathcal{W}$-homology groups of (X, Y) are defined ([41]) as

$$H_n(X, Y; \mathcal{W}) = \mathcal{W}_n(X, Y) = \pi_n^s(\mathcal{W} \wedge X \cup cY)\,. \tag{1.3}$$

These groups also satisfy all the Eilenberg-Steenrod axioms except the dimension axiom.

Let B_{SPL_n} be the classifying space for oriented PL-n-plane bundles ([19], [21]) and $\gamma_n \to B_{\mathrm{SPL}_n}$ the universal bundle. Let $M(\mathrm{SPL}_n)$ be the associated Thom space. In the standard way, these give the SPL-Thom spectrum $M(\mathrm{SPL})$. Recall

THEOREM 1.4 (Williamson [42]). $\Omega_*^{\mathrm{PL}}(X, Y) \cong H_*(X, Y; M(\mathrm{SPL}))$ *for* (X, Y) *a* CW-*pair.*

Whitney bundle sum induces the ring structure

$$\mu\colon M(\mathrm{SPL}) \wedge M(\mathrm{SPL}) \longrightarrow M(\mathrm{SPL})\,,$$

which makes $M(\mathrm{SPL})$ into an associative ring spectrum with unit U. (U is the embedding of the compactified fiber over a point $U: S^n \to M(\mathrm{SPL}_n)$, which gives $U: \mathcal{S} \to M(\mathrm{SPL})$ where $\mathcal{S}$ is the sphere spectrum and the composite

$$\mathcal{S} \wedge M(\mathrm{SPL}) \xrightarrow{U \wedge 1} M(\mathrm{SPL}) \wedge M(\mathrm{SPL}) \xrightarrow{\mu} M(\mathrm{SPL})$$

is the identity.)

μ gives the pairing $\Omega_*^{\mathrm{PL}}(pt) \times \Omega^{\mathrm{PL}}(X, Y) \to \Omega^{\mathrm{PL}}(X, Y)$ from the composition

$$\begin{aligned} &\pi_*[M(\mathrm{SPL})] \otimes \pi_*[M(\mathrm{SPL}) \wedge (X \cup cY)] \\ &\longrightarrow \pi_*[M(\mathrm{SPL}) \wedge M(\mathrm{SPL}) \wedge (X \cup cY)] \\ &\xrightarrow{(\mu \times 1)_*} \pi_*[M(\mathrm{SPL}) \wedge X \cup cY] . \end{aligned} \tag{1.5}$$

We have

$$H_*(X, Y; Z) = H_*(X, Y; K(Z, 0)) , \tag{1.6}$$

where $K(Z, 0)$ is the Eilenberg-MacLane spectrum with $K(Z, 0)_i = K(Z, i)$ ([41]). The Thom class of $M(\mathrm{SPL})$ gives a map $U^*: M(\mathrm{SPL}) \to K(Z, 0)$, so the composition $\mathcal{S} \xrightarrow{U} M(\mathrm{SPL}) \to K(Z, 0)$ represents the fundamental class. The Hurewicz homomorphism h is now described by the composite

$$\pi_*(\mathrm{MSPL} \wedge X \cup cY) \xrightarrow{U^* \wedge 1} \pi_*(K(Z, 0) \wedge X \cup cY) .$$

We recall three further results before turning to the definition of homology theories with coefficients.

THEOREM 1.7 (Atiyah-Dold [15]). *Let $\mathcal{W}$ be a spectrum. There is a spectral sequence $E^* \Rightarrow H_*(X, Y; \mathcal{W})$ with the $E^2_{i,j}$-term equal to $H_i(X, Y; \pi_j(\mathcal{W}))$. If $\mathcal{W}$ is a wedge of Eilenberg-MacLane spectra, then the sequence degenerates, and $E^2 = E^\infty$. Finally, if $\mathcal{W}$ is a ring spectrum, the action of $\pi_j(\mathcal{W})$ in E^2 extends to E^∞, and corresponds to the action of $\pi_j(\mathcal{W})$ on $H_*(X, Y; \mathcal{W})$.*

Let $Z_{(2)}$ denote the ring of fractions m/n, with n prime to 2. Then $H^i(X; Z_{(2)})$ can only have 2-torsion. Indeed, if X is locally finite, $H^m(X; Z_{(2)}) = Z_{(2)}^{(n)} \oplus \Gamma_2^m$ is a direct sum of a finite number of copies of $Z_{(2)}$ and a 2-group.

THEOREM 1.8 (Browder, Liulevicius, Peterson [6]). *There is a map f of $M(\mathrm{SPL})$ into a wedge of Eilenberg-MacLane spectra $\mathcal{W}$, so $f: M(\mathrm{SPL}) \to \mathcal{W}$ induces an isomorphism of $Z_{(2)}$-cohomology.*

This implies that the only k-invariants of $M(\mathrm{SPL})$ lie in p-torsion, with p an odd prime. Thus

COROLLARY 1.9. *The cokernel of $h: \Omega_*^{\mathrm{PL}}(X, Y) \to H_*(X, Y)$ is an odd torsion group for (X, Y), a locally finite* CW-*pair.*

Let G be a finite Abelian group and $M(G)_n$ the n^{th} Moore space for G. Thus $\tilde{H}_n(M(G); Z) = G$ and $\tilde{H}_i(M(G); Z) = 0$, $i \neq n$. The $M(G)_n$ form a spectrum $\mathfrak{M}(G)$.

DEFINITION 1.10. *Let $\mathfrak{W}$ be a spectrum and $\mathfrak{W}_*(\ )$ homology with respect to $\mathfrak{W}$. Then $\mathfrak{W}_*(\ ; G) = \mathfrak{W}$ homology with coefficients in G is defined to be homology with respect to $\mathfrak{W} \wedge \mathfrak{M}(G)$.*

COROLLARY 1.11. *Let G be a finite 2-group. Then $h: \Omega_*^{\text{PL}}(X, Y; G) \to H_*(X, Y; G)$ is onto.* (Of course, the same result holds for the ordinary differentiable bordism groups $\Omega_*(X, Y; G)$.)

Proof. From (1.8) and the Kunneth theorem,

$$f \wedge 1: M(\text{SPL}) \wedge M(G) \longrightarrow \mathfrak{W} \wedge M(G)$$

induces isomorphisms of *integral* homology. Hence they have the same homotopy type. But $\mathfrak{W} \wedge M(G)$ is again a product of Eilenberg-MacLane spaces if $\mathfrak{W}$ is, and (1.11) follows.

Consider $M_n(G) \wedge M_m(G)$. It has homology $H_{n+m}(M \wedge M) = G \otimes G$, $H_{n+m+1} = G \otimes G$, and all the remaining groups are zero.

LEMMA 1.12. *Let $\phi: G \otimes G \to G$ be any homomorphism, and suppose G contains no Z_2-direct summands. Then there is a map $\mu_\phi: M_n(G) \wedge M_m(G) \to M_{n+m}(G)$, so $\mu_\phi: H_{n+m}(M \wedge M) \to H_{n+m}(G)$ is exactly ϕ.*

This implies

COROLLARY 1.13. *Let $\mu: G \otimes G \to G$ be any pairing where G contains no Z_2-direct summands, and $\mathfrak{W}$ a ring spectrum of Hurewicz dimension n. Then $\mathfrak{W} \wedge M(G)$ is again a ring-spectrum with coefficient pairing μ and Hurewicz dimension n.*

Suppose, finally, that $\mu: G \otimes G \to G$ is associative. This implies that, on the level of homology, the diagram

$$\begin{array}{ccc} M(G) \wedge M(G) \wedge M(G) & \xrightarrow{\phi_\mu \wedge 1} & M(G) \wedge M(G) \\ \downarrow{\scriptstyle 1 \wedge \phi_\mu} & & \downarrow{\scriptstyle \phi_\mu} \\ M(G) \wedge M(G) & \xrightarrow{\phi_\mu} & M(G) \end{array}$$

commutes. Hence if $M(G) \wedge \mathfrak{W}$ is a wedge of Eilenberg-MacLane spectra, and $\mathfrak{W}$ is an associative ring-spectrum, then $M(G) \wedge \mathfrak{W}$ is again an associative ring-spectrum. Thus

COROLLARY 1.14. *Let G be a 2-group and $\mu: G \otimes G \to G$ an associative pairing. Then $\Omega_*^{\text{PL}}(\ ; G)$ has coefficients an associative ring. Moreover, in* *terms of the action of* $\Omega_*^{\mathrm{PL}}(\mathrm{pt}, G)$, *the kernel of the Hurewicz homomorphism consists of decomposables* $a \cdot b$, $h(b) \neq 0$, *and Massey products* $\langle a, n, b\rangle$ *with* $a \in \Omega_*^{\mathrm{PL}}(\mathrm{pt}, G)$ *and* $h(b) \neq 0$.*

Z_2-direct summands of G must be handled differently from other groups because there is no suitable map $M(Z_2) \wedge M(Z_2) \to M(Z_2)$. The map $\phi\colon B_{\mathrm{PL}} \to K(Z, 2)$ defined by taking the fundamental class of the Eilenberg-MacLane space to $\beta(v_1)$, the integral Bochstein of the first Stiefel-Whitney class, is an infinite loop map, so the fiber F of ϕ is an infinite loop space. The Thom space of the universal bundle restricted to F is $M(\mathrm{SPL}) \wedge M(Z_2)$, and the resulting identification of spectra makes $M(\mathrm{SPL}) \wedge M(Z_2)$ into an associative ring-spectrum.

This completes our discussion of the homotopy theoretic aspects of $\Omega_*^{\mathrm{PL}}(\ ; G)$. We now turn to the geometric aspects. From here on, it is most convenient to assume that G is a cyclic group Z_n, and the pairing $\mu\colon Z_n \otimes Z_n \to Z_n$ sends $a \otimes b$ to $a \cdot b$, and hence is the usual identification. It is clearly associative.

DEFINITION 1.15. *A closed* Z_n*-manifold* (M^m, k) *is an oriented m-manifold with boundary, together with an orientation-preserving* PL*-homeomorphism* $\kappa\colon N^{m-1} \times \{1, \cdots, n\} \to \partial M$. *Thus the boundary of* M *consists of n-ordered copies of* N.

A Z_n*-manifold* (M^n, κ) *with boundary is* M^n, *together with a manifold* N^{m-1} *with boundary and an oriented* PL*-homeomorphism (into)*

$$\kappa\colon N^{m-1} \times \{1, \cdots, n\} \longrightarrow \partial M^m .$$

Thus $\partial M^m - \kappa\big(\mathring{N} \times \{1, \cdots, n\}\big)$ *is a* Z_n*-manifold* W *with boundary* $(\partial N) \times \{1, \cdots, n\}$.

A map $f\colon (M^m, W, \kappa) \to (X, Y)$ *is a map of* Z_n*-manifolds if* $f \circ \kappa$ *factors as* $N \times \{1, \cdots, n\} \to N \xrightarrow{g} X$; i.e. *f is the same map on each component* N.

For convenience in the sequel, we use the notation

$$N_j \doteq \kappa[N \times j] .$$

Also, if $\big(M, \kappa\colon N \times \{1, \cdots, n\} \to \partial M\big)$ is a closed Z_n-manifold, we will write

$$\bar{\partial}(M) = N .$$

Bordisms are defined by setting $\{f, (M, W, \kappa)\} \simeq \{g, (M', W', \kappa')\}$ if there is a manifold P, a set $Z \subset \partial P$, and a map

$$H\colon (P, \partial P - \mathring{M} - \mathring{M}') \longrightarrow (X, Y) ,$$

* For G of odd order and prime to 3, the diagram above always homotopy commutes, and $\Omega^{\mathrm{PL}}(\mathrm{pt}, G)$ is always an associative ring [43]. A different argument applies at the prime 3 [29].

satisfying the expected conditions. This gives rise to a homology theory as before, and, using transversality, it is directly seen to be $\Omega_*^{\mathrm{PL}}(\ , Z_n)$.

Indeed, in homotopy consider the cofiber sequence

$$\cdots \longrightarrow S^m \xrightarrow{(\times n)} S^m \longrightarrow M(Z_n) \longrightarrow S^{m+1} \xrightarrow{(\times n)} S^{m+1} \longrightarrow .$$

Smashing with $M(\mathrm{SPL})$ gives a cofiber sequence of spectra and the long exact sequence

$$\cdots \longrightarrow \Omega_*^{\mathrm{PL}} \xrightarrow{(\times n)} \Omega_*^{\mathrm{PL}}(\ ) \longrightarrow \Omega_*^{\mathrm{PL}}(\ ; Z_n) \xrightarrow{\partial} .$$

Suppose now that (M^m, f) represents a class of order k in $\Omega_*^{\mathrm{PL}}(X)$. Then $(kM^m, kf) = \partial(W^{m+1}, F)$, and, if $k \mid n$ [or $n \mid k$], we have two classes in $\Omega_*^{\mathrm{PL}}(X, Z_n)$, $(n/k(W^{m+1}), n/k(F))$, $[(W^{m+1}, F)]$ and (M^m, f). Indeed, the sequence $0 \to \Omega_*(X) \otimes Z_n \to \Omega_*(X, Z_n) \to \mathrm{Tor}(\Omega_{*-1}(X), Z_n) \to 0$ is split exact.

Next, consider the effect of changing coefficients. Given the exact sequence

$$0 \longrightarrow Z_n \xrightarrow{i} Z_{nm} \xrightarrow{j} Z_m \longrightarrow 0 ,$$

we have the long exact sequence

$$\xrightarrow{\partial} \Omega(\ ; Z_n) \xrightarrow{i} \Omega(\ ; Z_{nm}) \xrightarrow{j} \Omega(\ ; Z_m) \xrightarrow{\partial} \cdots .$$

Here $i\{M\} = \{m(M)\}$, $\partial\{W\} = \{M\}$ in $\Omega(Z_n)$ where $mM = \partial W$.

The geometric interpretation of the ring structure of (1.13) is not direct since *the Cartesian product of two Z_n-manifolds is not a Z_n-manifold.*

Note, however, that by the collar neighborhood theorem ([27]), we can assume a neighborhood of ∂M has the form $(0, 1] \times \partial M$. Using a second application of the collaring theorem, we can assume given for each $N \subset \partial M$ a neighborhood of the form $(0, 1] \times N$. Similarly, we have neighborhoods $(0, 1] \times N'$ of the N'.

(1.16)

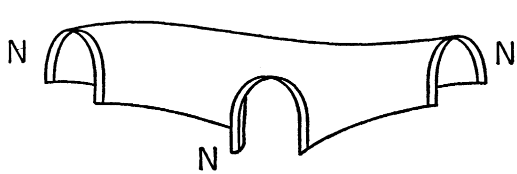

FIGURE 1.16.

DEFINITION 1.17. *Let M, M' be closed Z_n-manifolds. Then $M \underline{\times} M'$ is the Z_n-manifold*

$$(M \times M' \cup \textstyle\bigcup_{i<j} I^2 \times N_i \times N_j', \bar{N}, \bar{\kappa}) ,$$

where $(0, t, N_i \times N_j')$ is identified with

$$(2t, N_i) \times N_j' , \qquad t \leqq \frac{1}{2} ,$$

$$N_i \times (2 - 2t, N_j) , \qquad t \geqq \frac{1}{2} ,$$

and $(1, t, N_i \times N_j')$ *is identified with*

$$N_j \times (2t, N_i') , \qquad t \leqq \frac{1}{2} ,$$

$$(2 - 2t, N_j) \times N_i' , \qquad t \geqq \frac{1}{2} .$$

Here $\bar{N}_j$ *is* $M \times N_j' \cup N_j \times M'$, *where we identify* $N_i \times N'$ *with* $N \times N_i'$ *via the given* PL-*homeomorphism.*

(1.18)

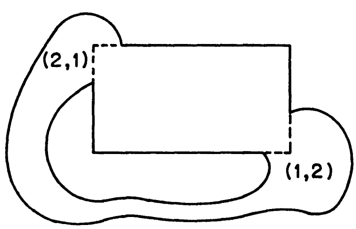

FIGURE 1.18.

By reparametrization, given Z_n-maps

$$f\colon (M, \kappa) \longrightarrow X ,$$
$$g\colon (M', \kappa') \longrightarrow X' ,$$

one obtains a Z_n-map

$$f \underline{\times} g\colon (M \underline{\times} M', \bar{\kappa}) \longrightarrow X \times X' ,$$

and these give the geometric interpretation of the product in (1.17).

DEFINITION 1.19. *Let* (M^m, κ) *be a* Z_n-*manifold. Then the associated singular manifold* $\bar{M}^m$ *is* $M^m \cup_{\pi\kappa^{-1}} N$ *where* $\pi\colon N \times \{1, \cdots, n\} \to N$ *is the projection.*

Example 1.20. If n is 2 and M is a closed Z_2-manifold, then $\bar{M}^m$ is a closed manifold. It is non-orientable, however, with the obstruction to orientation being N. There is a map $v_1\colon \bar{M}^m \to S^1$, defined by bicollaring N in $\bar{M}$, taking the resulting Thom map $\bar{M}^m \to S^1 \wedge N_+$, and projecting onto S^1. Clearly, $v_1^*(e^1)$ is the first Wu class of $\bar{M}^m$. Singular Z_2-manifolds, in fact, are characterized by the property that they are closed, with v_1 being the mod 2 restriction of an integral cohomology class ([37]); also compare the remark following (1.14).

Now we consider the Hurewicz homomorphism

$$h\colon \Omega_*^{\mathrm{PL}}(X, Y; Z_n) \longrightarrow H_*(X, Y; Z_n) . \tag{1.21}$$

Let (M, κ) be an open PL-Z_n-manifold, $\bar{M}$ its associated singular manifold, and

$$f\colon (M, \kappa, W) \longrightarrow (X, Y)$$

a map representing $x \in \Omega_*^{\mathrm{PL}}(X, Y; Z_n)$. f factors through $\bar{f}\colon (\bar{M}, \bar{W}) \to (X, Y)$ and, going one step further, $\bar{f}$ induces

$$\bar{\bar{f}}\colon \bar{M} \cup c(\bar{W}) \longrightarrow X \cup c(Y) . \tag{1.22}$$

Note that $\partial[\bar{M}] = n[N] + [W]$. Hence in (1.22), $\partial[\bar{M} \cup c\bar{W}] = n[N]$. Thus with Z_n as coefficients, $f_*[\bar{M} \cup c\bar{W}]$ is a well-defined class in $\tilde{H}_*(X \cup cY; Z_n) \cong H_*(X, Y; Z_n)$. It may be verified directly that $f_*[\bar{M} \cup c\bar{W}]$ does not depend on the choice of representative for x. In fact,

$$h\{f, M, \kappa, W\} = f_*[\bar{M} \cup c\bar{W}] .$$

The next thing that needs a comment is the normal bundle to a Z_n-manifold.

DEFINITION 1.23. *A Z_n-normal bundle to (M^n, κ) is the ordinary normal bundle ν to M^n, together with an oriented bundle isomorphism $\phi\colon \nu \mid \kappa(N \times \{1, \cdots, n\}) \to \nu(N) \times \{1, \cdots, n\}$ covering κ.*

Associated to the Z_n-normal bundle is a bundle $\bar{\nu}$ over the singular manifold $\bar{M}$, obtained by pasting together the bundles using ϕ. In the case of a Z_2-manifold, this is *not* the normal bundle to $\bar{M}$, as one can easily see by considering the Klein bottle. However, they differ only on N, where the difficulty is in the orientation.

$\bar{\nu}$ is induced from a map

$$\bar{\nu}\colon \bar{M} \longrightarrow B_{\mathrm{SPL}} ,$$

and hence has characteristic classes $\bar{\nu}^*(\lambda)$ for $\lambda \in H^*(B_{\mathrm{SPL}})$.

For example, in the case of a Z_2-manifold M, the total Wu class of M as an unoriented manifold is

$$(1 + v_1) \cup V(\bar{\nu}) , \tag{1.24}$$

where v_1 is the first Wu class, and $V(\bar{\nu})$ is the Wu class of the bundle $\bar{\nu}$. Indeed, on M, v_1 is the restriction of an integral class x_1 ([27]), and if we consider the map

$$(x_1)\colon M \longrightarrow S^1 ,$$

we see $v_1(\nu(M) + (x_1)^*(\xi_1)) = 0$, where ξ_1 is the Hopf bundle (Möbius band) over S^1. Thus $\nu(M) + x_1^*(\xi_1)$ is oriented. Now take $(\xi_1 + m\varepsilon)$ for m sufficiently large, and lift (x_1) to an embedding (y_1) in this bundle. Embed the bundle in Euclidean space E^{m+3}, delete a small neighborhood N of $*$ from S^1 on which f is regular, and $\pi^{-1}(\mathring{N})$ is $(\xi_1 + m\varepsilon)$-thickened to a neighborhood $D^1 \times D^1 \times D^{m+1}$. Then this neighborhood has a complimentary neighborhood $c = D^1 \times D^1 \times D^{m+1}$, and their union is $S^1 \times D^{m+2}$.

(1.25)

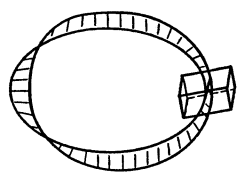

FIGURE 1.25.

Note that $(y_1)(M - f^{-1}(\mathring{N}))$ is an embedding of the Z_2-manifold in $D^1 \times D^1 \times D^{m+1}$. Moreover, from (1.30) we now see easily that $\bar{\nu}$ is the normal bundle to $(y_1)M$ in $\xi_1 + m\varepsilon$, and, since ξ_1 is the normal bundle to this embedding, we have $x_1^*(\xi_1) + \bar{\nu} = \nu$, and (1.24) follows.

2. Normal maps, surgery, and Sullivan's description of G/PL

In this section, we review the principal results of Sullivan's thesis.

Recall that a spherical fiber space is a map $f: E \to X$, for which the fiber has the homotopy type of a sphere. There are notions of Whitney sum, and a fiber homotopy trivial sphere bundle is one equivalent to $X \times S^n \xrightarrow{p_1} X$. The Whitney sums of trivial bundles are again trivial.

The equivalence classes of such bundles are preserved by the Whitney sum operation. Hence we can take the Grothendieck group $G(X)$ of such bundles over X.

THEOREM 2.1 (Stasheff [3]). *Let G_n be the set of homotopy equivalences of the sphere S^{n-1}, and B_{G_n} its classifying space. Let $i: G_n \subset G_{n+1}$ be the usual inclusion (suspension), and $B_{G_n} \hookrightarrow B_{G_{n+1}}$ the induced inclusion with $B_G = \lim_{n\to\infty} B_{G_n}$. Then, if X is a finite* CW-*complex,*

$$G(X) = [X, Z \times B_G] .$$

Consider the map

(2.2) $$B_{\mathrm{PL}} \longrightarrow B_G ,$$

induced by regarding the universal PL-sphere bundle as a fiber homotopy sphere bundle. Its fiber is the space G/PL. A homotopy class of maps $f: X \to G/\mathrm{PL}$ is exactly equivalent to specifying a PL-bundle γ over X, together with a fiber-homotopy trivialization of γ.

Following Sullivan, we describe the PL-bordism of G/PL.

DEFINITION 2.3. *Let M^n, $\tilde{M}^n$ be oriented* PL-*manifolds, both open or both closed. A degree* 1 *normal map $\rho\colon (\tilde{M}^n, \partial\tilde{M}^n) \to (M^n, \partial M^n)$ is a degree* 1 *map of pairs, together with a bundle isomorphism of the* PL-*normal bundle ν* ([26]) *to $(\tilde{M}^n)$, with $\rho^*(\gamma)$ for some $\gamma \in K_{\mathrm{PL}}(M^n)$.*

We say $\rho\colon (\tilde{M}^n, \partial\tilde{M}^n) \to (M^n, \partial M^n)$ is normally bordant to $\bar{\rho}\colon (\tilde{N}^n, \partial\tilde{N}) \to (N^n, \partial N)$ if there are manifolds $\tilde{W}^{n+1}$, W^{n+1}, with W a bordism from M to N, $\tilde{W}$ a bordism from $\tilde{M}$ to $\tilde{N}$, and a degree 1 normal map $F\colon (\tilde{W}, \partial) \to (W, \partial)$ so $F \mid \tilde{M}$ is ρ and $F \mid \tilde{N}$ is $\bar{\rho}$, together with the restrictions of the bundle isomorphism to ∂W being the original isomorphisms.

Normal bordism is an equivalence relation, and we can define the normal bordism groups $\mathfrak{NB}_*$ of *closed* oriented PL-manifolds in the evident way.

THEOREM 2.4 (Sullivan). $\mathfrak{NB}_* \cong \Omega_*^{\mathrm{PL}}(G/\mathrm{PL})$.

Proof. Let $f\colon M^n \to G/\mathrm{PL}$ be equivalent to $S^{l-1} \to \gamma \xrightarrow{\pi} M^n$, together with the fiberwise homotopy equivalence $h\colon \gamma \to M^n \times S^{l-1}$. Note that γ is a PL-manifold, and apply PL-transversality on $M^n \times *$. Thus we may assume $h^{-1}(M^n \times D^{l-1}) = \tilde{M}^n \times D^{l-1}$, and $\pi \mid \tilde{M}^n \to M^n$ is a degree 1 map. Note that the PL-normal bundle to γ is $\pi^*(\nu(M)) - \pi^*(\gamma)$, and the normal bundle to $\tilde{M}^n$ in γ is trivial. Hence the normal bundle of $\tilde{M}^n$ is stably $\pi^*(\nu(M) - \gamma)$, and π is a degree 1 normal map. The same construction works for bordisms. This gives a morphism $\lambda\colon \Omega_*^{\mathrm{PL}}(G/\mathrm{PL}) \to \mathfrak{NB}_*$. A similar construction proves the converse.

LEMMA 2.5. *Any element $x \in \Omega_n^{\mathrm{PL}}(G/\mathrm{PL})$ may be represented by a map of a simply-connected manifold M^n into G/PL for $n \geqq 4$* (see e.g. [5] or [39]).

Now consider a degree 1 normal map $f\colon \tilde{M}^n \to M^n$ of simply-connected manifolds and normal bordisms of the form

$$F\colon \tilde{W}^{n+1} \longrightarrow M^n \times I\,. \tag{2.6}$$

The main technical results of [5] or [18] can be summarized in

THEOREM 2.7. *$\rho\colon \tilde{M}^n \to M^n$, a normal degree* 1 *map of simply-connected closed manifolds, is normally bordant to a homotopy equivalence via a bordism of type* (2.7) *if and only if it is normally bordant to a homotopy equivalence via a general bordism. Moreover, if n is odd, such a bordism is always possible, while if n is $4k$ ($k > 1$), ρ is bordant to a homotopy equivalence if and only if* $\mathrm{Ind}(\tilde{M}^n) = \mathrm{Ind}(M^n)$, *and, if $n = 4k+2$, $k > 1$, the obstruction to making ρ bordant to a homotopy equivalence is a well-defined element κ of Z_2, called the Kervaire invariant of ρ.*

In the case $n = 4k$ in (2.7), $I(\tilde{M}^n) - I(M)$ is called the index of ρ and is always divisible by 8 ([18]). Again, from [18], there are maps $\phi\colon S^{4k} \to G/\mathrm{PL}$ with index $(\phi) = 8$, $k > 1$, or index $(\phi) = 16$ in case $k = 1$. Similarly, in each dimension $4k + 2$, there is a map $\phi\colon S^{4k+2} \to G/\mathrm{PL}$ having Kervaire invariant 1.

Sullivan sharpened (2.7) by observing first that the just cited results were strong enough to show that

$$\pi_i(G/\mathrm{PL}) = \begin{cases} Z, & i \equiv 0(4)\,, \\ Z_2, & i \equiv 2(4)\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{2.8}$$

Moreover, if $\mathfrak{N}_*^{\mathrm{PL}}(X)$ represents the unoriented PL-bordism ring, we have

COROLLARY 2.9. (a) *If* $n \equiv 0(4)$, *the map* $I\colon \{f\colon \tilde{M}^n \to M^n\} \to I(f)/8$ *factors through bordism and induces a homomorphism*

$$I\colon \Omega_n^{\mathrm{PL}}(G/\mathrm{PL}) \longrightarrow Z\,.$$

(b) *If* $n \equiv 2(4)$, *the Kervaire invariant induces a homomorphism*

$$K\colon \mathfrak{N}_n^{\mathrm{PL}}(G/\mathrm{PL}) \longrightarrow Z_2\,.^*$$

Now consider the effect on I and K of operating with $\Omega_*^{\mathrm{PL}}(\text{point})$ or $\mathfrak{N}_*^{\mathrm{PL}}(\text{point})$. We have

$$I(x(y)) = (\mathrm{Ind}\ x)I(y) \tag{2.10}$$

for $x \in \Omega_{4*}^{\mathrm{PL}}(\text{point})$ and

$$I(x(y)) = 0$$

otherwise. A more delicate argument ([32], [35]) now shows

THEOREM 2.11. $K(x(y)) = \langle V^2, [X]\rangle K(y)$, *where* V *is the total Wu class of a representative* X *of* x.

This is Sullivan's famous product formula. The original formula used W, the total Stiefel-Whitney class of X, in place of V^2. However, in the applications, all calculations become shorter if W is replaced by V^2. This observation is due to G. Brumfiel.

Also, while we do not explicitly prove (2.11) in the course of this paper, we develop sufficient machinery (notably in §§ 6 and 7) to make (2.11) routine.

(2.7) through (2.11) were sufficient to prove

COROLLARY 2.12. *The 2-localized homotopy type of* G/PL *is*

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* More precisely, in (2.9b), Rourke, Sullivan, and Wall have shown that there is a morphism

$$K\colon \mathfrak{N}_n^{\mathrm{PL}}(G/\mathrm{PL}) \longrightarrow Z_2\,,$$

so that the composition

$$\Omega_n^{\mathrm{PL}}(G/\mathrm{PL}) \longrightarrow \mathfrak{N}_n^{\mathrm{PL}}(G/\mathrm{PL}) \longrightarrow Z_2$$

is the Kervaire invariant.

$$E_2 \times \prod_{i=2}^{\infty} K(Z_2, 4i-2) \times K(Z, 4i)\ ,$$

where E_2 *is the unique stable 2-stage Postnikov system having* $\pi_2(E_2) = Z_2$, $\pi_4(E_2) = Z$, $\pi_i(E_2) = 0$ *otherwise and non-trivial k-invariant.*

For the proof, see [32] or [35].

Remark 2.13. (2.12) is proved by constructing classes K_*, (k_*) in $H^*(G/\mathrm{PL}, Z)$, $H^{4*+2}(G/\mathrm{PL}, Z_2)$, satisfying $I(M, f) = \langle L_*(M)f^*K_*, [M]\rangle$ $K(M, f) = \langle V^2(M)f^*(k_*), [M]\rangle$. There is a large ambiguity in the definition of K_* in Sullivan's proof of (2.12). However, since $\mathfrak{N}_*^{\mathrm{PL}}(X)/\overline{\mathfrak{N}}_*^{\mathrm{PL}}(\mathrm{point})\cdot\mathfrak{N}_*^{\mathrm{PL}}(X) \cong H_*(X, Z_2)$, and we have $H^*(X, Z_2) \cong \mathrm{Hom}(H_*(X, Z_2), Z_2)$, there is no ambiguity in the definition of k_*. From the point of view of cohomology theory, our efforts in the following sections will be directed toward removing the ambiguity from K_*. To do this, it is sufficient to define K_*, *together with its restrictions to* $H^*(G/\mathrm{PL}, Z_{2^r})$ for each r. Thus we shall have to talk of the analogue of the index homomorphism I for Z_{2^r}-manifolds.

3. Surgery on simply-connected Z_n-manifolds

Consider a closed Z_n-manifold (M^m, κ) and a map $f: (M^m, \kappa) \to G/\mathrm{PL}$. Let $S^{l-1} \to \gamma \to M^m$ be the associated bundle, together with the trivialization $t: \gamma \to M^m \times S^{l-1}$. Since f is a Z_n-map, we may assume t to be the same in a neighborhood of each identified component of ∂M. Making t transversal by first changing it equally on identified components and then extending transversality to the interior gives a Z_n-manifold $(\widetilde{M}^m, \tilde{\kappa})$ and a degree 1 normal map

$$(3.1) \qquad \rho: (\widetilde{M}^m, \tilde{\kappa}) \longrightarrow (M^m, \kappa)\ .$$

Moreover, by imitating the proof of (2.5), we have

LEMMA 3.2. *Bordism classes of degree* 1 *normal maps of* Z_n*-manifolds* (*as in* (3.1)) *correspond bijectively with* $\Omega_*^{\mathrm{PL}}(G/\mathrm{PL}; Z_n)$.

A closed Z_n-manifold (M^m, κ) is said to be simply-connected if $\pi_1(M^m) = \pi_1(N^{m-1}) = 0$. Analogous with (2.6), we have

LEMMA 3.3. *Any element* $x \in \Omega_m^{\mathrm{PL}}(G/\mathrm{PL}; Z_n)$ *may be represented by a map of a simply-connected* Z_n*-manifold* (M^m, κ) *into* G/PL.

Now, consider (3.1) for simply-connected Z_n-manifolds. We attempt to replace ρ by a homotopy equivalence of Z_n-manifolds by surgery. This means that surgeries on the boundary must be done equivariantly, but interior surgeries are arbitrary.

THEOREM 3.4. *There are homomorphisms*

$$I/8\colon \Omega^{PL}_{4m}(G/PL; Z_n) \longrightarrow Z_n ,$$
$$K_\partial\colon \Omega^{PL}_{4m-1}(G/PL; Z_n) \longrightarrow Z_n \otimes Z_2 ,$$
$$K\colon \Omega^{PL}_{4m-2}(G/PL, Z_n) \longrightarrow Z_n \otimes Z_2 ,$$
$$0\colon \Omega^{PL}_{4m-3}(G/PL, Z_n) \longrightarrow 0$$

so that the simply-connected surgery problem (3.1) *associated to* $x \in \Omega^{PL}_*(G/PL, Z_n)$ *is bordant to a homotopy equivalence* $m > 1$ *if and only if the appropriate homomorphism takes* x *to zero.*

Proof. In (3.1), look first at $\rho \mid \tilde{N}^m \to N^m$. Every surgery done on $\tilde{N}$ attaches n handles to $\tilde{M}$,

(3.5)

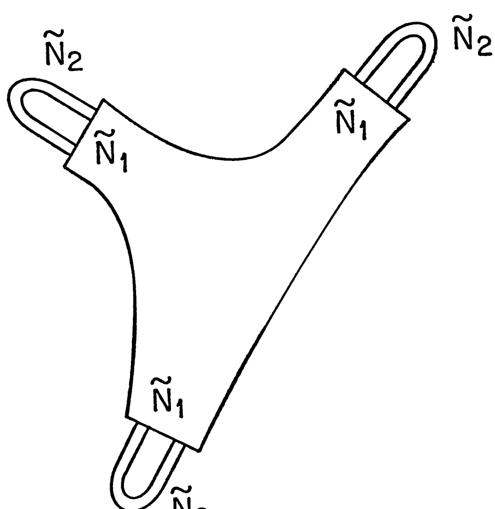


FIGURE 3.5.

but gives a problem bordant to the original one. Thus as a first step, we attempt to make $\rho \mid \tilde{N} \to N$ a homotopy equivalence.

If $\dim N \equiv 0(4)$, the obstruction to making $\rho \mid$ a homotopy equivalence is $I(\tilde{N}) - I(N)$. But $n(N) = \partial M$, hence $nI(N) = 0$ so $I(\tilde{N}) = I(N) = 0$ and $\rho \mid$ is equivalent to a homotopy equivalence. We now have an ordinary simply-connected surgery problem (3.1) with $\rho \mid$ a homotopy equivalence on the boundary. At this point, there is no further obstruction to completing the surgery on the interior ([18]) to make (3.1) an equivalence of Z_n-manifolds. This completes the case $* = 4k - 3$.

If $\dim N \equiv 1(4)$, there is no obstruction to making $\rho \mid$ a homotopy equivalence. However, the Kervaire invariant is an obstruction to making ρ an equivalence on the interior of M. Consider a bordism W of $\rho\colon (\tilde{M}, \tilde{\kappa}) \to (M, \kappa)$ to $\bar{\rho}\colon (\bar{M}, \bar{\kappa}) \to (M, \kappa)$, where we assume $\bar{\rho} \mid$ again a homotopy equivalence on the boundary. $\partial W = \tilde{M} \cup \bar{M} \cup n(Z)$, and the Kervaire invariant on $\partial(W)$ is zero. But $K(\partial W) = K(\tilde{M}) + K(\bar{M}) + n(K(Z))$. Hence $K(\bar{M}) = K(\tilde{M}) + n((Z))$, so if n is odd and Z is a bordism from N to N, having Kervaire invariant 1 ([5]), consider the bordism of Z_n-manifolds

(3.6)

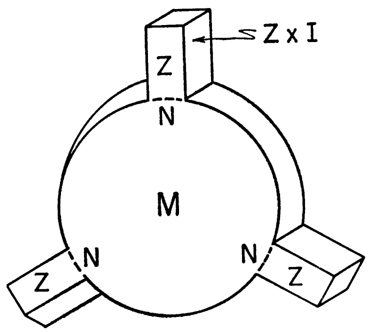


FIGURE 3.6.

where the back face is $\bar{M}$. Clearly, $K(\bar{M}) = 1 + K(M)$. Thus there is no obstruction in this case. However, if n is even, then $K(\bar{M}) = (\tilde{M})$, and the obstruction is well-defined. This completes the case $* = 4k - 2$.

If dim $N \equiv 2(4)$, there is a Kervaire obstruction K to making $\rho \mid \tilde{N} \rightarrow N$ into a homotopy equivalence. The invariant on $\partial(\tilde{M})$ is zero of course, but by equivariance $K(\partial\tilde{M}) = n \cdot K(\tilde{N})$. Thus $K(\tilde{N})$ is zero if n is odd, and we can make $\rho \mid$ a homotopy equivalence. There is then no further obstruction to making ρ an equivalence on the interior. So if n is odd, the obstruction is zero.

Consider the case n even. Let W be a normal bordism from $(\bar{M}, \bar{\kappa}) \rightarrow (M, \kappa)$ to $(\bar{M}, \bar{\kappa}) \rightarrow (M, \kappa)$. Then the Z's are normal bordisms from $\tilde{N}$ to $\bar{N}$, and $K(\partial Z) = K(\tilde{N}) + K(\bar{N}) = 0$, so K is well-defined and depends only on the bordism class. This completes the case $* = 4k - 1$.

If $\dim(N) = 4k - 1$, there is no obstruction to making $\rho \mid$ a homotopy equivalence. Hence the obstruction to completing the surgery is $I(\tilde{M}) - I(M)$ (with $\rho \mid$ a homotopy equivalence). This number may be changed, however. We have for W, as before,

(3.7) $$I\partial(W) = I(\tilde{M}) - I(\bar{M}) + nI(Z) = 0$$

by Novikov's index addition lemma ([30]). On the other hand, consider the degree 1 normal map $\bar{\rho}\colon Z \rightarrow I \times N$. It is a homotopy equivalence on the boundary, hence by [25], $I(Z) \equiv 0(8)$. From [5], there is a normal bordism from $\bar{\rho} \mid$ to $\rho \mid$, having index any multiple of 8. Using the construction of (3.6), we see that we can vary $I(\tilde{M}) - I(M)$ by any multiple of $8n$. But $I(\tilde{M}) - I(M)$ is itself divisible by 8. Hence $((\tilde{M}) - I(M))/8$ is a well-defined invariant of ρ, taken as a residue class mod n. This completes the proof of (3.4).

We now turn to the question of how one determines the value of I, given (3.1).

THEOREM 3.8. *Let* $\rho\colon \tilde{M}^{4k-1} \to M^{4k-1}$ *be a degree* 1 *normal map of closed oriented* PL-*manifolds, and suppose that* $W(\tilde{W})$ *is a bordism from* M *to* M' ($\tilde{M}$ *to* $\tilde{M}'$). *Assume* ρ *extends to a degree* 1 *normal map* $\bar{\rho}\colon \tilde{W} \to W$, *and* $\bar{\rho}\,|\,\tilde{M}'$ *is a homotopy equivalence. Then* $I(\tilde{W}) - I(W)$ *is a* mod 8 *invariant of the normal map* ρ.

Proof. Let $\tilde{W}'$, W' be a second set of bordisms, and $\bar{\bar{\rho}}$ a normal extension of ρ so $\bar{\bar{\rho}}\,|\,\tilde{M}''$ is again a homotopy equivalence. Then $\bar{\rho} \cup \bar{\bar{\rho}}\colon \tilde{W} \cup_{\tilde{M}} - \tilde{W}' \to W \cup_M - W'$ is a degree 1 normal map which is a homotopy equivalence on the boundary. Hence

$$I(\tilde{W} \cup - \tilde{W}') - I(W \cup - W') \equiv 0(8)\ . \tag{3.9}$$

But by the index addition theorem,

$$I(W \cup - W') = I(W) - I(W')\ ,$$
$$I(\tilde{W} \cup - \tilde{W}') = I(\tilde{W}) - I(\tilde{W}')\ ,$$

and (3.9) now implies $I(\tilde{W}) - I(W) \equiv I(\tilde{W}') - I(W')(8)$, as was to be shown.

DEFINITION 3.10. *Let* $\rho\colon \tilde{M}^{4k-1} \to M^{4k-1}$ *be a degree* 1 *normal map of closed oriented* PL-*manifolds. Then the semi-index of* ρ, *written* $\mathrm{SI}(\rho)$, *is the* mod 8 *index of any normal bordism of* ρ *to a homotopy equivalence.*

Thus we have

COROLLARY 3.11. $I/8(\{\rho\colon (\tilde{M}^{4k}, \tilde{k}) \to (M^{4k}, k)\})$ *is given as* $1/8\{I(\tilde{M}^{4k}) - I(M^{4k}) + n\mathrm{SI}(\rho\,|\,\tilde{N}^{4k-1} \to N^{4k-1})\}$.

Our object in the next three sections will be to give an effective method for calculating $\mathrm{SI}(\rho\,|)$.

We now consider the Kervaire invariant cases. If dimension (M) is $4k+2$, it is a result of Brumfiel and Wall (see e.g. [32]) that the obstruction obtained by first making $\rho\colon \tilde{M}^{4k+2} \to M^{4k+2}$ an equivariant homotopy equivalence on the boundaries and then attempting to do surgery on the interior is given for Z_2-manifolds by the formula

$$K(\rho) = \langle V^2(\bar{\nu}) \cup f^*(k_*),\ [M]\rangle = \langle V^2(\nu) \cup f^*(k_*),\ [M]\rangle\ . \tag{3.12}$$

The latter equality follows since

$$((1 + V_1)V(\nu)) = (1 + V_1)^2 V(\nu)^2 = V(\nu)^2$$

by (1.29). Note in particular that $K(\rho)$ is independent of the choice of the equivariant bordism to an equivariant homotopy equivalence.

Now we show that the result for Z_2-manifolds implies a similar result for Z_n-manifolds. Indeed, let n be even and

$$\pi: Z_n \longrightarrow Z_2$$

the non-trivial homomorphism. π converts a Z_n-manifold to a Z_2-manifold by defining $\partial_2 M = N \times \{\pi^{-1}(0)\} \cup N \times \{\pi^{-1}(1)\}$, and the homeomorphism identifies $(N \times i)$ with $(N \times (i+1))$.

We have the diagram of maps of singular manifolds (1.24) inducing the surgery problem on M,

$$M \longrightarrow \bar{M}_{(2)} \xrightarrow{\pi} \bar{M}_{(n)} \longrightarrow G/\mathrm{PL} \ . \tag{3.13}$$

Choose a bordism of the normal problem over N to a homotopy equivalence. This gives a bordism of $\partial_2(M)$ to a homotopy equivalence, and we see that the surgery problem for $\bar{M}_{(2)}$ is exactly the problem for M.

Now $\bar{\nu}(\bar{M}_{(2)}) = \pi^*\bar{\nu}(\bar{M}_{(n)})$, as is obvious. Hence

$$\begin{aligned} K(\rho) &= \langle \pi^* f^*(k_*) \cup \big(V\bar{\nu}(\bar{M}_{(2)})\big)^2, [\bar{M}_{(2)}]\rangle \\ &= \langle f^*(k_*) \cup (V\bar{\nu}\bar{M}_{(n)})^2, [\bar{M}_{(n)}] \otimes Z_2\rangle \\ &= \langle j\big(f^*(k_*) \cup V(\bar{\nu}\bar{M}_{(n)})^2\big), [\bar{M}_{(n)}]\rangle \ , \end{aligned} \tag{3.14}$$

where

$$j: H^*(\ ; Z_2) \longrightarrow H^*(\ ; Z_n)$$

is the coefficient homomorphism.

Finally, in the case when dimension M is $4k+3$, the surgery obstruction is given by

$$\begin{aligned} \langle \big(V\bar{\nu}(M)\big)^2 \cup f^*(k_*), [N]\rangle &= \langle j\big(V\bar{\nu}(M)^2 \cup f^*(k_*)\big), \partial_n[M]\rangle \\ &= \langle \beta_n j\big(V\bar{\nu}(M)^2 \cup f^*(k_*)\big), [M]\rangle \ . \end{aligned}$$

Remark 3.16. In the case of Z_2-manifolds, β_n becomes Sq^1, and $Sq^1\big(V\bar{\nu}(M)\big)^2 = 0$ so (3.15) becomes

$$K_\partial(\rho) = \langle \big(V\bar{\nu}(\bar{M})\big)^2 \cup f^*\big(Sq^1(k_*)\big), [M]\rangle \ .$$

4. The semi-index for surgery on $4k-1$-manifolds and Q/Z-quadratic forms

In this section, we obtain techniques for evaluating the semi-index (3.10) of a surgery problem,

$$\rho: \tilde{M}^{4k-1} \longrightarrow M^{4k-1} \ . \tag{4.1}$$

They involve a Gaussian invariant for rational quadratic forms.

Recall that, by doing framed surgeries on $\tilde{M}$ in (4.1), we can make ρ

$2k-2$-connected ([5, Chaps. 2, 3]). In the remainder of this section, we assume this condition satisfied. Note that now we need only make a series of modification on embedded S^{2k-1}'s in order to make ρ into a homotopy equivalence. This can be broken down even further. Let $K_{2k-1}(\widetilde{M})$ be the kernel, and split K as

(4.2) $$K^{\text{Free}} \oplus K^{\text{Tor}}_{2k-1}(\widetilde{M})$$

where K^{Free} is a free Abelian group and K^{Tor} is the torsion subgroup. Then we may first perform surgery to kill K^{Free}. This will reduce K_{2k-1} to $K^{\text{Tor}}_{2k-1}(\widetilde{M})$, and we have

LEMMA 4.3. *Let $\widetilde{W}$ be a bordism from $\widetilde{M}^{4k-1}$ to $\bar{M}^{4k-1}$ and*

$$F\colon \widetilde{W} \longrightarrow M \times I$$

a normal bordism of ρ to $\bar{\rho}$ so that:

(1) *$\widetilde{W}$ is obtained from $\widetilde{M}$ by adding $2k$-cells, and*

(2) *$K_{2k-1}(\bar{M}) \cong K^{\text{Tor}}_{2k-1}(\widetilde{M})$, in the sense that there is a direct sum embedding $i\colon K_{2k-1}(\bar{M}) \to K_{2k}(\bar{W}, \partial)$, and $\partial i(x) = \phi(x) - x$ defines the isomorphism. Then $I(\widetilde{W}) \equiv 0(8)$.*

Proof. Consider the exact sequence of kernels for F:

(4.4) $$0 \longrightarrow K_{2s}(\partial\widetilde{W}) \xrightarrow{r} K_{2s}(\widetilde{W}) \xrightarrow{j} K_{2s}(\widetilde{W}, \partial\widetilde{W}) \xrightarrow{\partial} K_{2s-1}(\partial\widetilde{W}) \xrightarrow{r} K_{2s-1}(\widetilde{W}) \longrightarrow 0\ .$$

Note that $K_{2s}(\partial\widetilde{W})$, $K_{2s}(\widetilde{W})$ are both free groups. By (4.3.2), and the fact that $\widetilde{W} \simeq \widetilde{M} \cup e^{2k} \cup \cdots \cup e^{2k}$, we see that j_* is a map onto a torsion-free direct summand in (4.4). Hence r_{2s} is an isomorphism. Now the self-interjection number of any element in $\operatorname{im}(r_{2s})$ is zero. Hence $I(\widetilde{W}) = I(\operatorname{im} j_*)$, but this admits a unimodular even form, and hence has index congruent to 0 mod (8).

From this, it follows that $\mathrm{SI}(\rho) = \mathrm{SI}(\bar{\rho}\colon \bar{M}^{4k-1} \to M^{4k-1})$, and the index of a bordism from ρ to a homotopy equivalence is determined by the process of doing surgery to kill $K^{\text{Tor}}_{2k-1}(\widetilde{M})$. Assume now that $\rho\colon \widetilde{M} \to M$ satisfies the additional condition imposed by (4.3.2). Let $F\colon W \to M \times I$ be a bordism from ρ to a homotopy equivalence, with W obtained from $\widetilde{M}$ by adding only $2k$-cells. The exact sequence of kernels (4.4) becomes

(4.5) $$0 \longrightarrow K_{2k}(W) \xrightarrow{A} K_{2k}(W, \partial W) \xrightarrow{\partial} K^{\text{Tor}}_{2k-1}(\widetilde{M}) \longrightarrow 0\ ,$$

where both $K_{2k}(W)$ and $K_{2k}(W, \partial W)$ are free. The intersection pairing identifies $K_{2k}(W; \partial W)$ with $\operatorname{Hom}_Z(K_{2k}(W); Z)$, and the self-intersection form on $K_{2k}(W)$ is given by

$$x \cdot x = A(x) \cap x = x \cap A(x) . \tag{4.6}$$

This form is even, rationally non-singular, and its index is $I(W) \equiv \mathrm{SI}(\rho)(8)$. The associated symmetric bilinear form is $x \cdot y = A(x) \cap y = x \cap A(y)$.

LEMMA 4.7. *Let* A^{-1}: $K_{2k}(W, \partial W) \otimes Q \to K_{2k}(W) \otimes Q$ *be the inverse of* $A \otimes 1$: $K_{2k}(W) \otimes Q \to K_{2k}(W, \partial W) \otimes Q$. *Then there is a well-defined non-singular quadratic pairing on* $K_{2k-1}^{\mathrm{Tor}}(\widetilde{M})$ *associated to* (4.6),

$$\lambda: K_{2k-1}^{\mathrm{Tor}}(\widetilde{M}) \longrightarrow Q/2Z ,$$

defined by taking the residue

$$\lambda(y) = \{x \cap (A^{-1}x)\} \qquad \textit{in } Q/2Z$$

where $\partial x = y$.

Proof. Indeed, if x' also satisfies $\partial x' = y$, we must have $x' = x + Az$. Thus

$$x' \cap A^{-1}x' = x \cap A^{-1}x + 2(x \cap z) + z \cap Az .$$

Since A is even, $z \cap Az$ is even, so, modulo twice an integer, $x' \cap A^{-1}x' = x \cap A^{-1}x^t$, and the form is well-defined. It remains to show λ non-singular.

The associated bilinear form $\varphi(y, y')$ is defined as $\{x \cap A^{-1}x'\}$ in Q/Z where

$$\partial x = y ,$$
$$\partial x' = y' .$$

Let μ: $K_{2k}(W) \to \mathrm{Hom}_Z(K_{2k}(W, \partial W); Z)$, μ': $K_{2k}(W, \partial W) \to \mathrm{Hom}_Z(K_{2k}(W); Z)$ be the isomorphism induced by Poincaré duality (cap product). We have the commutative diagram

$$\begin{array}{ccc} K_{2k}(W) & \xrightarrow[\cong]{\mu} & \mathrm{Hom}_Z(K_{2k}(W, \partial W); Z) \\ \downarrow A & & \downarrow A^* \\ K_{2k}(W, \partial W) & \xrightarrow[\cong]{\mu'} & \mathrm{Hom}_Z(K_{2k}(W); Z) , \end{array} \tag{4.8}$$

and A^* is injective since A is. Moreover, passing to Hom groups from (4.5), we have the short exact sequence

$$0 \to \mathrm{Hom}(K_{2k}(W, \partial W); Z) \to \mathrm{Hom}(K_{2k}(W); Z) \to \mathrm{Ext}(K_{2k-1}^{\mathrm{Tor}}(\widetilde{M}); Z) \to 0 . \tag{4.9}$$

But Pontrjagin duality identifies $\mathrm{Ext}(G; Z)$ with $\mathrm{Hom}(G; Q/Z)$ ([11, p. 139]). Thus (4.8) abuts above the square

$$\begin{array}{ccc} K_{2k}(W, \partial W) & \xrightarrow{\mu'} & \mathrm{Hom}_Z(K_{2k}(W); Z) \\ \downarrow \partial & & \downarrow \\ K_{2k-1}^{\mathrm{Tor}}(\widetilde{M}) & \xrightarrow{\nu} & \mathrm{Hom}(K_{2k-1}^{\mathrm{Tor}}(\widetilde{M}); Q/Z) \end{array} \tag{4.9}$$

where ν is the homomorphism corresponding to φ. Thus the fact that ν is an isomorphism follows from the 5-lemma, and (4.7) follows.

Remark 4.10. It follows from [18] that the form λ defined in (4.7) from (4.5) for our situation of kernels and surgery is in fact defined intrinsically on $\tilde{M}$ as a *self-linking* number. We do not need this fact in the sequel, but will show by another method in § 6 that λ is intrinsic to $\tilde{M} \to M$ and not to the choice of W, bording ρ: $\tilde{M} \to M$ to a homotopy equivalence.

LEMMA 4.11. *Let* λ: $K_{2k-1}^{\mathrm{Tor}}(\tilde{M}) \to Q/2Z$ *be given as in* (4.7), *and suppose* n *is the order of the finite group* $K_{2k-1}^{\mathrm{Tor}}(\tilde{M})$. *Then*

$$\mathcal{G}_\lambda = \sum\nolimits_{x \in K_{2k-1}^{\mathrm{Tor}}(\tilde{M})} e^{\pi i \lambda(x)} = e^{(\pi i/4) I(A)}(n^{1/2}) .$$

Proof. In [1], we find a similar result stated under the hypothesis that A is odd. However, the proof given there works without essential modification to show (4.11) for even A.*

Remark 4.12. I am indebted to G. Brumfiel for the proof given above. It replaces the considerably longer argument originally used. Similar results may be found in [2].

(4.11) gives us an explicit way to determine the semi-index of ρ once we are given the quadratic form λ on $K_{2k-1}^{\mathrm{Tor}}(\tilde{M})$.

We now consider the general non-singular quadratic form

$$\lambda\colon K \longrightarrow Q/2Z ,$$

defined on the finite Abelian group K, with associated bilinear form φ.

LEMMA 4.13. *Suppose* $K_1 \subset K$ *is a submodule of* K *on which* φ *is non-singular. Let* $K_2 = K_1^\perp$. *Then* $K = K_1 \oplus K_2$, *and* $\mathcal{G}_\lambda = \mathcal{G}_{\lambda_1} \cdot \mathcal{G}_{\lambda_2}$ *where* λ_i *is* λ-*restricted to* K_i.

Proof. That $K = K_1 \oplus K_2$ is well-known ([38]). Now note that $\lambda(k_1 + k_2) = \lambda(k_1) + \lambda(k_2) + 2\varphi(k_1, k_2) = \lambda(k_1) + \lambda(k_2)$. Thus

$$\sum e^{\pi i \lambda(k_1 + k_2)} = \sum e^{\pi i \lambda(k_1)} \sum e^{\pi i \lambda(k_2)} ,$$

and (4.13) follows

THEOREM 4.14. *Under the assumptions above, let* $\mathcal{G}_\lambda = \sum_{x \in K} e^{\pi i \lambda(x)}$. *Then*:

(a) *There is a finitely-generated free Abelian group* F, *a symmetric*

* Some questions have been raised about the convergence of the integrals used in [1]. However, P. Cohen points out that the usual way of interpreting such integrals is as distributions, in which case there is no further difficulty in [1]. Alternatively, the reader could introduce an integrating factor as is done in [2] or [45] and read off the answer as an asymptotic limit.

even map $A: F \to F$, *and an identification* μ *of* F *with* $\mathrm{Hom}_Z(F; Z)$ *so*

$$(4.15) \qquad 0 \longrightarrow F \xrightarrow{A} F \longrightarrow K \longrightarrow 0$$

is exact, and λ, φ *are induced from* (4.14) *as in* (4.7);

(b) *If* λ, λ' *are two quadratic forms on* K *with respect to the same bilinear form* φ, *then there is an* $a \in K$ *so that*

(1) $2a = 0$,

(2) $\lambda'(x) = \lambda(a + x) - \lambda(a)$

for each $x \in K$. *Moreover,* $\mathcal{G}_{\lambda'} = \mathcal{G}_\lambda e^{-\pi i \lambda(a)}$.

Proof. (a) is the $\varepsilon = +1$ part of [40; Th. 6, p. 296]. To show (b), note that $\lambda'(x) - \lambda(x)$ is a linear function of x. Thus there is an $a \in \mathrm{Hom}(K, Q/Z)$ so

$$\lambda'(x) - \lambda(x) = 2\varphi(x, a) \in Q/2Z ,$$

for all $x \in K$. Write $\lambda'(x) = \big(\lambda(x) + 2\varphi(x, a) + \lambda(a)\big) - \lambda(a) = \lambda(x + a) - \lambda(a)$, and (b.2) follows. Also, since

$$\lambda'(x - x) = 2\big(\lambda'(x) - \varphi(x, x)\big) = 2\big(\lambda(x) - \varphi(x, x)\big) + 4\varphi(x, a) = 0 ,$$

we have $4\varphi(x, a) \equiv 0$ in $Q/2Z$. Thus $2\varphi(x, a) \equiv 0$ in Q/Z, and $\varphi(x, 2a) = 0$. This being so for all x implies that $2a = 0$, so (b.1) follows.

Finally, note that

$$\mathcal{G}_{\lambda'} = \sum e^{\pi i \lambda'(x)} = \sum\nolimits_{x \in K} e^{\pi i \lambda(x+a)} e^{-\pi i \lambda(a)} = \mathcal{G}_\lambda e^{-\pi i \lambda(a)} ,$$

and (4.14b) follows.

Example 4.16. Let K be a direct sum of Z_2's and $\lambda: K \to Z_4 \subset Q/2Z$ an even quadratic form with respect to the non-singular bilinear form φ. Then $\dim(K)$ is even, K has a φ-symplectic basis $x_1 \cdots x_n$, $y_1 \cdots y_n$ with $\varphi(x_i, x_j) = \varphi(y_i, y_j) = 0$, $\varphi(x_i, y_j) = \delta_{ij}$ ([7]). Moreover, if K_i is the subgroup generated by (x_i, y_i), we have (4.13) $\mathcal{G}_\lambda = \prod_{i=1}^n \mathcal{G}_{\lambda_i}$. Now it is easily seen that, in K_i, we may assume $\lambda(x_i) = \lambda(y_i)$ is either 0 or 1. If $\lambda(x_i) = \lambda(y_i) = 1$, we find that $\mathcal{G}_{\lambda_i} = -2$. Similarly, if $\lambda(x_i) = \lambda(y_i) = 0$, we have $\mathcal{G}_{\lambda_i} = +2$. Thus $\mathcal{G}_\lambda = (-1)^r 2^n$ where r is the number of K_i for which $\lambda(x_i) = \lambda(y_i) = 1$; i.e., $\mathcal{G}_\lambda$ is negative if and only if the Arf invariant ([18]) of λ is 1.

COROLLARY 4.17. *Let* K *be a* Z_2*-vector space with non-singular bilinear form* φ, *and even quadratic form* λ *associated to* φ. *Let* $0 \to F \xrightarrow{A} F \to K \to 0$ *be a sequence* (4.15) *associated to* λ. *Then*

$$I(A) \equiv 4\varepsilon(8) ,$$

where ε *is the* Arf *invariant of* λ.

Example 4.18. Suppose again that K is a Z_2-vector space, but we no longer assume λ is even. Hence $K = K_1 \perp K_2 \perp K_3$ where K_1 has an orthogonal basis $x_1 \cdots x_r$ and $\lambda(x_1) = \cdots = \lambda(x_r) = 1/2$, while K_2 has an orthogonal basis $y_1 \cdots y_s$ with $\lambda(y_i) = 3/2$, and K_3 has a symplectic basis as in (4.16). We verify that, on Z_2 with $\lambda(1) = 1/2$, $\mathcal{G}_\lambda = e^{\pi i/4}\sqrt{2}$, while on Z_2 with $\lambda(1) = 3/2$, $\mathcal{G}_\lambda = e^{-\pi i/4}\sqrt{2}$. Thus on K we have

$$\mathcal{G}_\lambda = (e^{(\pi i/4)(r-s)})(-1)^{\mathrm{Arf}(K_3)}\,|K|^{1/2}\,.$$

Remark 4.19. Corresponding to the examples of (4.18), we can construct degree 1 normal maps $\rho\colon \tilde{M}^{4k-1} \rightarrow M^{4k-1}$, having $\mathrm{SI}(\rho) = \pm 1$ as follows. Let $\tau(S^{2k})$ be the tangent disc bundle to S^{2k}. Then $\tau(S^{2k})$, and hence $\partial\tau(S^{2k})$ are both stably parallelizable, so the degree 1 map of pairs

$$\rho\colon (\tau(S^{2k}),\, \partial\tau) \longrightarrow (D^{4k},\, S^{4k-1}) \tag{4.20}$$

is a degree 1 normal map. The self-intersection number of S^{2k} in $\tau(S^{2k})$ is $+2$, so we have the exact sequence of kernels

$$0 \longrightarrow Z \xrightarrow{\times 2} Z \longrightarrow Z_2 \longrightarrow 0 \tag{4.21}$$

and $\lambda(1) = 1/2$.

Similarly, if $\tilde{\tau}(S^{2k})$ is the stably trivial $2k$-disc bundle in which S^{2k} has self-intersection -2, then once more

$$\rho\colon (\tilde{\tau}(S^{2k}),\, \partial\tilde{\tau}) \longrightarrow (D^{4k},\, S^{4k-1})$$

is a degree 1 normal map, and this time the sequence of kernels

$$0 \longrightarrow Z \xrightarrow{\cdot(-2)} Z \longrightarrow Z_2 \longrightarrow 0$$

gives $\lambda(1) = 3/2$ in $\mathbf{Q}/2Z$.

In the general case of forms on arbitrary finite Abelian groups, we will find 4.22 very useful.

LEMMA 4.22 (Brumfiel-Knebusch). *Let*

$$\mathcal{S}\colon 0 \longrightarrow C \xrightarrow{\alpha} A \xrightarrow{\beta} B \longrightarrow 0$$

be a sequence (*not exact in general*) *of finite Abelian groups with* $\alpha\cdot\beta = 0$, α *an injection and* β *a surjection. Suppose a non-singular pairing* $\varphi\colon A \times A \rightarrow \mathbf{Q}/Z$ *is given so that* B *and* C *are dually paired* (i.e., $\varphi\,|\,C \times C$ is identically 0, and the pairing $\psi\colon C \times B \rightarrow \mathbf{Q}/Z$ defined by $\psi(c, b) = \varphi(c, g)$ for any g with $\beta(g) = b$ is non-singular). *Then there is a well-defined non-singular pairing* $\bar{\varphi}$ *on* $C^\perp/C = H_*(\mathcal{S})$. *Moreover, if* λ *is a quadratic form on* A *with respect to* φ *which vanishes on* C, *then* λ *induces a well-defined form* $\bar{\lambda}$, *quadratic with respect to* $\bar{\varphi}$ *on* $C^\perp/C$ *and*

$$\mathcal{G}_\lambda = \mathcal{G}_{\bar\lambda}|B| .$$

Proof. $C^\perp = \ker\beta$ due to the assumption of non-singularity on the pairing $C \times B \to \mathbf{Q}/Z$. Let $\pi: C^\perp \to C^\perp/C$ be the projection. Then for $a = \pi(a')$, $b = \pi(b')$, set $\bar\varphi(a, b) = \varphi(a', b')$. It is well defined, and non-singularity follows from the non-singularity of φ. A similar definition applies to $\bar\lambda$. It remains to calculate $\mathcal{G}_\lambda$. β splits A into cosets $\{C^\perp + b\}$, and $C^\perp$ in turn splits into cosets $\{C + a\}$. Write $\mathcal{G}_\lambda$ in terms of the double cosets as

$$\text{(4.23)} \qquad \sum_{a_j}\sum_{b_i}\sum_C e^{\pi i\lambda(b_i+a_j+C)} = \sum_{a_j,b_i}\sum_C e^{\pi i(\lambda(a_j+b_i)+2\varphi(b_i,C))} = \sum e^{\pi i\lambda(a_j+b_i)}\sum_C e^{2\pi i\varphi(b_i,C)} .$$

But for any $n>1$, we have $\sum_{t=0}^{n-1} e^{2\pi i(t/n)} = 0$. Thus summing over C in (4.23) gives 0 unless $b_i = 0$, so we have

$$\mathcal{G}_\lambda = \Big(\sum_{\bar a\in C^\perp/C} e^{\pi i\bar\lambda(\bar a)}\Big)|B| ,$$

and (4.22) follows.

We obtain sequences

$$0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0$$

as in (4.22) by simply taking for C any subgroup of A on which φ and λ vanish. Then $B = A/C^\perp$. For example, if $A = Z_q \oplus Z_q$ with symplectic bases e, f, and $\lambda(e) = 0$, then $\mathcal{G}_\lambda = |A|^{1/2} = q$. Again, if $\lambda(e) = 1$ and 4 divides q, then $2(e)$ generates C and $C^\perp/C = Z_2 \oplus Z_2$ with $\bar\lambda(\bar e) = 1$, $\bar\lambda(\bar f) = 0$ (since $\bar f = \pi(q/2\, f)$), so again $\mathcal{G}_\lambda = q$. Thus we have

COROLLARY 4.24. *Let A have quadratic form λ with respect to a sympletic φ, let $P \subset A$ be a maximal direct summand on which φ is non-singular and $2P = 0$, and let $\bar\lambda = \lambda$ restricted to P. Then*

$$\mathcal{G}_\lambda = \mathcal{G}_{\bar\lambda}(|A|/|P|)^{1/2} .$$

5. Generalizing the Browder-Brown approach to quadratic forms

In this section, we lay the homotopy theoretic groundwork for calculating the semi-index of a surgery problem

$$\rho: \widetilde{M}^{4k-1} \longrightarrow M^{4k-1}$$

in terms directly of the map ρ and a quadratic form on $K^{2k-1}_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z)$, without first making ρ highly connected.

We begin with some algebraic considerations. Consider an exact sequence of Abelian groups

$$\text{(5.1)} \qquad 0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ \pi\ } C \longrightarrow 0 .$$

The sequence $0 \to A \xrightarrow{i'} B' \xrightarrow{\pi'} C \to 0$ is equivalent to (5.1) if there is an isomorphism $j: B \to B'$ so $j \cdot i = i'$, $\pi' j = \pi$. The set of such equivalence classes is in 1-1 correspondence with $H^2(C, A)$ ([12]). This we see as follows. Let $s: C \to B$ be any map so $\pi \circ s = 1_c$. Then corresponding to s and (5.1), we construct a map $\varphi: C \times C \to A$ by

$$\varphi(c_1, c_2) = s(c_1 + c_2) - s(c_1) - s(c_2) . \tag{5.2}$$

φ satisfies

$$(5.3) \quad \begin{array}{ll} (1) & \varphi(a + b, c) - \varphi(a, b + c) - \varphi(b, c) + \varphi(a, b) = 0 , \\ (2) & \varphi(a, b) = \varphi(b, a) . \end{array}$$

Now recall the bar construction ([11], [12], [21]) as a resolution of C. $\bar{B}_1(C) = Z(C)$, $\bar{B}_2(C) = Z(C \times C)$, the free abelian group generated by C, $(C \times C)$. Similarly, $\bar{B}_3(C) = Z(C \times C \times C)$, and

$$(5.4) \quad \begin{aligned} \partial(a, b, c) &= (b, c) - (a + b, c) + (a, b + c) - (a, b) , \\ \partial(a, b) &= (a) + (b) - (a + b) . \end{aligned}$$

Thus we see that (5.3.1) is exactly the condition that φ be a 2-cocycle in $\mathrm{Hom}(\bar{B}_*(C), A)$. The equivalence class of φ in the resulting cohomology group $H^2(C, A)$ represents (5.1) and is easily checked to be independent of the choice of s.

Notice that a symmetric bilinear map $\varphi: C \times C \to A$ automatically satisfies (5.3.)

DEFINITION 5.5. *The exact sequence* (5.1) *admits a bilinear cocycle if there is a lifting s so φ_s is bilinear.*

PROPOSITION 5.6. *If the exact sequence* (5.1) *admits a symmetric bilinear cocycle $\varphi \in C^2(C, A)$, then in $H^2(C, A)$ we have $2\{\varphi\} = 0$.*

Proof. Let λ be the 1-cochain $\lambda(c) = -\varphi(c, c)$. Then (5.4) gives

$$\begin{aligned} (\delta\lambda)(a, b) &= +\lambda(a) - \lambda(a + b) + \lambda(b) \\ &= -\varphi(a, a) - \varphi(b, b) + \varphi(a + b, a + b) \\ &= 2\varphi(a, b) \end{aligned}$$

by bilinearity, and (5.6) follows.

Example 5.7. In terms of A, C, and φ, the group B has the form $C \times A$ with sum rule $(c_1, a_1) + (c_2, a_2) = (c_1 + c_2, \varphi(c_1, c_2) + a_1 + a_2)$. In particular, if $A = Z_{2^i} = C$ and φ is the bilinear form for which $\varphi(1, 1) = 1$, we have that $B \cong Z_{2^{i+1}} + Z_{2^{i-1}}$, and, if α, β are the generators, we see that, in (5.1), $i(1) = 2\alpha + \beta$, $\pi(\alpha) = 1$, $\pi(\beta) = -2$. Notice that in this case the cohomology class corresponding to φ is the non-zero class of order 2 in $H^2(Z_{2^i}; Z_{2^i}) \cong Z_{2^i}$.

We now turn to consideration of quadratic forms λ associated to a given symmetric bilinear form $\varphi: C \times C \to A$. We say λ is associated to φ if

(5.8) $$\lambda(x + y) = \lambda(x) + \lambda(y) + 2\varphi(x, y) .$$

We first set up the generic situation for a formula of type (5.8) to hold.

DEFINITION 5.9. *Let A be an Abelian group specified by giving generators $\langle g_1 \cdots g_n \cdots \rangle$ and relations $R_k = \sum a_{i,k} g_i$. The group $A_{\times 2}$ is defined by specifying generators $\bar{g}_i$ in $1-1$ correspondence with the generators of A and relations*

$$\bar{R}_k = \sum (2a_{i,k})\bar{g}_i$$

for each R_k satisfied in A. There is an injection (multiplication by 2)

(5.10) $$I: A \longrightarrow A_{\times 2}$$

defined by $I(g_i) = 2\bar{g}_i$.

In the sequel, we will assume that λ associated to φ in (5.8) takes its values in $A_{\times 2}$, and $2\varphi(x, y)$ will be understood to denote $I\varphi(x, y)$.

Example 5.11. The group $(Z_{2^i})_{\times 2} = Z_{2^{i+1}}$, and I is the usual injection. Similarly,

$$(\mathbf{Q}/Z)_{\times 2} = \mathbf{Q}/2Z \cong \mathbf{Q}/Z$$

and $I(a/b) = 2a/b$. Finally, $Z_{(\times 2)} = Z$, but $I: Z \to Z_{(\times 2)}$ is actual multiplication by 2.

PROPOSITION 5.12. *Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence of Abelian groups with bilinear cocycle $\varphi: C \times C \to A$. Then the set of quadratic forms*

$$\lambda: C \longrightarrow A_{(\times 2)}$$

associated to $-\varphi$ is in $1-1$ correspondence with the set of homomorphisms $h: B \to A_{\times 2}$ extending I.

Proof (compare [8, Prop. 1.3]). As a set, $B = C \times A$ with addition defined as in 5.7. Thus if $h: B \to A_{\times 2}$ is given, we have

$$h(c_1, 0) + h(c_2, 0) = h\big(c_1 + c_2, \varphi(c_1, c_2)\big) = h(c_1 + c_2, 0) + I\varphi(c_1, c_2) .$$

Hence $h(c_1, 0) + h(c_2, 0) - I\varphi(c_1, c_2) = h(c_1 + c_2, 0)$, and h defines a quadratic form on C associated to $-\varphi$. On the other hand, if $\lambda: C \to A_{\times 2}$ is given, then

$$h: B \longrightarrow A_{\times 2} ,$$

defined by

$$h(c, a) = I(a) + \lambda(c) ,$$

is a homomorphism and (5.12) follows.

We now turn to the homotopy-theoretic considerations needed.

Let X be a finite CW-complex, and suppose Y has the homotopy type of a locally finite complex. Let $\{X, Y\}_i = \lim_{n\to\infty} [\Sigma^{n+i}X, \Sigma^n Y]$ be the i^{th}-stable track group of homotopy classes of maps $f\colon X \to Y$. It has the structure of an Abelian group with addition generated by the composite

$$\Sigma^{i+n}X \xrightarrow{\Delta} \Sigma^{i+n}X \vee \Sigma^{i+n}X \xrightarrow{f\vee g} \Sigma^n Y \wedge \Sigma^n Y \xrightarrow{F} \Sigma^n Y . \tag{5.13}$$

Here F is the folding map and Δ is the diagonal approximation

$$(t, x) \longrightarrow \begin{cases} ((2t, x), \quad *), & t \leqq \frac{1}{2} \\ (* , (2t-1, x)), & t > \frac{1}{2} \end{cases} \qquad ([21]) .$$

The suspension

$$s\colon [X, Y] \longrightarrow \{X, Y\}_0 \tag{5.14}$$

is not generally a homomorphism, even when $[X, Y]$ has a group structure. In the circumstances of the next theorem, we see that s gives rise to a 2-cochain to which we can apply the algebraic remarks 5.1–5.12.

THEOREM 5.15. *Let X have dimension n, and Y be the L-fold loop space of Z ($L \gg n$). Then there is a space $\mathcal{F}(Z)$, a short exact sequence of Abelian groups*

$$0 \longrightarrow [X, \mathcal{F}(Z)] \xrightarrow{i} \{X, Y\}_0 \xrightarrow{\pi} [X, Y] \longrightarrow 0 , \tag{5.16}$$

and $\pi\circ s = 1_{[X,Y]}$. Hence s determines a 2-cocycle of (5.16) *in the cochains of $[X, Y]$ with coefficients in $[X, \mathcal{F}(Z)]$.*

Proof. $[\Sigma^L X, \Sigma^L Y] \cong \{X, Y\}_0$ since $L \gg n$. But $[\Sigma^L X, \Sigma^L Y] \cong [X, \Omega^L\Sigma^L Y]$. Let

$$\alpha\colon \Sigma^L Y \longrightarrow Z$$

be the adjoint of 1, and let

$$\beta\colon Y \longrightarrow \Omega^L\Sigma^L Y$$

be the usual inclusion. Then the composite

$$Y \xrightarrow{\beta} \Omega^L\Sigma^L Y \xrightarrow{\Omega^L\alpha} Y$$

is the identity. Let $F(Z)$ be the fiber of α, so $\mathcal{F}(Z) = \Omega^L F(Z)$ is the fiber in $\Omega^L(\alpha)$.

The cross-section β of the fibering α implies $\Omega^L\Sigma^L Y \simeq \mathcal{F}(Z) \times Y$. This is not an H-space splitting, but the inclusion $\mathcal{F}(Z) \hookrightarrow \Omega^L\Sigma^L Y$ and the projection $(\Omega^L\alpha)$ are both H-maps. The exact sequence (5.16) is thus obtained where i is induced by the inclusion $\mathcal{F}(Z) \hookrightarrow \Omega^L\Sigma^L Y$, and π is $\Omega^L(\alpha)_*$.

Remark 5.17. The cochain φ corresponding to (5.16) has a geometric interpretation. There is a map $J\colon Y\wedge Y \to \mathcal{F}(Z)$ ([31]) associated to the map $\mu\colon Y\times Y \to \Omega^L\Sigma^L Y$, defined by $\mu(y_1, y_2) = \beta(y_1\cdot y_2)*\chi(\beta y_1 * \beta y_2)$. Here $\chi(y)(t) = y(1-t)$, and $y_1\cdot y_2$ denotes their product in the H-space Y. Then $\varphi(\{f\}, \{g\})$ is represented by $J\cdot(f\wedge g)$.

PROPOSITION 5.18. *Let X have dimension n, and $Y = \Omega^L Z$. Suppose also that Y is $[(1/3)n] + 2$-connected. Then in dimensions less than $n+2$, $\mathcal{F}(Z) = S^{L-1}\ltimes_T Y\wedge Y$, and J is the inclusion of $Y\wedge Y$ in $S^{L-1}\ltimes_T Y\wedge Y$.* (This is immediate from [23, §§1 and 2].)

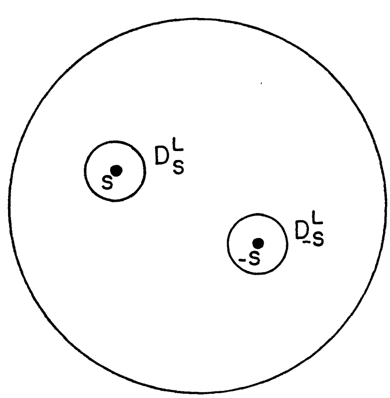
Here $S^{L-1}\ltimes_T Y\wedge Y$ is the space

$$S^{L-1}\times_T Y\wedge Y/(s, y_1, y_2)\sim(-s, y_2, y_1)$$
$$(s, *)\sim * .$$

The map $S^{L-1}\ltimes_T Y\wedge Y$ into $\mathcal{F}(Z)$ is given by first mapping $S^{L-1}\times Y\times Y/(s, y_1, y_2)\sim(-s, y_2, y_1)$ into $\Omega^L\Sigma^L Y$, as is done for example in [16] or [22]. A convenient visualization of the map has recently been given by J. P. May ([47]). In the unit disc D^L, let S^{L-1} be the sphere of radius 1/2, and for each point $s\in S^{L-1}$, let D_s^L be the disc of radius 1/4 centered at s. Then

$$(s, y_1, y_2)[t_1, \cdots, t_n] = \begin{cases} (4[(t_1, \cdots, t_n) - s], y_1) & \text{if } (t_1, \cdots, t_n)\in D_s \\ (4[(t_1, \cdots, t_n) + s], y_2) & \text{if } (t_1, \cdots, t_n)\in D_{-s} \\ * & \text{otherwise} \end{cases}$$

gives the desired map $S^{L-1}\ltimes_T Y\times Y \to \Omega^L\Sigma^L Y$.



If we define $\varphi'(s, y_1, y_2) = \varphi(s, y_1, y_2)*\chi(\beta y_1*\beta y_2)$, then φ' factors through a map $\bar{\varphi}: S^{L-1} \ltimes_T Y \wedge Y \to \mathcal{F}(Z)$. The reason this is a homotopy equivalence in dimensions less than $n+2$ is that from [22] we can construct a space $J_L(Y)$ having the homotopy type of $\Omega^L\Sigma^L Y$, and $J_L(Y)$ has the form $Y \cup_{F_2} (S^{L-1} \ltimes_T Y \times Y) \cup_{F_3} (C(3) \times_{\mathfrak{S}_3} Y \times Y \times Y) \cdots$ where F_2 projects $S^{L-1} \ltimes_T (Y \times * \cup * \times Y)$ onto Y, F_3 projects $C(3) \times (Y \times Y \times * \cup Y \times * \times Y \cup * \times Y \times Y)$ onto $(Y \cup_{F_2} S^{L-1} \times_T Y \times Y)$, etc. If Y is k-connected, then we can assume it is a CW-complex with no cells of dimension k or less. Looking at $J_L(Y)$, we can check that $F_2, F_3, \cdots$ are cellular, and passing from $(Y \cup_{F_2} S^{L-1} \ltimes_T Y \times Y)$ to the next stage adds new cells of dimension at least $3k+3$. Similarly, proceeding to the next stage adds cells of dimension not less than $4(k+1)$, etc. Hence $(Y \cup_{F_2} S^{L-1} \ltimes_T Y \times Y)$ is a $3k+2$-approximation of $\Omega^L\Sigma^L Y$.

In general, the φ described above is not bilinear. However, we have

COROLLARY 5.19. *Let Y (as above) be $k-1$-connected. Suppose $S^{L-1} \ltimes_T Y \wedge Y$ is $2k-1+t$-connected, and suppose also that X has dimension $2k+t$. Now let $\iota \in H^{2k+t}(S^{L-1} \ltimes_T Y \wedge Y)$, $\pi_{2k+t}(S^{L-1} \ltimes_T Y \wedge Y)$ be the fundamental class, and suppose $J^*(\iota) = r \otimes s \in H^*(Y \wedge Y)$. Then $\varphi(f, g)$ is bilinear, and is represented by the cohomology class $\varphi^*(\iota) = f^*(r) \cup g^*(s)$.*

All the cases we require of X's and Y's will satisfy the hypothesis of (5.19). Here are the specific examples:

(5.20) (1) $$Y = K(Z_2, n)\ .$$

Here $\pi_{2n}(\mathcal{F}(Z)) = Z_2$ and $J^*(\iota) = \iota \otimes \iota$.

(2) $$Y = K(Z, n)\ ,$$

so

$$\pi_{2n}(\mathcal{F}(Z)) = \begin{cases} Z, & n \text{ even} \\ Z_2, & n \text{ odd}\ , \end{cases}$$

and with appropriate coefficients,

$$J^*(\iota) = \iota \otimes \iota\ .$$

(3) $$Y = K(\mathbf{Q}/Z, 2n-1)\ ,$$

so $\pi_{4n-2}(\mathcal{F}(Z)) = 0$ but $\pi_{4n-1}(\mathcal{F}(Z)) \cong Q/Z$ while $J^*(\iota) = \iota \otimes \beta\iota = \beta\iota \otimes \iota$.

(4) $$Y = K(\mathbf{Q}/Z, 2n)\ ,$$

and $\pi_{2n}(\mathcal{F}(Z)) = \pi_{2n+1}(\mathcal{F}(Z)) = 0$ while $\pi_{2n+2}(\mathcal{F}(Z)) = Z_2$. Here $J^*(\iota) = \beta(\iota) \otimes \beta(\iota)$.

The k-invariants of the fibrations are given in these respective cases as

(1) $Sq^{n+1}(\iota)$,

(2) $\beta(Sq^n(\iota))$ for n even, and $Sq^{n+1}(\iota)$ for n odd.

(3) The system is given as

$$K(\mathbf{Q}/Z, 2n-1+s) \times K(\mathbf{Q}/Z, 4n-1+s) \longrightarrow K(Z_2, 4n+s),$$

where $k = Sq^{2n}\beta(I) + i\beta(J)$, and $i: H^*(\ ,Z) \to H^*(\ ,Z_2)$ is the coefficient homomorphism.

(4) Here the k-invariant is $Sq^{2n+2}\beta(I)$.

These results all follow routinely from [23, § 10]. The notation for $H^*(K(\mathbf{Q}/Z))$ is explained in the appendix. In particular, note that (5.20.1) is the case exploited by Browder and Brown ([3], [8]) in their work on the Kervaire invariant.

The following theorem generalizes [8, Cor. 1.1]. Suppose W is an arbitrary space of dimension $2n+t+L$, and $f: W \to \Sigma^L X$ is a *stable* map; that is to say, the homotopy classes of maps $[W, \Sigma^L X]$ are isomorphic under suspension with the stable track group $\{W, \Sigma^L X\}$. Then we have

THEOREM 5.21. *Suppose that f^* is an isomorphism in dimension $2n+t+L$ with coefficients in $\pi^s_{2n+t}(Y)$. Then if Y is one of the spaces in* (5.20), *the diagram*

$$\begin{array}{ccc} H^{2k+t}(X; \pi_{2n+t}(\mathcal{F}(Z))) & \longrightarrow & \{X; Y\} \\ f^* \big\downarrow \cong & & \big\downarrow f^* \\ H^{2k-t}(W; \pi_{2n+t}(\mathcal{F}(Z))) & \xrightarrow{\ I\ } & \{W; Y\} \end{array}$$

commutes, and $I \circ f^*$ is injective if and only if, in case

(1), (2) $Sq^{n+1}(x) = 0$, *all* $x \in H^{n+L-1}(W; Z_2)$;

(3) *for each* $x \in H^{2n-2+L}(W; Q/Z)$, *the* mod 2 *operation* $Sq^{2n}\beta(x)$ *is the* mod (2) *reduction of* $\beta(a_x)$ *for some* $a_x \in H^{4n+t+L-2}(W; \mathbf{Q}/Z)$;

(4) $Sq^{2n+2}(\beta x) = 0$ *for each* $x \in H^{2n+L}(W; \mathbf{Q}/Z)$.

Here the numbers (1)–(4) refer to the cases in (5.20). (The proof does not differ essentially from that of [8].)

We can now apply the techniques of [3], [8] to define quadratic forms on manifolds M^{2n+t} with appropriate orientations

(5.22) $$\sigma: M^{2n+t} \longrightarrow B_{\text{PL}}\langle v_m \rangle$$

and normal classes

(5.23) $$\alpha: S^{2n+t+L} \longrightarrow T(\nu(M)).$$

Specifically, in cases (1) and (2), $B_{\text{PL}}\langle v_m \rangle$ is the fiber in the map

$$v_{n+1}\colon B_{\mathrm{PL}} \longrightarrow K(Z_2, n+1)\ ,$$

which kills the $n+1^{\mathrm{st}}$ Wu class. In case (3), we use $B_{\mathrm{PL}}\langle v_{2n}\rangle$, and in case (4) the most convenient choice is $B_{\mathrm{PL}}\langle v_{2n+2}\rangle$. See [46] for a more complete discussion.

In all cases, Brown's original definition of the construction is unchanged. Thus, associated to σ there is a map of Thom spaces

(5.24) $$T(\sigma)\colon T(\nu) \longrightarrow T(\mathrm{PL}\langle v_n\rangle)\ .$$

Also, there is the duality isomorphism

(5.25) $$d\colon \pi^s_{2n+t}(T(\mathrm{PL}\langle v_m\rangle) \wedge Y) \cong \{DT(\mathrm{PL}\langle v_m\rangle), Y\}\ .$$

For $x \in H^n(M)$, consider the composite

(5.26) $$S^{L+2n+t} \longrightarrow T(\nu) \xrightarrow{\Delta} T(\nu) \wedge M_+ \xrightarrow{T(\sigma)\wedge(x)} T(\mathrm{PL}\langle v_m\rangle) \wedge Y\ .$$

Then, given any homomorphism

(5.27) $$h\colon \pi^s_{2n+t}(T(\mathrm{PL}\langle v_m\rangle) \wedge Y) \longrightarrow \pi^s_{2n+t}(Y)_{(\times 2)}\ ,$$

vanishing on the image of $\pi^s_{2n+t}(T(\mathrm{PL}\langle v_m\rangle)\wedge K(\mathrm{Q}, 2n-1))$ in $\pi^s_{2n+t}(T(\mathrm{PL}\langle v_m\rangle)\wedge Y)$ and satisfying the condition that, if

$$U \wedge 1\colon S^L \wedge Y \longrightarrow T(\mathrm{PL}\langle v_m\rangle) \wedge Y$$

is induced by including S^L as the Thom class in $T(\mathrm{PL}\langle v_m\rangle)$, then

$$h\circ(U \wedge 1)_*\colon \pi^s_{2n+t}(Y) \longrightarrow \pi^s_{2n+t}(Y)_{\times 2}$$

is exactly the map I defined in (5.10). We find that h defines a quadratic form

(5.28) $$(\alpha, \sigma, Y, h) = \lambda\colon H^n(M) \longrightarrow \pi^s_{2n+t}(Y)_{\times 2}\ .$$

Dually,we could define $\lambda = (\alpha, \sigma, Y, h)$ using the composite

(5.29) $$DT(\mathrm{PL}\langle v_m\rangle) \xrightarrow{D(\sigma)} \Sigma^L(M_+) \xrightarrow{\Sigma^L(x)} \Sigma^L Y_+$$

and a homomorphism

$$D(h)\colon \{DT(\mathrm{PL}\langle v_m\rangle), Y\} \longrightarrow \pi_{2n+t}(Y)_{\times 2}\ .$$

Remark 5.30. In the case when n is even and $Y = K(Z, n)$, it is easily seen that the quadratic form $(\alpha, \sigma, , Y, h)$ is *uniquely* determined on M by the cup product pairing

$$H^n(M) \otimes H^n(M) \longrightarrow H^{2n}(M)\ .$$

6. The generalized forms and surgery

In this section, we combine the results of §§4 and 5 to give an effective determination of the semi-index for the surgery problem 4.1 in terms directly of ρ without first doing surgery to make (4.1) $2k-2$-connected.

There is an exact sequence

$$\xrightarrow{\beta} K^*(\widetilde{M}, Z) \longrightarrow K^*(\widetilde{M}, \mathbf{Q}) \xrightarrow{\pi} K^*(\widetilde{M}, \mathbf{Q}/Z) \xrightarrow{\beta},$$

and we define $K^*_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z)$ to be the quotient group $K^*(\widetilde{M}, \mathbf{Q}/Z)/\mathrm{im}\,\pi$. It can also be identified with $\mathrm{im}(\beta)$, which is precisely the torsion subgroup in $K^{*+1}(\widetilde{M}, Z)$. Thus, by Poincaré duality, if $\widetilde{M}$ is a $4k-1$-manifold, then $K^{2k-1}_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z) = K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z)$. In the case where ρ is $2k-2$-connected, we showed in §4 ((4.7) and (4.10)) how to construct a quadratic form on $K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z)$, whose Gauss sum $\mathcal{G}_\lambda$ ((4.11)) determines the semi-index of ρ.

On the other hand, the techniques of § 5 ((5.20.3)) determine a quadratic form on $K^{2k-1}_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z)$. One of our main results in this section will show that the Poincaré duality isomorphism identifies the homotopically defined form on $K^{2k-1}_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z)$ to the surgery form on $K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z)$ if ρ is $2k-2$-connected. But this homotopically defined form does not depend on making ρ $2k-1$-connected, and is unchanged under surgeries on spheres of dimension less than $2k-1$ ((6.8)). Thus the Gauss sum is unchanged under such surgeries, and we obtain the semi-index of ρ without the necessity of first doing surgery! (For a more direct proof of this last fact, see (6.11) and [10].)

In (4.11), we mentioned that the quadratic form on $K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z)$ could be defined intrinsically if $\widetilde{M}$ is $2k-2$-connected. We briefly recall how this is done. Let $a \in K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z)$, and represent a by an embedding $S^{2k-1} \times D^{2k} \subset \widetilde{M}$ (the $S^{2k-1} \times D^{2k}$ suitably embedded so we can do normal surgery [18]). Suppose $n(a) = 0$, and consider $N = M - S^{2k-1} \times \mathring{D}^{2k}$ as a manifold with boundary $S^{2k-1} \times S^{2k-1}$. Let e_1, e_2 be the two generators of $H_{2k-1}(\partial N)$ corresponding to the core sphere $S^{k-1} \times *$ and the fiber sphere $* \times S^{2k-1}$, respectively. There is an integral homology class $A \in H_{2k}(N, \partial N, Z)$ which can actually be represented by an immersed disc D^{2k} with boundary sphere embedded in ∂N, and $\partial_* A = ne_1 + se_2$. Moreover, different choices of embeddings (obtained by changing the framing of the embedded core sphere), which are also suitable for doing surgery, change s by multiples of $2n$. $\lambda_*(a)$ is then defined as s/n in $\mathbf{Q}/2Z$. The bilinear form φ associated to λ_* is the linking form. On the level of cohomology, its expression is given by

$$\varphi(a, b) = \langle a \cup \beta b, [\widetilde{M}] \rangle \,. \tag{6.1}$$

An alternate description of λ_* is first to do surgery on the embedded S^{2k-1} representing $\alpha = \beta a \cap [\widetilde{M}]$, and thus attach a handle to $\widetilde{M}$. Let W be the resulting normal bordism, and let $\bar{D}^{2k}$ be the core disc of the attached handle. Then $n\bar{D} \cup D^{2k}$ (D^{2k} is the immersed disc of the preceding paragraph) represents a sphere S^{2k} immersed in W. With a little care in the choices made, we can assume S^{2k} stably framed in W, and hence its normal bundle is $s\tau(S^{2k})$ for some s. Then by [40, p.253, Lemma 5.3], $\lambda_*(\beta(a) \cap [\tilde{M}]) = s/n$. Alternately, in the cohomology group $H^{2k}(W, \partial W)$, there is a generator e dual to (D^{2k}, ∂), and an f dual to $n\bar{D}^{2k} \cup D^{2k}$ in $H^{2k}(W)$. f is partially characterized by the fact $i^*(f) = \beta(a)$ in $H^{2k}(M, Z)$, and we have

$$\lambda(a) = \langle f \cup e, [W, \partial W]\rangle / \langle j^*(e) \cup e, [W, \partial W]\rangle \ . \tag{6.2}$$

In view of (6.1) and (5.20.3), we give a quadratic form on $K^{2k-1}_{\text{Tor}}(\tilde{M}, \mathbf{Q}/Z)$ in terms of the sequence

$$0 \longrightarrow H^{4k-1}(\tilde{M}, \mathbf{Q}/Z) \longrightarrow \{\tilde{M}, K(\mathbf{Q}/Z, 2k-1)\} \longrightarrow [\tilde{M}, K(\mathbf{Q}/Z, 2k-1)] \longrightarrow 0 \tag{6.3}$$

and an orientation $\sigma\colon \tilde{M} \xrightarrow{\rho} M \longrightarrow B_{\text{PL}}\langle v_{2k}\rangle$. Precisely,

DEFINITION 6.4. *Let $\sigma\colon M \longrightarrow B_{\text{PL}}\langle v_{2k}\rangle$ be an orientation, and α any normal class for M. Then on $H^{2k-1}_{\text{Tor}}(M, \mathbf{Q}/Z)$, we define a quadratic form as*

$$\lambda = \big(T(\rho)\circ\alpha,\ \sigma,\ K(\mathbf{Q}/Z, 2k-1),\ h\big)\ ,$$

and on $H^{2k-1}_{\text{Tor}}(\tilde{M}, \mathbf{Q}/Z)$, we define a quadratic form as

$$\tilde{\lambda} = \big(\alpha,\ \sigma\circ\rho,\ K(\mathbf{Q}/Z, 2k-1),\ h\big)\ .$$

Here the notation is that of (5.28), *and h is an arbitrary homomorphism* (5.27)

We note that $\rho^*\colon H^{2k-1}_{\text{Tor}}(M; \mathbf{Q}/Z) \longrightarrow H^{2k-1}_{\text{Tor}}(\tilde{M}; \mathbf{Q}/Z)$ embeds $H^{2k-1}_{\text{Tor}}(M, \mathbf{Q}/Z)$ as a direct summand ([5, Chapter 1]), and, with respect to the bilinear form (6.1), the image is non-singular. K^{2k-1}_{Tor} is its orthogonal complement in $H^{2k-1}_{\text{Tor}}(\tilde{M}, \mathbf{Q}/Z)$. By the definition, we have

$$\tilde{\lambda}\big(\rho^*(x)\big) = \lambda(x)\ . \tag{6.5}$$

Thus, from (4.13) we have

$$\mathcal{G}_{\tilde{\lambda}}/\mathcal{G}_{\lambda} = \textstyle\sum_{x \in K^{2k-1}_{\text{Tor}}} e^{\pi i \tilde{\lambda}(x)}\ . \tag{6.6}$$

Now we can state

THEOREM 6.7. *Let M, $\tilde{M}$ be simply-connected, and assume $k > 1$* (*in* (4.1)). *Then for the quadratic forms defined in* (6.4), *have*

$$\mathcal{G}_{\tilde{\lambda}}/\mathcal{G}_{\lambda} = e^{(\pi i/4)\mathrm{SI}(\rho)} N\ ,$$

where N is a positive integer.

Proof. We begin by proving that the quotient (6.6) is unchanged when we do surgeries in dimensions less than $2k-1$. Actually, we show that $\mathcal{G}_{\tilde{\lambda}}$ itself is invariant under such surgeries.

LEMMA 6.8. *Let $M = \partial W$. Suppose an orientation $\sigma\colon M \longrightarrow B_{\text{PL}}\langle v_{2k}\rangle$ given, together with an extension to W. Let $b \in H^{2k-1}_{\text{Tor}}(M, \mathbf{Q}/Z)$ be $i^*(a)$ for* *some* $a \in H^{2k-1}(W)$. *Then* $\lambda(b) = 0$.

Proof. (compare Browder [3, Prop. 1.8, p. 163] or Brown [8, Lemma 1.14]). We have the diagram

$$\begin{array}{ccccccccc} M & \longrightarrow & W & \longrightarrow & W/M & \longrightarrow & \Sigma M & \longrightarrow & \Sigma W \longrightarrow \cdots \\ & {\scriptstyle (b)}\searrow & \downarrow{\scriptstyle (a)} & & & {\scriptstyle \Sigma(b)}\searrow & & & \downarrow{\scriptstyle \Sigma(a)} \\ & & K(\mathbf{Q}/Z, 2k-1) & & & & \Sigma K(\mathbf{Q}/Z, 2k-1) . & & \end{array}$$

Now by duality,

$$D(\sigma)\colon DT(\mathrm{PL}\langle v_{2k}\rangle) \longrightarrow \Sigma^L(M_+)$$

factors as

(6.10) $$DT(\mathrm{PL}\langle v_{2k}\rangle) \longrightarrow \Sigma^{L-1}(W/M) \longrightarrow \Sigma^L(M_+) ,$$

since W/M is the s-dual of $T(VW)$ and σ factored through W. Hence $(b) \in \{DT(\mathrm{PL}\langle v_{2k}\rangle), K(\mathbf{Q}/Z, 2k-1)\}$ given in (5.29) is obtained from the composition

$$DT(\mathrm{PL}\langle v^2{}_k\rangle) \longrightarrow \Sigma^{L-1}(W/M) \xrightarrow{\partial} \Sigma^L M \xrightarrow{\Sigma^L(i)} \Sigma^L(W) \xrightarrow{\Sigma^L a} \Sigma^L K(\mathbf{Q}/Z, 2k-1) ,$$

and, since $\Sigma^L(i)\circ\partial \simeq 0$, it follows that $(b) = 0$ and certainly $h(b) = 0$. (6.8) follows.

Now assume $\rho\colon \tilde{M} \to M$ is extended to a normal map $\bar{\rho}\colon W \to M \times I$, where W is obtained from $\tilde{M}$ by doing surgery on spheres of dimension less than $2k-1$. Then $\mathrm{im}(i^*) \cap H^{2k-1}_{\mathrm{Tor}}(\partial W, \mathbf{Q}/Z)$ provides an isomorphism

$$H^{2k-1}_{\mathrm{Tor}}(\tilde{M}, \mathbf{Q}/Z) \quad \text{with} \quad H^{2k-1}_{\mathrm{Tor}}(\tilde{M}', \mathbf{Q}/Z)$$

by writing $i^*(a) = a_1 - a_2$ (uniquely), $a_1 \in H^{2k-1}_{\mathrm{Tor}}(\tilde{M}, \mathbf{Q}/Z)$, $a_2 \in H^{2k-1}_{\mathrm{Tor}}(\tilde{M}', \mathbf{Q}/Z)$. The isomorphism identifies a_1 with a_2.

By (6.8), $\lambda(i^*(a)) = 0 = \lambda(a_1) + \lambda'(a_2)$ since the two boundary components are orthogonal under (6.1). Now $\lambda'(a_2)$ differs from the λ of (6.4) because $\partial W = [\tilde{M}] - [\tilde{M}']$, and so λ (6.9) is obtained from λ' by reversing the orientation of $\tilde{M}'$. This has the effect of changing $\lambda'(a_2)$ to $-\lambda'(a_2)$. The invariance of $\mathcal{G}_\lambda$ in this situation follows.

We now assume ρ to be $2k-2$-connected, and prove (6.7) in this case. In view of the preceding remarks, this will complete the proof.

Remark 6.11. The arguments which follow can be considerably simplified if the reader is interested only in (6.7), and not in the identification of the two forms. Indeed, it is possible to prove (6.7) directly without first making ρ highly connected by considering the diagram

$$\begin{array}{ccccccc} \tilde{M} & \longrightarrow & W & \longrightarrow & W/\partial W & \longrightarrow & \Sigma\tilde{M} \\ \downarrow{\scriptstyle (a)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma(a)} \\ K(\mathbf{Q}/Z, 2k-1) & \xrightarrow{\beta(\iota)} & K(\mathbf{Z}, 2k) & \longrightarrow & K(\mathbf{Q}, 2k) & \longrightarrow & \Sigma K(\mathbf{Q}/Z, 2k-1) \end{array}$$

and using (A14) of the appendix (compare the arguments following (6.19)). This is our procedure in [10]. So the geometric and homotopical arguments which follow, culminating in (6.19), should be read with the understanding that their intent is to relate the λ defined above with the refined self-linking invariant used in [18], [29], and [39].

Let $a \in K^{2k-1}_{\mathrm{Tor}}(\widetilde{M}, \mathbf{Q}/Z)$ be given, and, corresponding to it,

$$x = \beta a \cap [\widetilde{M}] \in K^{\mathrm{Tor}}_{2k-1}(\widetilde{M}, Z) .$$

Let $f_x\colon S^{2k-1} \to \widetilde{M}$ be an embedding so $(f_x)_*(e_{2k-1}) = x$. Moreover, we assume $\rho f_x \simeq 0$. Thus η_{f_x}, the normal PL-block bundle, is stably trivial, and f_x extends to an embedding

$$f'_x\colon S^{2k-1} \times D^{2k} \hookrightarrow \widetilde{M} .$$

We assume this map chosen so that the manifold W, obtained by attaching $D^{2k} \times D^{2k}$ over $f'_x(S^{2k-1} \times D^{2k})$, admits a normal extension of ρ over $M \times$ I. Precisely,

$$W = M \times I \bigcup\nolimits_{\mathrm{Im}(f'_x) \subset M \times 1} (D^{2k} \times D^{2k}) .$$

Our object now will be to verify that (6.2) holds for (f) in W. In order to do this, we must map D^{2k} into $\widetilde{M}$ so that $\partial D^{2k} = nf_x(S^{2k-1})$ in a good way in order to obtain a geometric hold on the cohomology classes e, f in (6.2).

LEMMA 6.12. *Let $N = \widetilde{M} - f'_x(S^{2k-1} \times \mathring{D}^{2k})$. Then there is a complex X having the homotopy type of $D^{2k} \vee S^2 \vee \cdots \vee S^2$ and an embedding $g\colon X \hookrightarrow N$ so that $\partial X = S^{2k-1}$ embeds in $\partial N = S^{2k-1} \times S^{2k-1}$ with degree (n, l), and trivial normal* PL*-block bundle. Moreover, there is a degree* 1 *map $h\colon S^{2k-1} \cup_n e^{2k} \to X \cup f'_x(S^{2k-1} \times D^{2k})$ so that $h \mid S^{2k-1} = f_x$ and $\rho \circ h \simeq 0$.*

Proof. $K_{2k-1}(\widetilde{M}) \cong \pi_{2k}(\rho)$ ([5]). Thus a map $g\colon S^{2k-1} \cup_n e^{2k} \to \widetilde{M}$ extending f_x exists if $nx = 0$, so $\rho \circ g \simeq 0$. We place $fg(\mathring{e}^{2k})$ in general position with respect to S^{2k-1}, and delete $g^{-1}f'_x(S^{2k-1} \times D^{2k})$ from e^{2k}. We may assume this is homeomorphic to D^{2k} minus a finite number of discs. Connect the boundaries of these discs together and delete tubular neighborhoods of the connecting lines, obtaining a disc e^{2k} with boundary S^{2k-1}. Change g slightly so that this S^{2k-1} is $= g^{-1}\partial(S^{2k-1} \times D^{2k})$, and assume $g \mid S^{2k-1} \to \partial(S^{2k-1} \times D^{2k})$ is now an embedding. Then from [17], g leaving $g \mid S^{2k-1}$ fixed is homotopic to a simple immersion. Such an immersion has circles of double points as its only singularities, and no double coverings occur. The homotopy type of such a space Y is

$$D^{2k} \vee (S^1 \vee S^2) \vee \cdots \vee (S^1 \vee S^2) ,$$

one pair $S^1 \vee S^2$ for each double point circle. Since $\widetilde{M}$ is simply-connected of dimension $\geqq 7$, each circle bounds a disc D^2 in the complement of Y. Adjoining these D^2, we get the desired complex X.

(6.12) now follows directly. Let $V = X \cup f'_x(S^{2k-1} \times D^{2k})$. Then $V \simeq S^{2k-1} \cup_n e^{2k} \vee S^2 \vee \cdots \vee S^2$ and is embedded in $\widetilde{M}$, so the composite

$$S^{2k-1} \cup_n e^{2k} \xrightarrow{r} V \xrightarrow{s} \widetilde{M}$$

represents x, and $\rho \circ s \circ r \simeq 0$. Let U be a regular neighborhood of V in $\widetilde{M}$, and set $\widetilde{Y} = U/\partial U$. By [44, p. 301, Prop. 3.2], the s-dual of $\widetilde{Y}$ is the Thom space $T(\nu)$ restricted to U, where ν is the normal bundle to $\widetilde{M}$. Thus, by using the Thom isomorphism, we see that $\widetilde{Y}$ is a CW-complex of the form

$$((S^{2k-1} \cup_n e^{2k}) \vee S^{4k-3} \vee \cdots \vee S^{4k-3}) \cup_\omega e^{4k-1} . \tag{6.13}$$

Here $\omega = \alpha + \beta$ $(\alpha \in \pi_{4k-2}(S^{2k-1} \cup_n e^{2k}),\ \beta \in \pi_{4k-2}(S^{4k-3} \vee \cdots \vee S^{4k-3}))$, with α stably trivial. The obvious map $p\colon \widetilde{M} \to \widetilde{Y}$ collapsing $\widetilde{M} - U$ to $*$ satisfies $p^*(\iota) = a$, where ι is the fundamental class in $H^{4k-1}(\widetilde{Y}; \mathbf{Q}/Z)$.

Now we turn to the classes e and f in (6.2). Consider again the W of (6.11). Define

$$U' = U \times I \cup D^{2k} \times D^{2k} \tag{6.14}$$

and

$$\widetilde{T} = U'/\partial U' . \tag{6.15}$$

U' is a regular neighborhood of $U \cup D^{2k} \times 0$ which has the homotopy type of

$$S^{2k} \vee S^2 \vee \cdots \vee S^2 .$$

Hence by again applying duality and noting that the composite

$$S^{2k} \longrightarrow U' \longrightarrow W \longrightarrow M$$

is homotopic to zero, we have

$$\widetilde{T} \simeq (S^{2k} \vee S^{4k-2} \vee \cdots \vee S^{4k-2}) \cup_\tau e^{4k} ,$$

where $\tau = \tau_1 + \tau_2$ with τ_1 again stably trivial. Note that the cup product $\langle (\iota_{2k}^2), e^{4k} \rangle$ determines τ_1. The map $p\colon W/\partial W \to \widetilde{T}$ satisfies $p^*(\iota) = e$.

Now consider the transverse sphere $(0 \times D^{2k}, 0 \times \partial D^{2k})$ embedded in $\widetilde{T}$. Let

$$q\colon S^{2k} \longrightarrow \widetilde{T}$$

represent this embedding. Then we have

LEMMA 6.16. *q_* is multiplication by n in integral homology in dimension $2k$. Moreover, the mapping cone of q is naturally homotopic to $\Sigma \widetilde{Y}$, and, using this homotopy equivalence, the diagram*

$$\begin{array}{ccc} W/\partial W & \longrightarrow & \Sigma\widetilde{M} \\ \downarrow p & & \downarrow \Sigma p \\ \widetilde{T} & \longrightarrow & \Sigma\widetilde{Y} \end{array}$$

commutes. (Thus, if $\bar{\iota}$ is the fundamental class in S^{2k}, we have that $\langle \bar{\iota}\cdot\iota, e^{4k}\rangle = (1/n)\langle \iota^2, e^{4k}\rangle$ again determines τ_1.)

Proof. Write

$$\widetilde{T} \cong \{I \times U/\{\partial U \times I \cup 0 \times U \cup 1 \times (U - S^{2k-1} \times \mathring{D}^{2k})\}\} \cup \{D^{2k} \times D^{2k}/\{S^{2k-1} \times D^{2k}\}\} . \tag{6.17}$$

The second set in the above decomposition of $\widetilde{T}$ has the homotopy type of S^{2k}, and $q\colon S^{2k} \to D^{2k} \times D^{2k}/S^{2k-1} \times D^{2k}$ is a homotopy equivalence. Moreover, if we collapse this set to $*$, we obtain $\Sigma\widetilde{Y}$.

Thus the sequence

$$S^{2k} \xrightarrow{q} \widetilde{T} \longrightarrow \Sigma\widetilde{Y}$$

is a cofibering. Again from (6.17), we see that, if we embed $\widetilde{Y}$ as $1 \times \widetilde{Y}$ in $\widetilde{T}$ and use the map

$$t\colon \widetilde{Y} \longrightarrow S^{2k}$$

defined by first collapsing $\widetilde{Y}$ to $(S^{2k-1} \times D^{2k}/\partial)$ and then projecting onto $(D^{2k}/\partial) = S^{2k}$, we can describe $\widetilde{T}$ as the cofiber in the map t. Moreover, the map

$$S^{2k} \longrightarrow S^{2k} \cup_t c\widetilde{Y}$$

is exactly the map q. Now (6.16) follows.

Remark 6.18. In the proof of (6.16), we have actually shown that the sequence

$$\widetilde{Y} \xrightarrow{t} S^{2k} \xrightarrow{q} \widetilde{T} \longrightarrow \Sigma\widetilde{Y} \longrightarrow \cdots$$

is a cofiber sequence.

Now consider the commutative diagram

$$\begin{array}{ccccccccccc} \widetilde{M} & & \xrightarrow{\quad l \quad} & & W/\partial W & \longrightarrow & \Sigma\widetilde{M} & & \longrightarrow & & \Sigma W/\partial W \\ \downarrow p & & & & \downarrow \bar{p} & & \downarrow & & & & \downarrow \\ \widetilde{Y} & \xrightarrow{t} & S^{2k} & \xrightarrow{q} & \widetilde{T} & \longrightarrow & \Sigma\widetilde{Y} & \longrightarrow & S^{2k+1} & \xrightarrow{\Sigma q} & \Sigma\widetilde{T} \\ \downarrow (\iota) & & \downarrow (\iota) & & \downarrow (\iota) & & \downarrow \Sigma(\iota) & & \downarrow \Sigma(\iota) & & \downarrow \\ K(Z_n, 2k-1) & \xrightarrow{\beta\iota} & K(Z, 2k) & \longrightarrow & X_n & \longrightarrow & \Sigma K(Z_n, 2k-1) & \longrightarrow & \Sigma K(Z, 2k) & \longrightarrow & \Sigma X_n \end{array} \tag{6.19}$$

where the bottom two horizontal lines are cofiber sequences, and l embeds $\widetilde{M}$ as $1 \times \widetilde{M} \subset W$.

The verification of (6.2) will follow from the properties of the bottom line in (6.19,) in particular the space X_n.

LEMMA 6.20. *For* $L \geqq 2$, *we have*

$$\pi_i(\Sigma^L X_n) = \begin{cases} Z, & i = 2k + L \text{ or } i = 4k + L \\ 0 \text{ otherwise}, & i \leqq 4k + L \,. \end{cases}$$

Moreover, the cofiber sequence of (6.19) (*for* X_n) *gives in homotopy in dimension* $4k + L$ *the exact sequence*

$$0 \longrightarrow Z \xrightarrow{\ j\ } Z \longrightarrow \pi_{L+4k}(\Sigma^{L+1}K(Z_n, 2k-1)) \longrightarrow 0 \,, \tag{6.21}$$

where j *is multiplication by* n *if* n *is odd, and by* $2n$ *if* n *is even* (direct from [23, §§1, 2, 10]).

Consider the map dual to $\bar{p}$ in (6.19) and the composite

$$T(\nu\,|\,U') \xrightarrow{D\bar{p}} T(\nu) \xrightarrow{T(\sigma)} T(\mathrm{PL}\langle v_{2k}\rangle) \,. \tag{6.22}$$

$T(\nu\,|\,U') \simeq S^{L+2k} \vee S^L \cup e_1^{L+2} \vee \cdots \vee e_t^{L+2}$, and $T(\nu)\circ D(\bar{p})$ on S^{L+2k} is homotopic to 0 from (6.12). Dualizing (6.22), we obtain the stable diagram

$$\begin{array}{ccccc} DT(\mathrm{PL}\langle v_{2k}\rangle) & \xrightarrow{D(T\sigma)} & \Sigma^L(W/\partial W) & \longrightarrow & \Sigma^{L+1}(\widetilde{M}) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma^{L+1}p} \\ Q = S^{L+4k-2} \cup (e_1^{L+4k} \cdots) & \longrightarrow & \Sigma^L \widetilde{T} & \longrightarrow & \Sigma^{L+1}\widetilde{Y} \,. \end{array} \tag{6.23}$$

Finally, combining (6.19) with (6.23), we have the commutative diagram

$$\begin{array}{ccccccc} D(T\mathrm{PL}\langle v_{2k}\rangle) & \longrightarrow & \Sigma^{L+1}(\widetilde{M}) & \xrightarrow{\Sigma^{L+1}(a')} & \Sigma^{L+1}K(Z_n, 2k-1) \,. \\ \Big| & \searrow{\scriptstyle DT(\sigma)} & \nearrow & \nearrow{\scriptstyle \Sigma^{L+1}(\iota)} & \uparrow \\ \Big| & \Sigma^L W/\partial W & & \Sigma^{L+1}\widetilde{Y} & \Big| \\ \Big| & \searrow & & \nearrow & \Big| \\ \downarrow & & & & \Big| \\ Q & \longrightarrow & \Sigma^L \widetilde{T} & \longrightarrow & \Sigma^L X_n \end{array} \tag{6.24}$$

Now, from (6.20) note that

(6.25) (1) the Hurewicz homomorphism

$$h\colon \pi_{4k+L}(\Sigma^L X_n) \longrightarrow H_{4k+L}(\Sigma^L X_n)$$

is injective;

(2) any map $f\colon Q \to \Sigma^L X_n$ factors up to homotopy as

$$Q \xrightarrow[\mu]{} S^{4k+L} \longrightarrow \Sigma^L X_n$$

where μ is the pinching map. In particular, $f_1 \simeq f_2$ if and only if $f_{1*} = f_{2*}$ in homology.

This means that the map $(a, 0)$ constructed as the top line in (6.24) factors as the composite

(6.26) $$D(TPL\langle v_{2k}\rangle) \xrightarrow{DU} S^{4k+L} \longrightarrow \Sigma^L X_n \longrightarrow \Sigma^{L+1}K(Z_n, 2k-1)\ .$$

LEMMA 6.27. *The generator α of the Z-component of $H^{4k}(X_n; Z)$ can be chosen so that $\bar{\iota}^2 = n\alpha$, where $\bar{\iota}$ is the generator of $H^{2k}(X_n; Z) = Z$.*

Proof. In the map $j\colon K(Z, 2k) \to X_n$, we have $j^*(\bar{\iota}) = n\iota$. Hence $j^*(\bar{\iota})^2 = n^2\iota^2$. But in the map $(\beta\iota)\colon K(Z_n, 2k-1) \to K(Z, 2k)$, we have $(\beta\iota)^*\iota^2 = (\beta\iota)^2 = \beta(\iota \cup \beta\iota)$ has order n in $H^k(K(Z_n, 2k-1))$. Thus $n\iota^2 = j^*\alpha$, and since no torsion element in $H^{4k}(X_n)$ has order greater than n, (6.27) follows.

Note finally that, if n is even, α evaluates 1 on the homotopy generator, and in any case $\alpha = \iota \cup \bar{\iota}$. This shows the result on comparing (6.26), the middle line of (6.19), and (6.2).

The proof of (6.7) is complete.

7. A product formula

Given a surgery problem

(7.1) $$\rho\colon \tilde{M} \longrightarrow M$$

for simply-connected Z_n-manifolds and a simply-connected Z_n-manifold P, we have the new problem

(7.2) $$\rho \underline{\times} 1\colon \tilde{M} \underline{\times} P \longrightarrow M \underline{\times} P$$

(see (1.17) for the definition of $M \underline{\times} P$). In the case where $\dim(M) + \dim(P) \equiv 0(4)$, the resulting surgery problem has a generalized index obstruction (3.4) and our object in this section is to evaluate it in terms of the original obstruction in (7.1).

THEOREM 7.3. *The surgery obstruction in dimensions greater than* 4 *for* (7.2) *is given by*

(1) $I(\rho)/8 \cdot \mathrm{ind}(P)$ *for* $\dim(P) \equiv \dim(M) \equiv 0(4)$;

(2) $\langle(\beta v_{2s})^2, [\bar{P}]\rangle \cdot K(M) = \langle v_{2s}Sq^1v_{2s}, \bar{\partial}P\rangle K(M)$ *for* $\dim P = 4s+2$, $\dim M = 4k+2$;

(3) $\langle v_{2s}\cdot Sq^1v_{2s}, [P] \otimes Z_2\rangle K_\partial(M)$ *for* $\dim P = 4s+1$, $\dim M = 4k+3$, *and* P *a closed, oriented manifold.*

(As a matter of notation, recall that $\bar{\partial}P$ is the Z_n-boundary of P as described in Definition 1.15 and the remarks which follow it.)

Proof. Assume ρ in (7.1) as highly connected as possible. Specifically, if $\dim(M) = 4k$, we assume $\rho \mid \bar{\partial}\tilde{M} \to \bar{\partial}M$ is a homotopy equivalence, and ρ is $2k-1$-connected on the interior of M. In case $\dim(M) = 4k+2$, we assume $\rho \mid \bar{\partial}\tilde{M} \to \bar{\partial}M$ is a homotopy equivalence, ρ is $2k$-connected on the interior of

M, and $K_{2k+1}(\tilde{M}) = Z \oplus Z$ if $K(M) = 1$. In case $\dim M = 4k + 3$, we assume $\rho | \bar{\partial}\tilde{M} \to \bar{\partial}\tilde{M}$ is $2k$-connected, ρ on the interior is $2k$-connected, and, if $K_\partial(M) = 1$, we have the exact sequence of kernels

$$(7.4)\quad 0 \longrightarrow K_{2k+2}(\tilde{M}, \partial\tilde{M}) \longrightarrow K_{2k+1}(\partial\tilde{M} \times \{1, \cdots, n\}) \longrightarrow K_{2k+1}(\tilde{M}) \longrightarrow 0 .$$

This is a sequence of free Z-modules and represents the entire kernel of ρ.

Take the product of (7.1) with P and attach the canonical handles to make $\tilde{M} \times P$ and $M \times P$ into Z_n-manifolds (1.17), and consider the Meyer-Vietoris sequences calculating the homology of $\tilde{M} \underline{\times} P$, $M \underline{\times} P$. By Poincaré duality, the kernels separate out to give a separate Meyer-Vietoris sequence calculating $K_*(\tilde{M} \underline{\times} P)$. In particular, when M has even dimension, $K_*(\bar{\partial}\tilde{M} \times \partial P) = 0$, so $K_*(\tilde{M} \underline{\times} P) = K_*(\tilde{M}) \otimes H_*(P)$ since $K_*(\tilde{M})$ is Z-free. Thus the interior index of $K(\tilde{M} \underline{\times} P)$ is $I/8(\tilde{M}) \cdot I(P)$ in case 1, and is zero in cases 2 and 3.

In order to complete the proof of (7.3), we must calculate the semi-index on a boundary component of $\tilde{M} \underline{\times} P$. We have

$$(7.5)\qquad \bar{\partial}(\tilde{M} \underline{\times} P) = \bar{\partial}\tilde{M} \times P \cup_{\bar{\partial}\tilde{M} \times \bar{\partial}P \times \{1 \cdots n\}} \tilde{M} \times \bar{\partial}P ,$$

where we identify $(\bar{\partial}\tilde{M}, i) \times \bar{\partial}P$ with $\bar{\partial}\tilde{M} \times (\bar{\partial}P, i)$. In case 1, the torsion kernel in dimension $2k + 2s - 1$ has the form

$$(7.6)\qquad K^{2k}(\tilde{M}) \otimes H^{2s-1}_{\mathrm{Tor}}(\bar{\partial}P, \mathbf{Q}/Z) .$$

In case 2, the kernel is

$$(7.7)\qquad K^{2k+1}(\tilde{M}) \otimes H^{2s}_{\mathrm{Tor}}(\bar{\partial}P, \mathbf{Q}/Z) .$$

LEMMA 7.8. *In case* 1, *let* $r \otimes s \in K^{2k}(\tilde{M}) \otimes H^{2s-1}_{\mathrm{Tor}}(\bar{\partial}P, \mathbf{Q}/Z)$, *and suppose* $\lambda_1 = (\alpha, \sigma \circ \rho, \tilde{M}, h_1)$, $\lambda_2 = (\alpha', \sigma', P, h_2)$ *are two quadratic forms* (5.28). *Then corresponding to these is an orientation* $\sigma \circ \rho \underline{\times} \sigma'$ *on* $\tilde{M} \underline{\times} P$, *and, with respect to this orientation, the form* $\lambda = (\alpha \wedge \alpha', \sigma\rho \underline{\times} \sigma', \tilde{M} \underline{\times} P, h_2)$ *satisfies*

$$\lambda(r \otimes s) = \lambda(r)\langle s \cdot \beta s, [\bar{\partial}P]\rangle .$$

(*Here* h_1 *is the homomorphism* $\pi_{4k}[M(S\mathrm{PL}\langle v_{2k+1}\rangle) \wedge K(Z, 2k)] \to Z$, *which gives the cup square* (5.20.2), *while* h_2 *is chosen as in* (5.20.3). *The assumption that* ρ *is* $2k - 1$-*connected makes the particular choices immaterial.*)

Proof. Let $f\colon S^{2k} \hookrightarrow \tilde{M}$ represent $r \cap [\tilde{M}/\partial]$. Consider the diagram

$$(7.9)\qquad \begin{array}{ccccc} DTS\mathrm{PL}\langle v_{2k+2s}\cdots\rangle \to DTS\mathrm{PL}\langle v_{2k+1}\cdots\rangle \wedge DTS\mathrm{PL}\langle v_{2s}\cdots\rangle & \xrightarrow{DU\wedge 1} & S^L \wedge DTS\mathrm{PL}\langle v_{2s}\cdots\rangle \\ \downarrow{\scriptstyle \sigma\wedge\sigma'} & & \downarrow{\scriptstyle l\wedge\sigma'} \\ (\Sigma^L M/\partial) \wedge \bar{\partial}P & \xrightarrow{D(f)\wedge 1} & \Sigma^L T(f) \wedge (\bar{\partial}P) \\ & & \downarrow{\scriptstyle \iota\wedge s} \\ & & \Sigma^L K(Z, 2k) \wedge K(\mathbf{Q}/Z, 2s-1) \\ & & \downarrow{\scriptstyle \Sigma^L\mu} \\ & & \Sigma^L K(\mathbf{Q}/Z, 2s+2k-1) . \end{array}$$

(Here $T(f)$ is the Thom space of the normal bundle to $f(S^{2k})$ while $D(f)$ is the L-fold suspension of the Pontrjagin-Thom map from $M/\partial \to T(f)$. The spaces $T(SPL\langle v_{2k+2s}, \cdots\rangle)$ are obtained as the Thom spaces of the universal bundles over the spaces $B_{SPL}\langle v_{2k+2s}, \cdots\rangle$, which are defined from B_{SPL} by killing all the Wu classes v_i for $i \geqq 2k + 2s$.) The composite represents $\lambda(r \otimes s)$. Precisely, $l \circ D(U)$ lifts to the fiber E in the fibration sequence

$$E \xrightarrow{j} \Sigma^L K(Z, 2k) \longrightarrow K(Z, 2k + L)\ . \tag{7.10}$$

Moreover, the map $\Sigma^L \mu$ restricted to $E \wedge K(Q/Z, 2s - 1)$ lifts to the fiber F in the map

$$F \xrightarrow{j} \Sigma^L K(Q/Z, 2s + 2k - 1) \longrightarrow K(Q/Z, 2s + 2k + L - 1)\ . \tag{7.11}$$

Thus λ on $r \otimes s$ is determined by the composite

$$\begin{array}{ccccc} D(TSPL\langle v_{2k+2s}, \cdots\rangle) \to & S^L \wedge D(TSPL\langle v_{2s}, \cdots\rangle) \to & E \wedge K(Q/Z, 2s-1) & \xrightarrow{\omega} & F \\ \downarrow D(U) & \downarrow 1 \wedge D(U) & & \nearrow \tau & \\ S^L \longrightarrow & S^L \wedge S^0 \to & E \wedge K(Q/Z, 2s - 1)/4s + 4k - 2\ . & & \end{array}$$

Here $/4s + 4k - 2$ means we pinch the $4s + 4k - 2$-skeleton of $E \wedge K$ to a point, and $D(U)$ is the dual of the inclusion $S^0 \subset T(SPL\langle v_{2k+2s}, \cdots\rangle)$ embedding the Thom sphere. We now need

PROPOSITION 7.13. *Let a be the fundamental class in $H^{4(s+k)-1+L}(F; \pi_{4(k+s)-1+L}(F))$. Then*

$$\omega^*(a) = \bar{a} \otimes (\iota \cdot \beta\iota)$$

in (7.12) *where $\bar{a}$ is the fundamental class in $H^{4k+L}(E; Z)$.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & \overset{i\wedge 1}{\nearrow} & Q(K(Z, 2k)) \wedge K(Q/Z, 2s-1) & \xrightarrow{\theta} & Q(K(Z, 2k) \wedge K(Q/Z, 2s-1)) \\ K(Z, 2k) \wedge K(Q/Z, 2s-1) & & & & \downarrow Q(\mu) \\ & \underset{\mu}{\searrow} & K(Q/Z, 2k+2s-1) & \xrightarrow{i} & QK(Q/Z, 2k+2s-1) \end{array} \tag{7.14}$$

where $Q(X) = \lim_{n\to\infty} \Omega^n \Sigma^n(X)$ ([16],[22]). In $H^{4(k+s)-1}(Q(K(Q/Z, 2k+2s-1)), Q/Z)$, there is a special class q satisfying: (1) $i^*(q) = 0$, and (2) $(*)^* q = \iota \otimes \beta\iota$ where

$$*\colon Q(X) \times Q(X) \longrightarrow Q(X)$$

is the loop product. Moreover, $j^*(q) = \sigma^L(a)$ on looping (7.11) L times ([23]). Next, using the $*$-product, we obtain the diagram

$$\begin{array}{ccc} (K(Z,2k) \times K(Z,2k)) \wedge K(Q/Z, 2s-1) & \xrightarrow{\text{shuff.}\ 1\wedge\Delta} & (K(Z,2k) \wedge K(Q/Z, 2s-1))^2 \\ \downarrow * \wedge 1 & & \downarrow * \\ Q(K(Z, 2k)) \wedge K(Q/Z, 2s-1) & \xrightarrow{\theta} & Q(K(Z, 2k) \wedge K(Q/Z, 2s-1))\ . \end{array} \tag{7.15}$$

Thus

$$\begin{aligned}(* \wedge 1)^*\theta^*Q(\mu)^*(q) &= (1 \wedge \triangle)^*\text{shuff.}(i\circ\mu)^{2*}Q(\mu)^*(\iota \otimes \beta\iota) \\ &= (1 \wedge \triangle)^*[(\iota \otimes \iota) \otimes \iota\cdot\beta\iota] \\ &= (\iota \otimes \iota) \otimes \iota \cup \beta\iota \, .\end{aligned}$$

But any element b satisfies $j^*(b) = \bar{a}$ if $(*)^*(b) = \iota \otimes \iota$ in $QK(Z, 2k)$ ([23]). (7.13) follows.

Clearly, Lemma 7.8 is an immediate consequence of (7.12) and (7.13).

We now have

LEMMA 7.16. *In case* 1, $\rho \underline{\times} 1 | \bar{\partial}(\tilde{M} \underline{\times} P) \to \bar{\partial}(M \underline{\times} P)$ *has semi-index equal to zero.*

Proof. For $\tilde{M}$, consider the exact sequence of kernels

$$0 \longrightarrow K^{2k}(\tilde{M}, \partial\tilde{M}) \xrightarrow{A} K^{2k}(\tilde{M}) \longrightarrow 0 \, .$$

We identify $K^{2k}(\tilde{M}, \partial\tilde{M})$ with $K^{2k}(\tilde{M})$ by Poincaré duality, and, with respect to this identification, choosing a basis for $K^{2k}(\tilde{M}, \partial\tilde{M})$ represents A as an $m \times m$ symmetric unimodular matrix with index divisible by 8.

Let ψ: $H^{2s-1}_{\text{Tor}}(\bar{\partial}P, \mathbf{Q}/Z) \to \mathbf{Q}/2Z$ be any form quadratic with respect to the bilinear pairing

$$\varphi(x, y) = \langle x \cup \beta y, [\bar{\partial}P]\rangle \, ,$$

and let a resolution (Th. 4.14(a)) of ψ be given as

$$0 \longrightarrow F \xrightarrow{B} F \longrightarrow H^{2s-1}_{\text{Tor}}(\bar{\partial}P, \mathbf{Q}/Z) \longrightarrow 0 \, .$$

Then

$$0 \longrightarrow K^{2k} \otimes F \xrightarrow{A^{-1}\otimes B} K^{2k} \otimes F \longrightarrow K^{2k} \otimes H^{2s-1}_{\text{Tor}}(\bar{\partial}P, \mathbf{Q}/Z) \longrightarrow 0$$

is a resolution of the λ obtained in (7.8). But $I(A^{-1}) = I(A)$ is congruent to zero mod 8, and $I(A^{-1} \otimes B) = I(A^{-1})\cdot I(B)$, which is also divisible by 8. Thus $\text{SI}(\rho \underline{\times} 1) = 0$, and (7.16) follows.

Note that (7.16) implies that the total obstruction in case 1 is the interior index, and (7.3.1) follows.

In case 2, we find for $r \otimes s$ in (7.7) that a result analogous to (7.8) is again true, and

$$\lambda(r \otimes s) = \lambda(r)\langle s\cdot\beta s, [\bar{\partial}P]\rangle \, . \tag{7.17}$$

Here, note that $\lambda(r) \in Z_2$ is the Kervaire form ([7]), $\langle s\cdot\beta s, [\bar{\partial}P]\rangle \in \mathbf{Q}/Z$ has order 2, and the indicated multiplication takes place in $Z_2 \subset \mathbf{Q}/2Z$. (The proof of (7.8) does not have to be changed; simply re-index the diagrams to take into account the different dimensions and interpretations for r, s, and λ.)

We assume $K^{2k+1}(M) = Z \oplus Z$ with generators e_1, e_2, respectively, and $\lambda(e_1) = \lambda(e_2) = 1$ in the group Z_2, while $\varphi(e_1, e_2) = 1$. Then $K \otimes H^{2s}_{\mathrm{Tor}}(\bar\partial P, \mathbf{Q}/Z)$ has the form $H^{2s}_{\mathrm{Tor}}(\bar\partial P, \mathbf{Q}/Z)\langle e_1\rangle \oplus H^{2s}_{\mathrm{Tor}}(\bar\partial P, \mathbf{Q}/Z)\langle e_2\rangle$, and $\bar\varphi$ satisfies

$$(7.18) \qquad \begin{cases} \bar\varphi(ae_i, be_i) = 0 \\ \bar\varphi(ae_1, be_2) = \langle a \cup \beta b, [\bar\partial P]\rangle . \end{cases}$$

Abstracting this situation, we have

LEMMA 7.19. *Let $\varphi: L \otimes L \to \mathbf{Q}/Z$ be a non-singular skew symmetric bilinear pairing. Define a quadratic form ψ on $L \oplus L \to \mathbf{Q}/Z$ by*

$$\psi(l_1, l_2) = 2A(l_1) + 2B(l_2) + 2\varphi(l_1, l_2) ,$$

where A, B are homomorphisms into $Z_2 \subset \mathbf{Q}/2Z$. Then ψ is non-singular, and if

$$A(x) = \varphi(\omega_A, x) ,$$
$$B(y) = \varphi(y, \omega_B) , \qquad \textit{for all } x, y \in L ,$$

then

$$\mathcal{G}_\psi = |L|\, e^{2\pi i \varphi(\omega_A, \omega_B)} .$$

Proof. The bilinear form $\bar\varphi$ associated to ψ is $\bar\varphi\big((l_1, l_2), (l_3, l_4)\big) = \varphi(l_1, l_4) - \varphi(l_2, l_3)$, which is symmetric and non-singular. Let ψ_0 be the form

$$\psi_0(l_1, l_2) = 2\varphi(l_1, l_2) .$$

Then $\mathcal{G}_{\psi_0} = |L|$. Moreover, the a in (4.14) which changes ψ_0 into ψ is (ω_B, ω_A). Hence

$$\mathcal{G}_\psi = \mathcal{G}_{\psi_0} e^{-\pi i \psi_0(\omega_B, \omega_A)} = \mathcal{G}_{\psi_0} e^{2\pi i \varphi(\omega_A, \omega_B)} .$$

Remark 7.20. From (4.14b), both ω_A and ω_B have order 2 so $\mathcal{G}_{\psi_0} = \pm|L|$, and is $-|L|$ if and only if $\varphi(\omega_A, \omega_B) = 1/2$.

Note that, in $H^{2s}_{\mathrm{Tor}}(X, \mathbf{Q}/Z)$, we have $a \cup \beta a = -\beta a \cup a$ so $a \cup \beta a$ is a Z_2-class. Moreover, looking in the universal example $K(\mathbf{Q}/Z, 2s)$, we see that $\iota \cup \beta\iota = \{Sq^{2s}\beta\iota\}$. This suggests that $\omega_A = \omega_B = v_{2s}$.

More exactly, let $i: Z_2 \to \mathbf{Q}/Z$ be the usual inclusion, inducing $i: H^{2s}(\bar\partial P; Z_2) \to H^{2s}(\bar\partial P; \mathbf{Q}/Z)$, and suppose v_{2s} given in $H^{2s}(\bar\partial P; Z_2)$. Then $\omega_A = \omega_B = i(v_{2s})$ and $\langle i(v_{2s}) \cup \beta i(v_{2s}), [\bar\partial P]\rangle = \varphi(\omega_A, \omega_B)$. Suppose now $i(v_{2s}) = 2x$. Then $i(v_{2s}) \cdot \beta i(v_{2s}) = 4\big(x\beta(x)\big) = 0$. Thus $\varphi(\omega_A, \omega_B) \neq 0$ only if $i(v_{2s})$ is not divisible by 2, i.e., only if $Sq^1(v_{2s}) \neq 0$. Moreover, in this case, $\langle v_{2s} Sq^1 v_{2s}, [\bar\partial P] \otimes Z_2\rangle \neq 0$ if and only if $\langle i(v_{2s})\beta i(v_{2s}), [\bar\partial P]\rangle = 1/2$. This proves case 2. (This paragraph does nothing more than identify $\varphi(\omega_A, \omega_B)$ with the Z_2-characteristic number $\langle v_{2s} Sq^1 v_{2s}, [\bar\partial P] \otimes Z_2\rangle$.)

We now prove case 3. If P is a closed, oriented manifold,

$\bar{\partial}(\tilde{M} \underline{\times} P) = \bar{\partial}(\tilde{M}) \times P$, and as a consequence,

$$K^*_{\text{Tor}}(\bar{\partial}(\tilde{M}\underline{\times}P) = K^*(\bar{\partial}\tilde{M}) \otimes H^*_{\text{Tor}}(P, \mathbf{Q}/Z) .$$

Applying the technique of (7.8), we have

$$(7.21) \qquad \lambda(r \otimes p) = \lambda(r)\langle p \cup \beta(p), [P]\rangle = \lambda(r)\langle Sq^{2s}\beta(p), [P]\rangle .$$

After doing surgery, we can assume that $K^{2k+1}(\bar{\partial}\tilde{M}) = Z \oplus Z$ with generators e, f, and $\lambda(e) = \lambda(f) = 1$. Then (7.19), (4.24), and (4.22) give the result.

This completes the proof of (7.3).

Remark 7.22. It is possible to prove (7.3.3) for a Z_n-manifold P or even for P, a Z_n-Poincaré duality space. However, the added complexities, bringing in covering spaces, and a generalized transfer would have added at least 10 pages to the exposition, so, in that the stronger result is not required in the sequel, it was thought best to leave those details to a forthcoming article.

8. The homotopy type of a map to G/PL

We use (7.3) to define cohomology classes K_{4i} in $H^{4i}(G/\text{PL}, Z_{(2)})$ so that, if $i\colon Z_{(2)} \to Z_{2^r}$ is a coefficient homomorphism, and $f\colon M^{4s} \to G/\text{PL}$ is a map of differentiable Z_{2^r}-manifold, then $\sum_i f^*(i(K_{4i}))$ determines $I/8$ for a surgery problem associated to f.

These classes are then used to classify the homotopy types of maps into G/PL.

In order to do things in proper generality, we first need to extend the $I/8$-invariant of (3.4) to the case of 4-dimensional manifolds as $I/8(\rho \times 1\colon \tilde{M} \times CP^2 \to M \times CP^2)$. In this situation, (6.7) is valid, but a result analogous to (7.8) shows that, on $\bar{\partial}\tilde{M} \times CP^2$, we have $\lambda(k \otimes e_2) = \lambda(k)$. Thus the machinery of § 6 evaluates $I/8$ (as an element of Z_n). Clearly, for all j, $I/8$ defines a homomorphism $\Omega^{\text{PL}}_{4j}(G/\text{PL}) \to Z_n$.

However, due to (7.3.3), we restrict attention to the differentiable bordism groups $\Omega_*(G/\text{PL})$ in the remainder of this section, because the indecomposable generator in $\Omega_{4s+1}(\text{point}, Z_{2^r})$ can always be taken orientable (see e.g. [34, pp. 181–182]), but we have no such assurances for generators in $\Omega^{\text{PL}}_{4s+1}(\text{point}, Z_{2^r})$.

Next we need

THEOREM 8.1. *Let V be a Z_n-normal bundle to M^{4s} ((1.28)), and suppose $V\colon M^{4s} \to B_{\text{SO}}$ is its classifying map. Then there is a class $L \in H^{4*,*}(B_{\text{SO}}; Z_{(2)})$ so that $i(L) \in H^{4*,*}(B_{\text{SO}}; Z_{2^j})$ satisfies*

$$\langle V^*(iL), [M]\rangle = I(M) ,$$

The Z_{2^j}-index of M. Moreover, if $n = 2$, then $L = (\sum V_{2i})^2$, and L is multiplicative. (This is easy and classical for differentiable M. It is also true for PL-manifolds, but considerably more is involved. We refer the reader to [29] for a proof.)

From this point on, we assume n is a power of 2.

THEOREM 8.2. *There are classes $K_{4i} \in H^{4i}(G/\mathrm{PL}; Z_{(2)})$ so that, if M^{4s} is an oriented Z_n-manifold and*

$$f\colon M^{4s} \longrightarrow G/\mathrm{PL}$$

is any map, then

$$I/8(f) = \langle f^*(i(K_*)) \cup L, [\bar{M}^{4s}]\rangle + \langle \textstyle\sum_j (\beta f^*(k_*)\cdot v_j + f^*(k_*)\beta v_j)\beta v_j, [\bar{M}^{4s}]\rangle\ ,$$

where k_* is the Kervaire class (2.14) *injected into* $\mathbf{Q}/Z$*-cohomology.*

The proof is modeled on (2.13). As the first step, we give an explicit description of the kernel of the Hurewicz homomorphism

$$(8.3)\qquad h\colon \Omega_*(G/\mathrm{PL}, Z_n) \longrightarrow H_*(G/\mathrm{PL}; Z_n) \longrightarrow 0\ .$$

Next, we inductively define a homomorphism

$$(8.4)\qquad K_{4r}^{(n)}\colon \Omega_{4r}(G/\mathrm{PL}, Z_n) \longrightarrow Z_n$$

by setting

$$(8.5)\qquad \begin{aligned}\langle K_{4r}^{(n)}, \{M, f\}\rangle = I/8(f) &- \textstyle\sum_{j<r} \langle f^*K_{4j}^{(n)} \cup L, [\bar{M}]\rangle \\ &- \langle \textstyle\sum (\beta f^*(k_*) \cup v_s + f^*(k_*) \cup \beta v_s)\beta v_s, [\bar{M}]\rangle\ .\end{aligned}$$

We then observe that $\langle K_{4r}^{(n)}, \{M, f\}\rangle = 0$ if $h\{M, f\} = 0$. Hence $K_{4r}^{(n)} \in \mathrm{Hom}(H_*(G/\mathrm{PL}; Z_n), Z_n)$. Finally, we fit all these $K_{4r}^{(n)}$ together to construct the desired K_{4r}.

Proof. We begin with a lemma.

LEMMA 8.6. *Let k divide n. Then a maximal direct summand of Ω_*(point, Z_n) which is isomorphic to a direct sum of Z_k's admits a basis $\{M_1\}, \cdots, \{M_t\}$ with the property that, if $\tau\colon Z_n \to Z_k$ is the surjection, $\tau\{M_j\} = g$ or $\tau\{M_j\} = (n/k)g$ where g is a generator. In particular,*

(1) *if Case* (1), *then M may be chosen orientable;*

(2) *if Case* (2), *then g is represented by a Z_k-manifold W, and $\{M_j\}$ is represented by n/k copies of W.*

(See the remarks following (1.19).)

In particular, Lemma 8.6 implies that, if $h(x) = 0$ for $x \in \Omega_*(X, Z_n)$ and x a generator of order k, x can be written as a sum of "decomposables"

$$(8.7)\qquad x \approx \{\textstyle\sum M_j \times X_j + \sum (n/k) W_k \underline{\times} Y_k\}\ ,$$

where M_j is closed and $h(X_j)$, $h(Y_k)$ are non-zero.
Now use (8.5) to define the $K_{4r}^{(n)} \in \operatorname{Hom}(\Omega_{4r}(G/\mathrm{PL}, Z_n), Z_n)$. Here, to be precise, choose a Thom class U (i.e., a map

$$U\colon K(Z_n, 0) \longrightarrow M(\mathrm{SO}) \wedge \mathfrak{M}(Z_n)$$

inducing an isomorphism in homotopy in dimension zero). This gives a lifting

$$U\colon H_*(X, Z_n) \longrightarrow \Omega_*(X, Z_n)\ ,$$

so $h \circ U = \mathrm{id}$. We now interpret the expression

$$\langle f^*(K_{4j}^{(n)}) \cup L, [\bar{M}]\rangle$$

in (8.5) to mean

$$\langle K_{4j}^{(n)}, f \circ U(L \cap [\bar{M}])\rangle\ . \tag{8.8}$$

It is a bordism invariant since $\partial(L \cap [\bar{W}]) = L \cap \partial[\bar{W}] = L \cap [\bar{M}_1] - L \cap [\bar{M}_2]$ $\bmod(n)$ for a bordism W from M_1 to M_2.

LEMMA 8.9. *$K_{4r}^{(n)}$ vanishes on* $\ker(h)$.

Proof. By (8.7), we may assume $x \in \ker(h)$ has the form $M \underline{\times} N$ and f is the composition $M \underline{\times} N \to \bar{M} \xrightarrow{f_1} G/\mathrm{PL}$. There are three cases to consider, depending on the dimension of M.

Case (1). $\dim M = 4s$. By (7.3.1), $I/8(f) = I/8(f_1)\cdot I(N)$ where by (8.1) $I(N) = \langle L, [\bar{N}]\rangle$. But from (8.5),

$$\begin{aligned}\langle K_{4t}^{(n)}, \{M \underline{\times} N\}\rangle &= I/8(f) - \langle \textstyle\sum_{r<t} K_{4r}^{(n)} \cup L, [\overline{M \underline{\times} N}]\rangle \\ &= I/8(f) - \langle K_{4r}^{(n)} \cup L, [\bar{M}]\rangle\langle L, [\bar{N}]\rangle\end{aligned} \tag{8.10}$$

by the multiplicative property of L. But this in turn is

$$I/8(f) - I/8(f_1)\langle L, [\bar{N}]\rangle = 0$$

from (8.5).

Case (2). $\dim M = 4s + 2$. By (7.3.2),

$$\begin{aligned}\langle K_{4t}^{(n)}, \{M \underline{\times} N, f\}\rangle &= I/8(f) - \langle \textstyle\sum (\beta(f^*(k_*))v_{2l} + f^*(k_*)\beta v_{2l})\beta v_{2l}, [\overline{M \underline{\times} N}]\rangle \\ &= I/8(f) - (n/2)\langle \textstyle\sum f^*(k_*)\cdot v_{2l}Sq^1 v_{2l}, [\bar{M}_2 \times \bar{\partial}N]\rangle \\ &= I/8(f) - (n/2)\langle \textstyle\sum f^*(k_*)v_{2l}Sq^1 v_{2l}, [\bar{M}_2] \times [\bar{\partial}N]\rangle \\ &= I/8(f) - (n/2)\langle \textstyle\sum f^*(k_*)v_{2l}^2, [\bar{M}_2]\rangle\langle v_{2s}Sq^1 v_{2s}, \bar{\partial}N]\rangle \\ &= 0\ .\end{aligned} \tag{8.11}$$

(Note here that v_{2l+1} is divisible by v_1, and v_1 is zero in our situation. Thus we were able to restrict to the v_{2l} in (8.11.))

Case (3). $\dim M = 4s + 3$. Note first that it is sufficient to prove this case if N is an indecomposable in Ω_{ij+1} (point). Then we take note of the fact that

$$(8.12)\qquad \beta i(f^*(k_*) \cup v_l) = \beta i f^*(k_*) i v_l + i f^*(k_*) \beta i v_l ,$$

which makes sense of the second expression in (8.2). We thus have

$$(8.13)\qquad \begin{aligned} \langle K_{4t}^{(n)}, \{M \underline{\times} N, f\}\rangle &= I/8(f) - \langle \textstyle\sum_l i(f^*(k_*) \cup v_l)\beta v_l, [\bar\partial M \underline{\times} N]\rangle \\ &= I/8(f) - (n/2)\langle \textstyle\sum_l f^*(k_*) \cup v_{2l} Sq^1 v_{2l}, [\bar\partial(M \underline{\times} N)] \otimes Z_2\rangle \\ &= I/8(f) - (n/2)\langle \textstyle\sum f^*(k_*) \cup v_{2l}^2, [\bar\partial M]\rangle\langle \textstyle\sum v_{2s} Sq^1 v_{2s}, [N] \otimes Z_2\rangle \\ &= 0 \end{aligned}$$

by (7.3.3). Thus in all three cases, (8.9) is verified, and the proof is complete.

Hence $K_{4t}^{(n)}$ factors through $\mathrm{Hom}(H_{4t}(G/\mathrm{PL}; Z_n), Z_n)$ as desired. Recall from the outline proof of (2.13) the definition of K_{4t}^0 in

$$\mathrm{Hom}(H_{4t}(G/\mathrm{PL}, Z), Z_{(2)}) .$$

LEMMA 8.14. *For X a locally finite complex,*

$$\mathrm{Hom}(H_*(X; Z_n), Z_n) \cong H^*(X; Z_n) .$$

Consider now the universal coefficient sequence

$$(8.15)\quad 0 \longrightarrow \mathrm{Ext}(H_{j-1}(X; Z), Z_n) \overset{\longleftarrow}{\longrightarrow} H^j(X; Z_n) \xrightarrow{\varphi} \mathrm{Hom}(H_j(X; Z), Z_n) \longrightarrow 0 .$$

By (8.14), this gives a map

$$h_n\colon \mathrm{Hom}(H_j(X; Z_n), Z_n) \longrightarrow \mathrm{Hom}(H_j(X; Z), Z_n) .$$

Also, from the coefficient map $\tau_r\colon Z_{(2)} \to Z_{2^r}$, we obtain a map

$$\tau_{r*}\colon \mathrm{Hom}(H_*(X; Z), Z_{(2)}) \longrightarrow \mathrm{Hom}(H_*(X; Z), Z_{2^r}) .$$

Again, if $\gamma_r\colon Z_{2^r} \to Z_{2^{r-1}}$ is the surjection, then we have maps

$$(8.16)\qquad \begin{aligned} \gamma_{r*}&\colon \mathrm{Hom}(H_j(X; Z_{2^r}), Z_{2^r}) \longrightarrow \mathrm{Hom}(H_j(X; Z_{2^{r-1}}), Z_{2^{r-1}}) , \\ \bar\gamma_{r*}&\colon \mathrm{Hom}(H_j(X; Z), Z_{2^r}) \longrightarrow \mathrm{Hom}(H_j(X; Z), Z_{2^{r-1}}) . \end{aligned}$$

Let $D = \varprojlim_r \mathrm{Hom}(H_j(X; Z_{2^r}), Z_{2^r})$, $E = \varprojlim_r \mathrm{Hom}(H_j(X; Z), Z_{2^r})$. There are natural maps

$$\begin{aligned} \mu&\colon H^*(X; Z_{(2)}) \longrightarrow D , \\ \mu'&\colon \mathrm{Hom}(H_*(X; Z), Z_{(2)}) \longrightarrow E \end{aligned}$$

from (8.14), and a map

$$\theta\colon D \longrightarrow E ,$$

defined using the maps φ in (8.15) and passing to limits.

The following result shows how to use D, E, θ, and μ' in order to calculate $H^*(X; Z_{(2)})$.

LEMMA 8.17. *The kernel of*

$$\theta - \mu' \colon D \oplus \operatorname{Hom}(H_*(X; Z), Z_{(2)}) \longrightarrow E$$

is exactly $H^*(X; Z_{(2)})$ *for* X *a locally finite complex.* (Indeed, the map

$$\mu \oplus \varphi \colon H^*(X; Z_{(2)}) \to D \oplus \operatorname{Hom}(H_*(X; Z); Z_{(2)})$$

maps $H^*(X; Z_{(2)})$ exactly onto this kernel, as can be verified on the level of cochains.)

Thus, in order to complete the proof of (8.2), we must show that the $\{K_{4t}^{(n)}\}$ define an element κ in D satisfying $\theta(\kappa) = \mu'(K_{4t}^{0})$. Explicity, we have

LEMMA 8.18. (1) $\gamma_{r*}(K_{4t}^{(2^r)}) = K_{4t}^{2^{r-1}}$,

(2) $\gamma_{r*}(K_{4t}^{(0)}) = h_{2^r}(K_{4t}^{2^r})$.

Proof. (2) is easy. If $x \in H_*(G/\mathrm{PL}; Z_{(2)})$, then there is a closed oriented manifold M and a map $f \colon M \to G/\mathrm{PL}$ so that $h\{M, f\}$ is some odd multiple of x. The surgery obstruction associated to $\{f, M\}$ depends only on the index of $\tilde{M}$ and not on any semi-index. Moreover, since M is oriented, the part of (8.5) involving the Kervaire classes vanishes, and the result follows by induction.

(1) Consider the coefficient sequence

$$(8.19)\quad 0 \to H_{4t}(G/\mathrm{PL}; Z) \otimes Z_{2^r} \to H_{4t}(G/\mathrm{PL}; Z_{2^r}) \overset{s}{\leftrightarrows} \operatorname{Tor}(H_{4t-1}(G/\mathrm{PL}; Z), Z_{2^r}) \to 0 .$$

The splitting of (8.19) by s gives us a method to write

$$(8.20)\qquad H_{4t}(G/\mathrm{PL}; Z_{2^r}) = H_{4t}(G/\mathrm{PL}; Z) \otimes Z_{2^r} \oplus \operatorname{Tor}(H_{4t-1}(G/\mathrm{PL}; Z), Z_{2^r}) .$$

In view of (8.18.2), the truth of (8.18.1), on the first summand of (8.20), follows. It remains to verify (8.18) on the second summand. To this end, look at an element $x \in H_{4t-1}^{\mathrm{Tor}}(G/\mathrm{PL}; Z_{(2)})$. Represent x by $f \colon M^{4t-1} \to G/\mathrm{PL}$ for some closed oriented M. Then if $2^r x = 0$, there is a W with $\partial W = 2^r M$ and $F \colon W \to G/\mathrm{PL}$, so $F \mid (\bar{\partial} W) = f$. Select a basis $\{x_1 \cdots x_s\}$ for $H_{4t-1}^{\mathrm{Tor}}(G/\mathrm{PL}; Z_{(2)})$, and associate with each x_i a pair $(W, \bar{\partial} W_i, F)$ as above. Then in terms of these explicit elements and (8.6.2), we build all the elements in the second summand in (8.20). Now, an easy induction on t using (8.5), and noting for the second term (involving the Kervaire classes) that the Bockstein is for the associated *singular* Z_n-manifold, give the result.

(8.2) now follows.

COROLLARY 8.21. *The two local homotopy type of a map* $f; X \to G/\mathrm{PL}$ *is completely determined in dimensions greater than* 4 *by picking*

(1) *a basis* $x_1^i, \cdots, x_r^j, \cdots$ *for* $\Omega_*(X; Z_{2^i})$ *as a module over* $\Omega_*(point; Z_{2^i})$ *for each* i, *and*

(2) *a representative* $g \colon M_r \to X$ *for* x_r^i *and evaluating the surgery in-*

variant for the problem associated to $f\circ g$. Conversely, given a sequence of compatible homomorphisms

$$\varphi_{i,r}\colon \Omega_i(X; Z_{2^r}) \longrightarrow (surgery\ obst.)_i$$

for $i > 4$, subject to (7.3), there is a unique map

$$f\colon X \longrightarrow G/\mathrm{PL}[6, 8, \cdots]$$

realizing the $\varphi_{i,r}$. (Here compatibility is in the sense of (8.6)).

The surgery classes in (8.2) determine the map $X \to G/\mathrm{PL}[6, 8, \cdots] = \prod_{i=2}^{\infty} K(Z_2\colon 4i-2) \times K(Z_{(2)}; 4i)$ since they determine ± 1 on the generators in $\pi_*(G/\mathrm{PL})$. But the situation in the total space G/PL is more involved. For E_2, we have an exact sequence

$$0 \longrightarrow H^4(X; Z) \longrightarrow [X; E_2] \longrightarrow H^2(X; Z_2) \xrightarrow{2\beta Sq^2} H^5(X; Z)\ , \tag{8.22}$$

and, in dimension 4, there are two generators ι^2, θ with $2(\delta\theta) = \delta\iota^2 = 4(y)$. Thus $\iota^2 - 2\theta$ represents an integral cohomology generator. (θ may be thought of as the fundamental class on the fiber $K(Z, 4)$ in E_2.) Two maps into E_2 agreeing in dimension 2 may differ on θ. Indeed, from (8.22) $f^*(\theta)$ may be changed by any integral cohomology class $x \in H^4(X; Z)$. Then $f^*(\iota^2 - 2\theta)$ is changed by $-2x$.

PROPOSITION 8.23. *There are cases when two homotopically distinct maps $f_1\colon X \to E_2$, $f_2\colon X \to E_2$ give the same map in cohomology.*

Proof. Suppose $H^4(X; Z)$ contains a Z_4-direct summand with generator x. Suppose also that X is 4-dimensional. Then changing f by $(2x)$ does not change the maps in cohomology for any coefficients.

Consequently, it cannot happen that considerations at the level of ordinary bordism can determine completely the 2-adic homotopy type of a map into G/PL in general.

Remark 8.24. Note that $H^4(E_2; Z) = Z$ with generator l, and if $\tau\colon Z \to Z_2$ is the coefficient homomorphism, then $\tau(l) = \iota^2$. Moreover, $K_4 = \pm l$. Also, if we look at the H-space structure of E_2 in G/PL, we find $\mu_*(\iota_* \otimes \iota_*) = \{\theta_*\}$ with Z_2-coefficients. This last suggests that, if we take the product of two 2-dimensional Kervaire invariant-one surgery problems, we obtain a surgery problem which has an unstable non-trivial bordism invariant associated with it. What the geometric interpretation of the invariant is the author has no idea.

Example. 8.25. In the case of certain spaces, homology type completely determines homotopy type for maps into G/PL. In particular, this is true for real projective spaces. From [20], $[\mathrm{RP}^{10}, G/\mathrm{PL}] = Z_4 \oplus Z_2 \oplus Z_2 \oplus Z_2$. We identify the various generators in terms of their effects in $\Omega_*(\mathrm{RP}^{10}; Z_2)$. As a module over $\Omega_*(\text{point}, Z_2)$, this set is free with 10-generators $e_1, e_2, \cdots, e_{10}$. The odd generators are represented by the non-trivial map $\mathrm{RP}^{2i+1} \to \mathrm{RP}^{10}$, while the even generators are represented by maps of the generalized Klein bottles $E^{2i} = S^{2i-1} \times_T S^1$ where $T(x, y) = (r(x), -y)$, with r the reflection of the upper and lower hemispheres of S^{2i-1}. Let q_1, q_2, q_3, q_4 be the generators of $[\mathrm{RP}^{10}, G/\mathrm{PL}]$. Then $q_1^*(k_2) = e^2$, $q_1^*(K_4) = e^4$, $q_1^*(K_{4i-2}) = q_1^*(K_{4i}) = 0$, $i > 1$, and

$$I/8(q_1 E^4) = 1 ,$$
$$I/8(q_1(E^8)) = 0 ,$$

while $K(q_1 E_2) = K(q_2 E_6) = K(q_3 E_{10}) = 1$ and $K_\partial(\mathrm{RP}^{2i-1}) = 0$. Similarly, for q_2, we have $q_2^*(k_6) = e^6$, $q_2 | \mathrm{RP}^5) \simeq 0$, and $q_2^*(K_8) = q_2^*(k_{10}) = 0$, so $I/8(q_2(E^8)) = 0$, $K(q_2 E^6) = K(q_2 E^{10}) = 1$. Also, $I/8(q_3(E^8)) = 1$, $K(q_3(E^{10})) = 0$, while $K(q_4(E^{10})) = 1$.

Appendix: Q/Z Cohomology

Define $Q_{(p)}$ to be the direct limit $\varinjlim Z_{p^i}$ where $\tau\colon Z_{p^i} \to Z_{p^{i+1}}$ sends the generator g_i to pg_{i+1}. There is a natural map

(A1) $$\varphi_p\colon Q_{(p)} \longrightarrow Q/Z$$

defined by $\varphi_p(g_i) = 1/p^i$, and, using these φ_p, we have

LEMMA A2. *$Q/Z \cong \sum_{p=\text{prime}} Q_{(p)}$, hence the dual group* $\mathrm{Hom}(Q/Z; Q/Z) = \mathrm{Hom}(Q/Z, S^1)$ *is isomorphic to*

$$\prod_{p=\text{prime}} Z_{(p^\infty)} .$$

(Recall that $Z_{(p^\infty)} = \varprojlim_{\gamma_i} Z_{p^i}$ where the map $\gamma_i\colon Z_{p^i} \to Z_{p^{i-1}}$ is the usual surjection.)

COROLLARY A3. (1) $K(Q_{(p)}, n) = {}_\tau\varinjlim K(Z_{p^i}, n)$,
(2) $K(Q/Z, n) = \text{weak limit} \prod^{\infty}_{\substack{p=\text{prime}\\ p<m}} K(Q_{(p)}, n)$.
Consequently, $K(Q_{(p)}, n)$ and $K(Q/Z, n)$ have the homotopy types of countable CW-*complexes.*

Hence we have from [28, Lemma 1]

COROLLARY A4. $H_*(K(Q_{(p)}, n); A) = \varinjlim H_*(K(Z_{p^i}, n); A)$.

In order to evaluate the cohomology of $K(Q/Z, n)$, we recall from [28] the definition of the $\lim^1$ functor: let $A_1 \overset{\pi_1}{\longleftarrow} A_2 \overset{\pi_2}{\longleftarrow} A_3 \longleftarrow \cdots$ be an inverse sequence of abelian groups. Then $\varprojlim_\pi (A_i)$ is defined as the kernel in the map

$$d\colon \prod (A_i) \longrightarrow \prod (A_i)$$

where $d(a_1, \cdots, a_n, \cdots) = (a_1 - \pi a_2, a_2 - \pi a_3, \cdots)$ and

(A5) $$\lim^1 (A_i) = \prod (A_i)/d \prod (A_i) .$$

LEMMA A6. (1) *$\mathrm{Lim}^1 (A_i) = 0$ if $\pi\colon A_i \to A_{i-1}$ is onto for all i.*

(2) *$\mathrm{Lim}^1 (A_i) = 0$ if there is a sequence of positive integers $n_1, n_2, \cdots, n_i, \cdots$, so the composites $\pi_i \circ \pi_{i+1} \circ \cdots \circ \pi_{i+n_i}$ are identically zero.*

Proof. In the first case,

$$\vec{a} = \bigl(0, -\tilde{a}_1, -(\tilde{\tilde{a}}_1 + \tilde{a}_2), \cdots, \cdots\bigr)$$

satisfies $d(\vec{a}) = (a_1, a_2, a_3, \cdots)$ where $\tilde{a}_i$ is chosen so $\pi\tilde{a}_i = a_i$. In the second case,

$$\vec{a} = (a_1 + \pi a_2 + \pi^2 a_3 + \cdots + \pi^{n_1} a_{n_1+1}, a_2 + \cdots + \pi^{n_2} a_{2+n_2}, \cdots)$$

satisfies

$$d(\vec{a}) = (a_1, a_2, \cdots) ,$$

and (A6) follows.

THEOREM A7. *$H^*\bigl(K(Q_{(p)}, n); Z\bigr) = \varprojlim H^*\bigl(K(Z_{p^i}, n); Z\bigr)$.*

Proof. We check $\lim^1 H^*\bigl(K(Z_{p^i}, n); Z\bigr) = 0$ in all dimensions. First, we consider the situation mod(p). $H^*\bigl(K(Z_{p^i}, n), Z_p\bigr)$ is the free commutative algebra (for p odd a polynomial algebra on even generators tensored with an exterior algebra on odd generators, and for $p = 2$ a polynomial algebra) on specific generators

$$\Lambda\bigl(\cdots, \mathcal{P}^I(\iota), \cdots, \mathcal{P}^I b(\iota), \cdots, b\mathcal{P}^I(\iota), \cdots, b\mathcal{P}^I b(\iota)\bigr) ,$$

where $b\colon H^i(X, Z_p) \to H^{i+1}(X, Z_p)$ is the Bockstein associated to the coefficient sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$. Looking at the integral cohomology of $X_i = K(Z_{p_i}, n)$, we can make the following observations:

(1) If n is even, then, except for a summand $Z_{p^{i_{\nu(k)}}}$ in $H^{nk+1}(X_i, Z)$ corresponding to $b[(\iota)^k]$, the order of any class in $H^r(X_i, Z)$ is bounded by a power of p which depends only on r and not on i ([11]).

(2) If n is odd, then, except for a class in dimension $(n + 1)k$ of order p^i corresponding to $[b(\iota)]^k$, the same statement is true.

(3) $\tau(\iota_{p^i}) = p(\iota_{p^{i-1}})$, so

$$\tau\bigl(b(\iota_{p^i})\bigr) = b(\iota_{p^{i-1}}) .$$

(4) $\tau(\mathcal{P}^I\iota) = \tau\bigl(b(\mathcal{P}^I\iota)\bigr) = 0 \bmod (p)$ for $I = (i_1, \varepsilon_1, \cdots, i_{n-1}, \varepsilon_{n-1}, i_n)$ not identically zero.

Thus, the elements in $H^r(X_i, Z)$, aside from the special elements in (1) and (2) above, can be split into direct sums $A_{r,i} + B_{r,i}$ where $\tau(A_{r,i}) \subset A_{r,i+1}$ as a direct summand, while $\tau(B_{r,i}) \subset pB_{r,i+1}$. Since $\lim^1(A_{r,i} \oplus B_{r,i}) = \lim^1 A_{r,i} \oplus \lim^1 B_{r,i}$, this $\lim^1$-term is zero by (A6). The exceptional elements associated to $(\iota)^k$ or $(b(\iota))^k$ are handled similarly, and (A7) follows.

Remark A8. The proof of (A7) shows that

$$H^*(K(Q_{(p)}, n); Z_p) = \Lambda(\cdots \mathcal{P}^I b(\iota) \cdots) ,$$

where $\mathcal{P}^I b$ is a permissible monomial in the (mod p) Steenrod algebra $\mathcal{A}(p)$ of excess less than n. Note also that each generator except $b(\iota)$ has order exactly p.

Remark A9. Of course, by Kunneth's theorem, $H^*(K(Q/Z, n); Z)$ is now evaluated directly by using (A3.2). In particular,

$$H^{n+1}(K(Q/Z, n); Z) \cong \mathrm{Hom}(Q/Z, Q/Z) .$$

We set $\beta(\iota) \in H^{n+1}(K(Q/Z, n); Z)$ to be the class corresponding to the identity.

In $H^n(K(Q/Z, n); Q/Z) = \mathrm{Hom}(H_n(K, Z); Q/Z) = \mathrm{Hom}(Q/Z, Q/Z)$, we have the class ι corresponding to the identity. Then $\beta\iota$ serves as a *universal Bockstein operator* in the following sense: let a coefficient sequence

$$0 \longrightarrow Z_s \longrightarrow Z_{sm} \longrightarrow Z_m \longrightarrow 0$$

be given. This gives a Bockstein operation

$$\beta_{m,s}\colon H^j(\ ; Z_m) \longrightarrow H^{j+1}(\ ; Z_s) .$$

On the other hand, the embedding $\rho\colon Z_m \hookrightarrow Q/Z$ and the surjection $\lambda\colon Z \rightarrow Z_s$ give rise to

$$K(Z_m, n) \xrightarrow{\lambda} K(Q/Z, n) \xrightarrow{j} K(Z_s, n+1)$$

where $j^*(\iota) = \rho(B\iota)$. Moreover,

$$(j \circ \lambda)^*(\iota) = \beta_{m,s}(\bar{\iota}) .$$

Remark A10. Suppose X is a complex. Let $H^n_{\mathrm{Tor}}(X; Z)$ be given. Then

$$\beta\colon H^{n-1}_{\mathrm{Tor}}(X; Q/Z) \longrightarrow H^n_{\mathrm{Tor}}(X; Z)$$

is an isomorphism.

We now turn to some basic considerations about the pairing $Q/Z \otimes Z \rightarrow Q/Z$. In particulur, this defines an operation

$$H^r(\ ; Q/Z) \otimes H^s(\ ; Z) \longrightarrow H^{r+s}(\ ; Q/Z) ,$$

which we denote $a \cup \beta b$.

PROPOSITION A11. $a \cup \beta b = (-1)^{(\dim(a)+1)(\dim b+1)} b \cup \beta a$.

Proof. $\beta(a \cup \beta b) = \beta(a) \cup \beta(b) = (-1)^{(\dim(a)+1)(\dim(b)+1)}(\beta(b) \cup \beta(a)$
$= \beta(b \cup \beta a))$.

Similarly, there is the pairing $Z \otimes Q/Z \to Q/Z$, which gives the operation $\beta a \cup b$.

PROPOSITION A12. $\beta(a) \cup b = (-1)^{\dim(a)+1} a \cup \beta(b)$.

Proof. Indeed, we have $\beta(\beta a \cup b) = (-1)^{\dim a+1}\beta(a) \cup \beta(b)$ on using our direct limit arguments, and (A12) follows from the proof of (A11).

In particular, we have

COROLLARY A13. $\iota_{2s} \cup \beta\iota_{2s} = \{Sq^{2s}\beta\iota_{2s}\}$.

Proof. We look in $K(Q_{(2)}, 2s)$. Note that $\beta(\iota_{2s} \cup \beta\iota_{2s}) = (\beta\iota_{2s})^2 = \beta\{Sq^{2s}\beta\iota_{2s}\}$. Thus their difference lies in the kernel of β, and hence is zero by (A10).

Finally, we point out a mysterious result first suspected by Brumfiel and later proved in a joint conversation. The reader would do well to compare our techniques in § 6 for proving (6.7) (notably (6.19)-(6.27)) with

THEOREM A14. *The fibration*

$$K(Q/Z, n) \xrightarrow{(\beta\iota)} K(Z, n+1) \xrightarrow{\iota} K(Q, n+1)$$

is also a cofibration.

Proof. Consider the directed system of cofiberings and the maps

$$\begin{array}{ccccc} K(Z_m, n) & \xrightarrow{(\beta_m\iota)} & K(Z, n+1) & \longrightarrow & X_m \\ \downarrow j & & \downarrow \mathrm{id} & & \downarrow r_m \\ K(Q/Z, n) & \longrightarrow & K(Z, n+1) & \longrightarrow & K(Q, n+1) \end{array}$$

where j embeds Z_m in Q/Z and $\langle r^*(\iota), e_m\rangle = 1/m$ where e_m is the generator of $H_{n+1}(X_m, Z) = Z$. The directed system has as limit

$$K(Q/Z, n) \xrightarrow{(\beta\iota)} K(Z, n+1) \longrightarrow \text{Mapping cone } (\beta(\iota))$$

by (9), and the limit of the maps r_m is a map (r): (Mapping cone) $\to K(Q, n+1)$. We check now that r induces isomorphisms in homology. For this, use (8) which implies that, except on $(\iota)^t$, the map $(\beta_m\iota)^*$ is injective. On the other hand, (A7) shows the cokernels of $\beta_m(\iota)$ have limit zero. Thus the only homology of (Mapping cone) is due to the $(\iota)^t$, and the result follows easily.

Question A15. Are there any other fiberings which are also cofiberings?

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