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TRIANGULATION OF MANIFOLDS

VIA A STUDY OF THE RELATION BETWEEN POLYHEDRAL HOMOLOGY MANIFOLDS AND TOPOLOGICAL MANIFOLDS

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Is any (metrizable) topological manifold homeomorphic to some locally finite simplicial complex? This triangulation problem is still unsolved, although Kirby and Siebenmann established a fine obstruction theory for the existence of a *combinatorial* triangulation of manifolds of dimension $\geqq 5$. Recall that a locally finite simplicial complex M is called a polyhedral (integral) homology n-manifold if the local homology group $H_*(M, M - x; Z)$ is isomorphic to $H_*(R^n, R^n - 0; Z)$ for any point x of M.

Our study starts from the converse question: Does every polyhedral homology manifold admit an underlying structure of topological manifold? Of course, there are many polyhedral homology manifolds which are not topological manifolds; for example, the (single) suspension of a non-simply-connected combinatorial homology sphere. The answer is, however, "yes" in the following sense. From every polyhedral homology n-manifold M, we can construct, in a standard way, a topological $(n + 1)$-manifold N unique up to a notion of blocked homeomorphism, and a simple homotopy equivalence $g\colon N \to M \times S^1$. This construction is accomplished by translating the s-decomposition of $M \times S^1$ given by the products of dual cells of M with S^1 into the topological manifold category. The argument is analogous to one used by Sullivan, Cohen and Sato for PL resolution of polyhedral homology manifolds, but it needs no obstructions in this context. Moreover, if dim $M \neq 4$ and dim $\partial M \neq 4$, we obtain a topological n-manifold M_{TOP} unique up to homeomorphism by a standard splitting of the projection $N \simeq_s M \times S^1 \to S^1$. We can regard M_{TOP} with a preferred simple homotopy equivalence

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$f\colon M_{\mathrm{TOP}} \to M$ as an underlying topological manifold for M. As a consequence, one can introduce a standard map $\mu\colon BHML_r \to BTOP_r$, $r \geqq 3$, from the classifying space of normal homology cobordism bundles into that of topological (regular) neighborhoods of codimension $r \geqq 3$. When stabilized, we have also $\mu\colon BHML \to BTOP \simeq BTOP$, where $BTOP$ is the classifying space of stable topological microbundles.

We will use the surgery theory of Wall's type on compact polyhedral homology manifolds. Since this theory as given previously by the others, for example, by Matsui or Maunder, is not fully satisfactory for our purpose, we present a simple proof of its existence as a good application of M_{TOP}.

A surgery datum x consists of a Poincaré complex X of formal dimension n, a homology cobordism bundle ξ over X, a compact polyhedral homology n-manifold M and a degree one map $\phi\colon M \to X$ with a stable equivalence $F\colon \tau(M) \cong \phi^*\xi$ of the homology cobordism bundles. The obstruction $\theta(x)$ in the Wall group $L_n(\pi_1(X))$ is defined to be the obstruction $\theta(x_{\mathrm{TOP}})$ of the underlying topological datum x_{TOP}.

THEOREM. *Let $n \geqq 5$. The datum x is normally cobordant to a simple homotopy equivalence if and only if $\theta(x) = 0$.*

With the help of the stability theorem proved by Matumoto-Matsumoto for $\mu_*\colon HML/HML_r \xrightarrow{\cong} TOP/TOP_r$ $(r \geqq 3)$, the following embedding lemma reduces all the steps of the homology manifold surgery to that of topological manifold surgery which is established by Wall and Kirby-Siebenmann.

LEMMA. *Let $\gamma \in \pi_{r+1}(\phi)$ and $r \leqq n-2$. Assume that an embedding $j\colon S^r \times D^{n-r} \to M_{\mathrm{TOP}}$ is in the regular homotopy class for doing the surgery on γ_{TOP}. Then, there exists a homology manifold s-cobordism $(W; M, \bar{M})$ with $\bar{M} \supset S^r \times D^{n-r}$ so that the natural inclusion $S^r \times D^{n-r} \to \bar{M}_{\mathrm{TOP}} = M_{\mathrm{TOP}}$ is regular homotopic to j.*

In fact, if $r \leqq 2$, then γ is representable by a PL embedding in the PL submanifold which is the neighborhood of the dual 3-skeleton. If $r \geqq 3$, we consider the homotopy inverse of f, $\psi\colon M \to M_{\mathrm{TOP}}$. By the Williamson's simplicial transversality theorem, ψ may be supposed to be transverse regular to the axis of $S^r \times D^{n-r}$ and we put $K = \psi^{-1}(S^r \times 0)$. Then, the tangent homology cobordism bundle of K is stably trivial so that K is s-cobordant to a PL manifold. By modifying M by an s-cobordism we may assume that K is a PL submanifold. Moreover, we can observe that $\psi|K\colon K \to S^r \times 0$ completes to a PL surgery datum whose obstruction vanishes and $\psi|K$ is normally cobordant to the identity even when $r \leqq 4$. If the cobordism contains only handles of index $\leqq 2$, each of these handles can be realized in the zero-codimensional PL submanifold in M as before. If not, $r \geqq 3$ and we identify M_{TOP} with $(K \times D^{n-r}) \cup (M - K \times \operatorname{int} D^{n-r})_{\mathrm{TOP}}$ and embed each normally framed handle $D^s \times D^{r-s}$ in M_{TOP}; we get the PL surgery datum for $(K_1, \partial D^s) \to (D^s, \partial D^s)$ relative to the boundary with $s + 1 \leqq r$. The situation is the same as before when we wanted to do ambient surgery on K in M and hence by an induction on the dimension of the submanifold K in the ambient surgery data we can achieve the ambient surgery on K in M. This completes the proof of the lemma and consequently the surgery theorem.

In this setting, we can classify the polyhedral homology manifold structures on a given topological manifold: Let $\mathscr{H}^3$ be the smooth H_*-cobordism classes of oriented smooth homology 3-spheres and let $\alpha: \mathscr{H}^3 \to Z_2$ be the Milnor-Kervaire-Rohlin homomorphism.

THEOREM. *A topological manifold V of dimension $\geqq 5$ is homeomorphic to M_{TOP} for some polyhedral homology manifold M if and only if $\beta k(V) \in H^5(V; \ker \alpha)$ vanishes, where $k(V)$ is the Kirby-Siebenmann obstruction class to admit a combinatorial triangulation and β is the Bockstein operation associated to the exact sequence* $0 \to \ker \alpha \to \mathscr{H}^3 \to^{\alpha} Z_2 \to 0$.

For the proof, let X be the Poincaré complex with a simple homotopy equivalence $\nu: V \to X$. We consider the following homotopy-commutative diagram each sequence of which is a homotopy fibration:

$$\begin{array}{ccccc} BPL & \longrightarrow & BHML & \longrightarrow & B(HML/PL) \\ \| & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \alpha_*} \\ BPL & \longrightarrow & BTOP & \xrightarrow{k} & B(TOP/PL) \\ \downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \beta_*} \\ * & \longrightarrow & B(TOP/HML) & = & B(TOP/HML) \end{array}$$

Since HML/PL and TOP/PL are the Eilenberg-Mac Lane spaces $K(\mathscr{H}^3, 3)$ and $K(Z_2, 3)$ respectively and the canonical map: $HML/PL \to TOP/PL$ is identified with the map induced by α, β_* in the diagram corresponds to the Bockstein operation associated to α. So, we understand that the obstruction for the stable bundle reduction through μ is $\nu(\tau(V)) = \beta k(\tau(V))$.

Using the transversality theorem due to Williamson, we can prove that the normal cobordism classes of homology manifold surgery data correspond to the concordance classes $\mathscr{T}_{HML}(X)$ of the homology tangential structures on X. So, we have the following commutative diagram:

$$\begin{array}{ccccccc} L_{n+1}(\pi_1(X)) & \longrightarrow & \mathscr{S}_{HML}(X) & \xrightarrow{\eta} & \mathscr{T}_{HML}(X) & \xrightarrow{\theta} & L_n(\pi_1(X)) \\ \| & & \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} & & \| \\ L_{n+1}(\pi_1(X)) & \longrightarrow & \mathscr{S}_{TOP}(X) & \xrightarrow{\eta} & \mathscr{T}_{TOP}(X) & \xrightarrow{\theta} & L_n(\pi_1(X)) \end{array}$$

The horizontal Sullivan-Wall sequences are exact in the sense that $L_{n+1}(\pi_1(X))$ operates on $\mathscr{S}_{CAT}(X)$ and $\ker \theta = \text{image } \eta \cong \mathscr{S}_{CAT}(X)/L_{n+1}(\pi_1(X))$ where $CAT = HML$ or TOP. Now, we do diagram chasing, assuming $\beta k(V) = 0$. Let $a \in \mathscr{T}_{TOP}(X)$ be the class of $\nu: V \to X$. Then, by the observation above, $\eta(a)$ has a lifting b such that $\mu_* b = \eta(a)$. There exist $c \in \mathscr{S}_{HML}(X)$ and $\lambda \in L_{n+1}(\pi_1(X))$ so that $\eta(c) = b$ and $\lambda \cdot \mu_* c = a$. Hence, $\mu_*(\lambda \cdot c) = a$. Thus, corresponding to $\lambda \cdot c$ there exists a polyhedral homology manifold M and a simple homotopy equivalence $m: M \to X$ such that the induced $m_{\mathrm{TOP}}: M_{\mathrm{TOP}} \to X$ is equivalent to $\nu: V \to X$; in particular, V is homeomorphic to M_{TOP}. End of proof!

Now we remark that the standard construction $M \mapsto M_{\mathrm{TOP}}$ gives $M = M_{\mathrm{TOP}}$ if the polyhedral homology manifold M is itself a topological manifold. And, if all the double suspensions of the polyhedral homology manifolds, that are homology spheres, are homeomorphic to the standard sphere, then every polyhedral homology manifold of dimension $\geqq 5$ is locally homeomorphic to euclidean space except at the vertexes with non-simply-connected links. Because R. D. Edwards gave an affirmative answer to the double suspension problem for combinatorial homology spheres of dimension $\geqq 4$, we can reduce the question of simplicial triangulation of topological manifolds to (1) the structure of ker α and (2) the multiple suspension problem for smooth homology 3-spheres.

In fact, we can state as follows (in a simplified form).

THEOREM. *Let $n \geqq 6$. Then, all the topological n-manifolds without boundary are simplicially triangulable if and only if there exists a smooth homology 3-sphere H^3 which satisfies the following three conditions:*

(i) *H^3 bounds a parallelizable smooth 4-manifold with signature* 8,

(ii) *the $(n-3)$-ple suspension of H^3 is homeomorphic to the n-sphere, and*

(iii) *the oriented connected sum $H^3 \# H^3$ bounds an acyclic smooth 4-manifold.*

We can remark that the existence of H^3 with (i) and (ii) is enough to triangulate V when $\beta_0 k(V) = 0$, where β_0: $H^4(V; Z_2) \to H^5(V; Z)$ is the integral Bockstein operation. In the case $n = 5$, we need possibly some modification of the condition (iii). And the case $n = 4$ is still mysterious. Recently D. Galevski and R. Stern obtained a similar result by different methods, which will be described in these PROCEEDINGS.

This note summerizes the results obtained in my thesis, which is supervised by Professor L. C. Siebenmann to whom I would like to express my deepest thanks.

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