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AN ELEMENTARY PROOF OF ROCHLIN'S SIGNATURE THEOREM AND ITS EXTENSION BY GUILLOU AND MARIN

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This paper stems from a lecture delivered in the I.A.S. geometric topology seminar (1976-77) and is expository in nature. In §§1-3 we will give an elementary proof of Rochlin's theorem on the signature of closed spin 4-manifolds. Our proof does not require any hard homotopy theory and is based on the Arf invariant of characteristic surfaces and the 4-dimensional cobordism group Ω_4. So it is similar, in spirit, to the recent geometric proof due to Freedman and Kirby [3], but we think, simpler than theirs. The subsequent sections extend these considerations to non-orientable characteristic surfaces of closed 4-manifolds and obtain Guillou and Marin's congruence (modulo 16) [4] which involves the signature of the 4-manifolds, the normal Euler number (= the self-intersection number) of the characteristic surfaces and the $\mathbf{Z}/8$ Arf invariant of E. H. Brown [1].

The author is deeply grateful to Professor L. C. Siebenmann for many valuable comments on the earlier drafts.

After finishing the second draft of this paper, the author was shown A. Casson's lecture notes [2] by C. Gordon. In the notes he gave a proof of Rochlin's theorem based on the Arf invariant of torus links.

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The details of his proof are considerably different from ours.

§1. The Arf invariant.

Let V be a finite dimensional vector space over $\mathbf{Z}/2$ which is given a non-singular symmetric bilinear form $(x, y) \mapsto x \cdot y \in \mathbf{Z}/2$. A function $q : V \to \mathbf{Z}/2$ is said to be quadratic (with respect to "$\cdot$") if it satisfies $q(x + y) = q(x) + q(y) + x \cdot y$ for all $x, y \in V$. Such a quadratic space $(V, \cdot, q)$ admits a symplectic basis $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r \in V$ satisfying $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$, $\alpha_i \cdot \beta_j = \delta_{ij}$ (Kronecker's delta). As is well known, the Arf invariant $\mathrm{Arf}(q) \in \mathbf{Z}/2$ is defined to be $\Sigma_{i=1}^{r} q(\alpha_i)q(\beta_i)$.

In this section we will recall the definition of the Arf invariant of a characteristic surface of an orientable 4-manifold. To deal with the Robertello-Arf invariant of knots simultaneously, we will not exclude 4-manifolds with non-empty boundary.

Let M^4 be a (compact) connected, smooth and orientable 4-manifold. Let F^2 be an orientable surface properly embedded in M^4. The boundary ∂F^2 is assumed to be $\cong \emptyset$ or $\cong S^1$. F^2 is called a characteristic surface if the homology class $[F^2, \partial F] \in H_2(M^4, \partial M; \mathbf{Z}/2)$ is dual to the 2-nd Stiefel-Whitney class $w_2(M)$. An equivalent condition is that the intersection number (mod 2) $F \cdot x$ is equal to the self-intersection number $x \cdot x$ for every $x \in H_2(M^4; \mathbf{Z}/2)$.

From now on we will assume that $H_1(M^4; \mathbf{Z}) = \{0\}$. Suppose that we are given a characteristic surface F^2, then we can define a quadratic function $q : H_1(F^2; \mathbf{Z}/2) \to \mathbf{Z}/2$ as follows [10], [3]:

Let C be a (generically) immersed circle in F^2. Since $H_1(M^4; \mathbf{Z}) = \{0\}$, C bounds a connected orientable surface D immersed in M^4. D may be assumed not to be tangent to F^2 at any point. The normal bundle ν_D of D is orientable and so trivial, because $D \simeq S^1 \vee \dots \vee S^1$. Note that any trivialization $\tau : \nu_D \cong D \times \mathbf{R}^2$ induces a unique trivialization $\nu_D|\partial D \cong \partial D \times \mathbf{R}^2$ on the boundary. In fact, the "difference" of any two trivializations τ_1, τ_2 of ν_D corresponds to a continuous map $g : D \longrightarrow SO(2)$. Since ∂D is homologous to zero in D, the induced map $g|\partial D : \partial D \longrightarrow SO(2)$ is homotopic to zero (because $SO(2) \simeq K(\mathbf{Z}, 1)$). Thus the induced trivializations coincide.

The normal line bundle ν_C of C in F^2 determines an orientable sub-line bundle in $\nu_D|\partial D$. Let $\mathcal{O}(D)$ be the number (modulo 2) of the full twists of ν_C in $\nu_D|\partial D$ with respect to the unique trivialization above. Let $D \cdot F$ be the number of the intersection points of $\mathrm{Int}(D)$ and F. Finally let $\mathrm{Self}(C)$ be the number of the self-intersection points of C on F^2.

Define $q(C)$ $(\in \mathbf{Z}/2)$ by

$$q(C) = \mathcal{O}(D) + D \cdot F + \mathrm{Self}(C) \pmod 2 .$$

<u>Lemma 1.1</u>. <u>The above definition gives a well defined function</u> $q : H_1(F^2; \mathbf{Z}/2) \longrightarrow \mathbf{Z}/2$ <u>which is quadratic with respect to the intersection pairing</u>: $H_1(F^2; \mathbf{Z}/2) \otimes H_1(F^2; \mathbf{Z}/2) \longrightarrow \mathbf{Z}/2$.

This lemma will be proved in §5 in a more general setting. (Cf. Lemma 5.1.)

Let $\mathrm{Arf}(F^2)$ be the Arf invariant of q.

<u>Lemma 1.2</u>. $\mathrm{Arf}(F^2)$ <u>depends only on the relative integral homology class</u> $[F^2, \partial F]$ <u>and the concordance class of the embedding</u> $\partial F^2 \longrightarrow \partial M^4$.

In particular, if $\partial M^4 = \emptyset$, $\mathrm{Arf}(F^2)$ is determined by the integral class $[F^2]$ and we can speak of the Arf invariant of an <u>integral characteristic homology class</u> ξ(i.e. the class whose mod 2 reduction is dual to $w_2(M)$). * We will denote it by $\mathrm{Arf}(\xi)$.

Another specific case is the case of knots in which $M^4 = D^4$ and $\partial F^2 = S^1$. In this case $[F^2, \partial F] = 0$, and $\mathrm{Arf}(F^2)$ is determined by the knot $K = \{\partial F \to \partial M\}$. This is nothing but the Robertello-Arf invariant of the knot K [8]. We will denote it by $\mathrm{Arf}(K)$.

<u>Proof of 1.2</u>. For simplicity assume that $\partial F = \emptyset$. (The other case will be dealt with similarly.) Let F_0 and F_1 be closed characteristic surfaces satisfying $[F_0] = [F_1] \in H_2(M^4;\mathbb{Z})$. Since $K(\mathbb{Z},2) = MSO(2)$, F_0 and F_1 are L-equivalent in Thom's sense. In other words, there is an orientable 3-manifold V^3 in $(\mathrm{Int}\, M^4) \times [0,1]$ such that $V^3 \cap M \times \{i\} = F_i$, $i = 0, 1$. Perturbing V^3 slightly, if necessary, we can assume that the projection $p : M^4 \times [0,1] \to [0,1]$ gives a Morse function $p' : V^3 \to [0,1]$ on V^3. Then F_1 is obtained from F_0 by successively attaching 0, 1, 2 and 3-handles within M^4.

It is easy to see that attaching 0-handles or 1-handles to different connected components does not change the Arf invariant. Consider the effect of attaching a 1-handle to the same component of F_0. The resulting quadratic function q' is the orthogonal sum of $q : H_1(F^2;\mathbb{Z}/2) \to \mathbb{Z}/2$ and $s : \mathbb{Z}/2(m) \oplus \mathbb{Z}/2(\ell) \to \mathbb{Z}/2$ with m and ℓ corresponding to the co-core and core of the attached 1-handle. Since $m.m = \ell.\ell = 0$, $\ell.m = 1$ and $s(m) = 0$, $\mathrm{Arf}(s)$ is zero. Thus

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* Any integral characteristic homology class is represented by a characteristic surface.

$\mathrm{Arf}(q') = \mathrm{Arf}(q)$.

Therefore attaching 0 and 1-handles does not change the Arf invariant. By duality attaching 2 and 3-handles does not change it either. □

§2. The proof of Rochlin's theorem.

We wish to prove Rochlin's theorem in the form he gave in [10]:

Theorem 2.1 (Rochlin). *Let M^4 be a closed oriented 4-manifold with $H_1(M^4; \mathbf{Z}) = \{0\}$, ξ an integral characteristic homology class. Then we have*

$$\mathrm{Arf}(\xi) = (\sigma(M^4) - \xi\cdot\xi)/8 \pmod 2 ,$$

where $\sigma(M^4)$ is the signature of M^4 and $\xi\cdot\xi$ is the self-intersection number.

Note that $\sigma(M^4) - \xi\cdot\xi$ is divisible by 8. (Cf. [7].)

If M^4 is spin (i.e., $w_2(M^4) = 0$), we can take 0 (zero) as a characteristic homology class. Then by the theorem, $\sigma(M)$ is divisible by 16. This is the classical Rochlin's theorem [9].

For the proof, we need two facts.

Facts: 1) *Let $K(p, q)$ be the classical torus knot of type (p, q) with p odd, q even. Then $\mathrm{Arf}(K(p, q)) = (1 - p^2)/8 \pmod 2$.*

2) *Let M^4 be a connected, 1-connected, closed and oriented 4-manifold. Then there exist integers ℓ, m, $n \geq 0$ such that $M^4 \# (\ell + 1)P \# \ell Q \cong mP \# nQ$, where P and Q are the complex projective plane and the one with the opposite orientation, respectively.*

Fact 1) is the "germ" of Rochlin's theorem. Using Fact 2), we can globalize it to obtain the Rochlin congruence.

These two facts are more or less standard. However, we will give some explanations in the next section.

Proof of Theorem 2.1. Let F^2 be a characteristic surface representing ξ. By doing framed surgery on $M^4 - F^2$, we can (and will) make M^4 connected and 1-connected without affecting the fact that F^2 is characteristic or altering the value of $\mathrm{Arf}(F^2)$.

The both sides of the formula we want to prove are additive with respect to the connected sum of the manifolds and the direct sum of the characteristic homology classes. Thus if the congruence holds for any two of three pairs (M_1, ξ_1), (M_2, ξ_2) and $(M_1 \# M_2, \xi_1 \oplus \xi_2)$ then it also holds for the remaining pair.

Let $\eta \in H_2(P; \mathbf{Z})$ and $\bar{\eta} \in H_2(Q; \mathbf{Z})$ be the generators represented by projective lines. They are characteristic, and the formula is easily checked for the pairs (P, η) and $(Q, \bar{\eta})$. Thus the above remark implies that we have only to prove the formula for the pair $(M^4 \# (\ell + 1)P \# \ell Q,$ $\xi \oplus \eta_1 \oplus \dots \oplus \eta_{\ell+1} \oplus \bar{\eta}_1 \oplus \dots \oplus \bar{\eta}_\ell)$ with some $\ell \geq 0$. But by Fact 2), $M^4 \# (\ell + 1)P \# \ell Q$ is diffeomorphic to a connected sum $mP \# nQ$ if ℓ is sufficiently large.

Note that every characteristic homology class of the connected sum $mP \# nQ$ is written as $s_1\eta_1 \oplus \dots \oplus s_m\eta_m \oplus t_1\bar{\eta}_1 \oplus \dots \oplus t_n\bar{\eta}_n$ with s_i, t_j odd. Then by the additivity of the formula again, the proof is reduced to that for pairs $(P, s\eta)$ and $(Q, t\bar{\eta})$ with odd s, t. As is easily seen, the orientations of the manifold M^4 and the characteristic surface are irrelevant to the proof. Therefore we have only to consider the case of $(P, s\eta)$ with odd $s > 0$.

To compute $\mathrm{Arf}(s\eta)$ we will take the algebraic curve C of degree $s > 0$: $C = \{(x : y : z); x^s + y^{s-1}z = 0\}$. C is homeomorphic to S^2 and is smoothly embedded in P except at the point $(0 : 0 : 1)$. At this point C has a cusp of type $x^s + y^{s-1} = 0$. Let B^4 be a small ball whose center is the singular point. Then $C \cap \partial B^4$ is a torus knot of type $(s, s-1)$. Taking a smooth surface G^2 in B^4 bounded by the knot, we get a smooth surface $F^2 = G^2 \cup (C - C \cap \mathrm{Int}\, B^4)$ representing $s\eta$. Now $\mathrm{Arf}(s\eta)$ is equal to $\mathrm{Arf}(F^2) = \mathrm{Arf}(G^2)$ $= \mathrm{Arf}(K(s, s-1))$. By Fact 1) this is equal to $(1 - s^2)/8 \pmod 2$ which is nothing but $(\sigma(P) - (s\eta)\cdot(s\eta))/8 \pmod 2$ as required. □

§3. <u>Explanation of Facts</u> 1) <u>and</u> 2).

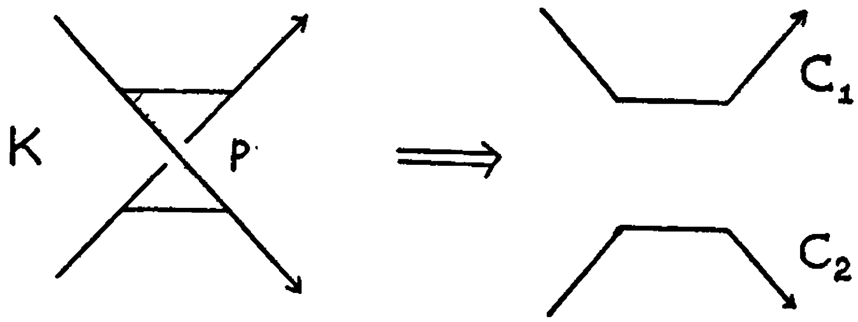
Fact 1): The following convenient way to get Fact 1) was suggested by Siebenmann. Recall that the Alexander polynomial of $K(p, q)$, $A(K(p, q); t)$, is equal to $(1 - t)(1 - t^{pq})/(1 - t^p)(1 - t^q)$. (Cf. [11, p. 178].) Then assuming p to be odd and q even we have $A(K(p, q); -1) = p$. Levine's result [6] says that $\mathrm{Arf}(K(p, q)) = 0$ or 1 according as $A(K(p, q); -1) \equiv \pm 1$ or $\pm 3 \pmod 8$. However, this is reformulated as $\mathrm{Arf}(K(p, q)) = (1 - p^2)/8 \pmod 2$.

There is an alternative and more elementary way. This method applies most neatly to the case of $K(s, s-1)$ $(s > 0)$ which is the only relevant case to our proof. We need a lemma.

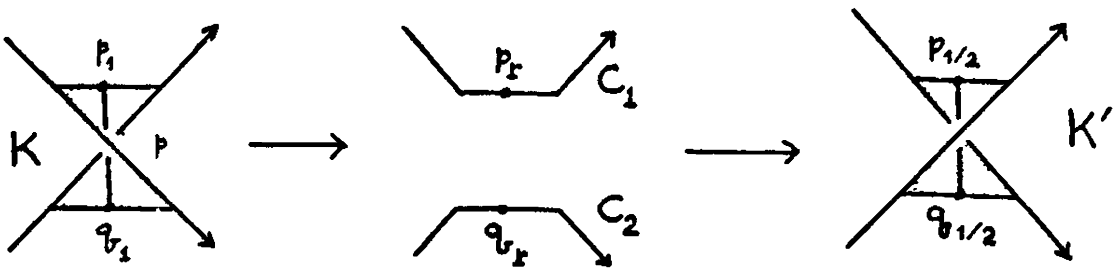
<u>Lemma 3.1. Let</u> p <u>be a double point of a regular projection of an oriented knot</u> K. <u>Let</u> C_1 <u>and</u> C_2 <u>be the two components of the link which is obtained by the surgery at</u> p <u>along the twisted rectangle</u>

shown in the figure below, $L(p;\,K)$ *the linking number of* C_1 *and* C_2. *If* K' *is the knot obtained from* K *by interchanging over and under crossing paths at* p, *then*

$$\mathrm{Arf}(K) = \mathrm{Arf}(K') + L(p;\,K) \pmod 2 \quad .$$



Proof. Let $B^4(r)$ be the standard 4-ball in $\mathbb{R}^4$ of radius $r \geq 0$. Let $S^3(r) = \partial B^4(r)$. Let us construct a surface G^2 in $B^4(1) - \mathrm{Int}\, B^4(1/2)$ by the following movie (plus smoothing):

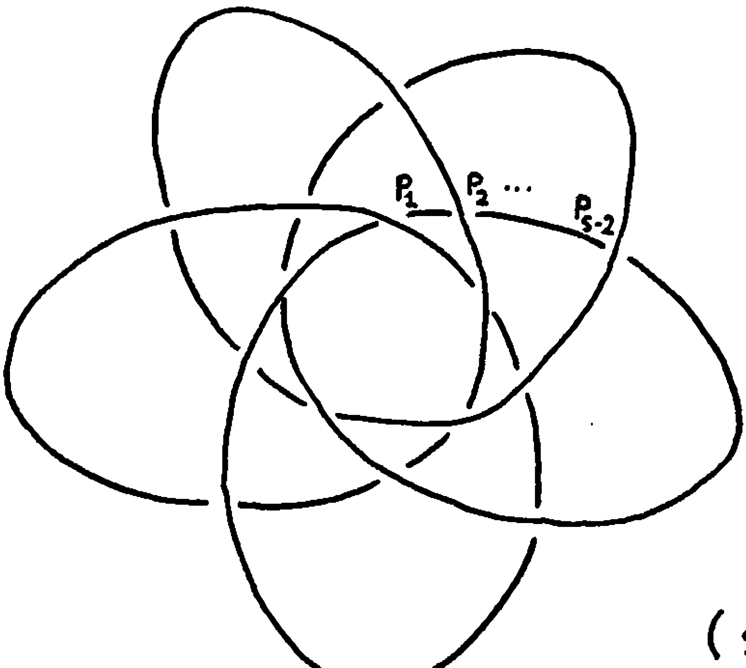


$(r = 1)$ $\qquad$ $(1 > r > 1/2)$ $\qquad$ $(r = 1/2)$

The boundary ∂G^2 consists of K in $S^3(1)$ and K' in $S^3(1/2)$. Let A and B be the two circles in G^2 defined by
$A = \overline{p_1q_1} \cup \{q_r;\ 1/2 \leq r \leq 1\} \cup \overline{q_{1/2}p_{1/2}} \cup \{p_r;\ 1/2 \leq r \leq 1\}$, $B = C_1$ (in $S^3(3/4)$, say). Take a surface F^2 in $B^4(1/2)$ such that $\partial F^2 = K'$. Then $G^2 \cup F^2$ is a surface in $B^4(1)$ bounded by K. Since

$H_1(G^2 \cup F^2; \mathbf{Z}/2) = H_1(F^2; \mathbf{Z}/2) \oplus \mathbf{Z}/2(A) \oplus \mathbf{Z}/2(B)$, the difference $\mathrm{Arf}(K) - \mathrm{Arf}(K')$ is equal to the Arf invariant of the quadratic function q on $\mathbf{Z}/2(A) \oplus \mathbf{Z}/2(B)$. By the figure above, q is computed as follows: $q(A) = \mathcal{O}(\Delta) = 1$ (Δ being a surface in $B^4(1) - \mathrm{Int}\, B^4(1/2)$ bounded by A), $q(B) = V \cdot G = \mathrm{Link}(C_1, C_2)$ (V being a Seifert surface of C_1 in $S^3(3/4)$), $A \cdot A = B \cdot B = 0$ and $A \cdot B = 1$. Thus $\mathrm{Arf}(q) = q(A) \cdot q(B) = \mathrm{Link}(C_1, C_2) \pmod 2$. □

To compute $\mathrm{Arf}(K(s, s-1))$ consider the regular projection of $K(s, s-1)$:



$(s = 5)$

Let $K^{(i)}$ be the knot obtained from $K(s, s-1)$ by interchanging over and under crossing paths at the points $p_1, p_2, \ldots, p_i$ in the projection. Then $K^{(s-2)} = K(s-1, s-2)$. This allows us an inductive calculation. Inspecting the figure above, we see that $L(p_i; K^{(i-1)}) = i\cdot(s-1-i)$, $1 \leq i \leq s-2$. Thus by applying Lemma 3.1 inductively,

$$\mathrm{Arf}(K(s, s-1)) = \mathrm{Arf}(K(s-1, s-2)) + \Sigma_{i=1}^{s-2} i\cdot(s-1-i) \quad .$$

An elementary consideration shows

$$\Sigma_{i=1}^{s-2} i\cdot(s-1-i) \equiv \begin{cases} 0 \pmod 2, & s-2 \not\equiv 1 \pmod 4 \quad , \\ 1 \pmod 2, & s-2 \equiv 1 \pmod 4 \quad . \end{cases}$$

Therefore, starting from $\mathrm{Arf}(K(1, 0)) = 0$ we have

$$\mathrm{Arf}(K(s, s-1)) = \begin{cases} 0, & s \equiv 0, 1, 2, 7 \pmod 8 \quad , \\ 1, & s \equiv 3, 4, 5, 6 \pmod 8 \quad . \end{cases}$$

Confining ourself to odd $s > 0$, we get the desired formula $\mathrm{Arf}(K(s, s-1)) \equiv (1 - s^2)/8 \pmod 2$.

Fact 2): This is a consequence of Wall's theorem [14]. However, since his theorem is far from being elementary we will give a direct proof described to the author by Siebenmann. Cf. [5].

Suppose that connected, 1-connected, closed and oriented 4-manifolds M^4 and N^4 are cobordant to each other. Let W^5 be an oriented cobordism between them. Decompose W^5 into a handle-body starting from $M \times [0, 1]$:

$$W^5 = M \times [0, 1] \cup \lambda_0 H^0 \cup \lambda_1 H^1 \cup \lambda_2 H^2 \cup \lambda_3 H^3 \cup \lambda_4 H^4 \cup \lambda_5 H^5 \quad .$$

(λ_i = the number of i-handles.) We can cancel 0 and 5-handles. A 1-handle is trivially attached to $M^4 \times [0, 1]$ and can be considered as a boundary connected sum of $M^4 \times [0, 1]$ and an embedded $S^1 \times D^4$. We do surgery on W^5 along the embedded $S^1 \times D^4$. Then $S^1 \times D^4$ is replaced by a 3-handle trivially attached to $M^4 \times [0, 1]$.
In the dual way we can surger out 4-handles and replace them by 2-handles. Now we can assume that there are only 2 and 3-handles.

Since M^4 and N^4 are 1-connected, by looking at the middle level we have

$$M^4 \# pS^2 \times S^2 \# qS^2 \underset{\sim}{\times} S^2 \cong N^4 \# rS^2 \times S^2 \# sS^2 \underset{\sim}{\times} S^2 \quad ,$$

where $S^2 \underset{\sim}{\times} S^2$ is the twisted S^2 bundle over S^2. The fact $S^2 \times S^2 \# P \cong S^2 \underset{\sim}{\times} S^2 \# P \cong 2P \# Q$ [13] implies the existence of $\ell, m \geq 0$ such that

$$M^4 \# (\ell + 1)P \# \ell Q \cong N \# (m + 1)P \# mQ \quad .$$

The signature $\sigma : \Omega_4 \longrightarrow \mathbf{Z}$ gives the isomorphism so every M^4 is cobordant to $\sigma(M^4)P$. Now Fact 2) follows from the above observation.

§4. Brown's invariant.

We will recall Brown's $\mathbf{Z}/8$ Arf invariant from [1]. Let V be a finite dimensional vector space over $\mathbf{Z}/2$ provided with a non-singular symmetric bilinear form $(x, y) \longmapsto x \cdot y \in \mathbf{Z}/2$. By a $\mathbf{Z}/4$-quadratic function is meant a function $\varphi : V \longrightarrow \mathbf{Z}/4$ satisfying $\varphi(x + y) = \varphi(x) + \varphi(y) + 2(x \cdot y)$ for all $x, y \in V$, where $2 : \mathbf{Z}/2 \longrightarrow \mathbf{Z}/4$ is the homomorphism sending $1 \longmapsto 2$. Such a triple $(V, \cdot, \varphi)$ is called a $\mathbf{Z}/4$-quadratic space.

The orthogonal sum $X \oplus Y$ of two $\mathbb{Z}/4$-quadratic spaces X, Y is defined as usual. A quadratic space is <u>indecomposable</u> if it is not isomorphic to the orthogonal sum of two non-trivial quadratic spaces. Then there are only four indecomposable spaces.

Namely,

$$P_+ = (\mathbb{Z}/2(a), \cdot, \varphi), \quad a \cdot a = 1, \quad \varphi(a) = 1 \ .$$

$$P_- = (\mathbb{Z}/2(a), \cdot, \varphi), \quad a \cdot a = 1, \quad \varphi(a) = -1 \ .$$

$$T_0 = (\mathbb{Z}/2(b) \oplus \mathbb{Z}/2(c), \cdot, \varphi), \quad b \cdot b = c \cdot c = 0, \\ b \cdot c = 1, \quad \varphi(b) = \varphi(c) = 0 \ .$$

$$T_4 = (\mathbb{Z}/2(b) \oplus \mathbb{Z}/2(c), \cdot, \varphi), \quad b \cdot b = c \cdot c = 0, \\ b \cdot c = 1, \quad \varphi(b) = \varphi(c) = 2 \ .$$

Following [7 , p. 112] we say that a $\mathbb{Z}/4$-quadratic space $X = (V, \cdot, \varphi)$ is <u>split</u> if V contains a subspace H with $\varphi(H) = \{0\}$, $H \cdot H = \{0\}$ and $\dim H = (1/2)\dim V$. For instance, T_0 and $P_+ \oplus P_-$ are split. Two $\mathbb{Z}/4$-quadratic spaces X and Y belong to the same <u>Witt class</u> if $X \oplus S_1 \cong Y \oplus S_2$ where the S_i are split. The Witt classes of $\mathbb{Z}/4$-quadratic spaces form the <u>Witt group</u> W. We denote the Witt class of X by $[X]$. Then $[T_0] = 0$ and $[P_+] + [P_-] = 0$ for instance.

Note the two relations: $P_+ \oplus T_4 \cong P_- \oplus P_- \oplus P_-$ and $P_- \oplus T_4 \cong P_+ \oplus P_+ \oplus P_+$. (Under each isomorphism the standard generators on the right are mapped to the elements $a + b$, $a + c$, $a + b + c$ of the space on the left.) These relations are written in terms of Witt classes as $[T_4] = 4[P_-]$ and $[T_4] = 4[P_+]$. Thus W is generated by $[P_+]$ and $8[P_+] = 2[T_4] = 4([P_+] + [P_-]) = 0$. In fact, W is shown to be isomorphic to $\mathbb{Z}/8$ by Brown's invariant as we see below.

Let X be a $\mathbb{Z}/4$-quadratic space $(V, \cdot, \varphi)$. E. H. Brown [1] considered the Monsky sum

$$\lambda(X) = \Sigma_{x \in V}\, i^{\varphi(x)}, \quad i = \sqrt{-1} \ .$$

The complex number $\lambda(X)$ takes the form $\sqrt{2}^{\dim V}(1 + i/\sqrt{2})^m$ $(m \in \mathbb{Z})$. For λ is multiplicative: $\lambda(X \oplus Y) = \lambda(X)\lambda(Y)$ and it takes the required form on each of the indecomposable spaces: $\lambda(P_+) = 1 + i$ $= \sqrt{2}(1 + i/\sqrt{2})$, etc.

Since the complex number $1 + i/\sqrt{2}$ is an 8-th root of unity, the integer m modulo 8 is well-defined. It is called <u>Brown's</u> <u>invariant</u> of X and is denoted by $\beta(X) \in \mathbb{Z}/8$.

<u>Lemma 4.1</u>. ([1 , Thm. 1.20, ix]) <u>If</u> X <u>is</u> <u>split,</u> <u>then</u> $\beta(X) = 0$.

By 4.1, β gives a homomorphism $W \to \mathbb{Z}/8$, and since $\beta(P_+) = 1$ it is an isomorphism.

<u>Remarks</u>. 1) ([1 , Thm. 1.20, vii]) Let $(V, \cdot, q)$ be a quadratic space in the sense of §1. Then $(V, \cdot, 2q)$ is a $\mathbb{Z}/4$-quadratic space and $\beta(V, \cdot, 2q) = 4\,\mathrm{Arf}(q)$, where $4 : \mathbb{Z}/2 \to \mathbb{Z}/8$ is the homomorphism sending $1 \mapsto 4$.

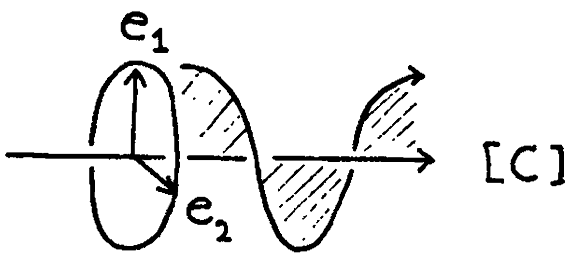
2) ([1 , Thm. 1.20, vi]) For every $\mathbb{Z}/4$-quadratic space $X = (V, \cdot, \varphi)$ we have $\beta(X) \equiv \dim V \pmod 2$.

§5. <u>Guillou</u> <u>and</u> <u>Marin's</u> <u>congruence</u>.

Guillou and Marin [4] extended Rochlin's congruence 2.1 by allowing non-orientable characteristic surfaces. In this section we will formulate their congruence. A proof will be given in §6.

Let M^4 be an oriented, connected and closed 4-manifold with $H_1(M^4; \mathbb{Z}) = \{0\}$. Let F^2 be a closed characteristic surface of M^4 (in the sense of §1) which is not necessarily orientable. Then we can define a $\mathbb{Z}/4$-quadratic function $\varphi : H_1(F^2; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/4$ as follows:

Let C be an immersed circle in F^2. Since $H_1(M^4; \mathbb{Z}) = \{0\}$, C bounds a connected orientable surface D immersed in M^4. We may assume that D is not tangent to F^2 at any point. As in §1, the normal bundle ν_D is trivial and on $\nu_D|C$ is induced a unique trivialization $\nu_D|C \cong C \times \mathbb{R}^2$. The normal bundle of C in F defines a sub-line bundle ν_C of $\nu_D|C$, and with the unique trivialization above we count the number $n(D)$ of <u>right-handed half twists</u> of ν_C. The picture illustrates the right-handed twists.



Here $[C]$ is a direction of C arbitrarily chosen, and $\{e_1, e_2\}$ is the basis of the fiber $(\cong \mathbb{R}^2)$ of $\nu_D|C$ which satisfies

$$(*) \qquad [r_D] \times [C] \times e_1 \times e_2 = [M^4] \quad ,$$

$[r_D]$ being the outward 'radial' direction of D.

Now the required form φ is defined by

$$\varphi(D) = n(D) + 2D \cdot F + 2 \operatorname{Self}(C) \pmod 4 \quad .$$

<u>Lemma 5.1</u>. $\varphi(D) \in \mathbb{Z}/4$ <u>depends only on the</u> $\mathbb{Z}/2$-<u>homology class of</u> $C = \partial D$. <u>The function</u> $\varphi : H_1(F^2; \mathbb{Z}/2) \to \mathbb{Z}/4$ <u>is</u> $\mathbb{Z}/4$-<u>quadratic</u>.

Let $\beta(F^2)$ be Brown's invariant of the $\mathbb{Z}/4$-quadratic space $(H_1(F^2; \mathbb{Z}/2), \cdot, \varphi)$.

<u>Theorem 5.2</u>. (Guillou and Marin) <u>With the notation above we have</u>

$$\sigma(M^4) \equiv F\cdot F + 2\beta(F^2) \pmod{16} \quad ,$$

<u>where</u> $F\cdot F$ <u>is the self-intersection number of</u> F (cf. [15]) <u>and</u> $2 : \mathbb{Z}/8 \to \mathbb{Z}/16$ <u>is the homomorphism sending</u> $1 \mapsto 2$.

<u>Corollary</u>. (Generalized Whitney's congruence [10]) <u>We have</u>

$$\sigma(M^4) \equiv F\cdot F + 2\chi(F^2) \pmod{4} \quad ,$$

<u>where</u> $\chi(F^2)$ <u>is the Euler characteristic of</u> F^2.

The corollary follows from 5.2 by Remark 2) of the previous section. Also by Remark 1) there we see that 5.2 reduces to 2.1 in the case when F^2 is orientable.

The rest of this section is devoted to the proof of Lemma 5.1.

<u>Proof of 5.1</u>. The proof is divided into four steps.

1) $\varphi(D)$ <u>depends only on the immersion</u> C.

Let us take another immersed surface D' with $\partial D' = C$ and show that $\varphi(D') = \varphi(D)$. By spinning D' around C (cf. [3 , Fig. 1]) if necessary, we may assume that the union $D \cup D'$ is a smoothly immersed closed surface Σ. (We are assuming that the outward 'radial' vectors of D and D' are exactly in opposite directions to each other at their boundary points.) D and D' determine the trivializations τ and τ' of $\nu_D|C = \nu_{D'}|C$, but they induce the opposite orientations on the fibers ($\cong \mathbb{R}^2$). Let $d(-\tau, \tau') \in \mathbb{Z} = \pi_1(SO(2))$ be the difference between $-\tau$ and τ'. Then

$$\text{(A)} \qquad \Sigma\cdot\Sigma = d(-\tau, \tau') - 2D\cdot D' \equiv d(-\tau, \tau') \pmod 2 \ .$$

Since F^2 is characteristic, $\Sigma\cdot\Sigma \equiv \Sigma\cdot F \pmod 2$, but $\Sigma\cdot F$ is equal to

$$\text{(B)} \qquad D\cdot F + D'\cdot F + w_1(\nu_C)[C] \pmod 2 \ .$$

The explanation of $w_1(\nu_C)[C]$ is this: If ν_C is orientable ($w_1(\nu_C)[C] = 0$) we can push Σ off from F^2 near C, but if ν_C is not orientable ($w_1(\nu_C)[C] = 1$) to put Σ in general position with respect to F^2 we necessarily create an odd number of intersection points of F^2 and Σ near C.

From (A) and (B) we have

$$\text{(C)} \qquad d(-\tau, \tau') \equiv D\cdot F + D'\cdot F + w_1(\nu_C)[C] \pmod 2 \ .$$

Finally

$$\text{(D)} \qquad n(D') \equiv n(D) + 2d(-\tau, \tau') + 2w_1(\nu_C)[C] \pmod 4 \ .$$

For the number (mod 4) of the right-handed half twists of ν_C in $\nu_D|C$ with respect to $-\tau$ (instead of τ) is equal to $-n(D)$. But $-n(D) \equiv n(D) + 2w_1(\nu_C)[C] \pmod 4$. (Proof: If $n(D) \equiv 1$ or 3 mod 4, ν_C is non-orientable so $w_1(\nu_C)[C] = 1$ and $n(D) + 2w_1(\nu_C)[C] \equiv 3$ or 1 mod 4. If $n(D) \equiv 0$ or 2 mod 4, ν_C

is orientable and $w_1(\nu_C)[C] = 0$.) Therefore $n(D') \equiv -n(D) + 2d(-\tau, \tau')$ $\equiv n(D) + 2w_1(\nu_C)[C] + 2d(-\tau, \tau')$ as required.

(C) and (D) imply that $n(D) + 2D\cdot F \equiv n(D') + 2D'\cdot F \pmod 4$. This is what we wanted to prove.

By 1) we can write $\varphi(C)$ in place of $\varphi(D)$.

2) $\varphi(C)$ depends only on the homotopy class of C.

φ is clearly regular homotopy invariant of C. If C' is homotopic to C, C' is regularly homotopic to a curve which is obtained from C by introducing a certain number of Whitney's double points (small figure eights), [12, § 7]. But for a small figure 8 on F^2 we can take a small disk D with ∂D = the figure 8 and $D\cdot F = 0$. It is seen that $n(D) = 2$. Obviously Self(a figure 8) = 1. Thus $\varphi(\text{a figure } 8) = 2 + 2\cdot 0 + 2\cdot 1 \equiv 0 \pmod 4$. This implies that $\varphi(C') = \varphi(C)$.

Now φ defines a map $\pi_1(F^2) \longrightarrow \mathbf{Z}/4$.

3) $\varphi : \pi_1(F^2) \longrightarrow \mathbf{Z}/4$ is $\mathbf{Z}/4$-quadratic.

In other words if $C \bullet C'$ denotes the composition of loops then $\varphi(C \bullet C') = \varphi(C) + \varphi(C') + 2(C\cdot C')$. The proof is straightforward from the definition of φ.

4) $\varphi : \pi_1(F^2) \longrightarrow \mathbf{Z}/4$ splits through $H_1(F^2; \mathbf{Z}/2)$.

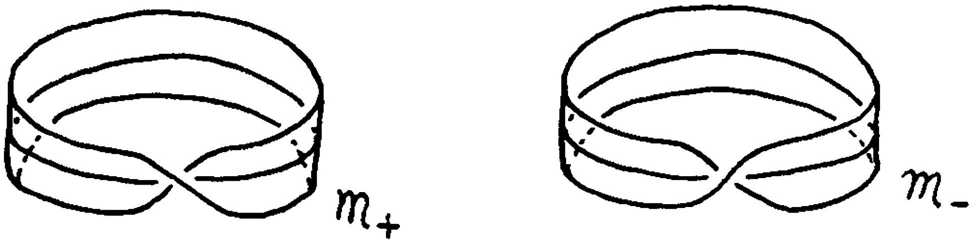
By 3) $\varphi(C \bullet C') = \varphi(C' \bullet C)$. Thus it splits through $H_1(F^2; \mathbf{Z})$. To prove 4) we have only to note that $\varphi(C + C) = 2\varphi(C) + 2(C\cdot C)$ $= 2(w_1(\nu_C)[C]) + 2(w_1(\nu_C)[C]) = 0 \pmod 4$.

This completes the proof of 5.1.

§6. Proof of Theorem 5.2.

We begin by checking the formula 5.2 on the key examples.

Let $\mathcal{M}_{\pm}$ be the two Möbius strips in $\mathbb{R}^3$.



Cap off the boundaries of $\mathcal{M}_{\pm}$ in a smooth way using disks $\Delta_{\pm}$ in $\mathbb{R}^4_- = \{(x, y, z, w);\ w \leq 0\}$. Then we obtain the two embeddings of $\mathbb{R}P^2 : \mathbb{R}P^2_{\pm} \subset \mathbb{R}^4$ $(= S^4 - \{\infty\})$.

Assertion. $\beta(\mathbb{R}P^2_{\pm}) = \pm 1$, $\mathbb{R}P^2_{\pm} \cdot \mathbb{R}P^2_{\pm} = \mp 2$ and $\sigma(S^4) = 0$. Thus Theorem 5.2 is true for the surfaces $\mathbb{R}P^2_{\pm} \subset S^4$.

Proof. Let D be a disk in $\mathbb{R}^4_+ = \{(x, y, z, w);\ w \geq 0\}$ which meets $\mathbb{R}^3 = \partial\mathbb{R}^4_+$ perpendicularly along $\partial D = C$, the central circle of $\mathcal{M}_+$ or $\mathcal{M}_-$. The restriction $\nu_D|C$ is identified with the normal bundle of C in $\mathbb{R}^3$, which has an untwisted framing. Let $\{e_1, e_2\}$ be such a normal framing that obeys the orientation rule (*) in §5: $[r_D] \times [C] \times e_1 \times e_2 = [\mathbb{R}^4]$ $(= [x] \times [y] \times [z] \times [w])$. In the present case, $[r_D] = -[w]$ so $[C] \times e_1 \times e_2$ coincides with the usual right-handed orientation $[x] \times [y] \times [z]$ of $\mathbb{R}^3$. Observing this, we can read the number $n(D)$ from the picture of $\mathcal{M}_{\pm}$: $n(D) = +1$ or -1 according as C belongs to $\mathcal{M}_+$ or $\mathcal{M}_-$. Since $D \cdot \mathbb{R}P^2_{\pm} = 0$ and $\mathrm{Self}(C) = 0$, we have $\varphi(C) = \pm 1$ thus $\beta(\mathbb{R}P^2_{\pm}) = \pm 1$ as asserted.

Next we will prove $\mathbb{R}P^2_{\pm} \cdot \mathbb{R}P^2_{\pm} = \mp 2$. For convenience, consider

$\mathbb{R}P^2_+$. Lift $\mathcal{M}_+$ to $\mathbb{R}^3 \times \{r\} = \{(x, y, z, w);\ w = r > 0\}$ and make it slightly wider to obtain $\mathcal{M}'$. Let $\partial\mathcal{M}' \times [0, r]$ be the "vertical annulus" in $\mathbb{R}^3 \times [0, r]$. Here $\partial\mathcal{M}' \times \{r\} = \partial\mathcal{M}'$, and $\partial\mathcal{M}' \times \{0\}$ $(\subset \mathbb{R}^3)$ is a closed curve parallel to the boundary $\partial\mathcal{M}_+$. Take a disk Δ' in $\mathbb{R}^4_-$ with $\partial\Delta' = \partial\mathcal{M}' \times \{0\}$. We assume that Δ' is isotopic in $\mathbb{R}^4_-$ to Δ_+ $(= \mathbb{R}P^2_+ \cap \mathbb{R}^4_-)$ and intersects it in general position. Let $\mathbb{R}P' = \mathcal{M}' \cup \partial\mathcal{M}' \times [0, r] \cup \Delta'$.

$\mathbb{R}P^2_+ \cdot \mathbb{R}P^2_+$ is equal to $\mathbb{R}P' \cdot \mathbb{R}P^2_+ = \Delta' \cdot \Delta_+$. If $\mathbb{R}^4_-$ were oriented so that the orientation $[\mathbb{R}^4_-]$ is consistent with [outward direction] $\times$ [the orientation of $\partial\mathbb{R}^4_-] = [w] \times [\mathbb{R}^3]$, then by the well-known relation the number $\Delta' \cdot \Delta_+$ would be equal to the linking number $\mathrm{Link}(\partial\Delta', \partial\Delta_+)$. But in the present case, $\mathbb{R}^4_-$ is oriented contrarily: $[\mathbb{R}^4_-]$ is induced from $[\mathbb{R}^4] = [\mathbb{R}^3] \times [w] = -[w] \times [\mathbb{R}^3]$. Thus $\Delta' \cdot \Delta_+ = -\mathrm{Link}(\partial\Delta', \partial\Delta_+) = -\mathrm{Link}(\partial\mathcal{M}' \times \{0\}, \partial\mathcal{M}_+) = -2$ as asserted.

The proof of $\mathbb{R}P^2_- \cdot \mathbb{R}P^2_- = 2$ is similar. $\square$

Now consider the general case. If $F^2 \subset M^4$ is orientable, Theorem 5.2 reduces to 2.1. Suppose F^2 is not orientable. Then F^2 is diffeomorphic to the connected sum $\mathbb{R}P^2 \# \ldots \# \mathbb{R}P^2$ (say, m-copies). Let C_1 be an embedded circle in the first copy which represents the generator of $H_1(\mathbb{R}P^2; \mathbb{Z}/2)$. As in §5, take a disk D_1 satisfying (i) $\partial D_1 = C_1$, (ii) D_1 is not tangent to F^2 and (iii) $\mathrm{Int}\, D_1 \cap C_1 = \emptyset$. Since the normal bundle of C_1 in F^2 is non-orientable, $\varphi(C_1) = n(D_1) + 2D_1 \cdot F$ is equal to $\pm 1 \pmod 4$. By spinning D_1 around C_1, we can accomplish the condition that (iv) $D_1 \cdot F^2 \equiv 0 \pmod 2$ and (v) $n(D_1) = \pm 1 \in \mathbb{Z}$ (without taking modulus 4).

For instance suppose $n(D_1) = 1$. There exists a diffeomorphism

$f : N_1 \to N$ of a tubular neighbourhood N_1 of C_1 in M^4 to a tubular neighbourhood N of C (the central circle of $\mathfrak{M}_+$) in S^4. We can assume that $f(N_1 \cap F^2) = N \cap \mathfrak{M}_+$ and $f(N_1 \cap D_1) = N \cap D$, where D is the disk in the proof of the assertion.

Construct a new 4-manifold $M' = (M - \mathrm{Int}\, N_1) \cup_f (S^4 - \mathrm{Int}\, N)$ and a new surface $F' = (F - \mathrm{Int}\, N_1 \cap F) \cup_f (\mathbb{R}P^2_+ - \mathrm{Int}\, N \cap \mathbb{R}P^2_+)$. Then $H_*(M'; \mathbb{Z}) = H_*(M \# S^2 \times S^2; \mathbb{Z})$. In particular,
$H_2(M'; \mathbb{Z}/2) = H_2(M; \mathbb{Z}/2) \oplus \mathbb{Z}/2(\alpha) \oplus \mathbb{Z}/2(\beta)$, where $\alpha = [D_1 \cup D]$ and $\beta = [$a fiber S of the 2-sphere bundle associated to $N]$.

F' is a characteristic surface of M'. In fact, since F is characteristic, $x \cdot F' = x \cdot F \equiv x \cdot x \pmod 2$ for every $x \in H_2(M; \mathbb{Z}/2)$. $\alpha \cdot F'$ is equal to $D_1 \cdot F^2 \equiv 0 \pmod 2$ (Condition (iv)). $\alpha \cdot \alpha$ is also $\equiv 0 \pmod 2$. Finally $\beta \cdot F' = S \cdot \mathfrak{M}_+$ but $S \cdot \mathfrak{M}_+ = \pm 2 \equiv 0 \pmod 2$. Clearly $\beta \cdot \beta \equiv 0 \pmod 2$. Thus $\beta \cdot F' \equiv \beta \cdot \beta$. Therefore, for every $y \in H_2(M'; \mathbb{Z}/2)$ we have $y \cdot F' \equiv y \cdot y \pmod 2$ as asserted.

Emphasizing the ambient 4-manifold, we will denote Brown's invariant by $\beta(F^2, M)$. Then $\beta(F', M') = \beta(F, M) - 1$. For we have surgered out the first copy of $\mathbb{R}P^2$ in $\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2 = F^2$ for which $\varphi(C_1) = 1$ as we assumed.

Give the orientation to M' which is consistent with $[M]$ and is opposite to $[S^4]$. Then $F' \cdot F' = F \cdot F - \mathbb{R}P_+ \cdot \mathbb{R}P_+ = F \cdot F + 2$ (cf. Assertion). Thus we have $F' \cdot F' + 2\beta(F', M') = F \cdot F + 2\beta(F, M)$. (The same equality is obtained also in the case $n(D_1) = -1$.)

Obviously $\sigma(M') = \sigma(M)$ and $F' = \mathbb{R}P^2 \# \dots \# \mathbb{R}P^2$ ($m - 1$ copies). Therefore, Guillou and Marin's congruence (5.2) is proved by induction on m.

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