



Noncommutative localization in algebraic L -theory

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Abstract

Given a noncommutative (Cohn) localization $A \rightarrow \sigma^{-1}A$ which is injective and stably flat we obtain a lifting theorem for induced f.g. projective $\sigma^{-1}A$ -module chain complexes and localization exact sequences in algebraic L -theory, matching the algebraic K -theory localization exact sequence of Neeman–Ranicki [Amnon Neeman, Andrew Ranicki, Noncommutative localisation in algebraic K -theory I, *Geom. Topol.* 8 (2004) 1385–1425] and Neeman [Amnon Neeman, Noncommutative localisation in algebraic K -theory II, *Adv. Math.* 213 (2007) 785–819].

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0. Introduction

The series of papers [2,3], studied the algebraic K -theory of the noncommutative (Cohn) localization $\sigma^{-1}A$ of a ring A inverting a collection σ of morphisms of f.g. projective left A -modules. By definition, $\sigma^{-1}A$ is stably flat if

$$\text{Tor}_i^A(\sigma^{-1}A, \sigma^{-1}A) = 0 \quad (i \geq 1).$$

An (A, σ) -module is an A -module T which admits an f.g. projective A -module resolution

$$0 \longrightarrow P \xrightarrow{s} Q \longrightarrow T \longrightarrow 0$$

with $s : \sigma^{-1}P \longrightarrow \sigma^{-1}Q$ an isomorphism of the induced $\sigma^{-1}A$ -modules. For $A \longrightarrow \sigma^{-1}A$ which is injective and stably flat we obtained an algebraic K -theory localization exact sequence

$$\cdots \longrightarrow K_n(A) \longrightarrow K_n(\sigma^{-1}A) \longrightarrow K_{n-1}(H(A, \sigma)) \longrightarrow K_{n-1}(A) \longrightarrow \cdots$$

with $H(A, \sigma)$ the exact category of (A, σ) -modules.

Let C be a bounded $\sigma^{-1}A$ -module chain complex such that each $C_i = \sigma^{-1}P_i$ is induced from an f.g. projective A -module P_i . The *chain complex lifting problem* is to decide if C is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A -modules. The problem has a trivial affirmative solution for a commutative or Ore localization, by the clearing of denominators, when C is actually isomorphic to $\sigma^{-1}D$. In general, it is not possible to lift chain complexes: the injective noncommutative localizations $A \longrightarrow \sigma^{-1}A$ which are not stably flat constructed in Neeman, Ranicki and Schofield [4, Remark 2.13] provide examples of induced f.g. projective $\sigma^{-1}A$ -module chain complexes of dimensions ≥ 3 which cannot be lifted.

In Section 1 we solve the chain complex lifting problem in the injective stably flat case, obtaining the following results (Theorems 1.4, 1.5):

Theorem 0.1. *For a stably flat injective noncommutative localization $A \longrightarrow \sigma^{-1}A$ every bounded chain complex C of induced f.g. projective $\sigma^{-1}A$ -modules is chain equivalent to $\sigma^{-1}D$ for a bounded chain complex D of f.g. projective A -modules. Moreover, if C is n -dimensional*

$$C : \cdots \longrightarrow 0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0 \longrightarrow \cdots$$

then D can be chosen to be n -dimensional.

In Section 2 we consider the algebraic L -theory of a noncommutative localization, obtaining the following results (Theorems 2.4, 2.5, 2.9):

Theorem 0.2. *Let $A \longrightarrow \sigma^{-1}A$ be a noncommutative localization of a ring with involution A , such that σ is invariant under the involution.*

(i) *There is a localization exact sequence of quadratic L -groups*

$$\cdots \longrightarrow L_n(A) \longrightarrow L_n^I(\sigma^{-1}A) \xrightarrow{\partial} L_n(A, \sigma) \longrightarrow L_{n-1}(A) \longrightarrow \cdots$$

with $I = \text{im}(K_0(A) \rightarrow K_0(\sigma^{-1}A))$, and $L_n(A, \sigma)$ the cobordism group of $\sigma^{-1}A$ -contractible $(n - 1)$ -dimensional quadratic Poincaré complexes over A .

(ii) If $\sigma^{-1}A$ is stably flat over A there is a localization exact sequence of symmetric L -groups

$$\dots \longrightarrow L^n(A) \longrightarrow L^n_I(\sigma^{-1}A) \xrightarrow{\partial} L^n(A, \sigma) \longrightarrow L^{n-1}(A) \longrightarrow \dots$$

with $L^n(A, \sigma)$ the cobordism group of $\sigma^{-1}A$ -contractible $(n - 1)$ -dimensional symmetric Poincaré complexes over A .

(iii) If $A \rightarrow \sigma^{-1}A$ is injective then $L^n(A, \sigma)$ (resp. $L_n(A, \sigma)$) is the cobordism group of n -dimensional symmetric (resp. quadratic) Poincaré complexes of (A, σ) -modules.

The L -theory exact sequences of Theorem 0.2 for an injective Ore localization $A \rightarrow \sigma^{-1}A$ (which is flat and hence stably flat) were obtained in Ranicki [5]. The quadratic L -theory exact sequence of 0.2(i) for arbitrary injective $A \rightarrow \sigma^{-1}A$ was obtained by Vogel [8,9]. The symmetric L -theory exact sequence of 0.2(ii) is new.

We refer to [6,7] for some of the applications of the algebraic L -theory of noncommutative localizations to topology.

Amnon Neeman used to be a coauthor of the paper, but decided to withdraw in May 2007.

1. Lifting chain complexes

If $A \rightarrow \sigma^{-1}A$ is a stably flat localization, we know from [3, Theorem 0.4, Proposition 4.5 and Theorem 3.7] that the functor $Ti : \frac{D^{\text{perf}}(A)}{\mathcal{R}c} \rightarrow D^{\text{perf}}(\sigma^{-1}A)$ is just an idempotent completion; it is fully faithful and all objects in $D^{\text{perf}}(\sigma^{-1}A)$ are, up to isomorphisms, direct summands of objects in the image of Ti . A fairly easy consequence of this is the following. Let $C \in D^{\text{perf}}(\sigma^{-1}A)$ be the complex

$$0 \rightarrow \sigma^{-1}C^m \rightarrow \sigma^{-1}C^{m+1} \rightarrow \dots \rightarrow \sigma^{-1}C^{n-1} \rightarrow \sigma^{-1}C^n \rightarrow 0,$$

with C^i all finitely generated, projective A -modules. Then there is complex $X \in D^{\text{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. That is, C is homotopy equivalent to the tensor product with $\sigma^{-1}A$ of a perfect complex over the ring A . In Section 1 we prove this (Theorem 1.4), and then refine the result to show that X may be chosen to be a complex of the form

$$0 \rightarrow X^m \rightarrow X^{m+1} \rightarrow \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0.$$

(Proof in Theorem 1.5.)

Remark 1.1. The proof of Theorem 1.4 relies on the following fact about triangulated categories. Suppose \mathcal{A} is a full, triangulated subcategory of a triangulated category \mathcal{B} , and suppose all objects in \mathcal{B} are direct summands of objects of \mathcal{A} . An object $X \in \mathcal{B}$ belongs to $\mathcal{A} \subset \mathcal{B}$ if and only if $[X] \in K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$. This fact may be found, for example, in [1, Proposition 4.5.11], but for the reader’s convenience its proof is included here in Lemma 1.2 and Proposition 1.3.

We begin by reminding the reader of some basic facts about Grothendieck groups. For any additive category \mathcal{A} we define $K_0^{\text{add}}(\mathcal{A})$ to be the Grothendieck group of the split exact category \mathcal{A} . This means that the short exact sequences in \mathcal{A} are precisely the split sequences. It is well known that every element of $K_0^{\text{add}}(\mathcal{A})$ can be expressed as

$$[X] - [Y]$$

for X and Y objects of \mathcal{A} . The expressions $[X] - [Y]$ and $[X'] - [Y']$ are equal in $K_0^{\text{add}}(\mathcal{A})$ if and only if there exists an object $P \in \mathcal{A}$ and an isomorphism

$$X \oplus Y' \oplus P = X' \oplus Y \oplus P.$$

If \mathcal{A} happens to be a triangulated category, then $K_0(\mathcal{A})$ means the quotient of $K_0^{\text{add}}(\mathcal{A})$ by a certain subgroup we will denote $T(\mathcal{A})$. The subgroup $T(\mathcal{A})$ is defined as the group generated by all

$$[X] - [Y] + [Z],$$

where there exists a distinguished triangle in \mathcal{A}

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

We prove:

Lemma 1.2. *Suppose \mathcal{B} is a triangulated category. Let \mathcal{A} be a full, triangulated subcategory of \mathcal{B} . Assume further that every object of \mathcal{B} is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

Then the map $f : K_0^{\text{add}}(\mathcal{A}) \longrightarrow K_0^{\text{add}}(\mathcal{B})$ induces a surjection $T(\mathcal{A}) \longrightarrow T(\mathcal{B})$. In symbols: $f(T(\mathcal{A})) = T(\mathcal{B})$.

Proof. Let $[X] - [Y] + [Z]$ be a generator of $T(\mathcal{B}) \subset K_0^{\text{add}}(\mathcal{B})$. We need to show it lies in the image of $T(\mathcal{A}) \subset K_0^{\text{add}}(\mathcal{A})$. Suppose therefore that

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

is a distinguished triangle in \mathcal{B} . Because every object of \mathcal{B} is a direct summand of an object in \mathcal{A} , we can choose objects C and D with

$$X \oplus C, \quad Z \oplus D$$

both lying in \mathcal{A} . But then we have two distinguished triangles in \mathcal{B}

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X,$$

$$C \longrightarrow C \oplus D \longrightarrow D \xrightarrow{0} \Sigma C$$

and their direct sum is a distinguished triangle

$$X \oplus C \longrightarrow Y \oplus C \oplus D \longrightarrow Z \oplus D \longrightarrow \Sigma(X \oplus C).$$

Two of the objects lie in \mathcal{A} . Since the subcategory $\mathcal{A} \subset \mathcal{B}$ is full and triangulated, the entire distinguished triangle lies in \mathcal{A} . Thus

$$[X \oplus C] - [Y \oplus C \oplus D] + [Z \oplus D] = [X] - [Y] + [Z]$$

lies in the image of $T(\mathcal{A})$. \square

The next proposition is well known; again, the proof is included for the convenience of the reader.

Proposition 1.3. *Let the hypotheses be as in Lemma 1.2. That is, suppose \mathcal{B} is a triangulated category. Let \mathcal{A} be a full, triangulated subcategory of \mathcal{B} . Assume further that every object of \mathcal{B} is a direct summand of an object in $\mathcal{A} \subset \mathcal{B}$.*

If X is an object of \mathcal{B} and $[X]$ lies in the image of the natural map $f : K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$, then $X \in \mathcal{A}$.

Proof. If we consider $[X]$ as an element of $K_0^{\text{add}}(\mathcal{B})$, then saying that its image in $K_0(\mathcal{B})$ lies in the image of $K_0(\mathcal{A}) \rightarrow K_0(\mathcal{B})$ is equivalent to saying that, modulo $T(\mathcal{B})$, $[X]$ lies in the image of $K_0^{\text{add}}(\mathcal{A})$. That is,

$$[X] \in T(\mathcal{B}) + f(K_0^{\text{add}}(\mathcal{A})) \subset K_0^{\text{add}}(\mathcal{B}).$$

By Lemma 1.2 we have that $f(T(\mathcal{A})) = T(\mathcal{B})$. Thus

$$\begin{aligned} T(\mathcal{B}) + f(K_0^{\text{add}}(\mathcal{A})) &= f(T(\mathcal{A})) + f(K_0^{\text{add}}(\mathcal{A})) \\ &= f(K_0^{\text{add}}(\mathcal{A})). \end{aligned}$$

That means there exist objects C and D in $\mathcal{A} \subset \mathcal{B}$ and an identity in $K_0^{\text{add}}(\mathcal{B})$

$$[X] = [C] - [D].$$

There must therefore be an object $P \in \mathcal{B}$ and an isomorphism

$$X \oplus D \oplus P \simeq C \oplus P.$$

But P is an object of \mathcal{B} , hence a direct summand of an object of \mathcal{A} . There is an object $P' \in \mathcal{B}$ with $P \oplus P' \in \mathcal{A}$. We have an isomorphism

$$X \oplus D \oplus P \oplus P' \simeq C \oplus P \oplus P'.$$

Putting $D' = D \oplus P \oplus P'$ and $C' = C \oplus P \oplus P'$ we have objects C', D' in \mathcal{A} , and a (split) distinguished triangle

$$D' \longrightarrow C' \longrightarrow X \longrightarrow \Sigma D'.$$

Since $\mathcal{A} \subset \mathcal{B}$ is triangulated we conclude that $X \in \mathcal{A}$. \square

The relevance of these results to our work here is

Theorem 1.4. *Let $A \rightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose we are given a perfect complex C over $\sigma^{-1}A$. Suppose further that $C \in D^{\text{perf}}(\sigma^{-1}A)$ is of the form*

$$0 \rightarrow \sigma^{-1}C^m \rightarrow \sigma^{-1}C^{m+1} \rightarrow \dots \rightarrow \sigma^{-1}C^{n-1} \rightarrow \sigma^{-1}C^n \rightarrow 0$$

where each C^i is a finitely generated, projective A -module. Then C is homotopy equivalent to $\{\sigma^{-1}A\}^L \otimes_A X$, for some $X \in D^{\text{perf}}(A)$.

Proof. The localization is stably flat. By [3, Theorem 0.4] the functor $T : \mathcal{T}^c \rightarrow D^{\text{perf}}(\sigma^{-1}A)$ is an equivalence of categories. By [3, Proposition 4.5 and Theorem 3.7] we also know that the functor $i : \frac{D^{\text{perf}}(A)}{\mathcal{R}^c} \rightarrow \mathcal{T}^c$ is fully faithful, and that every object in \mathcal{T}^c is isomorphic to a direct summand of an object in the image of i . Next we apply Proposition 1.3, with $\mathcal{B} = D^{\text{perf}}(\sigma^{-1}A)$ and \mathcal{A} the full subcategory containing all objects isomorphic to $Ti(x)$, for any $x \in \frac{D^{\text{perf}}(A)}{\mathcal{R}^c}$.

Now C is an object of $D^{\text{perf}}(\sigma^{-1}A)$, and in $K_0(D^{\text{perf}}(\sigma^{-1}A))$ we have an identity

$$[C] = \sum_{\ell=-\infty}^{\infty} (-1)^\ell [\sigma^{-1}C^\ell]$$

with

$$[\sigma^{-1}C^\ell] = [\{\sigma^{-1}A\} \otimes_A C^\ell] = [TiC^\ell]$$

certainly lying in the image of the map

$$K_0(Ti) : K_0\left(\frac{D^{\text{perf}}(A)}{\mathcal{R}^c}\right) \rightarrow K_0(D^{\text{perf}}(\sigma^{-1}A)).$$

Proposition 1.3 therefore tells us that C is isomorphic to an object in the image of the functor Ti . There exists a perfect complex $X \in D^{\text{perf}}(A)$ and a homotopy equivalence $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$. \square

The problem with Theorem 1.4 is that it gives us no bound on the length of the complex X with $\{\sigma^{-1}A\}^L \otimes_A X \simeq C$. We really want to know

Theorem 1.5. *Let $A \rightarrow \sigma^{-1}A$ be a stably flat localization of rings. Suppose $C \in D^{\text{perf}}(\sigma^{-1}A)$ is the complex*

$$0 \rightarrow \sigma^{-1}C^m \rightarrow \sigma^{-1}C^{m+1} \rightarrow \dots \rightarrow \sigma^{-1}C^{n-1} \rightarrow \sigma^{-1}C^n \rightarrow 0.$$

Then the complex $X \in D^{\text{perf}}(A)$ with $C \simeq \{\sigma^{-1}A\}^L \otimes_A X$, whose existence is guaranteed by Theorem 1.4, may be chosen to be a complex

$$0 \rightarrow X^m \rightarrow X^{m+1} \rightarrow \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow 0.$$

If $m = n$ this is easy. For $m < n$ we need to prove something. Our proof will appeal to the results of [3, Section 4]. We remind the reader that this was the section which dealt with the

subcategories $\mathcal{K}[m, n]$ of complexes in \mathcal{R}^c vanishing outside the range $[m, n]$. First we need a lemma.

Lemma 1.6. *Let M and N be any finitely generated projective A -modules. We may view M and N as objects in the derived category $D^{\text{perf}}(A)$, concentrated in degree 0. Then any map in $\mathcal{T}^c(\pi M, \pi N)$ can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\text{perf}}(A)$ as below*

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N.$$

The map $\alpha : N \rightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \longrightarrow \Sigma X$$

and X may be chosen to lie in $\mathcal{K}[0, 1]$.

Proof. By [3, Proposition 4.5 and Theorem 3.7] we know that the map

$$i : \frac{D^{\text{perf}}(A)}{\mathcal{R}^c} \rightarrow \mathcal{T}^c$$

is fully faithful. Therefore

$$\mathcal{T}^c(\pi M, \pi N) = \frac{D^{\text{perf}}(A)}{\mathcal{R}^c}(M, N).$$

That is, any map $\pi M \rightarrow \pi N$ can be written as $\pi(\alpha)^{-1}\pi(\beta)$, for some α, β morphisms in $D^{\text{perf}}(A)$ as below

$$M \xrightarrow{\beta} Y \xleftarrow{\alpha} N.$$

The map $\alpha : N \rightarrow Y$ fits in a triangle

$$X \longrightarrow N \xrightarrow{\alpha} Y \xrightarrow{\beta} \Sigma X$$

and X may be chosen to lie in \mathcal{R}^c . What is not clear is that we may choose X in $\mathcal{K}[0, 1] \subset \mathcal{R}^c$.

The easy observation is that we may certainly modify our choice of X to lie in $\mathcal{K} \subset \mathcal{R}^c$. This follows from [2, Lemma 4.5], which tells us that for any choice of X as above there exists an X' with $X \oplus X'$ isomorphic to an object in \mathcal{K} . We have a distinguished triangle

$$X \oplus X' \longrightarrow N \xrightarrow{\binom{\alpha}{0}} Y \oplus \Sigma X' \xrightarrow{\beta \oplus 1} \Sigma(X \oplus X')$$

and a diagram

$$M \xrightarrow{\binom{\beta}{0}} Y \oplus \Sigma X' \xleftarrow{\binom{\alpha}{0}} N,$$

and replacing our original choices by these we may assume $X \in \mathcal{K}$. Now we have to shorten X . By [2, Lemma 4.7], there exists a triangle in \mathcal{K}^c

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[1, \infty)$ and $X'' \in \mathcal{K}(-\infty, 1]$. The composite $X' \longrightarrow X \longrightarrow N$ is a map from $X' \in \mathcal{K}[1, \infty)$ to $N \in \mathcal{S}^{\leq 0}$, which must vanish. Hence we have that $X \longrightarrow N$ factors as $X \longrightarrow X'' \longrightarrow N$. We complete to a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'' & \longrightarrow & N & \xrightarrow{\gamma\alpha} & Y'' & \longrightarrow & \Sigma X'' \end{array}$$

and another representative of our morphism is the diagram

$$M \xrightarrow{\gamma\beta} Y'' \xleftarrow{\gamma\alpha} N.$$

We may, on replacing Y by Y'' , assume $X \in \mathcal{K}(-\infty, 1]$.

Applying [2, Lemma 4.7] again, we have that any $X \in \mathcal{K}(-\infty, 1]$ admits a triangle

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \Sigma X'$$

with $X' \in \mathcal{K}[0, 1]$ and $X'' \in \mathcal{K}(-\infty, 0]$. Form the octahedron

$$\begin{array}{ccccccc} X' & \longrightarrow & N & \xrightarrow{\alpha'} & Y' & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & N & \xrightarrow{\alpha} & Y & \longrightarrow & \Sigma X \\ & & & & \downarrow & & \downarrow \\ & & & & \Sigma X'' & \xrightarrow{1} & \Sigma X'' \end{array}$$

The composite $M \longrightarrow Y \longrightarrow \Sigma X''$ is a map from the projective module M , viewed as a complex concentrated in degree 0, to $\Sigma X'' \in \mathcal{K}(\infty, -1]$. This composite must vanish. The map $\beta : M \longrightarrow Y$ therefore factors as $M \xrightarrow{\beta'} Y' \xrightarrow{\gamma} Y$, and our morphism in \mathcal{T}^c has a representative

$$M \xrightarrow{\beta'} Y' \xleftarrow{\alpha'} N$$

so that in the triangle

$$X' \longrightarrow N \xrightarrow{\alpha'} Y' \longrightarrow \Sigma X'$$

X' may be chosen to lie in $\mathcal{K}[0, 1]$. \square

Now we are ready for

Proof of Theorem 1.5. We are given a complex $C \in D^{\text{perf}}(\sigma^{-1}A)$ of the form

$$0 \longrightarrow \sigma^{-1}C^m \longrightarrow \sigma^{-1}C^{m+1} \longrightarrow \dots \longrightarrow \sigma^{-1}C^{n-1} \longrightarrow \sigma^{-1}C^n \longrightarrow 0.$$

To eliminate the trivial case, assume $m \leq n + 1$. Shifting, we may assume $m = 0$ and $n \geq 1$. Theorem 1.4 guarantees that C is homotopy equivalent to $\{\sigma^{-1}A\}^L \otimes_A D$, with $D \in D^{\text{perf}}(A)$. But D need not be supported on the interval $[0, n]$. We need to show how to shorten D . Assume therefore that D is supported on $[-1, n]$. We will show how to replace D by a complex supported on $[0, n]$. Shortening a complex supported on $[0, n + 1]$ is dual, and we leave it to the reader.

We may suppose therefore that $D \in D^{\text{perf}}(A)$ is the complex

$$\dots \longrightarrow 0 \longrightarrow D^{-1} \longrightarrow D^0 \longrightarrow \dots \longrightarrow D^n \longrightarrow 0 \longrightarrow \dots$$

and that there is a homotopy equivalence of $\sigma^{-1}D$ with a shorter complex, that is a commutative diagram

$$\begin{array}{ccccccccccc} \longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \sigma^{-1}C^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}C^n & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & \sigma^{-1}D^{-1} & \xrightarrow{\partial} & \sigma^{-1}D^0 & \longrightarrow & \dots & \longrightarrow & \sigma^{-1}D^n & \longrightarrow & 0 & \longrightarrow \end{array}$$

so that the composite is homotopic to the identity. In particular, there is a map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$ so that $d\partial : \sigma^{-1}D^{-1} \rightarrow \sigma^{-1}D^{-1}$ is the identity.

By [2, Proposition 3.1] the map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$ lifts uniquely to a map $d' : \pi D^0 \rightarrow \pi D^{-1}$. By Lemma 1.6 the map d' can be represented as $\pi(\alpha)^{-1}\pi(\beta)$, where α and β are, respectively, the chain maps

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^{-1} & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

and

$$\begin{array}{ccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 & \longrightarrow & 0 & \longrightarrow \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & X & \xrightarrow{r} & Y & \longrightarrow & 0 & \longrightarrow \end{array}$$

The fact that $\sigma^{-1}\alpha$ is an equivalence tells us that the map $\sigma^{-1}r : \sigma^{-1}X \rightarrow \sigma^{-1}Y$ is injective, with cokernel $\sigma^{-1}D^{-1}$. The fact that $\alpha^{-1}\beta$ agrees with d' means that the composite

$$\sigma^{-1}D^0 \xrightarrow{\sigma^{-1}g} \sigma^{-1}Y \twoheadrightarrow \text{Coker}(\sigma^{-1}r)$$

is just the map $d : \sigma^{-1}D^0 \rightarrow \sigma^{-1}D^{-1}$. Let X be the chain complex

$$\longrightarrow 0 \longrightarrow D^0 \oplus X \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} D^1 \oplus Y \longrightarrow \cdots \longrightarrow D^n \longrightarrow 0 \longrightarrow \cdot$$

Let $f : X \rightarrow D$ be the natural map of chain complexes

$$\begin{array}{ccccccccccccccc} \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & D^0 \oplus X & \xrightarrow{\begin{pmatrix} \partial & 0 \\ g & r \end{pmatrix}} & D^1 \oplus Y & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow & 0 & \longrightarrow \\ & & & \downarrow & & \pi_1 \downarrow & & \downarrow \pi_1 & & & & & & \downarrow & \\ \longrightarrow & 0 & \longrightarrow & D^{-1} & \longrightarrow & D^0 & \xrightarrow{\partial} & D^1 & \longrightarrow & \cdots & \longrightarrow & D^n & \longrightarrow & 0 & \longrightarrow \end{array}$$

where the vertical maps labelled π_1 are the projections to the first factor of the direct sum. The map $\sigma^{-1}f$ is easily seen to be homotopy equivalence. Thus $\sigma^{-1}X$ is homotopy equivalent to $\sigma^{-1}D \cong C$. \square

2. Algebraic L-theory

An *involution* on a ring A is an anti-automorphism

$$A \longrightarrow A; \quad r \longmapsto \bar{r}.$$

The involution is used to regard a left A -module M as a right A -module by

$$M \times A \longrightarrow M; \quad (x, r) \longmapsto \bar{r}x.$$

The *dual* of a (left) A -module M is the A -module

$$M^* = \text{Hom}_A(M, A), \quad A \times M^* \longrightarrow M^*; \quad (r, f) \longmapsto (x \longmapsto f(x)\bar{r}).$$

The *dual* of an A -module morphism $s : P \rightarrow Q$ is the A -module morphism

$$s^* : Q^* \longrightarrow P^*; \quad f \longmapsto (x \longmapsto f(s(x))).$$

If M is f.g. projective then so is M^* , and

$$M \longrightarrow M^{**}; \quad x \longmapsto (f \longmapsto \overline{f(x)})$$

is an isomorphism which is used to identify $M^{**} = M$.

Hypothesis 2.1. In this section, we assume that

- (i) A is a ring with involution,
- (ii) the duals of morphisms $s : P \rightarrow Q$ in σ are morphisms $s^* : Q^* \rightarrow P^*$ in σ ,
- (iii) $\epsilon \in A$ is a central unit such that $\bar{\epsilon} = \epsilon^{-1}$ (e.g. $\epsilon = \pm 1$).

The noncommutative localization $\sigma^{-1}A$ is then also a ring with involution, with $\epsilon \in \sigma^{-1}A$ a central unit such that $\bar{\epsilon} = \epsilon^{-1}$.

We review briefly the chain complex construction of the f.g. projective ϵ -quadratic L -groups $L_*(A, \epsilon)$ and the ϵ -symmetric L -groups $L^*(A, \epsilon)$. Given an A -module chain complex C let the generator $T \in \mathbb{Z}_2$ act on the \mathbb{Z} -module chain complex $C \otimes_A C$ by the ϵ -transposition duality

$$T_\epsilon : C_p \otimes_A C_q \rightarrow C_q \otimes_A C_p : x \otimes y \mapsto (-1)^{pq} \epsilon y \otimes x.$$

Let W be the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z}

$$W : \dots \rightarrow \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2].$$

The ϵ -symmetric (resp. ϵ -quadratic) Q -groups of C are the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of $C \otimes_A C$

$$Q^n(C, \epsilon) = H^n(\mathbb{Z}_2; C \otimes_A C) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_A C)),$$

$$Q_n(C, \epsilon) = H_n(\mathbb{Z}_2; C \otimes_A C) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_A C)).$$

The Q -groups are chain homotopy invariants of C . There are defined forgetful maps

$$1 + T_\epsilon : Q_n(C, \epsilon) \rightarrow Q^n(C, \epsilon); \quad \psi \mapsto (1 + T_\epsilon)\psi,$$

$$Q^n(C, \epsilon) \rightarrow H_n(C \otimes_A C); \quad \phi \mapsto \phi_0.$$

For f.g. projective C the function

$$C \otimes_A C \rightarrow \text{Hom}_A(C^*, C); \quad x \otimes y \mapsto (f \mapsto \overline{f(x)}y)$$

is an isomorphism of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes, with $T \in \mathbb{Z}_2$ acting on $\text{Hom}_A(C^*, C)$ by $\theta \mapsto \epsilon\theta^*$. The element $\phi_0 \in H_n(C \otimes_A C) = H_n(\text{Hom}_A(C^*, C))$ is a chain homotopy class of A -module chain maps $\phi_0 : C^{n-*} \rightarrow C$.

An n -dimensional ϵ -symmetric complex over A (C, ϕ) is a bounded f.g. projective A -module chain complex C together with an element $\phi \in Q^n(C, \epsilon)$. The complex (C, ϕ) is Poincaré if the A -module chain map $\phi_0 : C^{n-*} \rightarrow C$ is a chain equivalence.

Example 2.2. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over A is essentially the same as a nonsingular ϵ -symmetric form (M, λ) over (A, σ) , with $M = (C_0)^*$ an f.g. projective A -module and

$$\lambda = \phi_0 : M \times M \rightarrow A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^*; \quad x \longmapsto (y \longmapsto \lambda(x, y))$$

is an A -module isomorphism.

See pp. 210–211 of [6] for the notion of an ϵ -symmetric (Poincaré) pair. The boundary of an n -dimensional ϵ -symmetric complex (C, ϕ) is the $(n - 1)$ -dimensional ϵ -symmetric Poincaré complex

$$\partial(C, \phi) = (\partial C, \partial\phi)$$

with $\partial C = C(\phi_0 : C^{n-*} \longrightarrow C)_{*+1}$ and $\partial\phi$ as defined on p. 218 of [6]. The n -dimensional ϵ -symmetric L -group $L^n(A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over A with C n -dimensional. In particular, $L^0(A, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric forms over A .

An n -dimensional ϵ -symmetric complex (C, ϕ) over A is $\sigma^{-1}A$ -Poincaré if the $\sigma^{-1}A$ -module chain map $\sigma^{-1}\phi_0 : \sigma^{-1}C^{n-*} \longrightarrow \sigma^{-1}C$ is a chain equivalence, in which case $\sigma^{-1}(C, \phi)$ is an n -dimensional ϵ -symmetric Poincaré complex over $\sigma^{-1}A$.

The n -dimensional ϵ -symmetric Γ -group $\Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}A$ -Poincaré complexes (C, ϕ) over A such that $\sigma^{-1}C$ is chain equivalent to an n -dimensional induced f.g. projective $\sigma^{-1}A$ -module chain complex. The n -dimensional ϵ -symmetric L -group $L^n(A, \sigma, \epsilon)$ is the cobordism group of $(n - 1)$ -dimensional ϵ -symmetric Poincaré complexes over A (C, ϕ) such that C is $\sigma^{-1}A$ -contractible, i.e. $\sigma^{-1}C \simeq 0$.

Similarly in the ϵ -quadratic case, with groups $L_n(A, \epsilon)$, $\Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon)$, $L_n(A, \sigma, \epsilon)$. The ϵ -quadratic L - and Γ -groups are 4-periodic

$$\begin{aligned} L_n(A, \epsilon) &= L_{n+2}(A, -\epsilon) = L_{n+4}(A, \epsilon), \\ \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) &= \Gamma_{n+2}(A \longrightarrow \sigma^{-1}A, -\epsilon) = \Gamma_{n+4}(A \longrightarrow \sigma^{-1}A, \epsilon), \\ L_n(A, \sigma, \epsilon) &= L_{n+2}(A, \sigma, -\epsilon) = L_{n+4}(A, \sigma, \epsilon). \end{aligned}$$

Proposition 2.3. *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ there is defined a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots$$

Similarly in the ϵ -quadratic case, with an exact sequence

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. The relative group of $L^n(A, \epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric $\sigma^{-1}A$ -Poincaré pairs over A $(f : C \longrightarrow D, (\delta\phi, \phi))$ with (C, ϕ) Poincaré. The effect of algebraic surgery on (C, ϕ) using this pair is a cobordant $(n - 1)$ -dimensional ϵ -symmetric Poincaré complex (C', ϕ') with $C' \sigma^{-1}A$ -contractible. The function

$(f : C \rightarrow D, (\delta\phi, \phi)) \mapsto (C', \phi')$ defines an isomorphism between the relative group and $L^n(A, \sigma, \epsilon)$. \square

Define

$$I = \text{im}(K_0(A) \rightarrow K_0(\sigma^{-1}A)),$$

the subgroup of $K_0(\sigma^{-1}A)$ consisting of the projective classes of the f.g. projective $\sigma^{-1}A$ -modules induced from f.g. projective A -modules. By definition, $L^n_I(\sigma^{-1}A, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes over $\sigma^{-1}A(B, \theta)$ such that $[B] \in I$. There are evident morphisms of Γ - and L -groups

$$\begin{aligned} \sigma^{-1}\Gamma^* : \Gamma^n(A \rightarrow \sigma^{-1}A, \epsilon) &\rightarrow L^n_I(\sigma^{-1}A, \epsilon); & (C, \phi) &\mapsto \sigma^{-1}(C, \phi), \\ \sigma^{-1}\Gamma_* : \Gamma_n(A \rightarrow \sigma^{-1}A, \epsilon) &\rightarrow L^n_I(\sigma^{-1}A, \epsilon); & (C, \psi) &\mapsto \sigma^{-1}(C, \psi). \end{aligned}$$

In general, the morphisms $\sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ need not be isomorphisms, since a bounded f.g. projective $\sigma^{-1}A$ -module chain complex D with $[D] \in I$ need not be chain equivalent to $\sigma^{-1}C$ for a bounded f.g. projective A -module chain complex C .

It was proved in Chapter 3 of Ranicki [5] that if $A \rightarrow \sigma^{-1}A$ is an injective Ore localization then the morphisms $\sigma^{-1}Q^*, \sigma^{-1}Q_*, \sigma^{-1}\Gamma^*, \sigma^{-1}\Gamma_*$ are isomorphisms, so that there are defined localization exact sequences for both the ϵ -symmetric and the ϵ -quadratic L -groups

$$\begin{aligned} \cdots \longrightarrow L^n(A, \epsilon) &\longrightarrow L^n_I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots, \\ \cdots \longrightarrow L_n(A, \epsilon) &\longrightarrow L^n_I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots. \end{aligned}$$

Special cases of these sequences were obtained by Milnor and Husemoller, Karoubi, Pardon, Smith, Carlsson and Milgram.

Let $G\pi : D(A) \rightarrow D(A)$ be the functor of Proposition 6.1 of [3], with $D(A)$ the derived category of A . For any bounded f.g. projective A -module chain complex C the natural A -module chain map

$$\varinjlim_{(B, \beta)} B = G\pi(C) \rightarrow \sigma^{-1}C$$

induces morphisms

$$\begin{aligned} \sigma^{-1}Q^* : \varinjlim_{(B, \beta)} Q^n(B, \epsilon) &= Q^n(G\pi(C), \epsilon) \rightarrow Q^n(\sigma^{-1}C, \epsilon), \\ \sigma^{-1}Q_* : \varinjlim_{(B, \beta)} Q_n(B, \epsilon) &= Q_n(G\pi(C), \epsilon) \rightarrow Q_n(\sigma^{-1}C, \epsilon) \end{aligned}$$

with the direct limits taken over all the bounded f.g. projective A -module chain complexes B with a chain map $\beta : C \rightarrow B$ such that $\sigma^{-1}\beta : \sigma^{-1}C \rightarrow \sigma^{-1}B$ is a $\sigma^{-1}A$ -module chain

equivalence. The natural projection $D \otimes_A D \rightarrow D \otimes_{\sigma^{-1}A} D$ is an isomorphism for any bounded f.g. projective $\sigma^{-1}A$ -module chain complex D (since this is already the case for $D = \sigma^{-1}A$), so the Q -groups of $\sigma^{-1}C$ are the same whether $\sigma^{-1}C$ is regarded as an A -module or $\sigma^{-1}A$ -module chain complex.

Theorem 2.4. (See Vogel [9, Theorem 8.4].) *For any ring with involution A and noncommutative localization $\sigma^{-1}A$ the morphisms*

$$\sigma^{-1}\Gamma_* : \Gamma_n(A \rightarrow \sigma^{-1}A, \epsilon) \rightarrow L_n^I(\sigma^{-1}A, \epsilon); \quad (C, \psi) \mapsto \sigma^{-1}(C, \psi)$$

are isomorphisms, and there is a localization exact sequence of ϵ -quadratic L -groups

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. By algebraic surgery below the middle dimension it suffices to consider only the special cases $n = 0, 1$. In effect, it was proved in [9] that $\sigma^{-1}Q_*$ is an isomorphism for 0- and 1-dimensional C . \square

It was claimed in Proposition 25.4 of Ranicki [6] that $\sigma^{-1}\Gamma^*$ is also an isomorphism, assuming (incorrectly) that the chain complex lifting problem can always be solved. However, we do have:

Theorem 2.5. *If $\sigma^{-1}A$ is a noncommutative localization of a ring with involution A which is stably flat over A , there is a localization exact sequence of ϵ -symmetric L -groups*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow L_n^I(\sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots$$

Proof. For any bounded f.g. projective A -module chain complex C the natural A -module chain map $G\pi(C) \rightarrow \sigma^{-1}C$ induces isomorphisms in homology

$$H_*(G\pi(C)) \cong H_*(\sigma^{-1}C).$$

Thus the natural $\mathbb{Z}[\mathbb{Z}_2]$ -module chain map

$$G\pi(C) \otimes_A G\pi(C) \rightarrow \sigma^{-1}C \otimes_A \sigma^{-1}C = \sigma^{-1}C \otimes_{\sigma^{-1}A} \sigma^{-1}C$$

induces isomorphisms of ϵ -symmetric Q -groups

$$\sigma^{-1}Q^* : \varinjlim_{(B, \beta)} Q^n(B, \epsilon) \rightarrow Q^n(\sigma^{-1}C, \epsilon)$$

(and also isomorphisms $\sigma^{-1}Q_*$ of ϵ -quadratic Q -groups). By Theorem 0.1 every n -dimensional induced f.g. projective $\sigma^{-1}A$ -module chain complex D is chain equivalent to $\sigma^{-1}C$ for an n -dimensional f.g. projective A -module chain complex C , with

$$Q^n(D, \epsilon) = Q^n(\sigma^{-1}C, \epsilon) = \varinjlim_{(B, \beta)} Q^n(B, \epsilon).$$

It follows that the morphisms of ϵ -symmetric Γ - and L -groups

$$\sigma^{-1}\Gamma^* : \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \longrightarrow L^n_I(\sigma^{-1}A, \epsilon); \quad (C, \phi) \longmapsto \sigma^{-1}(C, \phi)$$

are also isomorphisms, and the localization exact sequence is given by Proposition 2.3. \square

Hypothesis 2.6. For the remainder of this section, we assume Hypothesis 2.1 and also that $A \longrightarrow \sigma^{-1}A$ is an injection.

As in Proposition 2.2 of [2] it follows that all the morphisms in σ are injections.

We shall now generalize the results of Ranicki [5] and Vogel [8] to prove that under Hypotheses 2.1, 2.6 the relative L -groups $L^*(A, \sigma, \epsilon)$, $L_*(A, \sigma, \epsilon)$ in the L -theory localization exact sequences are the L -groups of $H(A, \sigma)$ with respect to the following duality involution.

Define the *torsion dual* of an (A, σ) -module M to be the (A, σ) -module

$$M^\wedge = \text{Ext}_A^1(M, A),$$

using the involution on A to define the left A -module structure. If M has f.g. projective A -module resolution

$$0 \longrightarrow P_1 \xrightarrow{s} P_0 \longrightarrow M \longrightarrow 0$$

with $s \in \sigma$ the torsion dual M^\wedge has the dual f.g. projective A -module resolution

$$0 \longrightarrow P_0^* \xrightarrow{s^*} P_1^* \longrightarrow M^\wedge \longrightarrow 0$$

with $s^* \in \sigma$.

Proposition 2.7. Let $M = \text{coker}(s : P_1 \longrightarrow P_0)$, $N = \text{coker}(t : Q_1 \longrightarrow Q_0)$ be (A, σ) -modules.

(i) *The adjoint of the pairing*

$$M \times M^\wedge \longrightarrow \sigma^{-1}A/A; \quad (g \in P_0, f \in P_1^*) \longmapsto fs^{-1}g$$

defines a natural A -module isomorphism

$$M^\wedge \longrightarrow \text{Hom}_A(M, \sigma^{-1}A/A); \quad f \longmapsto (g \longmapsto fs^{-1}g).$$

(ii) *The natural A -module morphism*

$$M \longrightarrow M^\wedge; \quad x \longmapsto (f \longmapsto \overline{f(x)})$$

is an isomorphism.

(iii) *There are natural identifications*

$$\begin{aligned} M \otimes_A N &= \text{Tor}_0^A(M, N) = \text{Ext}_A^1(M^\wedge, N) = H_0(P \otimes_A Q), \\ \text{Hom}_A(M^\wedge, N) &= \text{Tor}_1^A(M, N) = \text{Ext}_A^0(M^\wedge, N) = H_1(P \otimes_A Q). \end{aligned}$$

The functions

$$M \otimes_A N \longrightarrow N \otimes_A M; \quad x \otimes y \longmapsto y \otimes x,$$

$$\text{Hom}_A(M^\wedge, N) \longrightarrow \text{Hom}_A(N^\wedge, M); \quad f \longmapsto f^\wedge$$

determine transposition isomorphisms

$$T : \text{Tor}_i^A(M, N) \longrightarrow \text{Tor}_i^A(N, M) \quad (i = 0, 1).$$

(iv) For any finite subset $V = \{v_1, v_2, \dots, v_k\} \subset M \otimes_A N$ there exists an exact sequence of (A, σ) -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \bigoplus_k M^\wedge \longrightarrow 0$$

such that $V \subset \ker(M \otimes_A N \longrightarrow M \otimes_A L)$.

Proof. (i) Apply the snake lemma to the morphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_A(P_0, A) & \longrightarrow & \text{Hom}_A(P_0, \sigma^{-1}A) & \longrightarrow & \text{Hom}_A(P_0, \sigma^{-1}A/A) \longrightarrow 0 \\ & & \downarrow s^* & & \downarrow s_1^* & & \downarrow s_2^* \\ 0 & \longrightarrow & \text{Hom}_A(P_1, A) & \longrightarrow & \text{Hom}_A(P_1, \sigma^{-1}A) & \longrightarrow & \text{Hom}_A(P_1, \sigma^{-1}A/A) \longrightarrow 0 \end{array}$$

with s^* injective, s_1^* an isomorphism and s_2^* surjective, to verify that the A -module morphism

$$M^\wedge = \text{coker}(s^*) \longrightarrow \text{Hom}_A(M, \sigma^{-1}A/A) = \ker(s_2^*)$$

is an isomorphism.

(ii) Immediate from the identification

$$s^{**} = s : (P_0)^{**} = P_0 \longrightarrow (P_1)^{**} = P_1.$$

(iii) Exercise for the reader.

(iv) Lift each $v_i \in M \otimes_A N$ to an element

$$v_i \in P_0 \otimes_A Q_0 = \text{Hom}_A(P_0^*, Q_0) \quad (1 \leq i \leq k).$$

The A -module morphism defined by

$$u = \begin{pmatrix} s^* & 0 & 0 & \dots & 0 \\ 0 & s^* & 0 & \dots & 0 \\ 0 & 0 & s^* & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & v_3 & \dots & t \end{pmatrix} : U_1 = \left(\bigoplus_k P_0^* \right) \oplus Q_1 \longrightarrow U_0 = \left(\bigoplus_k P_1^* \right) \oplus Q_0$$

is in σ , so that $L = \text{coker}(u)$ is an (A, σ) -module with an f.g. projective A -module resolution

$$0 \longrightarrow U_1 \xrightarrow{u} U_0 \longrightarrow L \longrightarrow 0.$$

The short exact sequence of 1-dimensional f.g. projective A -module chain complexes

$$0 \longrightarrow Q \longrightarrow U \longrightarrow \bigoplus_k P^{1-*} \longrightarrow 0$$

is a resolution of a short exact sequence of (A, σ) -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow \bigoplus_k M^\wedge \longrightarrow 0.$$

The first morphism in the exact sequence

$$\text{Tor}_1^A\left(M, \bigoplus_k M^\wedge\right) \longrightarrow M \otimes_A N \longrightarrow M \otimes_A L \longrightarrow M \otimes_A \left(\bigoplus_k M^\wedge\right) \longrightarrow 0$$

sends $1_i \in \text{Tor}_1^A(M, \bigoplus_k M^\wedge) = \bigoplus_k \text{Hom}_A(M^\wedge, M^\wedge)$ to $v_i \in \ker(M \otimes_A N \longrightarrow M \otimes_A L)$. \square

Given an (A, σ) -module chain complex C define the ϵ -symmetric (resp. ϵ -quadratic) torsion Q -groups of C to be the \mathbb{Z}_2 -hypercohomology (resp. \mathbb{Z}_2 -hyperhomology) groups of the ϵ -transposition involution $T_\epsilon = \epsilon T$ on the \mathbb{Z} -module chain complex $\text{Tor}_1^A(C, C) = \text{Hom}_A(C^\wedge, C)$

$$Q_{\text{tor}}^n(C, \epsilon) = H^n(\mathbb{Z}_2; \text{Tor}_1^A(C, C)) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \text{Tor}_1^A(C, C))),$$

$$Q_n^{\text{tor}}(C, \epsilon) = H_n(\mathbb{Z}_2; \text{Tor}_1^A(C, C)) = H_n(W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (\text{Tor}_1^A(C, C))).$$

There are defined forgetful maps

$$1 + T_\epsilon : Q_n^{\text{tor}}(C, \epsilon) \longrightarrow Q_{\text{tor}}^n(C, \epsilon); \quad \psi \longmapsto (1 + T_\epsilon)\psi,$$

$$Q_n^{\text{tor}}(C, \epsilon) \longrightarrow H_n(\text{Tor}_1^A(C, C)); \quad \phi \longmapsto \phi_0.$$

The element $\phi_0 \in H_n(\text{Tor}_1^A(C, C))$ is a chain homotopy class of A -module chain maps $\phi_0 : C^{n-\wedge} \longrightarrow C$.

An n -dimensional ϵ -symmetric complex over (A, σ) (C, ϕ) is a bounded (A, σ) -module chain complex C together with an element $\phi \in Q_n^{\text{tor}}(C, \epsilon)$. The complex (C, ϕ) is Poincaré if the A -module chain maps $\phi_0 : C^{n-\wedge} \longrightarrow C$ are chain equivalences.

Example 2.8. A 0-dimensional ϵ -symmetric Poincaré complex (C, ϕ) over (A, σ) is essentially the same as a nonsingular ϵ -symmetric linking form (M, λ) over (A, σ) , with $M = (C_0)^\wedge$ an (A, σ) -module and

$$\lambda = \phi_0 : M \times M \longrightarrow \sigma^{-1}A/A$$

a sesquilinear pairing such that the adjoint

$$M \longrightarrow M^\wedge; \quad x \longmapsto (y \longmapsto \lambda(x, y))$$

is an A -module isomorphism.

The n -dimensional torsion ϵ -symmetric L -group $L_{\text{tor}}^n(A, \sigma, \epsilon)$ is the cobordism group of n -dimensional ϵ -symmetric Poincaré complexes (C, ϕ) over (A, σ) , with C n -dimensional. In particular, $L_{\text{tor}}^0(A, \sigma, \epsilon)$ is the Witt group of nonsingular ϵ -symmetric linking forms over (A, σ) .

Similarly in the ϵ -quadratic case, with torsion L -groups $L_n^{\text{tor}}(A, \sigma, \epsilon)$. The ϵ -quadratic torsion L -groups are 4-periodic

$$L_n^{\text{tor}}(A, \sigma, \epsilon) = L_{n+2}^{\text{tor}}(A, \sigma, -\epsilon) = L_{n+4}^{\text{tor}}(A, \sigma, \epsilon).$$

Theorem 2.9. *If $A \longrightarrow \sigma^{-1}A$ is injective the relative L -groups in the localization exact sequences of Proposition 2.3*

$$\cdots \longrightarrow L^n(A, \epsilon) \longrightarrow \Gamma^n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L^n(A, \sigma, \epsilon) \longrightarrow L^{n-1}(A, \epsilon) \longrightarrow \cdots,$$

$$\cdots \longrightarrow L_n(A, \epsilon) \longrightarrow \Gamma_n(A \longrightarrow \sigma^{-1}A, \epsilon) \xrightarrow{\partial} L_n(A, \sigma, \epsilon) \longrightarrow L_{n-1}(A, \epsilon) \longrightarrow \cdots$$

are the torsion L -groups

$$L^*(A, \sigma, \epsilon) = L_{\text{tor}}^*(A, \sigma, \epsilon),$$

$$L_*(A, \sigma, \epsilon) = L_*^{\text{tor}}(A, \sigma, \epsilon).$$

Proof. For any bounded (A, σ) -module chain complex T there exists a bounded f.g. projective A -module chain complex C with a homology equivalence $C \longrightarrow T$. Working as in [8] there is defined a distinguished triangle of $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complexes

$$\Sigma \text{Tor}_1^A(T, T) \longrightarrow C \otimes_A C \longrightarrow T \otimes_A T \longrightarrow \Sigma^2 \text{Tor}_1^A(T, T)$$

with \mathbb{Z}_2 acting by the ϵ -transposition T_ϵ on the \mathbb{Z} -module chain complex $\text{Tor}_1^A(T, T)$ and by the $(-\epsilon)$ -transpositions $T_{-\epsilon}$ on $C \otimes_A C$ and $T \otimes_A T$, inducing long exact sequences

$$\cdots \longrightarrow Q_{\text{tor}}^n(T, \epsilon) \longrightarrow Q^{n+1}(C, -\epsilon) \longrightarrow Q^{n+1}(T, -\epsilon) \longrightarrow Q_{\text{tor}}^{n-1}(T, \epsilon) \longrightarrow \cdots.$$

$$\cdots \longrightarrow Q_n^{\text{tor}}(T, \epsilon) \longrightarrow Q_{n+1}(C, -\epsilon) \longrightarrow Q_{n+1}(T, -\epsilon) \longrightarrow Q_{n-1}^{\text{tor}}(T, \epsilon) \longrightarrow \cdots.$$

Passing to the direct limits over all the bounded (A, σ) -module chain complexes U with a homology equivalence $\beta : T \longrightarrow U$ use Proposition 2.7(iv) to obtain

$$\varinjlim_{(U, \beta)} Q^{n+1}(U, -\epsilon) = 0,$$

$$\varinjlim_{(U, \beta)} Q_{n+1}(U, -\epsilon) = 0$$

and hence

$$\varinjlim_{(U, \beta)} Q_{\text{tor}}^n(U, \epsilon) = Q^{n+1}(C, -\epsilon),$$

$$\varinjlim_{(U, \beta)} Q_n^{\text{tor}}(U, \epsilon) = Q_{n+1}(C, -\epsilon). \quad \square$$

Remark 2.10. The identification $L_*(A, \sigma, \epsilon) = L_*^{\text{tor}}(A, \sigma, \epsilon)$ for noncommutative $\sigma^{-1}A$ was first obtained by Vogel [8].

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