

# A Survey of Classical Knot Concordance

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### **Abstract**

This survey provides an overview of the concordance group of knots in three-dimensional space. It begins with a review of the definitions of knots and concordance and then presents aspects of the algebraic theory of concordance. Following this, Casson–Gordon invariants are examined in detail. Recent results from the topological locally flat category are presented, as are new applications from smooth geometry. A discussion of the interplay between 3-dimensional knot properties and concordance is presented. The survey concludes with a brief list of open problems.

In 1926, Artin [3] described the construction of certain knotted 2-spheres in  $\mathbf{R}^4$ . The intersection of each of these knots with the standard  $\mathbf{R}^3 \subset \mathbf{R}^4$  is a nontrivial knot in  $\mathbf{R}^3$ . Thus, a natural problem is to identify which knots can occur as such slices of knotted 2-spheres. Initially it seemed possible that every knot is such a *slice knot* and it was not until the early 1960s that Murasugi [86] and Fox and Milnor [24,25] succeeded at proving that some knots are not slice.

Slice knots can be used to define an equivalence relation on the set of knots in  $S^3$ : knots  $K$  and  $J$  are equivalent if  $K\# - J$  is slice. With this equivalence the set of knots becomes a group, the *concordance group* of knots. Much progress has been made in studying slice knots and the concordance group, yet some of the most easily asked questions remain untouched.

There are two related theories of concordance, one in the smooth category and the other topological. Our focus will be on the smooth setting, though the distinctions and main results in the topological setting will be included. Related topics must be excluded, in particular the study of link concordance. Our focus lies entirely in the classical setting; higher dimensional concordance theory is only mentioned when needed to understand the classical setting.

## 1. Introduction

Two smooth knots,  $K_0$  and  $K_1$ , in  $S^3$  are called *concordant* if there is a smooth embedding of  $S^1 \times [0,1]$  into  $S^3 \times [0,1]$  having boundary the knots  $K_0$  and  $-K_1$  in  $S^3 \times \{0\}$  and  $S^3 \times \{1\}$ , respectively. Concordance is an equivalence relation, and the set of equivalence classes forms a countable abelian group,  $\mathcal{C}$ , under the operation induced by connected sum. A knot represents the trivial element in this group if it is *slice*; that is, if it bounds an embedded disk in the 4-ball.

The concordance group was introduced in 1966 by Fox and Milnor [25], though earlier work on slice knots was already revealing aspects of its structure. Fox [24] described the use of the Alexander polynomial to prove that the figure eight knot is of order two in  $\mathcal{C}$  and Murasugi [86] used the signature of a knot to obstruct the slicing of a knot, thus showing that the trefoil is of infinite order in  $\mathcal{C}$ . (These results, along with much of the introductory material, are presented in greater detail in the body of this article.) The application of abelian knot invariants (those determined by the cohomology of abelian covers or, equivalently, by the Seifert form) to concordance culminated in 1969 with Levine's classification of higher dimensional knot concordance, [62,63], which applied in the classical dimension to give a surjective homomorphism,  $\phi : \mathcal{C} \rightarrow \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ .

In 1975, Casson and Gordon [8,9] proved that Levine's homomorphism is not an isomorphism, constructing nontrivial elements in the kernel, and Jiang [42] expanded on this to show that the kernel contains a subgroup isomorphic to  $\mathbf{Z}^\infty$ . Along these lines it was shown in [72] that the kernel also contains a subgroup isomorphic to  $\mathbf{Z}_2^\infty$ . The 1980s saw two significant developments in the study of concordance. The first was based on Freedman's work [26,27] studying the structure of topological 4-manifolds. One consequence was that methods of Levine and those of Casson–Gordon apply in the

topological locally flat category, rather than only in the smooth setting. More significant, Freedman proved that all knots with trivial Alexander polynomial are in fact slice in the topological locally flat category.

The other important development concerns the application of differential geometric techniques to the study of smooth 4-manifolds, beginning with the work of Donaldson [20,21] and including the introduction of Seiberg–Witten invariants and their application to symplectic manifolds, the use of the Thurston–Bennequin invariant [2,101], and recent work of Ozsváth and Szabó [96]. This work quickly led to the construction of smooth knots of Alexander polynomial one that are not smoothly slice, along with a much deeper understanding of related issues, such as the 4-ball genus of knots. Using these methods it has recently been shown that the results of Ozsváth and Szabó [96] imply that the kernel of Levine’s homomorphism contains a summand isomorphic to  $\mathbf{Z}$  and thus contains elements that are not divisible [78]. References are too numerous to enumerate here; a few will be included as applications are mentioned.

Recent work of Cochran et al. [14,15] has revealed a deeper structure to the knot concordance group. In that work a filtration of  $\mathcal{C}$  is defined:

$$\cdots \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{.5} \subset \mathcal{F}_0 \subset \mathcal{C}.$$

It is shown that  $\mathcal{F}_0$  corresponds to knots with trivial Arf invariant,  $\mathcal{F}_{.5}$  corresponds to knots in the kernel of  $\phi$  and all knots in  $\mathcal{F}_{1.5}$  have vanishing Casson–Gordon invariants. Using von Neumann  $\eta$ -invariants, it has been proved in [16] that each quotient is infinite. This work places Levine’s obstructions and those of Casson–Gordon in the context of an infinite sequence of obstructions, all of which reveal a finer structure to  $\mathcal{C}$ .

*Outline:* Section 2 is devoted to the basic definitions related to concordance and algebraic concordance. In Section 3, algebraic concordance invariants are presented, including the description of Levine’s homomorphism. Sections 4 and 5 present Casson–Gordon invariants and their application. In Section 6, the consequences of Freedman’s work on topological surgery in dimension four are described. Section 7 concerns the application of the results of Donaldson and more recent differential geometric techniques to concordance. In Section 8, the recent work of Cochran, Orr and Teichner on the structure of the topological concordance group is outlined. Section 9 relates 3-dimensional knot properties and concordance. Finally, Section 10 presents a few outstanding problems in the study of knot concordance.

## 2. Definitions

We will work in the smooth setting. In Section 6, there will be a discussion of the necessary modifications and main results that apply in the topological locally flat category. Knots are usually thought of as isotopy classes of embeddings of  $S^1$  into  $S^3$ . However, to simplify the discussion of orientation and symmetry issues, it is worthwhile to begin with the following precise definitions of knots, slice knots, concordance and Seifert surfaces.

## 2.1. Knot theory and concordance

DEFINITION 2.1.

- (1) A knot is an oriented diffeomorphism class of a pair of oriented manifolds,  $K = (\Sigma^3, \Sigma^1)$ , where  $\Sigma^n$  is diffeomorphic to the  $n$ -sphere.
- (2) A knot is called slice if there is a pair  $(B^4, D^2)$  with  $\partial(B^4, D^2) = K$ , where  $B^4$  is the 4-ball and  $D^2$  is a smoothly embedded 2-disk.
- (3) Knots  $K_1$  and  $K_2$  are called concordant if  $K_1 \# -K_2$  is slice. (Here  $-K$  denotes the knot obtained by reversing the orientation of each element of the pair and connected sum is defined in the standard way for oriented pairs.) The set of concordance classes is denoted  $\mathcal{C}$ .
- (4) A Seifert surface for a knot  $K$  is an oriented surface  $F$  embedded in  $S^3$  such that  $K = (S^3, \partial F)$ .

The basic theorem in the subject is the following.

**THEOREM 2.2.** *The set of concordance classes of knots forms a countable abelian group, also denoted  $\mathcal{C}$ , with its operation induced by connected sum and with the unknot representing the identity.*

Related to the notion of slice knots there is the stronger condition of being a ribbon knot.

**DEFINITION 2.3.** A knot  $K$  is called ribbon if it bounds an embedded disk  $D$  in  $B^4$  for which the radial function on the ball restricts to be a smooth Morse function with no local maxima in the interior of  $D$ .

There is no corresponding group of ribbon concordance. Casson observed that for every slice knot  $K$  there is a ribbon knot  $J$  such that  $K \# J$  is ribbon. Hence, if any equivalence relation identifies ribbon knots, it also identifies all slice knots. There is however a notion of ribbon concordance, first studied in [39].

## 2.2. Algebraic concordance

An initial understanding of  $\mathcal{C}$  is obtained via the algebraic concordance group, defined by Levine in terms of Seifert pairings.

**DEFINITION 2.4.** A Seifert pairing for a knot  $K$  with Seifert surface  $F$  is the bilinear mapping

$$V : H_1(F) \times H_1(F) \rightarrow \mathbf{Z}$$

given  $V(x, y) = \text{lk}(x, i_* y)$ , where  $\text{lk}$  denotes the linking number and  $i_*$  is the map induced by the positive pushoff,  $i : F \rightarrow S^3 - F$ .

(Here and throughout, homology groups will be taken with integer coefficients unless indicated otherwise.) A Seifert matrix is the matrix representation of the Seifert pairing with respect to some free generating set for  $H_1(F)$ .

If the transpose pairing  $V^T$  is defined by  $V^T(x, y) = V(y, x)$  then  $V - V^T$  represents the unimodular intersection form on  $H_1(F)$ . Hence, in general we define an abstract Seifert pairing on a finitely generated free  $\mathbf{Z}$ -module  $M$  to be a bilinear form  $V : M \times M \rightarrow \mathbf{Z}$  satisfying  $V - V^T$  is unimodular. (In order for this to make sense for the trivial knot with Seifert surface  $B^2$ , the Seifert form on the 0-dimensional  $\mathbf{Z}$ -module is defined to be unimodular.)

**DEFINITION 2.5.** An abstract Seifert form  $V$  on  $M$  is metabolic if  $M = M_1 \oplus M_2$  with  $M_1 \cong M_2$  and  $V(x, y) = 0$  for all  $x$  and  $y \in M_1$ . Such an  $M_1$  is called a metabolizer for  $V$ .

**THEOREM 2.6.** *If  $K$  is slice and  $F$  is a Seifert surface for  $K$ , then the associated Seifert form is metabolic.*

**PROOF.** Let  $D$  be a slice disk for  $K$ . The union  $F \cup D$  bounds a 3-manifold  $R$  embedded in  $B^4$ . Such an  $R$  can be constructed explicitly, or an obstruction theory argument can be used to construct a smooth mapping  $B^4 - D \rightarrow S^1$  which has  $F \cup D$  as the boundary of the pull-back of a regular value. (Note that this construction depends on the triviality of the normal bundle to  $D$ .)

A duality argument implies that  $\text{rank}(\ker(H_1(F) \rightarrow H_1(R))) = (1/2)\text{rank}(H_1(F))$ . For any  $x$  and  $y$  in that kernel,  $V(x, y) = 0$ : since  $x$  bounds a 2-chain in  $R$ ,  $i_*(x)$  bounds a 2-chain in  $B^4 - R$  which is disjoint from the chain bounded in  $R$  by  $y$ .

Since  $V$  vanishes on this kernel, it vanishes on the summand  $M$  generated by the kernel, and hence  $V$  is metabolic.  $\square$

**COROLLARY 2.7.** *If  $K_1$  is concordant to  $K_2$  and these knots have Seifert forms  $V_1$  and  $V_2$  (with respect to arbitrary Seifert surfaces), then  $V_1 \oplus -V_2$  is metabolic.*

In general, abstract Seifert forms  $V_1$  and  $V_2$  are called *algebraically concordant* if  $V_1 \oplus -V_2$  is metabolic. This is an equivalence relation. (The proof is based on cancellation: if  $V$  and  $V \oplus W$  are metabolic, then so is  $W$ . See [49].)

**THEOREM 2.8.** *The set of algebraic concordance classes forms a group, denoted  $\mathcal{G}$ , with its operation induced by direct sum. The trivial 0-dimensional  $\mathbf{Z}$ -module serves as the identity.*

In the following theorem, defining Levine's homomorphism, we temporarily use the notation  $[K]$  to denote concordance class of a knot and  $[V_F]$  to represent the algebraic concordance class of a Seifert form associated to an arbitrarily chosen Seifert surface  $F$  for  $K$ .

**THEOREM 2.9.** *The map  $\phi : \mathcal{C} \rightarrow \mathcal{G}$  defined by  $\phi([K]) = [V_F]$  is a surjective homomorphism.*

PROOF. That this map is well-defined follows from the previous discussion and, in particular, Corollary 2.7. Surjectivity follows from an explicit construction of a surface with desired Seifert form [6,102].  $\square$

### 3. Algebraic concordance invariants

Levine [62] defined a collection of homomorphisms from  $\mathcal{G}$  to the groups  $\mathbf{Z}$ ,  $\mathbf{Z}_2$  and  $\mathbf{Z}_4$ . These can be properly combined to give an isomorphism  $\Phi$  from  $\mathcal{G}$  to the infinite direct sum  $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ . The proof of this will be left to [62].

We should remark that what Levine actually did was to classify the rational algebraic concordance group, based on rational matrices. He also showed that the integral group injects into the rational group, with image sufficiently large to contain  $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ . Stolz [104] completed the classification of the integral concordance group.

In this section, we will describe a collection of invariants that are sufficient to show that  $\mathcal{G}$  contains a summand isomorphic to  $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$ . The invariants will be applied to a particular family of knots, which we now describe.

Figure 1 illustrates a basic knot that we denote  $K(a, b, c)$ ; the curves  $J_1$  and  $J_2$  can be ignored for now. The integers  $a$  and  $b$  indicate the number of full twists in each band. The integer  $c$  is odd and represents the number of half twists between the bands; those twists between the bands are so placed as to not add twisting to the individual bands. Figure 2 illustrates a particular example,  $K(2, 0, 3)$ , along with a basis for the first homology of the Seifert surface, indicated with oriented dashed curves on the surface.

The knot  $K(a, b, c)$  bounds a genus one Seifert surface with Seifert form represented by the following matrix with respect to the indicated basis of  $H_1(F)$ .

$$\begin{pmatrix} a & (c + 1)/2 \\ (c - 1)/2 & b \end{pmatrix}$$

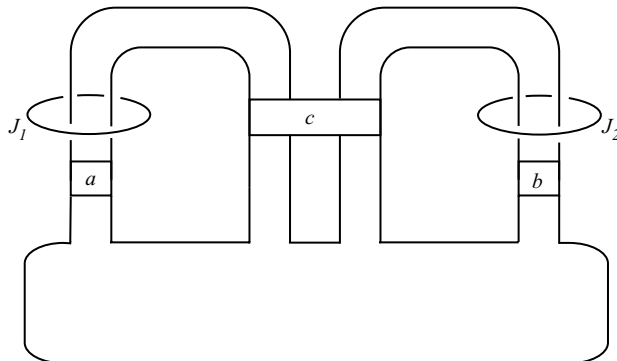
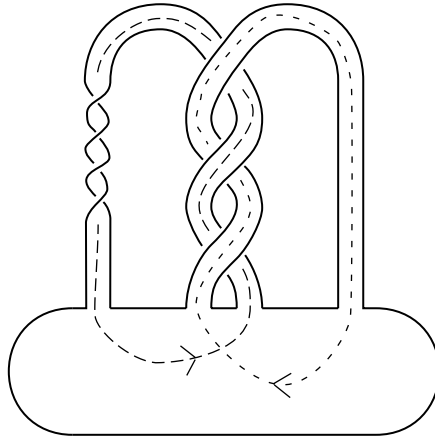


Fig. 1. The knot  $K(a, b, c)$ .

Fig. 2. The knot  $K(2, 0, 3)$ .

### 3.1. Integral invariants, signatures

Let  $V$  be a Seifert matrix and  $V^T$  its transpose. If  $\omega$  is a unit complex number that is not a root of the Alexander polynomial of  $V$ ,  $\Delta_V(t) = \det(V - tV^T)$ , then the form  $V_\omega = \frac{1}{2}(1 - \omega)V + \frac{1}{2}(1 - \bar{\omega})V^T$  is nonsingular. In this case, if  $V$  is metabolic, the signature of  $V_\omega$  is 0. To adjust for the possibility of the  $V_\omega$  being singular, for general  $\omega$  on the unit circle the signature  $\sigma_\omega(V)$  is defined to be the limiting average of the signatures of  $V_{\omega_+}$  and  $V_{\omega_-}$ , where  $\omega_+$  and  $\omega_-$  are unit complex numbers approaching  $\omega$  from different sides. For all  $\omega$ ,  $\sigma_\omega$  defines a homomorphism from  $\mathcal{G}$  to  $\mathbf{Z}$ . It is onto  $2\mathbf{Z}$  if  $\omega \neq 1$ . For the set of  $\omega$  given by roots of unity  $e^{2\pi i/p}$  where  $p$  is a prime, the functions  $\sigma_\omega$  are independent on  $\mathcal{G}$  (this can be seen using the  $b$ -twisted doubles of the unknot,  $K(1, b, 1)$ ,  $b > 0$ ), and hence together these give a map of  $\mathcal{G}$  onto  $\mathbf{Z}^\infty$ . In the case of  $\omega = -1$ , this signature, defined by Trotter [111], was shown to be a concordance invariant by Murasugi [86]. The more general formulation is credited to Levine and Tristram [110] and is referred to as the Levine–Tristram signature.

In [43,44] the identification of these signatures with signatures of the branched covers of  $B^4$  branched over a pushed in Seifert surface of a knot was made. In [12] it was shown that the set of  $\sigma_\omega$  over all  $\omega$  with positive imaginary part are independent.

### 3.2. The Arf invariant: $\mathbf{Z}_2$

Given a  $(2g) \times (2g)$  Seifert matrix  $V$  one defines a  $\mathbf{Z}_2$ -valued quadratic form on  $\mathbf{Z}_2^{2g}$  by  $q(x) = xVx^T$ . This is a nonsingular quadratic form in the sense that  $q(x+y) - q(x) - q(y) = x \cdot y$  where the nonsingular bilinear pairing  $x \cdot y$  is given by the matrix  $V + V^T$ . (Recall that the determinant of  $V + V^T$  is odd.)



The simplest definition of the Arf invariant of a nonsingular quadratic form on a  $\mathbf{Z}_2$ -vector space  $W$  is that  $\text{Arf}(q) = 0$  or  $\text{Arf}(q) = 1$  depending on whether  $q$  takes value 0 or 1, respectively, on a majority of elements in  $W$ . See, for instance [5,41]. The Arf invariant defines a homomorphism on the Witt group of  $\mathbf{Z}_2$  quadratic forms and in particular vanishes on metabolic forms. Hence, the Arf invariant gives a well-defined  $\mathbf{Z}_2$ -valued homomorphism from  $\mathcal{G}$  to  $\mathbf{Z}_2$ .

This invariant was first defined by Robertello [97]. Murasugi [87] observed that  $\text{Arf}(V) = 0$  if and only if  $\Delta_V(-1) = \pm 1 \pmod 8$ .

**3.3. Polynomial invariants:  $\mathbf{Z}_2$**

The Alexander polynomial of a Seifert matrix is defined to be  $\Delta_V(t) = \det(V - tV^T) \in \mathbf{Z}[t, t^{-1}]$ . If different Seifert matrices associated to the same knot are used to compute an Alexander polynomial, the resulting polynomials will differ by multiplication by a unit in  $\mathbf{Z}[t, t^{-1}]$ , that is by  $\pm t^n$  for some  $n$ . Hence, two Alexander polynomials are considered equivalent if they differ by multiplication by  $\pm t^n$  for some  $n$ .

If  $V$  is metabolic, then  $\Delta_V(t) = \pm t^n f(t)f(t^{-1})$  for some integral polynomial  $f$ . For concordance considerations, if  $p(t)$  is an irreducible symmetric polynomial ( $p(t^{-1}) = \pm t^n p(t)$ ) then the exponent of  $p(t)$  in the irreducible factorization of  $\Delta_V(t)$  taken modulo 2 yields a  $\mathbf{Z}_2$  invariant of  $\mathcal{G}$ . Fox and Milnor [25] used this to define a surjective homomorphism of  $\mathcal{G}$  to  $\mathbf{Z}_2^\infty$ . The knots  $K(a, -a, 1)$  (see Figure 1) are of order at most 2 in  $\mathcal{C}$  since for each,  $K(a, -a, 1) = -K(a, -a, 1)$ . On the other hand, these have distinct irreducible Alexander polynomials if  $a > 0$ . The existence of an infinite summand of  $\mathcal{G}$  isomorphic to  $\mathbf{Z}_2^\infty$  follows. Note that the knot  $K(1, -1, 1)$  is the figure eight knot.

**3.4.  $W(\mathbf{Q})$ :  $\mathbf{Z}_2$  and  $\mathbf{Z}_4$  invariants**

The matrix  $V + V^T$  defines an element in the Witt group of  $\mathbf{Q}$ ,  $W(\mathbf{Q})$ . We will now summarize the theory of this Witt group and associated Witt groups of finite fields. Details can be found in [41]. Notice that the determinant of  $V + V^T$  is odd; hence, in the following discussion we restrict attention to odd primes  $p$ .

Recall that the Witt group of an arbitrary field  $\mathbf{F}$  consists of finite dimensional  $\mathbf{F}$ -vector spaces with nonsingular symmetric forms and forms  $W_1$  and  $W_2$  are equivalent if  $W_1 \oplus -W_2$  is metabolic. Addition is via direct sums.

There is a surjective homomorphism  $\bigoplus \partial_p : W(\mathbf{Q}) \rightarrow \bigoplus W(\mathbf{F}_p)$ . Here  $\mathbf{F}_p$  is the field with  $p$  elements, and the direct sums are over the set of all primes. For  $p$  odd, the group  $W(\mathbf{F}_p)$  is isomorphic to either  $\mathbf{Z}_2$  or  $\mathbf{Z}_4$ , depending on whether  $p$  is 1 or 3 modulo 4. We next define  $\partial_p$  and then discuss the invariants of  $W(\mathbf{F}_p)$ . (For completeness, we note here that the kernel of  $\bigoplus \partial_p$  is  $W(\mathbf{Z})$  which is isomorphic to  $\mathbf{Z}$  via the signature [41].)

**3.4.1. Reducing to finite fields.** There is a simple algorithm giving the map  $\partial_p$ . A symmetric rational matrix  $A$  can be diagonalized using simultaneous row and column

operations, and this form decomposes as the direct sum of forms:  $\bigoplus_{i=1}^n (a_i p^{\epsilon_i})$  where  $\gcd(a_i, p) = 1$ ,  $\epsilon_i = 1$  for  $i \leq m$  and  $\epsilon_i = 0$  for  $m + 1 \leq i \leq n$ . The map  $\partial_p$  takes  $A$  to the  $\mathbb{F}_p$  form represented by the direct sum  $\bigoplus_{i=1}^n (a_i)$ .

**3.4.2.  $W(\mathbb{F}_p)$ :  $\mathbf{Z}_2$  and  $\mathbf{Z}_4$  invariants.** For  $p$  odd, any form on a finite dimensional  $\mathbb{F}_p$ -vector space can be diagonalized with  $\pm 1$  as the diagonal entries. In the Witt group the form represented by the matrix  $(1) \oplus (-1)$  is trivial. A little more work shows that the form  $4(1)$  is Witt trivial: find elements  $a$  and  $b$  such that  $a^2 + b^2 = -1$  and consider the subspace spanned by  $(1, 0, a, b)$  and  $(0, 1, b, -a)$ . Hence,  $W(\mathbb{F}_p)$  is generated by  $(1)$ , an element of order 2 or 4.

In the case that  $p \equiv 1$  modulo 4,  $-1$  is a square. It follows quickly that  $W(\mathbb{F}_p) \cong \mathbf{Z}_2$ . On the other hand, in the case that  $p \equiv 3$  modulo 4,  $-1$  is not a square, and  $W(\mathbb{F}_p) \cong \mathbf{Z}_4$ .

As a simple example, if one starts with the Seifert form for the knot  $K(1, -5, 1)$ ,

$$V = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix}, \quad V + V^\tau = \begin{pmatrix} 2 & 1 \\ 1 & -10 \end{pmatrix}.$$

Diagonalizing over the rationals yields

$$V = \begin{pmatrix} 2 & 0 \\ 0 & -(2)(3)(7) \end{pmatrix}.$$

With  $p = 3$  this form maps to the element  $(-14)$  of  $W(\mathbb{F}_3)$ , which is equivalent to the form  $(1)$ , a generator of order 4. The same is true working with  $p = 7$ .

As a consequence of the next theorem we will see that this particular form  $V$  is actually of order four in  $\mathcal{G}$ .

**3.5. Quadratic polynomials**

A special case of a theorem of Levine (Section 23 of [63]) gives the following result, which implies in particular that the form just described is of order 4 in  $\mathcal{G}$ .

**THEOREM 3.1.** *Suppose that  $\Delta_V(t)$  is an irreducible quadratic. Then  $V$  is of finite order in the algebraic concordance group if and only if  $\Delta_V(1)\Delta_V(-1) < 0$ . In this case  $V$  is of order 4 if  $|\Delta_V(-1)| = p^a q$  for some prime  $p$  congruent to 3 modulo 4,  $a$  odd, and  $p$  and  $q$  relatively prime; otherwise it is of order 2.*

**3.6. Other approaches to algebraic invariants**

There are alternative approaches to algebraic obstructions to a knot being slice that do not depend on Seifert forms. For instance, Milnor [83] described signature invariants based on

his duality theorem for the infinite cyclic cover of a knot complement. The equivalence of these signatures and those of Tristram and Levine is proved in [82]. There is also an interpretation of the algebraic concordance group in terms of the Blanchfield pairing of the knot.

#### 4. Casson–Gordan invariants

In the case that  $K$  is algebraically slice, Casson–Gordon invariants offer a further obstruction to a knot being slice. We follow the basic description of [8].

##### 4.1. Definitions

We begin by reviewing the *linking form* on torsion ( $H_1(M)$ ) for an oriented 3-manifold  $M$ . If  $x$  and  $y$  are curves representing torsion in the first homology, then  $\text{lk}(x, y)$  is defined to be  $(d \cap y)/n \in \mathbf{Q}/\mathbf{Z}$ , where  $d$  is a 2-chain with boundary  $nx$ . Intersections are defined via transverse intersections of chains, and of course one must check that the value of the linking form is independent of the many choices in its definition. For a closed oriented 3-manifold the linking form is nonsingular in the sense that it induces an isomorphism from torsion ( $H_1(M)$ ) to  $\text{hom}(\text{torsion}(H_1(M)), \mathbf{Q}/\mathbf{Z})$ .

Such a symmetric pairing on a finite abelian group,  $l : H \times H \rightarrow \mathbf{Q}/\mathbf{Z}$ , is called *metabolic* with *metabolizer*  $L$  if the linking form vanishes on  $L \times L$  for some subgroup  $L$  with  $|L|^2 = |H|$ .

Let  $M_q$  denote the  $q$ -fold branched cover of  $S^3$  branched over a given knot  $K$ , and let  $\bar{M}_q$  denote 0-surgery on  $M_q$  along  $\tilde{K}$ , where  $\tilde{K}$  is the lift of  $K$  to  $M_q$ . Here  $q$  will be a prime power.

Let  $x$  be an element of self-linking 0 in  $H_1(M_q)$  and suppose that  $x$  is of prime power order, say  $p$ . Linking with  $x$  defines a homomorphism  $\chi_x : H_1(M_q) \rightarrow \mathbf{Z}_p$ . Furthermore,  $\chi_x$  extends to give a  $\mathbf{Z}_p$ -valued character on  $H_1(\bar{M}_q)$  which vanishes on the meridian of  $\tilde{K}$ . In turn, this character extends to give  $\bar{\chi}_x : H_1(\bar{M}_q) \rightarrow \mathbf{Z}_p \oplus \mathbf{Z}$ . Since  $x$  has self-linking 0, bordism theory implies that the pair  $(\bar{M}_q, \bar{\chi}_x)$  bounds a 4-manifold, character, pair,  $(W, \eta)$ .

More generally, for any character  $\chi : H_1(M_q) \rightarrow \mathbf{Z}_p$ , there is a corresponding character  $\bar{\chi} : H_1(\bar{M}_q) \rightarrow \mathbf{Z}_p \oplus \mathbf{Z}$ . This character might not extend to a 4-manifold, but since the relevant bordism groups are finite, for some multiple  $r\bar{M}_q$  the character given by  $\bar{\chi}$  on each component does extend to a 4-manifold, character pair,  $(W, \eta)$ .

Let  $Y$  denote the  $\mathbf{Z}_p \times \mathbf{Z}$  cover of  $W$  corresponding to  $\eta$ . Using the action of  $\mathbf{Z}_p \times \mathbf{Z}$  on  $H_2(Y, \mathbf{C})$  one can form the twisted homology group  $H_2^t(W, \mathbf{C}) = H_2(Y, \mathbf{C}) \otimes_{\mathbf{C}[\mathbf{Z}_p \times \mathbf{Z}]} \mathbf{C}(t)$ . (The action of  $\mathbf{Z}_p$  on  $\mathbf{C}(t)$  is given by multiplication by  $e^{2\pi i/p}$ .) There is a nonsingular Hermitian form on  $H_2^t(W, \mathbf{C})$  taking values in  $\mathbf{C}(t)$ . The Casson–Gordon invariant is defined to be the difference of this form and the intersection form of  $H_2(W, \mathbf{C})$ , both tensored with  $1/r$ , in  $W(\mathbf{C}[t, t^{-1}]) \otimes \mathbf{Q}$ . (In showing that this Witt class yields a well-defined obstruction to slicing a knot, the fact that  $\Omega_4(\mathbf{Z}_p \oplus \mathbf{Z})$  is nonzero appears, and as a

consequence one must tensor with  $\mathbf{Q}$  to arrive at a well-defined invariant, even in the case of  $\chi_x$  in which it is possible to take  $r = 1$ .)

DEFINITION 4.1. The Casson–Gordon invariant  $\tau(M_q, \chi)$  is the class  $(H_2^1(W, \mathbf{C}) - H_2(W, \mathbf{C})) \otimes 1/r \in W(\mathbf{C}(t)) \otimes \mathbf{Q}$ .

**4.2. Main theorem**

The main theorem of [8] states:

THEOREM 4.2. *If  $K$  is slice, there is a metabolizer  $L$  for the linking form on  $H_1(M_q)$  such that, for each prime power  $p$  and each element  $x \in L$  of order  $p$ ,  $\tau(M_q, \chi_x) = 0$ .*

The proof shows that if  $K$  is slice with slice disk  $D$ , then covers of  $B^4 - D$  can be used as the manifold  $W$ , and for this  $W$  the invariant vanishes.

COMMENT. There are a number of extensions of this theorem. With care the definition of the Casson–Gordon invariant can be refined and  $\tau$  can be viewed as taking values in  $W(\mathbf{Q}[\zeta_p](t)) \otimes \mathbf{Z}[1/p]$ . This yields finer invariants (see, for instance [34]). The observation that  $L$  can be assumed to be equivariant with respect to the deck transformation of  $M_q$  can give stronger constraints (see, for example [57]). In [50] it is demonstrated that a factorization of the Alexander polynomial of a knot yields further constraints on the metabolizer  $L$ .

**4.3. Invariants of  $W(\mathbf{C}(t)) \otimes \mathbf{Q}$**

In the next section, we will describe examples of algebraically slice knots which can be proved to be nonslice using Casson–Gordon invariants. We conclude this section with a description of the types of algebraic invariants associated to the Witt group  $W(\mathbf{C}(t)) \otimes \mathbf{Q}$ .

**4.3.1. Signatures.** Let  $\xi$  be a unit complex number. Let  $A \otimes a/b \in W(\mathbf{C}(t)) \otimes \mathbf{Q}$ . Then  $A$  can be represented by a matrix of rational functions,  $A(t)$ . The signature  $\sigma_\xi(A \otimes a/b)$  is defined, roughly, to be  $(a/b)\sigma(A(\xi))$  where  $\sigma$  denotes the standard Hermitian signature. There is the technical point arising that  $A(\xi)$  might be singular, so the precise definition of  $\sigma_\xi(A \otimes a/b)$  takes the two-sided average over unit complex numbers close to  $\xi$ . This limit is defined to be the *Casson–Gordon signature invariant*,  $\sigma_\xi(K, \chi)$ . For  $\xi = 1$  this is abbreviated as  $\sigma(K, \chi)$ .

**4.3.2. Discriminants.** If the matrix  $A(t)$  represents  $0 \in W(\mathbf{C}(t))$ , the discriminant,  $\text{dis}(A(t)) = (-1)^k \det(A(t))$  (where  $k$  is half the dimension of  $A$ ) will be of the form  $f(t)\bar{f}(t)$  for some rational function  $f$ . Let  $g(t) = t^2 + \lambda t + 1, |\lambda| > 2$  be an irreducible real symmetric polynomial. It follows that for a matrix  $A(t)$ , the exponent of  $g(t)$  in the factorization of  $\text{dis}(A(t))$  gives a  $\mathbf{Z}_2$ -valued invariant of the Witt class of  $A(t)$ .

More generally, in the case that  $p$  is odd, the exponent of  $g$  in the determinant of  $A(t)^a$  gives a  $\mathbf{Z}_2$  invariant of the class represented by  $A(t) \otimes a/b$  in  $W(\mathbf{C}(t)) \otimes \mathbf{Z}[1/p]$ .

These discriminants were first discussed in unpublished work of Litherland [68]. Later developments and applications are included in [34].

In [56,57] a 3-dimensional approach to the definition of Casson–Gordon discriminant invariants is presented. In short, the representation  $\bar{\chi}: H_1(\bar{M}_q) \rightarrow \mathbf{Z}_p \times \mathbf{Z}$  determines the twisted homology group:  $H_1^t(\bar{M}_q, \mathbf{Q}(\zeta_p)[t])$ . This is a  $\mathbf{Q}[\zeta_p][t]$  module, and the discriminant of the Casson–Gordon invariant is given by the order of this module. Although this 3-dimensional approach does not give the signature invariant, it has the advantage of being completely algorithmic in computation via a procedure first developed in [66,112] and applied in [56–58]. A computer implementation of that algorithm facilitated the classification of the order of low-crossing number knots in concordance [108] and the proof that most low-crossing number knots which are not reversible are not concordant to their reverses, in [107].

In a different direction, we note that some effort has been made in removing the restriction on prime power covers and characters. In the case of ribbon knots, it was known that stronger results could be attained. Recent work of Kim [52] has developed examples of nonslice algebraically slice knots for which all prime power branched covers are homology spheres. Other work in this realm includes that of Letsche [61] and recent work of Friedl [28,29].

## 5. Companionship and Casson–Gordon invariants

In Casson and Gordon’s original work the computation of Casson–Gordon invariants was quite difficult, largely limited to restricted classes of knots. Litherland [69] studied the behavior of these invariants under companionship and, independently, Gilmer [31] found interpretations of particular Casson–Gordon invariants in terms of signatures of simple closed curves on a Seifert surface for a knot. Further work addressing companionship and Casson–Gordon invariants includes [1]. In this section, we describe the general theory and its application to genus one knots.

### 5.1. Construction of companions

Let  $U$  be an unknotted circle in the complement of a knot  $K$ . If  $S^3$  is modified by removing a neighborhood of  $U$  and replacing it with the complement of a knot  $J$  in  $S^3$  (via a homeomorphism of boundaries that identifies the meridian of  $J$  with the longitude of  $U$  and vice versa) then the resulting manifold is again diffeomorphic to  $S^3$ . The image of  $K$  in this manifold will be denoted  $K(J)$  (the choice of  $U$  will be suppressed in the notation). In the language of classical knot theory,  $K(J)$  is a satellite knot with companion  $J$  and satellite  $K$ .

If  $M_q$  is the  $q$ -fold branched cover of  $S^3$  branched over  $K$ , then  $U$  has  $q'$  lifts, denoted  $U_i$ ,  $i = 1, \dots, q'$ , where  $q' = \gcd(q, \text{lk}(U, K))$ . It follows that  $M'_{q'}$ , the  $q'$ -fold branched cover of  $S^3$  branched over  $K(J)$ , is formed from  $M_q$  by removing neighborhoods of the

$U_i$  and replacing each with the  $q/q'$ -cyclic cover of the complement of  $J$ . If  $\chi$  is a  $\mathbf{Z}_p$ -valued homomorphism on  $H_1(M_q)$ , there is a naturally associated homomorphism  $\chi'$  on  $H_1(M'_q)$ .

**5.2. Casson–Gordon invariants and companions**

In the case that  $\text{lk}(U, K) = 0$ , we have the following theorem of Litherland [69] (see also [35]).

**THEOREM 5.1.** *In the situation just described, with  $\text{lk}(U, K) = 0$*

$$\sigma(K(J), \chi') = \sigma(K, \chi) + \sum_{i=1}^q \sigma_{\chi(U_i)/p}(J).$$

The main idea of the proof is fairly simple. If  $(W, \eta)$  is the chosen pair bounding  $(\bar{M}_q, \bar{\chi})$  in the definition of the Casson–Gordon invariant, then for the new knot  $K'$  a 4-manifold  $W'$  can be built from  $W$  by attaching copies of a 4-manifold with character  $(Y, \eta)$  bounding 0-surgery on  $J$  with its canonical representation to  $\mathbf{Z}$ . Signatures of cyclic covers of  $Y$  are related to the signatures of  $J$ . A similar analysis can be done for the discriminant of the Casson–Gordon invariant. This was detailed in [34], and further explored in [58] where it was no longer assumed that  $J$  was null homologous.

**EXAMPLE.** Consider the knot illustrated in Figure 1 with  $a = 0$ ,  $b = 0$  and  $c = 3$ . The bands have knots  $J_1$  and  $J_2$  tied in them. (We will also refer to the pair of unknotted circles as  $J_1$  and  $J_2$  in this situation, as the meaning is unambiguous.) Call the resulting knot  $K(J_1, J_2)$ . The homology of the 2-fold cover is isomorphic to  $\mathbf{Z}_3 \oplus \mathbf{Z}_3$  with the linking form vanishing on the two summands. Call generators of the summands  $x_1$  and  $x_2$ . An analysis of the cover shows that  $\chi_{x_1}$  is a  $\mathbf{Z}_3$ -valued character that vanishes on the lifts of  $J_1$  and takes value  $\pm 1$  on the two lifts of  $J_2$ . Similarly for  $\chi_{x_2}$ .

Since  $K$  is slice, by the Casson–Gordon theorem, either  $\sigma(K, \chi_{x_1})$  or  $\sigma(K, \chi_{x_2})$  must vanish. Hence, using Theorem 5.1, if  $K(J_1, J_2)$  is slice, either  $2\sigma_{1/3}(J_1)$  or  $2\sigma_{1/3}(J_2)$  must vanish. By choosing  $J_1$  and  $J_2$  so that this is not the case, one constructs basic examples of algebraically slice knots which are not slice.

**5.3. Genus one knots and the Seifert form**

Gilmer [31,32] observed that for genus one knots the computation of Casson–Gordon invariants is greatly simplified. Roughly, he interpreted the Casson–Gordon signature invariants of an algebraically slice genus one knot in terms of the signatures of knots tied in the bands of the Seifert surface. The previous example offers an illustration of the

appearance of these signatures. This work is now most easily understood via the use of companionship just described.

In short, if an algebraically slice knot  $K$  bounds a genus one Seifert surface  $F$ , then some nontrivial primitive class in  $H_1(F)$  has trivial self-linking with respect to the Seifert form. If that class is represented by a curve  $\alpha$ , the surface can be deformed to be a disk with two bands attached, one of which is tied into the knot  $\alpha$ . If a new knot is formed by adding the knot  $-\alpha$  to the band, the knot becomes slice and certain of its Casson–Gordon invariants will vanish. However, the previous results on companionship determine how the modification of the knot changes the Casson–Gordon invariant. The situation is made somewhat more delicate in that  $\alpha$  is not unique: for genus one algebraically slice knots there are two metabolizers. The following represents the sort of result that can be proved.

**THEOREM 5.2.** *Let  $K$  be a genus one slice knot. The Alexander polynomial of  $K$  is given  $(at - (a + 1))((a + 1)t - a)$  for some  $a$ . For some simple closed curve  $\alpha$  representing a generator of a metabolizer of the Seifert form and for some infinite set of prime powers  $q$ , one has*

$$\sum_{i=1}^q \sigma_{bmi/p}(\alpha) = 0$$

for all prime power divisors  $p$  of  $(a - 1)^q - a^q$ , and for all integers  $b$ .

(The appearance of the term  $(a - 1)^q - a^q$  represents the square root of the order of the homology of the  $q$ -fold branched cover.) Since the sum is taken over a coset of the multiplicative subgroup of  $\mathbf{Z}_p$ , by combining these cosets one has the following.

**COROLLARY 5.3.** *If  $K$  is a genus one slice knot with nontrivial Alexander polynomial, then for some simple closed curve  $\alpha$  representing a generator of a metabolizer of the Seifert form, there is an infinite set of prime powers  $p$  for which*

$$\sum_{i=1}^{p-1} \sigma_{i/p}(\alpha) = 0.$$

A theorem of Cooper [17] follows quickly:

**COROLLARY 5.4.** *If  $K$  is a genus one slice knot with nontrivial Alexander polynomial, then for some simple closed curve  $x$  representing a generator of a metabolizer of the Seifert form*

$$\int_0^{1/2} \sigma_t(x) dt = 0.$$

(This theorem reappears in [15] where the integral is reinterpreted as a metabelian von Neumann signature of the original knot  $K$ , giving a direct reason why it is a concordance invariant. For more on this, see Section 8.)

EXAMPLE. Consider the knot  $K(0, 0, 3)$ , as in Figure 1. Replacing the curves labeled  $J_1$  and  $J_2$  with the complements of knots  $J_1$  and  $J_2$  yields a knot for which the metabolizers of the Seifert form are represented by the knots  $J_1$  and  $J_2$ . The knot is algebraically slice, but by the previous corollary, if both of the knots have signature functions with nontrivial integral, the knot is not slice.

## 6. The topological category

Freedman [26] developed surgery theory in the category of topological 4-manifolds, proving roughly that for manifolds with fundamental groups that are not too complicated (in particular, finitely generated abelian groups) the general theory of higher dimensional surgery descends to dimension 4. The most notable consequence of this work was the proof the 4-dimensional Poincaré conjecture: a closed topological 4-manifold that is homotopy equivalent to the 4-sphere is homeomorphic to the 4-sphere.

Two significant contributions to the study of concordance quickly followed from Freedman's original paper. The first of these, proved in [27], is that a locally flat surface in a topological 4-manifold has an embedded normal bundle. The use of such a normal bundle was implicit in the proof that slice knots are algebraically slice. It is also used in a key step in the proof of the Casson–Gordon theorem, as follows. Casson–Gordon invariants of slice knots are shown to vanish via the observation that for a slice knot  $K$ , if 0-surgery is performed on  $K$ , the resulting 3-manifold  $M(K, 0)$  bounds a homology  $S^1 \times B^3$ ,  $W$ . This  $W$  is constructed by removing a tubular neighborhood of a slice disk for  $K$  in the 4-ball. The existence of the tubular neighborhood is equivalent to the existence of the normal bundle.

In a different direction, Freedman's theorem implied that in the topological locally flat category all knots of Alexander polynomial one are slice. To understand why this is a consequence, note first the following.

**THEOREM 6.1.** *For a knot  $K$ , if  $M(K, 0)$  bounds a homology  $S^1 \times B^3$ ,  $W$ , with  $\pi_1(W) = \mathbf{Z}$  then  $K$  is slice.*

PROOF. We have that  $M(K, 0)$  is formed from  $S^3$  by removing a solid torus and replacing it with another solid torus. Performing 0-surgery on the core,  $C$ , of that solid torus returns  $S^3$ . Attach a 2-handle to  $W$  with framing 0 to  $C$ . The resulting manifold is a homotopy ball with boundary  $S^3$ , and hence, by the Poincaré conjecture, is homeomorphic to  $B^4$ . The cocore of that added 2-handle is a slice disk for the boundary of the cocore, which can be seen to be the original  $K$ .  $\square$

Freedman observed that a surgery obstruction to finding such a manifold  $W$  is determined by the Seifert form, and for a knot of Alexander polynomial one that is the only obstruction, and it vanishes.



### 6.1. Extensions

Is it possible that more delicate arguments using 4-dimensional surgery might yield stronger results, showing that other easily identified classes of algebraically slice knots are slice, based only on the Seifert form of the knot? The following result indicates that the answer is no.

**THEOREM 6.2.** *If  $\Delta_K(t)$  is nontrivial then there are two nonconcordant knots having that Alexander polynomial.*

This result was first proved in [76] where there was the added constraint that the Alexander polynomial is not the product of cyclotomic polynomials  $\phi_n(t)$  with  $n$  divisible by three distinct primes. The condition on Alexander polynomials is technical, assuring that some prime power branched cover is not a homology sphere. Kim [52] has shown this condition is not essential in particular cases, and in unpublished work he has shown that the result applies for all nontrivial Alexander polynomials.

### 7. Smooth knot concordance

In 1983, Donaldson [20] discovered new constraints on the intersection forms of smooth 4-manifolds. This and subsequent work soon yielded the following theorem.

**THEOREM 7.1.** *Suppose that  $X$  is a smooth closed 4-manifold and  $H_1(X, \mathbf{Z}_2) = 0$ . If the intersection form on  $H_2(X)$  is positive definite then the form is diagonalizable. If the intersection form is even and definite, and hence of the type  $nE_8 \oplus mH$ , where  $H$  is the standard 2-dimensional hyperbolic form, then if  $n > 0$ , it follows that  $m > 2$ .*

This result is sufficient to prove that many knots of Alexander polynomial one are not slice. The details of any particular example cannot be presented here, but the connections with Theorem 7.1 are easily explained.

Let  $M(K, 1)$  denote the 3-manifold constructed as 1-surgery on  $K$ . Then  $M(K, 1)$  bounds the 4-manifold  $W$  constructed by adding a 2-handle to the 4-ball along  $K$  with framing 1. If  $K$  is slice, the generator of  $H_2(W)$  is represented by a 2-sphere with self-intersection number 1. A tubular neighborhood of that sphere can be removed and replaced with a 4-ball, showing that  $M(K, 1)$  bounds a homology ball,  $X$ . If  $M(K, 1)$  also bounds a 4-manifold  $Y$  (say simply connected) with intersection form of the type obstructed by Theorem 1, then a contradiction is achieved using the union of  $X$  and  $Y$ .

As an alternative approach, notice that if  $K$  is slice, the 2-fold branched cover of  $S^3$  branched over  $K$ ,  $M_2$ , bounds the  $\mathbf{Z}_2$ -homology ball formed as the 2-fold branched cover of  $B^4$  branched over the slice disk. Hence, if  $M_2$  is known to bound a simply connected 4-manifold with one of the forbidden forms of Theorem 7.1, then again a contradiction is achieved.

It seems that prior to Donaldson's work it was known that either of these approaches would be applicable to proving that particular polynomial one knots are not slice, but these arguments were not published. In particular, following the announcement of Donaldson's theorem it immediately was known that the pretzel knot  $K(-3, 5, 7)$  and the untwisted double of the trefoil (Akbulut) are not slice. Early papers presenting details of such arguments include [36] where it was shown that there are topologically slice knots of infinite order in smooth concordance. See [13] for further examples.

### 7.1. Further advances

Continued advances in smooth 4-manifold theory have led to further understanding of the knot slicing problem. In particular, proving that large classes of Alexander polynomial one knots are not slice has fallen to algorithmic procedures. Notable among this work is that of Rudolph [99–101]. Here, we outline briefly the approach using Thurston–Bennequin numbers, as described by Akbulut and Matveyev in the paper [2].

The 4-ball has a natural complex structure. If a 2-handle is added to the 4-ball along a knot  $K$  with appropriate framing, which we call  $f$  for now, the resulting manifold  $W$  will itself be complex. According to Lisca and Matić [67],  $W$  will then embed in a closed Kahler manifold  $X$ . Further restrictions on the structure of  $X$  are known to hold, and with these constraints the adjunction formula of Kronheimer and Mrowka [59,60] applies to show that no essential 2-sphere in  $X$  can have self-intersection greater than or equal to  $-1$ .

On the other hand, if  $K$  were slice and the framing  $f$  of  $K$  were greater than  $-2$ , such a sphere would exist. The appropriate framing  $f$  mentioned above depends on the choice of representative of  $K$ , not just its isotopy class. If the representative is  $\mathbf{K}$ , then  $f = tb(\mathbf{K}) - 1$ , where  $tb(\mathbf{K})$  is the Thurston–Bennequin number, easily computed from a diagram for  $\mathbf{K}$ .

Applying this, both Akbulut–Matveyev [2] and Rudolph [101] have given simple proofs that, for instance, all iterated positive twisted doubles of the right handed trefoil are not slice.

Although these powerful techniques have revealed a far greater complexity to the concordance group than had been expected, as of yet they seem incapable of addressing some of the basic questions: for instance, the slice implies ribbon conjecture and problems related to torsion in the concordance group.

## 8. Higher order obstructions and the filtration of $\mathcal{C}$

Recent work of Cochran, Orr and Teichner has demonstrated a deep structure to the topological concordance group. This is revealed in a filtration of the concordance group by an infinite sequence of subgroups:

$$\cdots \mathcal{F}_{2.0} \subset \mathcal{F}_{1.5} \subset \mathcal{F}_1 \subset \mathcal{F}_{.5} \subset \mathcal{F}_0 \subset \mathcal{C}.$$

This approach has successfully placed known obstructions to the slicing problem – the Arf invariant, algebraic sliceness, and Casson–Gordon invariants – as the first in an infinite sequence of invariants. Of special significance is that each level of the induced filtration of the concordance group has both an algebraic interpretation and a geometric one. Here, we can offer a simplified view of the motivations and consequences of their work, and in that interest will focus on the  $\mathcal{F}_n$  with  $n$  a nonnegative integer.

To begin, suppose that  $M(K, 0)$ , 0-surgery on a knot  $K$ , bounds a 4-manifold  $W$  with the homology type and intersection form of  $S^1 \times B^3 \#_n S^2 \times S^2$ . Such a  $W$  will exist if and only if the Arf invariant of  $K$  is trivial. Constructing one such  $W$  is fairly simple in this case. Push a Seifert surface  $F$  for  $K$  into  $B^4$  and perform surgery on  $B^4$  along a set of curves on  $F$  representing a basis of a metabolizer for its intersection form, with the additional condition that it represents a metabolizer for the  $\mathbf{Z}_2$ -Seifert form. (Finding such a basis is where the Arf invariant condition appears.) When performing the surgery, the surface  $F$  can be ambiently surgered to become a disk, and the complement of that disk is the desired  $W$ .

If a generating set of a metabolizer for the intersection form on  $H_2(W)$  could be represented by disjoint embedded 2-spheres, then surgery could be performed on  $W$  to convert it into a homology  $S^1 \times B^3$ . It would quickly follow that  $K$  would be slice in a homology 4-ball bounded by  $S^3$ .

In the higher dimensional analog (of the concordance group of knotted  $(2k - 1)$ -spheres in  $S^{2k+1}$ ,  $k > 1$ ), there is an obstruction (to finding this family of spheres) related to the twisted intersection form on  $H_{k+1}(W, \mathbf{Z}[\pi_1(W)])$ , or, equivalently, related to the intersection form on the universal cover of  $W$ . In short, the intersection form of  $W$  should have a metabolizer that lifts to a metabolizer in the universal cover of  $W$ . In this higher dimensional setting, if the obstruction vanishes then, via the Whitney trick, the metabolizer for  $W$  can be realized by embedded spheres and  $W$  can be surgered as desired. This viewpoint on knot concordance has its roots in the work of Cappell and Shaneson [7].

Whether in high dimensions or in the classical setting, the explicit construction of a  $W$  described earlier in this section yields a  $W$  with cyclic fundamental group. This obstruction is thus determined solely by the infinite cyclic cover and vanishes for algebraically slice knots. Of course, in higher dimensions algebraically slice knots are slice. Clearly something more is needed in the classical case.

In light of the Casson–Freedman approach to 4-dimensional surgery theory, in addition to finding immersed spheres representing a metabolizer for  $W$ , one needs to find appropriate dual spheres in order to convert the immersed spheres into embeddings. The Cochran–Orr–Teichner filtration can be interpreted as a sequence of obstructions to finding a family of spheres and dual spheres. To describe the filtration, we denote  $\pi^{(0)} = \pi = \pi_1(W)$  and let  $\pi^{(n)}$  be the derived subgroup:  $\pi^{(n+1)} = [\pi^{(n)}, \pi^{(n)}]$ .

**DEFINITION 8.1.** A knot  $K$  is called  $n$ -solvable if there exists a (spin) 4-manifold  $W$  with boundary  $M(K, 0)$  such that: (a) the inclusion map  $H_1(M(K, 0)) \rightarrow H_1(W)$  is an isomorphism; (b) the intersection form on  $H_2(W, \mathbf{Z}[\pi/\pi^{(n)}])$  has a dual pair of self-annihilating submodules (with respect to intersections and self-intersections),  $L_1$  and  $L_2$ ; and (c) the images of  $L_1$  and  $L_2$  in  $H_2(W)$  generate  $H_2(W)$ .

(Here and in what follows we leave the description of  $n.5$ -solvability to [14].)  
There are the following basic corollaries of the work in [14].

**THEOREM 8.2.** *If the Arf invariant of a knot  $K$  is 0, then  $K$  is 0-solvable. If  $K$  is 1-solvable,  $K$  is algebraically slice. If  $K$  is 2-solvable, Casson–Gordon type obstructions to  $K$  being slice vanish. If  $K$  is slice,  $K$  is  $n$ -solvable for all  $n$ .*

One of the beautiful aspects of [14] is that this very algebraic formulation is closely related to the underlying topology. For those familiar with the language of Whitney towers and gropes, we have the following theorem from [14].

**THEOREM 8.3.** *If  $K$  bounds either a Whitney tower or a grope of height  $n + 2$  in  $B^4$ , then  $K$  is  $n$ -solvable.*

Define  $\mathcal{F}_n$  to be the subgroup of the concordance group consisting of  $n$ -solvable knots. One has the filtration (where we have dropped the  $n.5$ -subgroups).

$$\cdots \mathcal{F}_3 \subset \mathcal{F}_2 \subset \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{C}.$$

Beginning with [14] and culminating in [16], there is the following result.

**THEOREM 8.4.** *For all  $n$ , the quotient group  $\mathcal{F}_n/\mathcal{F}_{n+1}$  is infinite and  $\mathcal{F}_2/\mathcal{F}_3$  is infinitely generated.*

Describing the invariants that provide obstructions to a knot being in  $\mathcal{F}_n$  is beyond the scope of this survey. However, two important aspects should be mentioned. First, Cochran et al. [14] identifies a connection between  $n$ -solvability and the structure and existence of metabolizers for linking forms on

$$H_1(M(K, 0), \mathbf{Z}[\pi_1(M(K, 0))/\pi_1(M(K, 0))^{(k)}]), \quad k \leq n,$$

generalizing the fact that for algebraically slice knots the Blanchfield pairing of the knot vanishes.

The second aspect of proving the nontriviality of  $\mathcal{F}_n/\mathcal{F}_{n+1}$  is the appearance of von Neumann signatures for solvable quotients of the knot group. Though difficult to compute in general, Cochran et al. [14] demonstrates that if  $K$  is built as a satellite knot, then in special cases, as with the Casson–Gordon invariant, the value of this complicated invariant is related to the Tristram–Levine signature function of the companion knot. More precisely, if a knot  $K$  is built from another knot by removing an unknot  $U$  that lies in  $\pi^{(n)}$  of the complement and replacing it with the complement of a knot  $J$ , then the change in a particular von Neumann  $\eta$ -invariant of the  $\pi^{(n)}$ -cover is related to the integral of the Tristram–Levine signature function of  $J$ , taken over the entire circle. The Cheeger–Gromov estimate for these  $\eta$ -invariants can then be applied to show the nonvanishing of the invariant by choosing  $J$  in a way that the latter integral exceeds the estimate. This construction generalizes in a number of ways the one used in applications of the Casson–Gordon invariant described earlier, which applied only in the case that  $U \in \pi^{(1)}$

and  $U \notin \pi^{(2)}$ . Furthermore, the Casson–Gordon invariant is based on a finite dimensional representation where here the representation becomes infinite dimensional. In the construction of [16] it is also required that one work with a family of unknots; a single curve  $U$  will not suffice.

## 9. Three-dimensional knot properties and concordance

### 9.1. Primeness

The first result of the sort to be discussed here is the theorem of Kirby and Lickorish [55]:

**THEOREM 9.1.** *Every knot is concordant to a prime knot.*

Shorter proofs of this were given in [70,94]. In these constructions it was shown that the concordance can be chosen so that the Seifert form, and hence the algebraic invariants, of the knot are unchanged. Myers [89] proved that every knot is concordant to a knot with hyperbolic complement, and hence to one with no incompressible tori in its complement. Later, Soma [103] extended Myers’s result by showing that fibered knots are (fibered) concordant to fibered hyperbolic knots.

In the reverse direction, one might ask if every knot is concordant to a composite knot, but the answer here is obviously yes:  $K$  is concordant to  $K\#J$ , for any slice knot  $J$ . However, when the Seifert form is taken into consideration the question becomes more interesting. Here we have the following example, the proof of which is contained in [74 version 1].

**THEOREM 9.2.** *There exists a knot  $K$  with Seifert form  $V_K = V_{J_1} \oplus V_{J_2}$ , but  $K$  is not concordant to a connected sum of knots with Seifert forms  $V_{J_1}$  and  $V_{J_2}$ .*

Notice that by Levine’s classification of higher dimensional concordance, such examples cannot exist in dimensions greater than 3.

### 9.2. Knot symmetry: amphicheirality

For the moment, view a knot  $K$  formally as a smooth oriented pair  $(S, K)$  where  $S$  is diffeomorphic to  $S^3$  and  $K$  is diffeomorphic to  $S^1$ . Equivalence is up to orientation preserving diffeomorphism. (In dimension three it does not matter whether the smooth or locally flat topological category is used.)

**DEFINITION 9.3.** A knot  $(S, K)$  is called reversible (or invertible), negative amphicheiral, or positive amphicheiral, if it is equivalent to  $K^r = (S, -K)$ ,  $-K = (-S, -K)$ , or  $-K^r = (-S, K)$  respectively. It is called strongly reversible, strongly positive amphicheiral, or strongly negative amphicheiral if there is an equivalence that is an involution.

Each of these properties constrains the algebraic invariants of a knot, and hence can constrain the concordance class of a knot. For instance, according to Hartley [40], if a knot  $K$  is negative amphicheiral, then its Alexander polynomial satisfies  $\Delta_K(t^2) = F(t)F(t^{-1})$  for some symmetric polynomial  $F$ . It follows quickly from the condition that slice knots have polynomials that factor as  $g(t)g(t^{-1})$  that if a knot  $K$  is concordant to a negative amphicheiral knot,  $\Delta_K(t^2)$  must factor as  $F(t)F(t^{-1})$ . Further discussion of amphicheirality and knot concordance is included in [18], where the focus is on higher dimensions, but some results apply in dimension three.

EXAMPLE. Let  $K$  be a knot with Seifert form

$$V_a = \begin{pmatrix} 1 & 1 \\ 0 & -a \end{pmatrix}.$$

If  $a$  is positive, it follows from Levine's characterization of knots with quadratic Alexander polynomial (Theorem 3.1) that  $K$  is of order two in the algebraic concordance group if every prime of odd exponent in  $4a + 1$  is congruent to 1 modulo 4. It follows as one example that any knot with Seifert form  $V_3$ , for instance the 3-twisted double of the unknot, is of order 2 in algebraic concordance but is not concordant to a negative amphicheiral knot.

This gives insight into the following conjecture based on a long standing question of Gordon [38]:

CONJECTURE 9.4. *If  $K$  is of order two in  $\mathcal{C}$ , then  $K$  is concordant to a negative amphicheiral knot.*

(Gordon's original question did not have the "negative" constraint in its statement.)

In a different direction, it was noted by Long [81] that the example of a knot  $K$  for which  $K\# - K'$  is not slice (described in the next subsection) yields an example of a nonslice strongly positive amphicheiral knot. Flapan [23] subsequently found a prime example of this type. It has since been shown that the concordance group contains infinitely many linearly independent such knots [73].

### 9.3. Reversibility and mutation

Every knot is algebraically concordant to its reverse. A stronger result, but the only proof in print, follows from Long [81]: if  $K$  is strongly positive amphicheiral then it is algebraically slice. For any knot,  $K\# - K'$  is strongly positive amphicheiral, so  $K$  and  $K'$  are algebraically concordant. It is proved in [71] that there are knots that are not concordant to their reverses. Further examples have been developed in [56,90,107].

Kearton [48] observed that since  $K\# - K'$  is a (negative) mutant of the slice knot  $K\# - K$ , an example of a knot which is not concordant to its reverse yield an example of

mutation changing the concordance class of a knot. Similar examples for positive mutants proved harder to find and were developed in [56,58].

#### 9.4. Periodicity

A knot  $K$  is called periodic if it is invariant under a periodic transformation  $T$  of  $S^3$  with the fixed point set of  $T$  a circle disjoint from  $K$ . Some of the strongest results concerning periodicity are those of Murasugi [88] constraining the Alexander polynomials of such knots. Naik [91] used Casson–Gordon invariants to obstruct periodicity for knots for which all algebraic invariants coincided with those of a periodic knot.

A theory of periodic concordance has been developed. Basic results in the subject include those of Cha and Ko [11] and Naik [92] obstructing knots from being periodically slice and those of Davis and Naik [19] giving a characterization of the Alexander polynomials of periodically ribbon knots.

#### 9.5. Genus

The 4-ball genus of a knot  $K$ ,  $g_4(K)$ , is the minimal genus of an embedded surface bounded by  $K$  in the 4-ball. It is a concordance invariant of a knot which is clearly bounded by its 3-sphere genus.

This invariant has been studied extensively. It is known to be bounded below by half the classical signature and the Tristram–Levine signature [43,44,86,110]. In the case that a knot is algebraically slice, Gilmer developed bounds on the 4-ball genus using Casson–Gordon invariants [30]. In [51] it is shown that for any pair of nonnegative integers  $m$  and  $n$  there is a knot  $K$  with a mutant  $K^*$  such that  $g_4(K) = m$  and  $g_4(K^*) = n$ ; a knot and its mutant are algebraically concordant. Beyond that, there are many results giving bounds on the 4-ball genus in the smooth setting based on differential geometric results. See, for instance [101,109].

Nakanishi [93] and Casson observed that there are knots that bound surfaces of genus one in the 4-ball but which are not concordant to knots of 3-sphere genus 1. In [77] this observation was the starting point of the definition of the concordance genus of a knot  $K$ : the minimum genus among all knots concordant to  $K$ . It is shown that this invariant can be arbitrarily large, even for knots of 4-ball genus 1, and even among algebraically slice knots.

#### 9.6. Fibering

A knot is called fibered if its complement is a surface bundle over  $S^1$ . It is relatively easy to see that not all knots are concordant to fibered knots, as follows. The Alexander polynomial of a fibered knot is monic. Consider a knot  $K$  with  $\Delta_K(t) = 2t^2 - 3t + 2$ . If  $K$  were concordant to a fibered knot, then  $\Delta_K(t)g(t) = f(t)f(t^{-1})$  for some monic polynomial  $g$  and integral  $f$ . However, since  $\Delta_K(t)$  is irreducible and symmetric, it would have to be a

factor of  $f(t)$  and of  $f(t^{-1})$ , giving it even exponent in  $\Delta_K(t)g(t)$ , implying it is a factor of  $g(t)$ , contradicting monotonicity.

As mentioned above, Soma [103] proved that fibered knots are concordant to hyperbolic fibered knots.

The most significant result associating fibering and concordance is the theorem of Casson and Gordon [10].

**THEOREM 9.5.** *If  $K$  is a fibered ribbon knot, then the monodromy of the fibration extends over some solid handlebody.*

### 9.7. Unknotting number

The unknotting number of a knot  $K$  is the least number of crossing changes that must be made in any diagram of  $K$  to convert it to an unknot. This is closely related to the 4-ball genus of a knot (see the discussion above) and questions regarding the slicing of a knot in manifolds bounded by  $S^3$  other than  $B^4$ , for instance, a once punctured connected sums of copies of  $S^2 \times S^2$ . A related invariant that is more closely tied to concordance was introduced by Askitas [4,95], which we call the slicing number of a knot:  $u_s(K)$  is the minimum number of crossing changes required to convert a knot into a slice knot. It is relatively easy to see that the 4-ball genus of a knot provides a lower bound on the slicing number; it was shown in [85] and later in [75] that these two need not be equal.

## 10. Problems

Past problem sets that include questions related to the knot concordance group include [38,54].

- (1) *Is every slice knot a ribbon knot?* A knot is ribbon if it bounds an embedded disk in  $B^4$  having no local maxima (with respect to the radial function) in its interior. In the topological category this is not defined, so one asks the following instead: *is every slice knot homotopically ribbon?* (That is, does  $K$  bound a disk  $D$  in  $B^4$  such that  $\pi_1(S^3 - K) \rightarrow \pi_1(B^4 - D)$  is surjective?) In the smooth setting one then has the additional question: *is every homotopically ribbon knot a ribbon knot?*

One has little basis to conjecture here. Perhaps obstructions will arise (in either category) but the lack of potential examples is discouraging. On the other hand, topological surgery might provide a proof in that category, but would give little indication concerning the smooth setting.

- (2) *Describe all torsion in  $\mathcal{C}$ .* Beginning with [25] the question of whether there is any odd torsion has been open. More generally, the only known torsion in  $\mathcal{C}$  is two torsion that arises from knots that are concordant to negative amphicheiral knots, and Conjecture 9.4 (first suggested in [38]) states that negative amphicheirality is the source of all (two) torsion in  $\mathcal{C}$ .



As described in Section 9, the Seifert form

$$V_3 = \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}$$

represents 2-torsion in  $\mathcal{G}$  but cannot be represented by a negative amphicheiral knot.

The prospects for understanding 4-torsion look better. A start has been made in [79,80] where it is shown, for instance, that no knot with Seifert form

$$V_5 = \begin{pmatrix} 1 & 1 \\ 0 & -5 \end{pmatrix}$$

can be of order 4 in  $\mathcal{C}$ , although every such knot is of order 4 in  $\mathcal{G}$ .

Closely related to questions of torsion is the question: *Does Levine's homomorphism split?* That is, is there a homomorphism  $\psi: \mathcal{G} \rightarrow \mathcal{C}$  such that  $\phi \circ \psi$  is the identity? An affirmative answer would yield elements of order 4 in  $\mathcal{C}$  as well as elements of order 2 that do not arise from negative amphicheiral knots.

See [45,84] for computations of the algebraic orders of small crossing number knots.

- (3) *If the knots  $K$  and  $K\#J$  are doubly slice, that is cross-sections of unknotted 2-spheres in  $R^4$ , is  $J$  doubly slice?* The study of double knot concordance has a long history, with some of the initial work appearing in [106]. Other references include [33,46,47,64,65,105]. The property of double sliceness can be used to define a double concordance group which maps onto  $\mathcal{C}$  and there is a corresponding algebraic double concordance group formed using quotienting by the set of hyperbolic Seifert forms rather than metabolic forms. Algebraic invariants show that the kernel is infinitely generated, and Casson–Gordon invariants and Cochran–Orr–Teichner methods apply in the case that algebraic invariants do not [33,53]. Although a variety of questions regarding double null concordance can be asked, this problem points to the underlying geometric difficulty of the topic.
- (4) *Describe the structure of the kernel of Levine's homomorphism,  $\mathcal{A} = \ker(\phi: \mathcal{C} \rightarrow \mathcal{G})$ .* It is known [42,72] that  $\mathcal{A}$  contains a subgroup isomorphic to  $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty$ . A reasonable conjecture is that  $\mathcal{A} \cong \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty$ . It has recently been shown by the author [78] that results of Ozsváth and Szabó [96] imply that  $\mathcal{A}$  has a summand isomorphic to  $\mathbf{Z}$ . This implies that  $\mathcal{A}$  contains elements that are not divisible and that  $\mathcal{A}$  is not a divisible group. There remains the unlikely possibility that  $\mathcal{A}$  does contain infinitely divisible elements, perhaps including summands isomorphic to  $\mathbf{Q}$  and  $\mathbf{Q}/\mathbf{Z}$ .
- (5) *Describe the kernel of the map from  $\mathcal{C}$  to the topological concordance group,  $\mathcal{C}_{\text{top}}$ .* It is known that the kernel is nontrivial, containing for instance nonsmoothly slice Alexander polynomial one knots. (See [13,36] for early references.) In fact it contains an infinitely generated such subgroup [22]. What more can be said about this kernel?

- (6) *Identify new relationships between the various unknotting numbers and genera of a knot.* Here is a problem that seems to test the limits of presently known techniques. If  $K$  can be converted into a slice knot by making  $m$  positive crossing changes and  $n$  negative crossing changes, then a geometric construction yields a surface bounded by  $K$  in the 4-ball of genus  $\max\{m, n\}$ . Conversely: *If the 4-ball genus of  $K$  is  $g_4$ , can  $K$  be converted into a slice knot by making  $g_4$  positive and  $g_4$  negative crossing changes?* A simpler question ask the same thing except for the 3-sphere genus  $g_3$  instead of  $g_4$ . (It is interesting to note that at this time it seems unknown if the classical unknotting number satisfies  $u(K) \leq 2g_3(K)$ .)

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