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A NEW FORMULA FOR WINDING NUMBER

ABSTRACT. We give a new formula for the winding number of smooth planar curves and show how this can be generalized to curves on closed orientable surfaces. This gives a geometric interpretation of the notion of winding number due to B. Reinhart and D. R. J. Chillingworth.

1. INTRODUCTION

The winding number of a smooth closed planar curve is just the number of complete turns the tangent vector to the curve makes as one passes once around the curve; that is to say, it is the degree of the curve's Gauss map (see, for example, [1]). As shown by Whitney [12], winding number is invariant under regular homotopy, and may thus be regarded as an integer-valued homomorphism from the group of regular homotopy classes of smooth planar curves. Moreover, again by Whitney [12], there is a simple formula for the winding number of normal planar curves in terms of the number of positive crossings and negative crossings.

For curves on closed orientable surfaces the situation is more complicated. Here there is no canonical definition of winding number (and in the non-orientable case there is no general definition at all [3]). However, as shown by Reinhart [7], [8], [9], and Chillingworth [3], for a chosen set of smooth generators of the fundamental group of the surface, there is a unique notion of winding number that gives zero for each of the chosen generators and gives the value 1 for small anti-clockwise contractible loops. (See [6], [5] and [4] for other works on winding numbers.) Here, by a winding number one again means a homomorphism from the group of regular homotopy classes of smooth curves on the surface. However, whereas for planar curves the winding number is an integer, for curves on a closed oriented surface M one can show that the winding number homomorphism must take its values in the finite group $\mathbb{Z}_{|\chi(M)|}$, where $\chi(M)$ is the Euler characteristic of M (see [7]; we give further discussion of this point below in Section 5). In [7] an algorithm is given for the computation of the winding number of a given curve, for a certain choice of generators; but the algorithm requires that the regular homotopy class of the curve be first expressed in terms of the generators, and this can often be troublesome.

The aim of this paper is to provide a new geometric method for the computation of winding numbers. In the special case of planar curves, this method also gives a new formula; let us first describe this case. Let γ be a

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smooth closed planar curve and suppose that γ is *normal*; that is, γ has only finitely many crossings, γ only passes through each crossing point twice, and at each crossing point the two tangent vectors to γ are linearly independent. (As Whitney showed in [12], every planar curve can be arbitrarily closely approximated by a normal curve.) Now each normal planar curve partitions its complement into a finite number of connected open regions. The 'outside' region is the only one that is not precompact. The following elementary lemma shows that the regions can be coherently numbered (see [2]).

LEMMA 1. *Given a closed oriented normal curve γ, one can associate integers to each of the regions such that at each segment of γ the number to the left of γ is* 1 *greater than the number to the right, and the outside region is numbered zero. Moreover, such a numbering is unique.*

Now for each positive (respectively negative) integer i, define S_i to be the closure of the union of the regions with a number greater (resp. less) than or equal to i. Each of the S_i is a polygonal domain with a piecewise smooth boundary comprising segments of γ. In particular, each S_i has a well-defined Euler characteristic $\chi(S_i)$.

THEOREM 1. *The winding number $\omega(\gamma)$ of γ is given by*

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i).$$

The proof of this theorem is quite simple. On the one hand, the winding number of γ is just the integral of its curvature, divided by 2π. On the other hand, by the Gauss–Bonnet formula (see, for instance, [1]), the Euler characteristic of each of the S_i is (up to a sign, and when multiplied by 2π) equal to the integral of the curvature of those segments of γ that form the boundary of S_i plus the sum of the external angles at the crossing points of γ that occur in the boundary of S_i. It suffices then to notice that each crossing of γ occurs in the boundary of precisely two domains S_i, and that the corresponding pairs of external angles each sum to zero.

In the rest of this paper we will show how an analogous formula may be obtained for curves on arbitrary closed orientable surfaces. We begin in Section 2 by proving a generalization of Lemma 1. This enables us to state the general formula in Theorem 2. Section 3 gives several examples. Theorem 2 is proved in Section 4. Once again the proof is simple. Section 5 gives a brief discussion.

Finally, it should be said that the key idea in Theorem 2 is essentially contained in Lemma 5 of [8], which B. Reinhart attributes to A. Haefliger. This lemma states the following: if on a closed surface M of genus g a simple bounding curve cuts off surfaces with p and $g - p$ handles, then its winding number is $\pm(2p - 1) \bmod (2g - 2)$. Theorem 2 may be regarded as a generalization of this result to the case of non-simple non-bounding curves.

Our thanks go to David Chillingworth, for having made a remark that improved the proof of Theorem 2.

2. THE GENERAL FORMULA

Consider a closed oriented Riemannian surface M, of genus g, and let $v_1, \dots, v_{2g}$ be a set of smooth curves whose homology classes, $[v_1], \dots, [v_{2g}]$ say, form a generating set for the first homology group $H_1(M)$ of M. Now let γ be a smooth oriented curve on M. We will suppose throughout the rest of this paper that the curves $\gamma, v_1, \dots, v_{2g}$ are normal curves and that, moreover, they intersect each other normally.

Now the homology class $[\gamma]$ of γ may be written as a sum:

$$[\gamma] = n_1[v_1] + \cdots + n_{2g}[v_{2g}]$$

where the coefficients $n_1, \dots, n_{2g}$ are integers. And as in the planar case, the complement C in M of the union

$$\gamma \cup v_1 \cup \cdots \cup v_{2g}$$

is a finite number of connected open regions. Choose an arbitrary point x_0 in C. We can now state the generalization of Lemma 1.

LEMMA 2. *One can associate integers to each of the regions such that at each segment of γ the number to the left of γ is 1 greater than the number to the right of γ, and for each $i = 1, \dots, 2g$, the number to the left of each segment of v_i is n_i less than the number to the right of v_i, and the region containing the point x_0 is numbered zero. Moreover, such a numbering is unique.*

Proof. The curves $\gamma, v_1, \dots, v_{2g}$ determine a natural CW-complex; its 0-skeleton is the set of crossing points of the curves, the 1-skeleton is the image of the curves, and the 2-skeleton is the entire set M. In particular, the 2-cells of this complex are just the regions determined by the curves $\gamma, v_1, \dots, v_{2g}$. This CW-complex may not be regular, but it has a natural orientation determined by the orientation of the curves and that of M.

Let $c_1, c_2, \dots, c_p$ denote the 1-cells of this complex, and let $e_1, e_2, \dots, e_q$ be the 2-cells. Suppose that the 2-cell containing the point x_0 is e_q. Now, the 1-cycle

$$c = \gamma - (n_1 v_1 + \cdots + n_{2g} v_{2g})$$

is exact and hence c is the boundary of a 2-chain:

$$c = \partial(m_1 e_1 + m_2 e_2 + \cdots + m_q e_q),$$

for some integers $m_1, \ldots, m_q$. Moreover, it is clear that these integers may be chosen in precisely one way such that $m_q = 0$.

The integers $m_1, \ldots, m_q$ define a numbering of the regions of γ which, by construction, satisfies the required properties. This completes the proof of Lemma 2.

Now, as in the planar case, for each positive (resp. negative) integer i, define S_i to be the closure of the union of the regions with a number greater (resp. less) than or equal to i. We will call the sets S_i the *domains* of γ. Each of them is a Riemannian surface with piecewise smooth polygonal boundary, and hence has a well-defined Euler characteristic $\chi(S_i)$. The general result is the same as that in Theorem 1.

THEOREM 2. *The winding number $\omega(\gamma)$ of γ is given by*

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i),$$

if M is the torus $\mathbb{T}^2$, and

$$\omega(\gamma) = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i) \pmod{|\chi(M)|},$$

otherwise.

3. Examples

In this section we give three examples. The first example is shown in Figure 1. This is a planar curve γ with winding number $\omega(\gamma)$. Only four numbers occur in the numbering of the regions of γ: $-1, 0, 1$ and 2. The Euler characteristic of S_1 is 0, while those of S_{-1} and S_2 are both 1. Hence, Theorem 1 reads; $\omega(\gamma) = 0 + 1 - 1 = 0$.

The next two examples are on the closed oriented surface M of genus 3. Figure 2 shows the choice of generators $v_1, \ldots, v_6$. In the example shown in Figure 3, the curve γ belongs to the homology class $[v_3] - [v_5]$. The only numbers occurring in the numbering of the regions of γ are 0 and 1. Figure 4 shows the domain S_1; it has Euler characteristic -3. Hence, Theorem 2 reads; $\omega(\gamma) = 1 \pmod 4$.

In the final example, in Figure 5, the curve γ belongs to the homology class $[v_3] + [v_4]$. The only numbers occurring in the numbering of the regions of γ are $0, 1$ and -1. Domain S_1 has Euler characteristic -1, while domain S_{-1} has Euler characteristic 1. Hence, Theorem 2 reads; $\omega(\gamma) = 2 \pmod 4$.

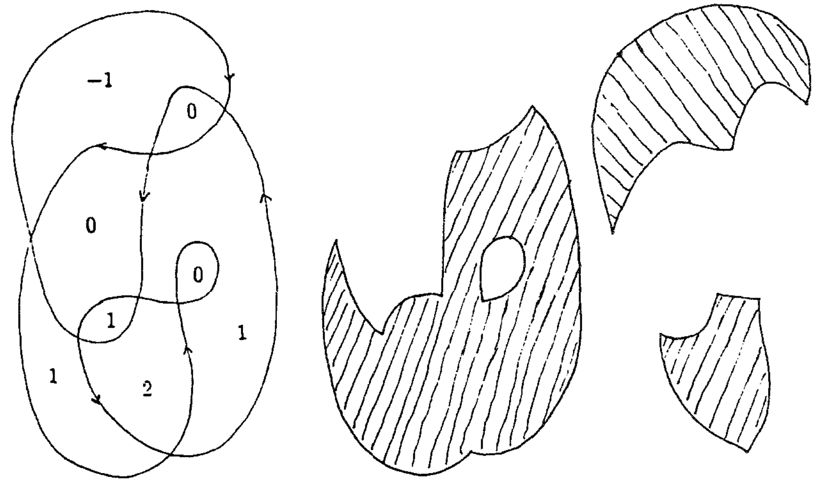

Fig. 1.

4. Proof of theorem 2

We will prove Theorem 2 for closed orientable surfaces M other than the torus. The case of the torus is quite analogous.

We assume, as above, that we have a chosen generating set $v_1, \ldots, v_{2g}$ of the fundamental group of M, and that the homology class $[\gamma]$ of the given curve γ is

$$[\gamma] = n_1[v_1] + \cdots + n_{2g}[v_{2g}],$$

where $n_1, \ldots, n_{2g} \in \mathbb{Z}$.

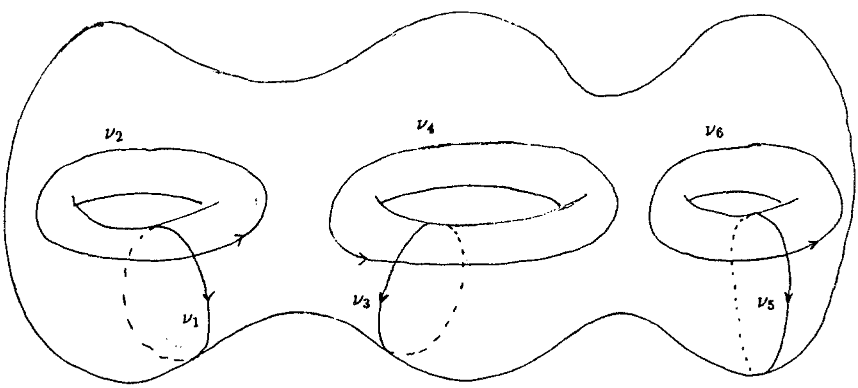

Fig. 2.

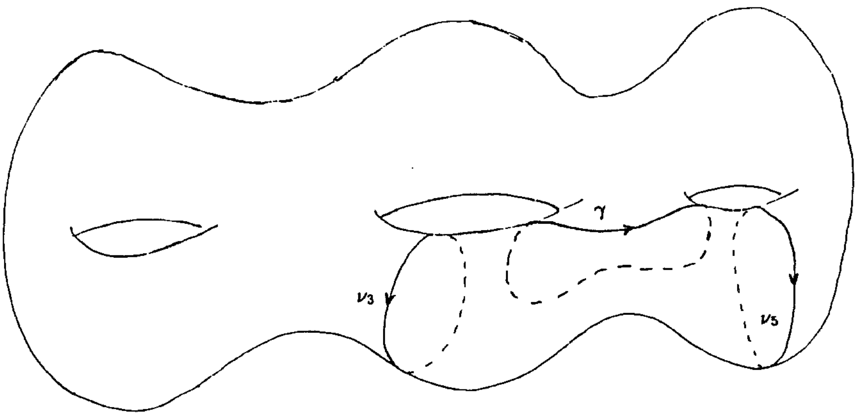

Fig. 3.

We first choose an arbitrary conformal structure on M. Then according to [7] there is a vector field X on M with only one singularity, at x_0, such that the following two conditions hold:

(a) for each $i = 1, \ldots, 2g$, the integral around v_i of the angle σ_i between X and v_i is zero,
(b) if θ denotes the angle between X and γ then

$$\omega(\gamma) \equiv \frac{1}{2\pi} \int_\gamma d\theta \quad (\mathrm{mod}\, |\chi(M)|).$$

We will show that

$$(1) \qquad \frac{1}{2\pi} \int_\gamma d\theta = \sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i),$$

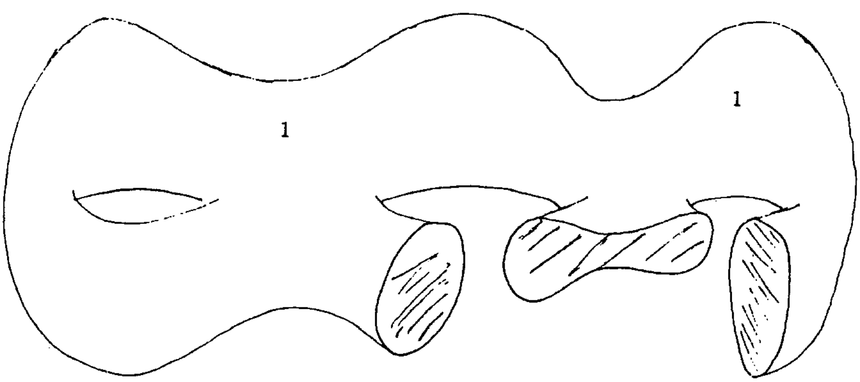

Fig. 4.

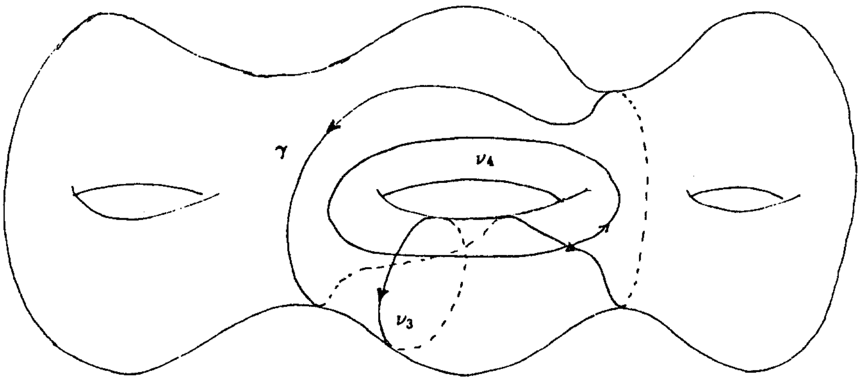

Fig. 5.

where, for each non-zero integer i, the set S_i is defined as in Section 2. Consider one of these domains S_i, for some non-zero integer i. The domain S_i has a piecewise smooth boundary with external corner angles $\phi_1, \ldots, \phi_k$ say. Let δ_i denote the angle between the vector field X and the tangent to the boundary of S_i. Now X has no singularity in S_i, since its only singularity occurs at x_0, which lies in a region numbered zero. It follows from the Poincaré–Hopf theorem (see, for instance, [11], see also Lemma 5.7 of [3]) that

$$2\pi \cdot \chi(S_i) = \int_{\partial S_i} \mathrm{d}\delta_i + \sum_{j=1}^{k} \phi_j. \tag{2}$$

Now it is clear that in the summation

$$\sum_{i>0} \chi(S_i) - \sum_{i<0} \chi(S_i),$$

the total contribution of the external corner angles of the boundaries of the different domains is zero. Indeed, consider a corner point x say, in the boundary of one of the domains. Then x is contained in the closure of four regions, numbered $n_1 \leqslant n_2 \leqslant n_3 \leqslant n_4$ say. (Note that at least three of these numbers are distinct.) Then x is a corner point of precisely two domains and the sum of the corresponding two external angles is zero. To see this, one needs to consider three cases, according to whether the numbers $n_1, \ldots, n_4$ are all positive, all negative, or neither all positive nor all negative. Each case is simple, and so we leave the details to the reader. As an example, for natural number n, Figure 6 shows the boundaries of the domains numbered n and $n + 1$ meeting at a certain crossing. In this figure the external angle to S_{n+1} is

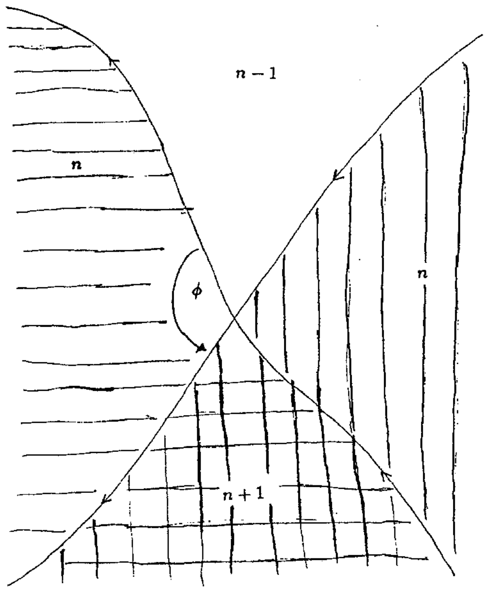

Fig. 6.

ϕ, while the external angle to S_n at the same point is $-\phi$. Evidently, the sum is zero.

In view of the above discussion, in order to prove (1) it suffices to prove

$$\int_\gamma d\theta = \sum_{i>0}\left(\int_{\partial S_i} d\delta_i\right) - \sum_{i<0}\left(\int_{\partial S_i} d\delta_i\right). \tag{3}$$

In order to prove (3), let us consider a domain S_i. The boundary of S_i is composed of segments, some of which are segments of γ and some of which are segments of the curves v_j. Now notice that for points on γ, one has $\delta_i = \theta$, and for points on v_j, one has $\delta_i = \sigma_j$. Notice also that for $i \geqslant 1$, the orientation of the boundary of S_i coincides with the positive direction of γ and the negative direction of the curves v_j. Similarly, for $i \leqslant 1$, the orientation of the boundary of S_i coincides with the negative direction of γ and the positive direction of the curves v_j. Finally, notice that each segment of γ occurs in the boundary of precisely one domain S_i, and for each j, each segment of v_j occurs in the boundary of precisely n_j domains S_i. Combining these observations, one has that

$$\begin{aligned}\sum_{i>0}\left(\int_{\partial S_i} d\delta_i\right) - \sum_{i<0}\left(\int_{\partial S_i} d\delta_i\right)\\ = \int_\gamma d\theta - \left(n_1\int_{v_1} d\sigma_1 + \cdots + n_{2g}\int_{v_{2g}} d\sigma_{2g}\right).\end{aligned} \tag{4}$$

But by the definition of X one has

$$\int_{v_i} d\sigma_i = 0,$$

for each $i = 1, \ldots, 2g$. Hence (4) implies (3), as required. This completes the proof of Theorem 2.

5. Discussion

We begin by considering an orientable closed surface M, of genus g, which for the moment we suppose to be other than the torus. Recall that the fundamental group $\pi_1(M)$ of M has a generating set $\mu_1, \ldots, \mu_{2g}$ with the one relation

$$\mu_1\mu_2\mu_1^{-1}\mu_2^{-1} \cdots \mu_{2g-1}\mu_{2g}\mu_{2g-1}^{-1}\mu_{2g}^{-1} = 1.$$

By a theorem of Smale [10], the group $\pi_R(M)$ of regular homotopy classes of smooth curves on M is isomorphic to the fundamental group $\pi_1(S^1M)$ of the unit tangent bundle S^1M of M. This latter group (see [7]) has generating set $\mu_1, \ldots, \mu_{2g}, h$ and the one relation

$$\mu_1\mu_2\mu_1^{-1}\mu_2^{-1} \cdots \mu_{2g-1}\mu_{2g}\mu_{2g-1}^{-1}\mu_{2g}^{-1} = h^{|\chi(M)|}.$$

Now, if by a 'winding number' one is to understand a homomorphism, ϕ say, from $\pi_R(M)$ to some abelian group, then ϕ must factor through the abelianization of $\pi_R(M) \cong \pi_1(S^1M)$. But this abelianization is, of course, nothing other than the first homology group $H_1(S^1M)$ of S^1M. From the above presentation of $\pi_1(S^1M)$ one has immediately that

$$H_1(S^1M) \cong \mathbb{Z}^{2g} \times \mathbb{Z}_{|\chi(M)|}.$$

Now the factor $\mathbb{Z}_{|\chi(M)|}$ in the above product is generated by the homology class of the generator h in $\pi_1(S^1M)$. And h is given by the homotopy class of the fibre of the natural fibration of S^1M over M, and this in turn corresponds to the regular homotopy class of a small contractible loop in M. But if a winding number is to have any meaning, it should be non-zero on small contractible loops. It follows therefore that if the winding number is to be a homomorphism into a cyclic group, then that group must necessarily be $\mathbb{Z}_{|\chi(M)|}$. In other words a winding number is just a homomorphism

$$\bar{\phi}: H_1(S^1M) \cong \mathbb{Z}^{2g} \times \mathbb{Z}_{|\chi(M)|} \to \mathbb{Z}_{|\chi(M)|}.$$

Seen in this light, it is clear that every winding number necessitates a preferred choice of generators for $\pi_1(M)$, and that every such choice determines a winding number homomorphism.

The above remarks clearly also apply to the torus. Here the group $H_1(S^1M)$ is just $\mathbb{Z}^3$ and winding numbers are integer valued. In fact, the above remarks clearly also apply to the case of non-compact surfaces. Here again the winding numbers are integer valued. However, for non-compact surfaces the formula given in Theorem 2 has no general meaning, because of the lack of definition of Euler characteristic for noncompact surfaces. This parallels the approach in [7] and [3], which also gives a method for computing winding numbers only in the compact case.

A comparison of the algorithm given in [7] and [3] with the method of Theorem 2, reveals certain advantages of the geometric approach. Firstly, as mentioned in the introduction, the algorithm of [7] and [3] first requires that the regular homotopy class of the given curve be expressed in terms of the generators of the group $\pi_R(M)$. Theorem 2, however, only requires that one express the homology of the given curve in terms of the generators of $H_1(M)$. In addition, the algorithm in [7] is stated only for simple curves, and that of [3] is applicable only to direct curves (that is, normal curves with no nullhomotopic loops). So, in order to find the winding number of an arbitrary curve, using [7] or [3], one must first find a curve which is regularly homotopic to the given curve and to which one can apply the algorithm. Of course this difficulty does not occur with the method of Theorem 2. On the down side, Theorem 2 requires the calculation of the Euler characteristic of a number of regions. It therefore seems less suited to calculation by computer than is the algorithm of [7] and [3].

It should be noted that there is a more direct proof of Theorem 2 than the one given above. In effect, it suffices to show that the formula in the statement of Theorem 2 defines a homomorphism from $H_1(S^1M)$. To see this one must prove that the evaluation of the formula is independent of the choice of base point x_0, that it is invariant under regular homotopy, and that it is additive. The first two points are straightforward. Additivity can be established by induction on the number of crossings.

It is perhaps also worth remarking that there is an alternate formulation of Theorems 1 and 2. Suppose that one has a normal curve γ in the plane or on a closed oriented surface M, and suppose that the corresponding regions are numbered as in Lemma 1 and Lemma 2. Then for each integer i let A_i denote the closure of the union of the regions with number i. So, for instance, if $i > 0$ one has

$$S_i = \bigcup_{j \geqslant i} A_j.$$

Then it is easy to prove that the winding number $\omega(\gamma)$ of γ can also be written as

$$\omega(\gamma) = \sum_{i \in \mathbb{Z}} i . \chi(A_i),$$

if γ is a curve in the plane or on the torus, and by the same expression modulo $|\chi(M)|$ if M is a closed orientable surface other than the torus. This follows from Theorems 1 and 2 since, for example, for positive i, one has $\chi(A_i) = \chi(S_i) - \chi(S_{i+1})$.

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