Preface.

In these notes, the Shale-Weil representation of the symplectic group is discussed, as well as some of its applications to number theory.

The monograph is composed of two parts:

In Part I, written by Gérard Lion and Michèle Vergne, we introduce the Shale-Weil representation and establish a relation between its cocycle and the Maslov index.

In Part II, written by Michèle Vergne, applications of θ-series to liftings of modular forms are given.

Although the results of the first part enlightens the exposition of the classical transformation properties of θ-functions, a reader mainly interested by these applications to liftings could read directly the second part with an eventual glance to earlier paragraphs. The two parts have separate introductions and bibliographical notes.

<u>The authors</u>

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Part I: The Shale-Weil representation and the Maslov Index.

by

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1.0. Introduction:

Relations between symplectic geometry, Maslov index, representations of the Heisenberg group and the Shale-Weil representation of the symplectic group are discussed in this chapter. We first give in 1.1 the basic definitions and properties of symplectic vector spaces, Lagrangian subspaces, the Heisenberg Lie algebra and the action of the symplectic group of those objects.

The Schrödinger representation of the Heisenberg group N associated to a Lagrangian plane ℓ is constructed in 1.2.

We prove in 1.3 the Stone-von Neumann theorem which asserts that all unitary representations of N whose restriction to the center of N acts by the same non-trivial character are essentially the same: two such irreducible representations are equivalent. Although this uniqueness theorem underlies the construction of the Shale-Weil representation R, we will give however a direct construction of R, independent of the proof of this theorem. Thus results of Section 1.3 will not be needed subsequently.

The Schrödinger representations W_ℓ and $W_{\ell'}$ of the Heisenberg group associated to the Lagrangian planes ℓ and ℓ' are equivalent: we give in 1.4 a canonical choice of an operator $\mathcal{F}_{\ell',\ell}$ such that:

$$W_\ell(n) = \mathcal{F}^{-1}_{\ell',\ell} \; W_{\ell'}(n) \; \mathcal{F}_{\ell',\ell} \quad \text{for every } n \in N.$$

(This operator in appropriate coordinates is a partial Fourier transform.)

Before going further, we have to introduce, in 1.5, the Maslov index of a triple of Lagrangian planes: Let (V,B) be a symplectic vector space, ℓ_1, ℓ_2 and ℓ_3 three Lagrangian planes, then

$$Q_{123}(x_1 \oplus x_2 \oplus x_3) = B(x_1,x_2) + B(x_2,x_3) + B(x_3,x_1)$$

is a quadratic form on $\ell_1 \oplus \ell_2 \oplus \ell_3$, which can be diagonalized with p times the eigenvalue 1 and q times the eigenvalue -1. A modified definition of the Maslov index $\tau(\ell_1,\ell_2,\ell_3)$ due to M. Kashiwara is $\tau(\ell_1,\ell_2,\ell_3) = \text{sign } Q_{123} = p - q$. We prove in 1.5 that this Maslov index verifies a fundamental chain property.

The symplectic group G acts on Lagrangian planes, on the Heisenberg Lie algebra $\mathfrak{n}$ and on the Heisenberg Lie group N. If W_ℓ is the Schrödinger representation of N associated to ℓ, the representation $W'(n) = W_\ell(g\cdot n)$ is equivalent to W_ℓ: The natural action $A(g)$ of the symplectic group on functions on N transforms the representation W_ℓ of N into the representation $n \to W_{g\cdot\ell}(g\cdot n)$. The Fourier operator $\mathcal{F}_{\ell,g\cdot\ell}$ intertwines the representation $W_{g\cdot\ell}$ with W_ℓ. Thus the canonical unitary operator $R_\ell(g) = \mathcal{F}_{\ell,g\cdot\ell}\ A(g)$ satisfies the fundamental relation:

$$W_\ell(g\cdot n) = R_\ell(g)\ \ W_\ell(n)\ \ R_\ell(g)^{-1}, \text{ for every } n \in N.$$

We prove in (1.6) that

$$\mathcal{F}_{\ell_1,\ell_2}\ \mathcal{F}_{\ell_2,\ell_3}\ \mathcal{F}_{\ell_3,\ell_1} = e^{\frac{i\pi}{4}\tau(\ell_1,\ell_2,\ell_3)}\ \text{Id}$$

and deduce from this formula that

$$R_\ell(g_1g_2) = c_\ell(g_1,g_2)\ R_\ell(g_1)\ R_\ell(g_2)$$

with $$c_\ell(g_1,g_2) = e^{-\frac{i\pi}{4}\tau(\ell,g_1\ell,g_1g_2\ell)}.$$

It is known that the Shale-Weil projective representation R_ℓ is not equivalent to a true representation of G. We construct in 1.7 a function $s(\tilde{\ell}_1,\tilde{\ell}_2)$, defined on couples $(\tilde{\ell}_1,\tilde{\ell}_2)$ of oriented Lagrangian planes, invariant under the action of the symplectic group. Let us write

$$c(\ell_1,\ell_2,\ell_3) = e^{\frac{i\pi}{4}\tau(\ell_1,\ell_2,\ell_3)},$$

we prove the relation

$$c(\ell_1,\ell_2,\ell_3)^2 = s(\tilde{\ell}_1,\tilde{\ell}_2)\ s(\tilde{\ell}_2,\tilde{\ell}_3)\ s(\tilde{\ell}_3,\tilde{\ell}_1).$$

This leads to the results of Shale and Weil that the projective representation R_ℓ is equivalent to a true representation of the metaplectic group, a double covering of G.

The Section 1.8 is devoted to the construction of the universal covering group G of $SL(2,\mathbb{R})$ by elementary means. Some explicit formulas for the Shale-Weil representation of G are given.

Let Λ be the manifold of all Lagrangian planes of (V,B). In Section 1.9, we use the chain property of τ to construct both universal coverings of Λ and of $Sp(B)$. We here use the fact that the function $(g_1,g_2) \to \tau(\ell, g_1\cdot\ell, g_1g_2\ell)$

is a $\mathbb{Z}$-valued cocycle of $Sp(B)$. We relate our construction of the Maslov index to the formulas in coordinates of Leray and Souriau. The results of this section are independent of the rest of the notes. We follow in the Section 1.9 (as in many other parts of these notes) an idea of M. Kashiwara, and we are happy to thank him.

The appendix extends the notions of Part I to a local field k. The Kashiwara index $\tau(\ell_1,\ell_2,\ell_3)$ is then defined to be the class of the form Q_{123} in the Witt group W_k. Thus we obtain a canonical cocycle of the symplectic group $G = Sp(n,k)$ with values in W_k. We define a function $s(\tilde{\ell}_1,\tilde{\ell}_2)$ on couples of oriented Lagrangian planes, invariant by the action of G. We describe then the metaplectic group, using this function s, and prove as in 1.7 that the Weil projective representation of G lifts to a representation of the metaplectic group. The results this appendix are due to Patrice Perrin. Similar results had been obtained independently by R. Rao.

Bibliographical notes are given at the end of Part I.

The second author gave a seminar in 1978-79 at the Massachusetts Institute of Technology on these subjects and thank the participants for several improvements of the text, especially Martin Andler, Victor Kac, Steve Paneitz, Carolyn Schröeder and Bob Styer. We thank Patrice Perrin for discussions on his results and Masaki Kashiwara for communicating some unpublished texts.

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The Shale-Weil Representation and the Maslov Index

1.1. Symplectic vector spaces and the Heisenberg Lie algebra.

Let V be a finite dimensional real vector space. Let B be a non-degenerate skew symmetric form on V. Hence $\dim V$ is even. Let $\dim V = 2n$. Then we can choose a basis $(P_1, P_2, \cdots, P_n, Q_1, Q_2, \cdots, Q_n)$ of V with the relations:

1.1.1. $$B(P_i,P_j) = 0 \qquad B(Q_i,Q_j) = 0$$
$$B(P_i,Q_j) = \delta_{ij} \qquad B(Q_i,P_j) = -\delta_{ij}.$$

We will call such a basis a symplectic basis of (V,B).

We consider the Lie algebra $\mathfrak{n} = V + \mathbb{R}E$, with the bracket law defined as follows:

$$[X,Y] = B(X,Y)E, \text{ if } X,Y \in V$$

$$[\mathfrak{n},E] = 0, \text{ i.e. } \mathbb{R}E \text{ is the center of } \mathfrak{n}.$$

$\mathfrak{n}$ is called the Heisenberg Lie algebra. If (P_i,Q_j) is a symplectic basis of (V,B), the Lie algebra $\mathfrak{n}$ has as basis (P_i,Q_j,E) with the Heisenberg commutation relations, or "canonical commutation relations:

1.1.2. $$[P_i, Q_j] = \delta_{ij}E$$
$$[P_i, P_j] = 0$$
$$[Q_i, Q_j] = 0.$$

Let A_n be the associative algebra of differential operator with polynomial coefficients on $\mathbb{C}^n$ and the corresponding bracket $[D_1, D_2] = D_1D_2 - D_2D_1$. The algebra A_n is generated by $p_i = \frac{\partial}{\partial x_i}$, $q_j = x_j$ satisfying the canonical relations $[p_i, q_j] = \delta_{ij}$. The corresponding representation $P_i \to p_i$, $Q_j \to iq_j$, $E \to i\,\mathrm{Id}$ of $\mathcal{n}$ will be of particular importance to us.

We will use some simple lemmas for symplectic spaces: If L is a subspace of (V,B) we will denote by $L^\perp$ the orthogonal complement of L in V relative to B, i.e.

$$L^\perp = \{x \in V;\ B(x,y) = 0 \quad \forall y \in V\}.$$

We then have:

1.1.3. a) $(\dim L) + \dim(L^\perp) = 2n$

b) $(L^\perp)^\perp = L$

c) $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$

d) $(L_1 \cap L_2)^\perp = L_1^\perp + L_2^\perp$.

a), b), c) are clear; d) is easily deduced from c) by using the relation $(L^\perp)^\perp = L$.

If a subspace ℓ of V is such that $\ell = \ell^\perp$, ℓ is called a Lagrangian subspace of V. We have then $B(x,y) = 0$ for every $x,y \in \ell$, hence ℓ is totally isotropic with respect to the form B; moreover if x is such that $B(x,\ell) = 0$, then

$x \in \ell$; i.e. ℓ is a maximal totally isotropic subspace of (V,B).

Lemma 1.1.4. Let ℓ be a Lagrangian subspace of (V,B). There exists a Lagrangian subspace ℓ' such that $\ell \oplus \ell' = V$.

Proof: Let ℓ_0' be maximal among the totally isotropic subspaces such that $\ell' \cap \ell = 0$. Hence we have $\ell_0'^\perp + \ell^\perp = V$. However we have $\ell_0'^\perp \subset \ell + \ell_0'$, for otherwise, we could choose $x \in \ell_0'^\perp$ and not in $\ell + \ell_0'$ and the subspace ℓ_0' contained in $\ell_0' + \mathbb{R}x$ would not be maximal. Hence we obtain $\ell_0' + \ell = V$, which is the equality required.

1.1.5. Hence given ℓ a Lagrangian subspace, we can find ℓ' such that $\ell \oplus \ell' = V$; the bilinear form B clearly induces a pairing between ℓ and $\ell' = V/\ell$. So we can choose a symplectic basis $(P_1, P_2, \cdots, P_n, Q_1, Q_2, \cdots, Q_n)$ such that $\ell = \sum_{i=1}^{n} \mathbb{R}P_i$, $\ell' = \sum_{i=1}^{n} \mathbb{R}Q_i$.

1.1.6. We define the symplectic group $G = Sp(B)$: By definition $g \in G$ if g is an invertible linear transformation of the vector space V preserving the form B, i.e. for every $x,y \in V$, $B(gx,gy) = B(x,y)$.

1.1.7. Let $V = \mathbb{R}P \oplus \mathbb{R}Q$, with $B(P,Q) = 1$, be the two dimensional canonical symplectic space. Then $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $Sp(B)$ if and only if $B(aP + cQ, bP + dQ) = B(P,Q) = 1$, i.e. $ad - bc = 1$. Hence the symplectic group for a two-dimensional symplectic vector space is isomorphic to $SL(2,\mathbb{R})$.

1.1.8. Let V be a $2n$ dimensional symplectic vector space. We choose a decomposition $V = \ell \oplus \ell'$ of V into two complementary Lagrangian spaces. We write x for an element of ℓ and y for an element of ℓ'. We write an element g of $G = Sp(B)$ as $g = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$, with

$$a: \ell \to \ell\,,\quad b: \ell' \to \ell$$
$$c: \ell \to \ell',\quad d: \ell' \to \ell' \quad .$$

The conditions for g to be in G are:

$$\begin{aligned} B(g\cdot x, g\cdot x') &= 0 & & x, x' \in \ell \\ B(g\cdot x, g\cdot y) &= B(x,y) & & x \in \ell,\ y \in \ell' \quad . \\ B(g\cdot y, g\cdot y') &= 0 & & y, y' \in \ell' \end{aligned}$$

We identify ℓ and ℓ'^* and ℓ' to $(\ell)^*$, via the bilinear map $B(x,y)$, $x \in \ell$, $y \in \ell'$.

We identify ℓ' with ℓ^* via $x \to B(x,y)$ for $y \in \ell'$.

1.1.9. Hence these conditions are equivalent to:

$$\begin{aligned} {}^t ca &= {}^t ac \\ {}^t ad - {}^t cb &= id \quad . \\ {}^t db &= {}^t bd \end{aligned}$$

In particular we have

$$g^{-1} = \left(\begin{array}{c|c} {}^t d & -{}^t b \\ \hline -{}^t c & {}^t a \end{array}\right) \quad .$$

As $g^{-1} \in G$ we have also ${}^t c d = {}^t d c$, ${}^t a b = {}^t b a$, ${}^t a d - {}^t b c = id$. If $x \in \ell$, $y \in \ell'$, and $u: \ell' \to \ell'$, then:

$$B(x, uy) = B({}^t u x, y).$$

1.1.10. From 1.1.5, we see that the group G acts transitively on the couples (ℓ, ℓ') of transverses Lagrangian planes, as they can be transformed to the canonical pair $\ell = \sum_{i=1}^{n} \mathbb{R}P_i$, $\ell' = \sum_{i=1}^{n} \mathbb{R}Q_i$ by a symplectic automorphism.

1.2. The Heisenberg group and the Schrödinger representation.

We consider the Heisenberg group N as being the simply connected Lie group of Lie algebra $\mathfrak{n}$. Via the exponential map exp, N is identified with the $2n+1$ vector space $V + \mathbb{R}E$ with the multiplication law:

$$\exp(v + tE)\cdot\exp(v' + t'E) = \exp(v + v' + (t + t' + \frac{B(v,v')}{2})E)$$

where $v, v' \in V$, $t, t' \in \mathbb{R}$.

For dv the euclidean measure on the vector space V, $dv\,dt$ is the Haar measure on N. ($dv\,dt$ is invariant by left and right translations.)

The subgroup $\{\exp tE\}$ is the center Z of N.

The group $G = Sp(B)$ acts as a group of automorphisms on N by $g\cdot(\exp v + tE) = \exp(gv + tE)$. In particular G preserves the center Z of N.

1.2.1. Let ℓ be a Lagrangian subspace of (V,B), then $\ell + \mathbb{R}E$ is an abelian subalgebra of $\mathfrak{n}$. We consider the group $L = \exp(\ell + \mathbb{R}E)$ which is an abelian subgroup of N.

1.2.2. Let us consider the function $f(\exp v + tE) = e^{2i\pi t}$ on the group N, f is a function with values in the torus $T = \{z \in \mathbb{C},\ |z| = 1\}$. It is immediate to see that f restricted to L defines a character of L, i.e.

$$f(h_1h_2) = f(h_1)f(h_2) \quad \text{for} \quad h_1, h_2 \in L.$$

(If we order the pairs (H,ψ) of a subgroup H of N together with a character ψ of H via the relation $(H,\psi) < (H_1,\psi_1)$ if $H \subset H_1$ and $\psi_1|H = \psi$, it is easy to see that the pair (L,f) is maximal for this order.)

1.2.3. Let us consider (ℓ,ℓ') as in (1.1.4). Every element n of N can be written uniquely as $n = \exp y' \exp(x + tE)$, with $y' \in \ell'$, $x \in \ell$, $t \in \mathbb{R}$. Hence the coset space N/L can be identified with ℓ'. The euclidean measure dv' on ℓ' defines on N/L a positive measure $d\dot{n}$ invariant by the left action of N on the homogeneous space N/L. We recall that such a measure $d\dot{n}$ is unique up to multiplication by a positive constant.

1.2.4. We consider for a given Lagrangian subspace ℓ the representation $W(\ell)$ induced by the character f of the group L. $W(\ell)$ is the Schrödinger representation of N associated to ℓ. We write

$$W(\ell) = \operatorname{Ind}_L^N \uparrow f.$$

By definition of induced representation, $W(\ell)$ is realized as follows: The Hilbert space $H(\ell)$ is the completion of the space of continuous functions φ on N such that

1.2.4. a) $\varphi(nh) = f(h)^{-1}\varphi(n)$, for every $n \in N$ and every $h \in L$.

1.2.4. b) The function $n \to |\varphi(n)|$ on N/L is square integrable with respect to the invariant measure $d\dot{n}$ on N/L.

The norm of φ is:

$$\|\varphi\|^2 = \int_{N/L} |\varphi(n)|^2 \, d\dot{n} .$$

The representation $W(\ell)$ is defined to be the representation of N in $H(\ell)$ given by left translations:

i.e. $(W(\ell)(n_0)\varphi)(n) = \varphi(n_0^{-1}n)$ for $\varphi \in H(\ell)$, $n_0 \in N$.

As $\exp tE$ is in the center of the group N, we have:

$$\begin{aligned}(W(\ell)(\exp tE)\varphi)(n) &= \varphi((\exp - tE)n)\\ &= \varphi(n \exp - tE)\\ &= e^{2i\pi t}\varphi(n)\end{aligned}$$

i.e. $W(\ell)(\exp tE) = e^{2i\pi t} Id_{H(\ell)}$,

where $Id_{H(\ell)}$ denotes the identity operator on $H(\ell)$.

1.2.5. Let us consider (ℓ, ℓ') as in 1.1.4. Each element of N is written uniquely as $n = \exp y \cdot \exp(x + tE)$ where $y \in \ell'$, $x \in \ell$. Hence, if $\varphi \in H(\ell)$, the condition 1.2.4 a) is written as $\varphi(\exp y \cdot \exp x \cdot \exp tE) = e^{-2i\pi t}\varphi(\exp y)$, $(y \in \ell'$ $x \in \ell$, $t \in E)$. So φ is completely determined by its restriction to $\exp \ell'$. Hence the application $\varphi \to \varphi(y) = \varphi(\exp$ defines an isometry R of $H(\ell)$ with $L^2(\ell')$. The representation $\widehat{W}(n) = RW(n)R^{-1}$ acts on $L^2(\ell')$ by the following formulas:

$$(\widetilde{W}(\exp x)\varphi)(y) = e^{2i\pi B(x,y)}\varphi(y) \qquad x \in \ell,\ y \in \ell'$$

$$(\widetilde{W}(\exp y_0)\varphi)(y) = \varphi(y - y_0) \qquad y,\ y_0 \in \ell'$$

$$\widetilde{W}(\exp tE) = e^{2i\pi t} Id.$$

Let us consider $x = \Sigma x_i P_i$, $y = \Sigma y_j Q_j$ as in 1.1.5. Then $L^2(\ell')$ is identified with $L^2(\mathbb{R}^n)$. We consider the space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions on $\mathbb{R}^n$. Then it is easy to see that if

$$X \in \mathfrak{n},\ f \in \mathcal{S},\ dW(X)\cdot f = \frac{d}{dt} W(\exp tX)\cdot f\Big|_{t=0}$$

defines a representation of $\mathfrak{n}$, which is the infinitesimal representation associated to $\widetilde{W}$. From the preceding formulas, we see immediately that

$$\begin{aligned} d\widehat{W}(P_j) &= 2i\pi x_j\\ d\widehat{W}(Q_i) &= -\frac{\partial}{\partial x_i}\\ d\widehat{W}(E) &= 2i\pi Id.\end{aligned}$$

In a sense to be made precise, this is the unique representation of $\mathfrak{n}$ which can be exponentiated to a unitary representation of N.

1.3. The Weyl transform and the Stone-Von Neumann theorem.

1.3.1. Let G be a topological group. A unitary representation T of G in the Hilbert space H is a homomorphism $g \to T(g)$ of G in the group of unitary operators on H (i.e. $T(g_1 g_2) = T(g_1) \cdot T(g_2)$). We also require the continuity of the maps from G to H given by $g \to T(g)x$ for every $x \in H$.

There is an obvious notion of equivalence: if (T,H) and (T',H') are two representations of G in H and H', we say that $T \sim T'$ if an isomorphism $I: H \to H'$ exists such that the diagram:

$$\begin{array}{ccc} H & \xrightarrow{I} & H' \\ \downarrow T_1(g) & & \downarrow T_2(g) \\ H & \xrightarrow{I} & H' \end{array}$$

is commutative for every $g \in G$.

If T_1 and T_2 are two representations of G in H_1 and H_2, we can form $T = T_1 \oplus T_2$ represented in $H = H_1 \oplus H_2$ by

$$\left(\begin{array}{c|c} T_1(g) & 0 \\ \hline 0 & T_2(g) \end{array}\right).$$

We say that T is irreducible, if T cannot be written as $T_1 \oplus T_2$, or equivalently if there is no closed subspace H_1 of H stable under all the operators $T(g)$.

Let H be a Hilbert space, we will denote by $\overline{H}$ the same space H, but with the multiplication law $t \cdot x = \overline{t}x$ and the

scalar product $<x,y>_{\overline{H}} = \overline{<x,y>_H}$. An isomorphism of H and $\overline{H}$ is then given by an antilinear map $\sigma: H \to H$, such that $\sigma(\lambda x) = \overline{\lambda}\sigma(x)$, $\|\sigma(x)\| = \|x\|$. If T is a unitary representation of G in H, the same formulas for $T(g)$ define a unitary representation of G in $\overline{H}$ that we denote by $\overline{T}$. If we have defined $\sigma: H \to \overline{H}$ an antilinear isomorphism between H and $\overline{H}$, the representation $\overline{T}$ is equivalent to the representation $\sigma^{-1} \circ T \circ \sigma$ on the Hilbert space H.

Let H_1, H_2 be two Hilbert spaces; we consider the Hilbert space $H_1 \otimes H_2$: $H_1 \otimes H_2$ is the completion of the vector space spanned by finite linear combinations $\sum_{i=1}^{N} v_i \otimes w_i$, $v_i \in H_1$, $w_i \in H_2$ for the natural inner product such that $<v_1 \otimes w_1, v_2 \otimes w_2> = <v_1,v_2><w_1,w_2>$. If (e_i) is a Hilbert basis of H_1 and (f_j) a Hilbert basis of H_2, then $e_i \otimes f_j$ is a Hilbert basis of $H_1 \otimes H_2$. Also $H_1 \otimes H_2$ is the Hilbert sum of the Hilbert subspaces $(H_1 \otimes \mathbb{C}f_j)$. If we are mainly interested in H_1, we will say that $H_1 \otimes H_2$ is a multiple of the space H_1, with multiplicity equal to $\dim H_2$, i.e. finite or $+\infty$.

Let T_1 be a unitary representation of G in H_1. We consider the representation T of G in $H_1 \otimes H_2$ given by $T(g) = T_1(g) \otimes Id_{H_2}$, where Id_{H_2} denotes the identity operator on H_2. We will then say that T is a multiple of the representation T_1 (with multiplicity $\dim H_2$): $H_1 \otimes H_2$

can be written as $\bigoplus_j (H_1 \otimes \mathbb{C}f_j)$ and $T = \bigoplus_j T_1(g) \otimes 1_{\mathbb{C}f_j}$.

Let T be a unitary representation of a Lie group, with a left invariant Haar measure dg, on a Hilbert space H. We define the space H^∞ of C^∞ vectors for T by the following condition: $x \in H^\infty$ if the map $g \to T(g)x$ is C^∞.

Let φ be a function with compact support. We can form the operator:

$$T(\varphi) = \int \varphi(g)\, T(g)\, dg, \text{ i.e.}$$

$$\langle T(\varphi)x,y\rangle = \int_G \varphi(g) \langle T(g)x,y\rangle\, dg .$$

This integral makes sense if φ is in $L^1(G,dg)$, or if x is in H^∞ and $\varphi(g)dg$ a distribution with compact support. For and φ_2 continuous functions with compact support, we form the convolution product $\varphi_1 *_G \varphi_2$ defined by:

$$(\varphi_1 * \varphi_2)(g) = \int_G \varphi_1(u)\, \varphi_2(u^{-1}g)\, du .$$

We have

$$\int_G (\varphi_1 * \varphi_2)(g)\psi(g)dg = \iint_{G\times G} \varphi_1(g)\varphi_2(h)\psi(gh)dg\, dh$$

if ψ is continuous with compact support. Thus we have defined a structure of algebra on the space of continuous functions. with compact support, as well as on $L^1(G)$ or on the space $\mathcal{E}'(G)$ of distributions with compact support.

The following proposition is immediate to prove:

1.3.2. Proposition: For every unitary representation T, we have: $T(\varphi_1 *_G \varphi_2) = T(\varphi_1) \circ T(\varphi_2)$.

Remark: If G is unimodular, and $\varphi^*(g) = \overline{\varphi(g^{-1})}$, we have $T(\varphi^*) = T(\varphi)^*$, where $T(\varphi)^*$ is the adjoint of $T(\varphi)$.

1.3.3. Let N be the Heisenberg group. We will prove the Stone-Von Neumann theorem (or uniqueness of the representation of the canonical Heisenberg commutation relations):

Theorem:

a) $W(\ell)$ is an irreducible representation of N.

b) Every unitary representation T of N on a Hilbert space H such that $T(\exp tE) = e^{2i\pi t}\, Id_H$ is a multiple of $W(\ell)$.

This theorem will be fundamental in our work.

Let N be the Heisenberg group. We consider continuous functions φ on N satisfying

1.3.3.a) $\varphi(n \exp tE) = e^{-2i\pi t}\varphi(n)$ for every $n \in N$, $t \in \mathbb{R}$.

Such functions can be considered as functions on the group $B = N/\Gamma$ where Γ is the discrete central subgroup

$$\Gamma = \{n = \exp kE;\ k \in \mathbb{Z}\} .$$

Every element $n \in B$ is written uniquely as $n = \exp v \exp tE$, with $t \in \mathbb{R}/\mathbb{Z}$. We obtain a Haar measure db on B by considering the product $dv\, dt$ where dv is a Lebesgue measure on V, and dt

a mass one Lebesgue measure on $\mathbb{R}E/\mathbb{Z}E$.

A function satisfying 1.3.3 a) is determined by its restric[tion] to exp V. Hence the space of continuous functions on N satisf[ying] 1.3.3. a) is identified with the space of continuous functions on V. We still denote by $\varphi(v) = \varphi(\exp v)$ the restriction of φ to V.

Let T be a unitary representation of N satisfying

$$T(\exp tE) = e^{2i\pi t}\mathrm{Id}_H .$$

Then T is a representation of B. We form

1.3.4. $\quad W_T(\varphi) = T(\varphi) = \int_B \varphi(b)T(b)db = \int_V \varphi(v)T(\exp v)dv$

for φ a continuous compactly supported function on V (or B) $W_T(\varphi)$ is the Weyl transform of the functions φ. We will also consider $W_T(\varphi)$ for φ a rapidly decreasing function on V.

Compactly supported continuous functions satisfying 1.3.3. form an algebra under the convolution product on B. We have then defined a structure of algebra on the space of continuous functions with compact support on V. We have:

$$(\varphi_1 *_B \varphi_2)(\exp v) = \int_V \varphi_1(\exp u)\, \varphi_2(\exp -u \exp v)du$$

$$= \int_V \varphi_1(\exp u)\, \varphi_2(\exp(v-u) \exp - \frac{B(u,v)}{2} E)du$$

$$= \int_V \varphi_1(\exp u)\, \varphi_2(\exp(v-u))\, e^{i\pi B(u,v)}du$$

i.e.

1.3.5. $\quad (\varphi_1 *_B \varphi_2)(v) = \int_V \varphi_1(u)\, \varphi_2(v-u)\, e^{i\pi B(u,v)}du$.

If φ satisfies 1.3.3 a), the function $\varphi^*(n) = \overline{\varphi(n^{-1})}$ also satisfies 1.3.3 a); thus identifying φ with a function on V, we have $\varphi^*(v) = \overline{\varphi(-v)}$.

If φ satisfies 1.3.3.a) the function $n \to \varphi(n_0^{-1}n)$ as well as the function $n \to \varphi(nn_0)$ satisfies 1.3.3.a) (for $n_0 \in N$); we will denote them by $n_0 *_B \varphi$ and $\varphi *_B n_0^{-1}$, (convolution with Dirac distributions).

If $n_0 = \exp u_0$, identifying φ with a function on V,

$$(n_0 *_B \varphi)(u) = e^{i\pi B(u_0,u)}\varphi(u-u_0)$$

$$(\varphi *_B n_0)(u) = e^{i\pi B(u,u_0)}\varphi(u-u_0) .$$

1.3.6. From Proposition 1.3.2, we deduce:

a) $W_T(\varphi_1 *_B \varphi_2) = W_T(\varphi_1) \circ W_T(\varphi_2)$

b) $W_T(\varphi^*) = W_T(\varphi)^*$

c) $W_T(n_0 *_B \varphi) = T(n_0) \circ W_T(\varphi)$

d) $W_T(\varphi *_B n_0) = W_T(\varphi) \circ T(n_0)$.

1.3.7. We consider $T = W(\ell)$; we will now see that the Weyl transform W extends to an isomorphism from the space $L^2(V)$ to the space of Hilbert-Schmidt operators on $H(\ell)$. We recall some facts on Hilbert-Schmidt operators:

Let H be a Hilbert space. We recall that a Hilbert-Schmidt operator $A: H \to H$ on H is an operator such that for some orthonormal basis (e_i) of H,

$$\sum_i \|Ae_i\|^2 = \sum_{i,j} |\langle Ae_i, e_j\rangle|^2 = \sum_{i,j} |\langle e_i, A^*e_j\rangle|^2 = \sum_{i,j} |a_{ij}|^2 < \infty .$$

This sum doesn't depend on the choice of the orthonormal basis (e_i) and is denoted by $\|A\|^2_{H.S.}$ For A and B Hilbert-Schmidt operators, B^*A is of trace class and $<A,B> = \mathrm{Tr}\, B^*A = \sum_i <B^*Ae_i,e_i> = \sum_i <Ae_i,Be_i>$ defines a scalar product on the space $\mathcal{J}\ell_2(H)$ of Hilbert-Schmidt operators on H; hence $\mathcal{J}\ell_2(H)$ is a Hilbert space, having as basis the operators $E_{i,j}(x) = <x,e_i>e_j$. Let $x,y \in H$, we define the rank one operator $E_{x,y}(v) = <v,y>x$ on H. Clearly $E_{x,y}$ is Hilbert-Schmidt and

$$<E_{x_1,y_1},E_{x_2,y_2}> = <x_1,x_2>\overline{<y_1,y_2>} .$$

Hence as the map $x,y \to E_{x,y}$ is linear in x, antilinear in y we obtain an isometry from $H \otimes \overline{H}$ onto $\mathcal{J}\ell_2(H)$.

Let us suppose now that $H = L^2(E,dy)$ where (E,dy) is a measure space: We will see that the space of Hilbert-Schmidt operators on $L^2(E,dy)$ is equivalent to the space $L^2(E \times E,dxdy)$ of square integrable functions $K(x,y)$ on $E \times E$ via

$$(K\varphi)(x) = \int K(x,y)\varphi(y)dy$$

Let (e_i) be a Hilbert basis of $L^2(E)$. We denote by

$$(e_i \otimes e_j)(\varphi) = <\varphi,e_j> e_i .$$

Hence

$$((e_i \otimes e_j)(\varphi))(x) = <\varphi,e_j> e_i(x)$$
$$= (\int \varphi(y)\, \overline{e}_j(y)dy)\, e_i(x)$$

$$= \int \varphi(y)\, e_i(x)\overline{e}_j(y)dy$$
$$= \int K_{ij}(x,y)\, \varphi(y)dy$$

with $K_{ij}(x,y) = e_i(x)\overline{e_j(y)}$.

The Hilbert-Schmidt operators are of the form $\Sigma\, a_{ij}\, e_i \otimes e_j$ with $\Sigma\, |a_{ij}|^2 < \infty$; hence $K(x,y) = \Sigma\, a_{ij}\, e_i(x)\overline{e_j(y)}$ is such that

$$\int |K(x,y)|^2\, dxdy = \Sigma\, |a_{ij}|^2 < \infty .$$

Thus the operator associated to the element $\Sigma\, a_{ij}\, e_i \otimes e_j$ of $H \otimes \overline{H}$ is the operator $\int K(x,y)\, \varphi(y)dy$ with $K(x,y) = \Sigma\, a_{ij}\, e_i(x)\overline{e_j(y)}$. Let us remark here for later reference: that if $\psi_0 \in L^2(V)$ is of norm 1, the projector $P: H \to \mathbb{C}\psi_0$ given by $P(\varphi) = <\varphi,\psi_0>\psi_0$ is given by the kernel $K = \psi_0(x)\overline{\psi_0(y)}$.

1.3.8. We now prove

<u>Proposition</u>: Let $T = W(\ell)$; the Weyl transform $W = W_T$ extends to an isomorphism from $L^2(V)$ to the space of Hilbert-Schmidt operators on $H(\ell)$.

<u>Proof</u>: We have

$$W_T(\varphi) = \int_V \varphi(v)\, T(v)dv .$$

We write $V = \ell' \oplus \ell$ according to the choice of ℓ,ℓ' as in 1.1.4. Hence

$$W_T(\varphi) = \iint \varphi(y+x) \quad T(\exp(y+x))\, dydx, \quad y \in \ell',\ x \in \ell$$
$$= \iint \varphi(y+x) \quad T(\exp y) \quad T(\exp x)e^{-i\pi B(y,x)}\, dxdy.$$

Let us identify $H(\ell)$ with $L^2(\ell')$ as in 1.2.5, and we write $W_T(\varphi)\cdot f$ for $f \in L^2(\ell')$

$$(W_T(\varphi)\cdot f)(\xi) = \iint e^{-i\pi B(y,x)}\varphi(y+x)(\widetilde{W}(\exp y)\widetilde{W}(\exp x)f)(\xi)\, dxdy$$

$$= \iint e^{-i\pi B(y,x)}\varphi(y+x)e^{2i\pi B(x,\xi-y)}f(\xi-y)\, dxdy$$

Changing y to $\xi - y$, we have

$$= \iint e^{-i\pi B(\xi-y,x)}\varphi(\xi-y+x)e^{2i\pi B(x,y)}f(y)\, dxdy$$

$$= \iint e^{i\pi B(x,\xi+y)}\varphi(\xi-y+x)f(y)\, dxdy.$$

Hence if we define:

$$K_\varphi(\xi,y) = \int e^{i\pi B(x,y+\xi)}\varphi(\xi-y+x)\, dx,$$

we have written $W_T(\varphi)$ as a kernel operator, i.e.

$$(W_T(\varphi)f(\xi) = \int_{\ell'} K_\varphi(\xi,y)f(y)dy.$$

To prove our proposition, we have to see that if $\varphi \in L^2(V)$ the corresponding kernel $K_\varphi \in L^2(\ell' \times \ell')$. The bilinear form $B(x,\xi)(x \in \ell,\ \xi \in \ell')$ defines a nondegenerate bilinear map on $\ell \times \ell'$. Hence the partial Fourier transform

$$(\mathcal{F}_x\varphi)(y,\xi) = \int e^{-2i\pi B(x,\xi)}\varphi(y+x)\, dx$$

is a unitary isomorphism from $L^2(\ell' \oplus \ell)$ onto $L^2(\ell' \times \ell')$. But our kernel is then

$$(K_\varphi)(\xi,y) = (\mathcal{F}_x\varphi)(\xi-y,\ -\frac{y+\xi}{2})$$

which is clearly in $L^2(\ell' \times \ell')$. Hence our proposition is prove

1.3.9. <u>Corollary</u>: If U is a bounded operator on $H(\ell)$ commuting with all the operators $W(\ell)(n)$ for $n \in N$ then U is a scalar operator.

<u>Proof</u>: Let $U: H(\ell) \to H(\ell)$ commuting with all the operators $W(\ell)(n)$ for every n, then U commutes with the operator $W_T(\varphi)$ for φ function on V which is compactly supported and hence by continuity with all the operators $W(\varphi)$ for $\varphi \in L^2(V)$, in particular with all the Hilbert-Schmidt operators. Taking a Hilbert basis (e_i), the relation $U(e_i \otimes e_j) = (e_i \otimes e_j)\ U$ for every (i,j) implies immediately that $U = \lambda\ \mathrm{Id}$ for some $\lambda \in \mathbb{C}$. It follows that there cannot be any closed invariant proper subspace H_1 of $H(\ell)$ (the projector P_{H_1} has to be Id_H by the preceding corollary). Hence the part a) of the theorem is proven.

1.3.10. Let us fix $\ell = \ell_0$, $\ell' = \ell_0'$, $H_0 = H(\ell_0) = L^2(\ell_0')$, $W_0 = W(\ell_0)$. We remark that, if φ_1, φ_2 belong to the space $\mathcal{S}(V)$ of rapidly decreasi[ng] functions on V, then $\varphi_1 *_B \varphi_2 \in \mathcal{S}(V)$. Hence we have defined a structure of (noncommutative) algebra on $\mathcal{S}(V)$. In the Weyl transform given by W_0 the corresponding kernel $K_\varphi(\xi,y)$ for $\varphi \in \mathcal{S}(V)$ is a rapidly decreasing function of (ξ,y), and we obtain this way all the operators with rapidly decreasing kernels.

Now we will prove the Stone-Von Neumann theorem. Let us first sketch the idea of the proof: via the Weyl transform W_{T_0}, we have identified the algebra $(\mathcal{S}(V), *_B)$ with a subalgebra $\mathcal{O}\!\!\!/_0$ of operators on H_0. If (T,H) is a representation of N

into a Hilbert space H satisfying $T(\exp tE) = e^{2i\pi t}\, Id_H$, the Weyl transform W_T defines a homomorphism of $(\mathcal{S}(V), *_B)$ into a subalgebra $\mathcal{A}$ of operators on H. Hence we obtain a homomorphism $\varphi: \mathcal{A}_0 \to \mathcal{A}$ of algebras such that $\varphi(A^*) = \varphi(A)^*$.

Let us recall that if V_0 and V are two finite-dimensional complex Hilbert spaces, and φ a homomorphism from $End_{\mathbb{C}} V_0$ to $End_{\mathbb{C}} V$ satisfying $\varphi(A^*) = \varphi(A)^*$, $\varphi(1) = 1$, then there exist V_1 and an isomorphism $I: V_0 \otimes V_1 \to V$ such that for

$$A \in End_{\mathbb{C}} V_0,\ \varphi(A) = I \circ (A \otimes Id_{V_1}) \circ I^{-1}.$$

We will give here the proof for the finite-dimensional case as our proof in the general case will be similar:
Let $x_1 \in V_0$ with $\|x_1\| = 1$; we consider P_1 the projector $P_1(x) = \langle x, x_1\rangle x_1$ on the one-dimensional space $\mathbb{C}x_1$; we have $P_1^2 = P_1$, $P_1 = P_1^*$.

Let us consider $\varphi(P_1)$. First we remark that $\varphi(P_1) \neq 0$. In fact we can find $(x_1, x_2, \cdots, x_n)$ an orthonormal basis of V_0 starting with x_1. We denote by P_i the projector on $\mathbb{C}x_i$. If g_i is any operator on V such that $g_i(x_1) = x_i$, then $g_i P_1 g_i^* = P_i$. Hence $\sum_i g_i P_1 g_i^* = Id_{V_0}$. Applying the homomorphism φ, we obtain $\sum_{i=1}^{n} \varphi(g_i)\varphi(P_1)\varphi(g_i)^* = Id_V$ so $\varphi(P_1) \neq 0$ and verify $\varphi(P_1)^2 = \varphi(P_1)$, $\varphi(P_1)^* = \varphi(P_1)$, i.e. $\varphi(P_1)$ is a projector on a subspace V_1 of V. From the relation $\sum_{i=1}^{n} \varphi(g_i)\varphi(P_1)\varphi(g_i)^* =$ we see that every element of V is of the form $\sum_{i=1}^{n} \varphi(g_i) w_i$ with $w_i \in V_1$. We consider the surjective map

$I: V_0 \otimes V_1 \to V$ defined by $I(A \cdot x_1 \otimes w) = \varphi(A)w$. This map is well defined as, if $(A-B)x_1 = 0$, $(A-B)P_1 = 0$, then $(\varphi(A)-\varphi(B))\varphi(P_1) = 0$ meaning that $\varphi(A)w = \varphi(B)w$ for $w \in V_1$. Clearly now $I(Ax \otimes w) = \varphi(A) I(x \otimes w)$ and it is easy to conclude the proof.

Now we come back to our case. Let us choose a function ψ_1 in $\mathcal{S}(\ell')$ such that $\|\psi_1\|_{L^2(\ell')} = 1$. We consider the projector P_1 on $\mathbb{C}\psi_1$, P_1 is given by the kernel $\overline{\psi_1(y)}\, \psi_1(x)$ and hence is of the form $W_0(\varphi_1)$ for $\varphi_1 \in \mathcal{S}(V)$. As P_1 is a projector on the one-dimensional subspace $\mathbb{C}\psi_1$ we have $P_1^2 = P_1$, $P_1^* = P_1$, $P_1 \circ W_0(n) \circ P_1 = \alpha(n) P_1$ with $\alpha(n) = \langle W_0(n)\psi_1, \psi_1\rangle$. Hence from 1.3.6 the function φ_1 satisfies

1.3.11. $\quad \varphi_1 *_B \varphi_1 = \varphi_1, \quad \varphi_1^* = \varphi_1, \ \varphi_1 *_B n *_B \varphi_1 = \alpha(n)\varphi_1.$

As $\varphi_1 \in \mathcal{S}(V)$, we can calculate $W_T(\varphi_1)$ for any unitary representation (T, H) of N satisfying $T(\exp tE) = e^{2i\pi t}\, Id_H$.

1.3.12. <u>Lemma</u>: The space H is generated by the elements $T(n) W_T(\varphi_1) \cdot x$ for $n \in N$, $x \in H$.

<u>Proof</u>: Let $y \in H$ be such that $\langle y, T(n) W_T(\varphi_1) \cdot x\rangle = 0$ for every $n \in N$, $x \in H$. We compute for $n_0 = \exp u$,

$$\langle y, T(n_0) W_T(\varphi_1) T(n_0)^{-1} . x\rangle = 0 = \int_V \langle y, T(\exp u) T(\exp v) T(\exp -u) \cdot x\rangle \varphi_1(v)\, dv$$
$$= \int_V \langle y, T(\exp(v + B(u,v)E)) \cdot x\rangle \varphi_1(v)\, dv$$
$$= \int_V \langle y, T(\exp v) x\rangle \varphi_1(v) e^{2i\pi B(u,v)}\, dv = 0.$$

Our function $\varphi_1(v) \langle y, T(\exp v)x\rangle$ is a continuous function in L^2 (as $\varphi_1(v) \in \mathcal{J}(V)$ and $|\langle y, T(\exp v)x\rangle| \leq \|y\|\|x\|$ is bounded. The preceding equality means that the Fourier transform of this function with respect to the bilinear form $B(u,v)$ is identical zero. Hence $\varphi_1(v) \langle y, T(\exp v)x\rangle \equiv 0$. As φ_1 is not identically zero, there exist v_0 such that $\langle y, T(\exp v_0)x\rangle = 0$ for every $x \in H$. This implies $\langle y, H\rangle = 0$ hence $y = 0$.

Now from the relations 1.3.11 and lemma 1.3.6, we deduce that $W_T(\varphi_1)$ is a projector on the subspace $H_1 = W_T(\varphi_1)H$ of H. As in the finite-dimensional case, we wish to define $I: H_0 \otimes H_1$ via the formula $I(W_0(n)\psi_1 \otimes w) = T(n)\cdot w$, with $n \in N$, $w \in H_1$. We first verify

1.3.13. For $w_1 = W_T(\varphi_1)x_1$, $w_2 = W_T(\varphi_1)x_2$, $n_1, n_2 \in N$

$$\langle T(n_1)w_1, T(n_2)w_2\rangle_H = \langle W_0(n_1)\psi_1, W_0(n_2)\psi_1\rangle_{H_0} \langle w_1, w_2\rangle_{H_1}.$$

<u>Proof</u>: We have

$$\langle T(n_1)W_T(\varphi_1)x_1, T(n_2)W_T(\varphi_1)x_2\rangle_H$$
$$= \langle W_T(\varphi_1)T(n_2)^{-1}T(n_1)W_T(\varphi_1)x_1, x_2\rangle_H .$$

Using relations 1.3.11 and lemma 1.3.6, this equals

$$= \alpha(n_2^{-1}n_1)\langle W_T(\varphi_1)x_1, x_2\rangle_H = \alpha(n_2^{-1}n_1)\langle W_T(\varphi_1)x_1, W_T(\varphi_1)x_2\rangle_H$$

which is the desired equality.

As the representation W_0 is irreducible, the set of linear combinations $\Sigma\, c_i W_0(n_i)\cdot\psi_1$ is a dense subspace of H_0.

It is clear now that we can define as isometry I from $H_0 \otimes H_1 \to H$ via the formula

$$I\left(\sum_{i=1}^{N} W_0(n_i)\psi_1 \otimes w_i\right) = \sum_{i=1}^{N} T(n_i)w_i .$$

This map is well defined as if $\sum_{i=1}^{N} W_0(n_i)\cdot\psi_1 \otimes w_i = 0$ the equality 1.3.13 implies $\|\sum_{i=1}^{N} T(n_i)w_i\|_H = \|\Sigma\, W_0(n_i)\psi_1 \otimes w_i\|_{H_0 \otimes H_1} = 0$. The operator I is surjective by 1.3.12. Hence I is a unitary isomorphism between $H_0 \otimes H_1$ and H. Clearly $I\,(W_0(n) \otimes Id_{H_1})\, I^{-1} = T(n)$. Hence T is a multiple of the representation W_0.

1.4. Fourier transforms and intertwining operators.

Let ℓ_1 and ℓ_2 be two Lagrangian planes. We can form the unitary representations $W_1 = W(\ell_1)$ and $W_2 = W(\ell_2)$ of N. By the Stone-Von Neumann theorem, we know that they are equivalent, i.e. there exists a unitary operator $\mathcal{F}_{2,1}: H(\ell_1) \to H(\ell_2)$ such that:

1.4.1. $$\mathcal{F}_{2,1}\ W_1(n) = W_2(n)\ \mathcal{F}_{2,1}, \text{ for every } n \in N.$$

$\mathcal{F}_{2,1}$ is determined by this relation up to a scalar of modulus one, as follows from 1.3.9.

1.4.2. Let us first compute $\mathcal{F}_{2,1}$ in the case where $\ell_1 = \mathbb{R}P_1 \oplus \cdots$ and $\ell_2 = \mathbb{R}Q_1 \oplus \cdots \oplus \mathbb{R}Q_n$. We adopt the following conventions: $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n)$, $x\cdot y = \sum_{i=1}^{n} x_i y_i$, $x\cdot P = \sum_{i=1}^{n} x_i P_i$, $y\cdot Q = \sum_{i=1}^{n} y_i Q_i$.

Then W_1 acts on $L^2(dy)$ and W_2 on $L^2(dx)$ by the follo[wing] formulas:

$$(W_1(\exp x_0P)\varphi)(y) = e^{2i\pi x_0\cdot y}\,\varphi(y)$$

$$(W_1(\exp y_0Q)\varphi)(y) = \varphi(y - y_0)$$

$$(W_2(\exp x_0P)\varphi)(x) = \varphi(x - x_0)$$

$$(W_2(\exp y_0Q)\varphi)(x) = e^{-2i\pi x\cdot y_0}\,\varphi(x).$$

We denote by $\mathcal{F}$ the Fourier transform from $L^2(dy) \simeq H(\ell_1)$ to $L^2(dx) \simeq H(\ell_2)$ given by

1.4.3. $$(\mathcal{F}\varphi)(x) = \int e^{-2i\pi x\cdot y}\,\varphi(y)dy.$$

Since $\mathcal{F}$ transforms translation operators into multiplication operators it is immediate that $\mathcal{F}\cdot W_1(n) = W_2(n)\circ\mathcal{F}$. Hence $\mathcal{F}_{2,1} = \mathcal{F}$.

1.4.4. Let ℓ_1 and ℓ_2 be two Lagrangian planes. Let $L_1 = \exp(\ell_1 + \mathbb{R}E)$ and $L_2 = \exp(\ell_2 + \mathbb{R}E)$ be the subgroups of N associated to ℓ_1 and ℓ_2 (1.2.1). We consider, as in 1.2.4, the Hilbert spaces $H(\ell_1)$ and $H(\ell_2)$ canonically associated to ℓ_1 and ℓ_2. We wish to find an operator from H_1 to H_2 intertwining the representations $W(\ell_1)$ and $W(\ell_2)$.

The formal construction is simple: We look for an operator commuting with left translations and transforming a function φ semi-invariant under the right action of L_1 (i.e. verifying 1.2.4 a)) into a function φ semi-invariant under L_2. Hence it is natural to "force" φ to be semi-invariant under L_2 by averaging right translates of φ under L_2, taking in account that φ verifies 1.2.4 a) for $h \in L_1 \cap L_2$.

Hence we will define formally:

1.4.5. $$(\mathcal{F}_{\ell_2,\ell_1}\varphi)(n) = \int_{L_2/L_1\cap L_2} \varphi(nh_2)f(h_2)d\dot{h}_2$$

where $d\dot{h}_2$ denotes a positive L_2-invariant measure on the homogeneous space $L_2/L_1\cap L_2$. As $d\dot{h}_2$ is unique up to multiplication by a positive scalar, we remark that $\mathcal{F}_{\ell_2,\ell_1}$ is therefore defined up to multiplication by a positive constant.

Let us compute $\mathcal{F}_{\ell_2,\ell_1}$ for the preceding example. We have

$L_2 \cap L_1 = \{ \exp tE \}$. Hence $(L_2/L_2 \cap L_1, d\dot{h}_2)$ is identified with (ℓ_2, dy), and

$$(\mathcal{F}_{\ell_2,\ell_1}\varphi)(n) = \int_{\ell_2} \varphi(n \exp y\cdot Q)dy.$$

We identify $H(\ell_1)$ with $L^2(dy)$ by $\varphi(y) = \varphi(\exp y\cdot Q)$, $H(\ell_2)$ with $L^2(dx)$ by $\varphi(x) = \varphi(\exp x\cdot P)$. Our operator $\mathcal{F}_{\ell_2,\ell_1}$ becomes

$$\begin{aligned}(\mathcal{F}_{\ell_2,\ell_1}\varphi)(\exp x\cdot P) &= \int_{\ell_2} \varphi(\exp x.P \exp y\cdot Q)dy \\ &= \int_{\ell_2} \varphi(\exp x\cdot P \exp y\cdot Q \exp -x\cdot P \exp x.P)dy \\ &= \int_{\ell_2} \varphi(\exp y\cdot Q \exp -x\cdot P \exp(x.y)E \\ &= \int_{\ell_2} \varphi(\exp yQ)e^{-2i\pi x\cdot y}dy, \\ &\qquad \text{as } \varphi \in H(\ell_1) \\ &= \int_{\ell_2} \varphi(y)e^{-2i\pi x\cdot y}dy = (\mathcal{F}\varphi)(x).\end{aligned}$$

Hence $\mathcal{F}_{2,1}$ defined formally by 1.4.3 is indeed a unitary operator given by the Fourier transform.

The following lemma shows that this is basically the only situation.

1.4.6. Lemma: Let ℓ_1 and ℓ_2 be two Langrangian subspaces of (V,B). There exists a symplectic basis $(P_1, P_2, \cdots, P_n, Q_1, Q_2, \cdots, Q_n)$ of V such that:

$$\ell_1 = \mathbb{R}P_1 \oplus \mathbb{R}P_2 \oplus \cdots \oplus \mathbb{R}P_k \oplus \mathbb{R}P_{k+1} \oplus \cdots \oplus \mathbb{R}P_n$$

$$\ell_2 = \mathbb{R}Q_1 \oplus \mathbb{R}Q_2 \oplus \cdots \oplus \mathbb{R}Q_k \oplus \mathbb{R}P_{k+1} \oplus \cdots \oplus \mathbb{R}P_n.$$

Proof: Let W be maximal in the collection of all totally isotropic subspaces S satisfying $(\ell_1+\ell_2) \cap S = 0$. Hence $W^\perp \subset \ell_1 + \ell_2 + W$ by the maximality condition. But we have $(\ell_1 \cap \ell_2) + W^\perp = V$ as $(\ell_1+\ell_2) \cap W = 0$ (1.1.3), hence $(\ell_1 \cap \ell_2) + \ell_1 + \ell_2 + W = V$, i.e. W is a complementary subspace to $\ell_1 + \ell_2$ in V. The bilinear form B defines a duality between $\ell_1 \cap \ell_2$ and $V/\ell_1+\ell_2 \simeq W$. Hence we can choose a basis $P_{k+1}, \cdots, P_n$ of $\ell_1 \cap \ell_2$ and a basis $Q_{k+1}, \cdots, Q_n$ of W such that $B(P_i,Q_j) = \delta_{ij}$, $B(P_i,P_j) = 0$, $B(Q_i,Q_j) = 0$ for $i,j \geq k+1$. Let us consider the subspace $M = \ell_1 \cap \ell_2 + W$. Clearly $V = M \oplus M^\perp$ as $M \cap M^\perp = 0$. We have

$$\ell_1 = \ell_1 \cap \ell_2 + \ell_1 \cap M^\perp$$

$$\ell_2 = \ell_1 \cap \ell_2 + \ell_2 \cap M^\perp.$$

It is immediate to see that $\ell_1 \cap M^\perp$ and $\ell_2 \cap M^\perp$ are transverse Lagrangian planes in $M^\perp$. We then choose $P_1, P_2, \cdots, P_k$ a basis of $\ell_1 \cap M^\perp$, $Q_1, Q_2, \cdots, Q_k$ a basis of $\ell_2 \cap M^\perp$ by duality, and we obtain the lemma.

Let us consider ℓ_1, ℓ_2 and the symplectic basis of V given by the lemma 1.4.6. We will write $x = (x', x'')$, with $x' = (x_1, x_2, \cdots, x_k)$, $x'' = (x_{k+1}, \cdots, x_n)$ and similar notations. Then $H(\ell_1)$ is identified with $L^2(dy'dy'')$ and

$H(\ell_2)$ is identified with $L^2(dx'dy'')$. Our formal operator $\mathcal{F}_{2,1}$ becomes then $\varphi(y',y'') \to (\mathcal{F}'\varphi)(x',y'')$ where $\mathcal{F}'$ denotes the partial Fourier transform in the first k-variables. If we compute $\mathcal{F}_{1,2}$ we see that $\mathcal{F}_{1,2} = (\mathcal{F}')^{-1}$. Hence we have proven the

1.4.7. Proposition: The operator

$$(\mathcal{F}_{\ell_2,\ell_1}\varphi)(n) = \int_{L_2/L_1\cap L_2} \varphi(nh_2)f(h_2)d\dot{h}_2$$

is an intertwining operator between $W(\ell_1)$ and $W(\ell_2)$. $\mathcal{F}_{\ell_2,\ell_1}$ is canonically defined up to multiplication by a positive constant and $\mathcal{F}_{\ell_2,\ell_1} = (\mathcal{F}_{\ell_1,\ell_2})^{-1}$.

We will now make specific the choice of $d\dot{h}_2$ so that $\mathcal{F}_{\ell_2,\ell_1}$ becomes a unitary operator.

1.4.8. Let us first recall the definition of α-densities on a vector space E. Let E be a k-dimensional vector space over $\mathbb{R}$, $\Lambda^k E$ the space of k-vectors in E. $\Lambda^k E$ is one-dimensional over $\mathbb{R}$. For any real number α, we call a density of order α a map $\rho: \Lambda^k E - \{0\} \to \mathbb{R}$ such that $\rho(\lambda v) = |\lambda|^\alpha \rho(v)$ for each $v \in \Lambda^k E - \{0\}$, $\lambda \in \mathbb{R} - \{0\}$. The space of all densities of order α is one-dimensional over $\mathbb{R}$, and is denoted by $\Omega_\alpha(E)$. If $w \in \Lambda^k E^*$ is a k-form, we denote by $|w|^\alpha$ the α density defined by $|w|^\alpha(v) = |(w,v)|^\alpha$, for $v \in \Lambda^k E - \{0\}$. $\Omega_1(E)$ is referred also as the space of volume forms. We denote by $|\Lambda^k E|$ the space $\Lambda^k E$, modulo the equivalence relation $e \sim -e$.

1.4.9. Let ℓ be a Lagrangian subspace of (V,B) and $e \in \Lambda^n \ell$ be a n-vector. We identify ℓ with $(V/\ell)^*$ via the bilinear map B. Hence each element $|e| \in |\Lambda^n \ell|$ gives a volume form $|e|$ on V/ℓ.

Let $L = \exp(\ell \oplus \mathbb{R}E)$ and $X = N/L$. The tangent space of X at the image $\dot{1}$ of the identity element 1 of N is canonically identified with V/ℓ. It is immediate to see that $|e|$ can be extended as an N-invariant volume-form on X. We denote by $d_{|e|}$ the associated measure on X. We denote by $H(\ell,|e|)$ the space $H(\ell)$ where the choice of the inner product on $H(\ell)$ is now determined by:

$$\|\varphi\|^2 = \int_X |\varphi(n)|^2 \, d_{|e|}\dot{n}.$$

1.4.10. Given (V,B), a symplectic vector space, we have a canonical element $\omega \in \Lambda^{2n} V^*$ defined by :

$$\omega = \underbrace{B \wedge B \wedge \cdots \wedge B}_{n\text{-times}}.$$

If E is a vector space, we denote the top-degree term of the graded vector space ΛE by $\Lambda^{max} E$.

Let E_1 be a subspace of E, we have then:

$$\Lambda^{max} E \simeq \Lambda^{max}(E/E_1) \otimes \Lambda^{max} E_1.$$

If $E = E_1 \oplus E_2$,

$$\Lambda^{max} E \simeq \Lambda^{max}(E/E_1) \otimes \Lambda^{max}(E/E_2).$$

1.4.11. Let us consider two Lagrangian planes ℓ_1 and ℓ_2. The bilinear form B defines a canonical symplectic form B' on $\ell_1+\ell_2/\ell_1\cap\ell_2$. The spaces $\ell_1/\ell_1\cap\ell_2$ and $\ell_2/\ell_1\cap\ell_2$ are transverse Lagrangian subspaces in $\ell_1+\ell_2/\ell_1\cap\ell_2$.

We have:

$$\Lambda^{max}(V/\ell_1\cap\ell_2) \simeq \Lambda^{max}(V/\ell_1+\ell_2) \otimes \Lambda^{max}(\ell_1+\ell_2/\ell_1\cap\ell_2)$$

$$\Lambda^{max}(\ell_1+\ell_2/\ell_1\cap\ell_2) \simeq \Lambda^{max}(\ell_1+\ell_2/\ell_1) \otimes \Lambda^{max}(\ell_1+\ell_2/\ell_2)$$

Thus $\Lambda^{max}(V/\ell_1\cap\ell_2) \simeq \Lambda^{max}(V/\ell_1+\ell_2) \otimes \Lambda^{max}(\ell_1+\ell_2/\ell_1) \otimes \Lambda^{max}(\ell_1$

From this, we deduce:

1.4.12. <u>Lemma</u>:

$$\Omega_1(V/\ell_1\cap\ell_2) \simeq \Omega_{1/2}(\ell_1+\ell_2/\ell_1\cap\ell_2) \otimes \Omega_{1/2}(V/\ell_2) \otimes \Omega_{1/2}(V/\ell_1).$$

<u>Proof</u>: Let $u_1 \wedge u_2 \wedge u_3$ be an element of $\Lambda^{max}(V/\ell_1\cap\ell_2)$, with

$$u_1 \in \Lambda^{max}(V/\ell_1+\ell_2),\ u_2 \in \Lambda^{max}(\ell_1+\ell_2/\ell_1) \text{ and } u_3 \in \Lambda^{max}(\ell_1+\ell_2/$$

Thus $u_1 \wedge u_2 \in \Lambda^{max}(V/\ell_1)$, $u_2 \wedge u_3 \in \Lambda^{max}(\ell_1+\ell_2/\ell_1\cap\ell_2)$ and $u_3 \wedge u_1 \in \Lambda^{max}(V/\ell_2)$. Hence for $\rho_1 \in \Omega_{1/2}(\ell_1+\ell_2/\ell_1\cap\ell_2)$, $\rho_2 \in \Omega_{1/2}(V/\ell_2)$ and $\rho_3 \in \Omega_{1/2}(V/\ell_1)$, we define

$$(\rho_1 \otimes \rho_2 \otimes \rho_3,\ u_1 \wedge u_2 \wedge u_3) = \rho_1(u_2 \wedge u_3)\ \rho_2(u_3 \wedge u_1)\ \rho_3(u_1 \wedge u_2)$$

which is homogeneous of degree 1, thus it is a volume form on $V/\ell_1\cap\ell_2$.

As $\ell_1+\ell_2/\ell_1\cap\ell_2$ is provided with a canonical half-density $|\omega'|^{1/2}$, we get a canonical map from

$\Omega_{1/2}(V/\ell_1) \otimes \Omega_{1/2}(V/\ell_2)$ to $\Omega_1(V/\ell_1\cap\ell_2)$ by $(\rho_1,\rho_2) \to |\omega'|^{1/2} \otimes \rho_2 \otimes \rho_1$.

1.4.13. Let $e_1 \in \Lambda^n\ell_1$ and $e_2 \in \Lambda^n\ell_2$. We recall that the choice of a volume form $\delta \in \Omega_1(\ell_2/\ell_1\cap\ell_2)$ defines a canonical operator $\mathcal{F}^{\delta}_{\ell_2,\ell_1}: H(\ell_1,e_1) \to H(\ell_2,e_2)$ by :

$$(\mathcal{F}^{\delta}_{\ell_2,\ell_1}\varphi)(n) = \int_{L_2/L_1\cap L_2} \varpi(nh_2)f(h_2)\delta\dot{h}_2,$$

where $\delta\dot{h}_2$ denotes the L_2 invariant positive measure on $L_2/L_1\cap L_2$ associated to δ.

We now give a canonical choice of δ in function of e_1 and e_2. We consider the identifications:

$$\Omega_1(V/\ell_1\cap\ell_2) \simeq \Omega_{1/2}(V/\ell_1) \otimes \Omega_{1/2}(V/\ell_2) \otimes |\omega'|^{1/2}$$

$$\Omega_1(V/\ell_1\cap\ell_2) \simeq \Omega_1(V/\ell_2) \otimes \Omega_1(\ell_2/\ell_1\cap\ell_2).$$

Given $e_1 \in \Lambda^n\ell_1$ and $e_2 \in \Lambda^n\ell_2$, there exist a unique $\delta \in \Omega_1(\ell_1/\ell_1\cap\ell_2)$ such that

$$|e_1|^{1/2} \otimes |e_2|^{1/2} \otimes |\omega'|^{1/2} = |e_2| \otimes \delta.$$

We denote symbolically $\delta = |e_1|^{1/2} \otimes |e_2|^{-1/2} \otimes |\omega'|^{1/2}$. If $\ell_1 = \ell_2$ and $e_1 = e_2$, then $\delta = 1$. If $\ell_1 \cap \ell_2 = 0$, identifying $\Omega_1(L_2) = \Omega_1(V/L_1)$, we have for $e_1 \wedge e_2 = c\omega$, $\delta = |c|^{-1/2}|e_1|$.

1.4.13. <u>Proposition</u>: Let $e_1 \in \Lambda^n\ell_1$, $e_2 \in \Lambda^n\ell_2$ and

$\delta = |e_1|^{1/2} \otimes |e_2|^{-1/2} \otimes |\omega'|^{1/2}$. Then $\mathcal{F}^{\delta}_{\ell_2,\ell_1}$ is a unitary operator from $H(\ell_1,e_1)$ to $H(\ell_2,e_2)$. With this choice $\mathcal{F}_{\ell_1,\ell_2} = \mathcal{F}^{-1}_{\ell_2,\ell_1}$.

Proof: It follows from our calculation in coordinates in 1.4.2. that our normalized operator is precisely $\mathcal{F}$. The general case is deduced from 1.4.6.

1.5. Maslov index.

In this section, we define the Maslov index and prove its properties following an unpublished text of M. Kashiwara.

Let ℓ_1,ℓ_2,ℓ_3 be 3 Lagrangian planes in V. We consider the 3n-dimensional vector space $\ell_1 \oplus \ell_2 \oplus \ell_3$. We define the Maslov index $\tau(\ell_1,\ell_2,\ell_3)$:

1.5.1. Definition (Kashiwara): $\tau(\ell_1,\ell_2,\ell_3)$ is the signature of the quadratic form $Q(x_1 + x_2 + x_3)$ on the vector space $\ell_1 \oplus \ell_2 \oplus \ell_3$ defined by:

$$Q(x_1+x_2+x_3) = B(x_1,x_2) + B(x_2,x_3) + B(x_3,x_1).$$

The signature of Q is defined as follows: In a certain basis of $\ell_1 \oplus \ell_2 \oplus \ell_3$, the matrix of Q is diagonal and contains p times the coefficient $+1$, q times the coefficient -1; the signature of Q is then $p - q$.

It is clear from the definition, that we have:

1.5.2. For any $g \in Sp(B)$, $\tau(g\ell_1,g\ell_2,g\ell_3) = \tau(\ell_1,\ell_2,\ell_3)$.

1.5.3. $\tau(\ell_1,\ell_2,\ell_3) = -\tau(\ell_2,\ell_1,\ell_3) = -\tau(\ell_1,\ell_3,\ell_2)$.

Let ℓ_1 and ℓ_2 be two transverse Lagrangian planes. We have seen that we can always choose a symplectic basis $P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_n$, such that $\ell_1 = \sum_{i=1}^{n} \mathbb{R}P_i$, and $\ell_2 = \Sigma\, \mathbb{R}Q_i$, i.e. for ℓ_1', ℓ_2', two other transverse Lagrangian planes, we can certainly find $g \in Sp(B)$ such that $g\ell_1 = \ell_1'$, $g\ell_2 = \ell_2'$. On the contrary, formula 1.5.2 shows that the symplectic group does not act transitively on triples of Lagrangian planes. In fact we will see that the configuration of three transverse Lagrangian planes is determined by their index.

Let ℓ_1, ℓ_2, ℓ_3 be three Lagrangian planes. We suppose first that ℓ_1 and ℓ_3 are transverse, i.e. $V = \ell_1 \oplus \ell_3$. We denote by p_{13} the projection of V on ℓ_1 perpendicular to ℓ_3, and p_{31} the projection of V on ℓ_3 perpendicular to ℓ_1. We have:

1.5.4. <u>Lemma</u>: If ℓ_1 and ℓ_3 are transverse, then $\tau(\ell_1,\ell_2,\ell_3)$ is the signature of the quadratic form on ℓ_2 which associates to $x \in \ell_2$, $B(p_{13}x, p_{31}x) = B(x, p_{31}x) = B(p_{13}x, x)$.

<u>Proof</u>: We have

$$Q(x_1 + x_2 + x_3) = B(x_1,x_2) + B(x_2,x_3) + B(x_3,x_1)$$
$$= B(x_1,p_{31}x_2) + B(p_{13}x_2,x_3) + B(x_3,x_1) \text{ as } \ell_1 \text{ and } \ell_3 \text{ are Lagrangians,}$$
$$= B(p_{13}x_2,p_{31}x_2) - B(x_1 - p_{13}x_2, x_3 - p_{31}x_2).$$

Let $y_1 = x_1 - p_{13}x_2$, $y_2 = x_2$, $y_3 = x_3 - p_{31}x_2$; with respect to these coordinates $Q(x_1 + x_2 + x_3) = B(p_{13}y_2,p_{31}y_2) - B(y_1,y_3)$ hence $\operatorname{sign} Q = \operatorname{sign} B(p_{13}y_2,p_{31}y_2) - \operatorname{sign} B(y_1,y_3)$. As the signature of the form $B(y_1,y_3)$ is equal to zero, we obtain the lemma.

1.5.5. <u>Remark</u>. The bilinear form $S(x,y) = B(p_{13}x,p_{31}y)$ on ℓ_2 is symmetric, since for $x,y \in \ell_2$, $B(x,y) = 0 = B(p_{13}x + p_{31}x, p_{13}y + p_{31}y) = B(p_{13}x,p_{31}y) + B(p_{31}x,p_{13}y)$, as ℓ_1 and ℓ_3 are Lagrangian.

1.5.6. The kernel of the bilinear form S on ℓ_2 is equal to $\ell_2 \cap \ell_1 + \ell_2 \cap \ell_3$: if $B(p_{13}x,p_{31}y) = 0$ for every $y \in \ell_2$,

then $B(p_{13}x,y) = 0$, hence $p_{13}x \in \ell_2 \cap \ell_1$. As $x = p_{31}x + p_{13}x$, we have $p_{31}x \in \ell_2 \cap \ell_3$, and $x \in \ell_1 \cap \ell_2 + \ell_3 \cap \ell_2$.

1.5.7. <u>Corollary</u>: Let (ℓ_1,ℓ_2,ℓ_3) be three mutually transverse Lagrangian planes, i.e. $\ell_1 \cap \ell_2 = 0$, $\ell_2 \cap \ell_3 = 0$, $\ell_1 \cap \ell_3 = 0$. Then there exists a symplectic basis $P_1, \dots, P_n, Q_1, \dots, Q_n$ and an integer k, $0 \le k \le n$, such that

$$\ell_1 = \mathbb{R}P_1 \oplus \cdots \oplus \mathbb{R}P_n$$
$$\ell_2 = \mathbb{R}Q_1 \oplus \cdots \oplus \mathbb{R}Q_n$$
$$\ell_3 = \mathbb{R}(P_1 \oplus \varepsilon_1 Q_1) \oplus \cdots \oplus \mathbb{R}(P_n \oplus \varepsilon_n Q_n)$$
$$\text{with } \varepsilon_i = +1 \text{ if } i \le k,\ \varepsilon_i = -1 \text{ if } i > k.$$

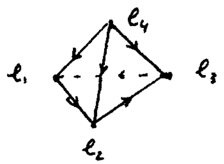
We then have $\tau(\ell_1,\ell_2,\ell_3) = n - 2k$.

<u>Proof</u>: The symmetric form $S(x,y)$ on ℓ_2 is non-degenerate, we hence can choose a basis $Q_1, Q_2, \dots, Q_n$ of ℓ_2, such that $S(Q_i,Q_j) = -\varepsilon_j\delta_{ij}$. As B gives a duality between ℓ_1 and ℓ_2 we can choose a basis $P_1, P_2, \dots, P_n$ of ℓ_1 such that $B(P_i,Q_j) = \delta_{ij}$, i.e. $(P_1, \dots, P_n, Q_1, \dots, Q_n)$ is a symplectic basis. Let $Z_i = p_{13}Q_i$, with respect to the decomposition $V = \ell_1 \oplus \ell_3$; then $Q_i - Z_i \in \ell_3$. By definition $S(Q_i,Q_j) = B(p_{13}Q_i,Q_j) = -\varepsilon_j\delta_{i,j}$, for every j. So $p_{13}Q_i = -\varepsilon_i P_i$. A basis of ℓ_3 is then $Q_i + \varepsilon_i P_i = \varepsilon_i(P_i + \varepsilon_i Q_i)$, and the lemma is proven.

1.5.8. Proposition. For $\ell_1,\ell_2,\ell_3,\ell_4$ four Lagrangian planes, τ verifies the chain condition:

$$\tau(\ell_1,\ell_2,\ell_3) = \tau(\ell_1,\ell_2,\ell_4) + \tau(\ell_2,\ell_3,\ell_4) + \tau(\ell_3,\ell_1,\ell_4).$$

We visualize this relation as follows:



Proof: Let us first suppose that ℓ_4 is such that $\ell_4 \cap \ell_i = 0$ for $i = 1,2,3$. We have that $\tau(\ell_1,\ell_2,\ell_3)$ is the signature of the form $Q(x_1,x_2,x_3)$ on $\ell_1 \oplus \ell_2 \oplus \ell_3$. The second member is the signature of the quadratic form Q' on $\ell_1 \oplus \ell_2 \oplus \ell_3$ given by

$$Q'(y_1,y_2,y_3) = B(p_{14}y_2,y_2) + B(p_{24}y_3,y_3) + B(p_{34}y_1,y_1).$$

The transformations:

$$x_1 = y_1 + p_{14}y_2 \qquad y_1 = \tfrac{1}{2}(x_1-p_{14}x_2+p_{14}x_3)$$
$$x_2 = y_2 + p_{24}y_3 \quad \text{and} \quad y_2 = \tfrac{1}{2}(x_2-p_{24}x_3+p_{24}x_1)$$
$$x_3 = y_3 + p_{34}y_1 \qquad y_3 = \tfrac{1}{2}(x_3-p_{34}x_1+p_{34}x_2)$$

are reciprocal (as $p_{14}p_{34}y_1 = y_1$ and similar relations). Let us prove that these transformations give the equivalence of Q and Q'.

We have:

$$B(x_1,x_2) = B(p_{14}y_2,y_2) + B(y_1,y_2) + B(y_1,p_{24}y_3) + B(p_{14}y_2,p_{24}y_3).$$

By cyclic permutation, we have to show that

$$B(y_1,y_2) + B(y_2,p_{34}y_1) + B(p_{34}y_1,p_{14}y_2) = 0.$$

Let us write $y_2 = p_{14}y_2 + p_{41}y_2$, this is

$$B(y_1,p_{41}y_2) + B(p_{41}y_2,p_{34}y_1), \text{ as } B(y_1,p_{14}y_2) = 0,$$

$$= B(y_1,p_{41}y_2) + B(p_{41}y_2,y_1) = 0, \text{ as } B(p_{41}y_2,p_{43}y_1) = 0.$$

This proves the proposition for this case.

Now let us take a Lagrangian plane m transverse to all the ℓ_j, $j = 1,2,3,4$, and let us express $\tau(\ell_i,\ell_j,\ell_k)$ as a function of $\tau(\ell_i,\ell_j,m)$. Then the result follows immediately.

1.5.9. Let ρ be an isotropic subspace of V, i.e. $B(\rho,\rho) = 0$. Then B defines a non-degenerate symplectic form on $\rho^\perp/\rho$. For W a subspace of V, we define

$$W^\rho = (W \cap \rho^\perp) + \rho = (W+\rho) \cap \rho^\perp \subset \rho^\perp.$$

We have $(W^\perp)^\rho = (W^\rho)^\perp$ as $(W^\rho)^\perp = ((W \cap \rho^\perp)+\rho)^\perp = \rho^\perp \cap (W^\perp+\rho) = (W^\perp)^\rho$. Hence if W is a Lagrangian plane of V, W^ρ/ρ is a Lagrangian plane in $\rho^\perp/\rho$.

We will prove:

1.5.10. Proposition: Let $\rho \subset (\ell_1 \cap \ell_2) + (\ell_2 \cap \ell_3) + (\ell_3 \cap \ell_1)$ then $\tau(\ell_1,\ell_2,\ell_3) = \tau(\ell_1^\rho,\ell_2^\rho,\ell_3^\rho)$.

We first prove the

1.5.11. <u>Lemma</u>: Let ℓ_1, ℓ_2 two Lagrangian subspaces. Then if $\ell = (\ell \cap \ell_1) + (\ell \cap \ell_2)$, we have $\tau(\ell_1, \ell, \ell_2) = 0$.

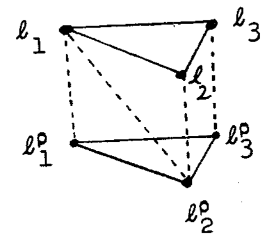
<u>Proof</u>: Let us choose $Y_1 \subset \ell \cap \ell_1$, $Y_2 \subset \ell \cap \ell_2$ such that $\ell = Y_1 \oplus Y_2$. For an element $x = (x_1, u+v, x_2)$ of the space $\ell_1 \oplus Y_1 \oplus Y_2 \oplus \ell_2$ with $x_1 \in \ell_1$, $u \in \ell \cap \ell_1$, $v \in \ell \cap \ell_2$, $x_2 \in \ell_2$, we have $Q(x) = B(x_1, v) + B(u, x_2) + B(x_2, x_1)$ $= B(x_2 - v, x_1 - u)$ as $B(u,v) = 0$. Hence the signature of Q is equal to the signature of the quadratic form $B(y_2, y_1)$ on $\ell_2 \oplus \ell_1$ which is obviously zero.

Now let us prove proposition 1.5.10. Let ρ be contained in $\ell_{123} = \ell_1 \cap \ell_2 + \ell_2 \cap \ell_3 + \ell_3 \cap \ell_1$. In particular, as ℓ_{123} is isotropic, $\ell_1 \cap \ell_2$, $\ell_2 \cap \ell_3$ and $\ell_3 \cap \ell_1$ are contained in $\rho^\perp$. We have $\ell_1^\rho = (\ell_1 \cap \rho^\perp) + \rho$ by definition. If $u = u_{12} + u_{23} + u_{31} \in \rho$ with $u_{12} \in \ell_1 \cap \ell_2$, $u_{23} \in \ell_2 \cap \ell_3$, $u_{31} \in \ell_3 \cap \ell_1$, we have $u_{23} = u - u_{12} - u_{31} \in \rho + (\ell_1 \cap \rho^\perp) = \ell_1^\rho$. Hence $u_{23} \in \ell_1^\rho \cap (\ell_2 \cap \rho^\perp)$. So $\ell_1^\rho = (\ell_1^\rho \cap \ell_1) + (\ell_2 \cap \ell_1^\rho) =$ $(\ell_1^\rho \cap \ell_1) + (\ell_2^\rho \cap \ell_1^\rho)$.

We conclude by the preceding formula that

$$\tau(\ell_1, \ell_1^\rho, \ell_2) = \tau(\ell_1, \ell_1^\rho, \ell_2^\rho) = 0.$$

From proposition 1.5.8, using the chain rule, it follows that $\tau(\ell_1, \ell_2, \ell_3) = \tau(\ell_1^\rho, \ell_2^\rho, \ell_3^\rho)$, as seen from the diagram:



1.5.12. We define, for a sequence $(\ell_1, \ell_2, \cdots, \ell_k)$ of Lagrangian spaces in (V,B), the Maslov index $\tau(\ell_1, \ell_2, \cdots, \ell_k)$ for $k \geq 4$, by:

$$\tau(\ell_1, \ell_2, \cdots, \ell_k) = \tau(\ell_1, \ell_2, \ell_3) + \tau(\ell_1, \ell_3, \ell_4) + \cdots + \tau(\ell_1, \ell_{k-1}, \ell_k)$$
$$= \tau(\ell_1, \ell_2, \ell) + \tau(\ell_2, \ell_3, \ell) + \cdots + \tau(\ell_{k-1}, \ell_k, \ell) + \tau(\ell_k, \ell_1, \ell),$$

where ℓ is an arbitrary Lagrangian space. (The equality follows from 1.5.8.) We have:

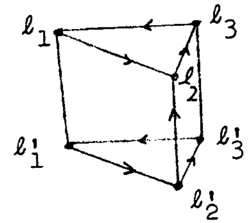
1.5.13. <u>Proposition</u>:

a) The index $\tau(\ell_1, \ell_2, \cdots, \ell_k)$ is invariant under the action of the symplectic group, and its value is unchanged under circular permutation.

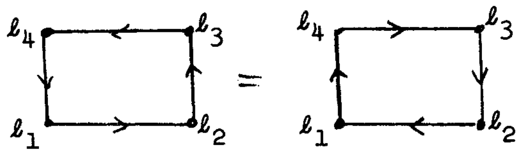
b) For any Lagrangian planes ℓ_1, ℓ_2, ℓ_3, $\ell_1', \ell_2', \ell_3'$, we have:

$$\tau(\ell_1', \ell_2', \ell_3') = \tau(\ell_1, \ell_2, \ell_3) + \tau(\ell_1', \ell_2', \ell_2, \ell_1)$$
$$+ \tau(\ell_2', \ell_3', \ell_3, \ell_2) + \tau(\ell_3', \ell_1', \ell_1, \ell_3)$$

as visualized by the graphic:



c) $\tau(\ell_1,\ell_2,\ell_3,\ell_4) = -\tau(\ell_2,\ell_1,\ell_4,\ell_3)$ as visualized by



1.6. The cocycle of the Shale-Weil representation and the Maslov index.

Let ℓ_1,ℓ_2,ℓ_3 be three Lagrangian planes in the symplectic vector space (V,B). We consider the canonical unitary intertwining operator (we leave implicit the choice of e_1,e_2,e_3):

$$\mathcal{F}_{i,j}: H(\ell_j) \longrightarrow H(\ell_i)$$

which intertwines $W(\ell_j)$ and $W(\ell_i)$ defined in 1.4.8. It is clear that the operator $\mathcal{F}_{1,3}\ \mathcal{F}_{3,2}\ \mathcal{F}_{2,1}$ is proportional to the identity operator on $H(\ell_1)$ as this operator intertwines the irreducible representation $W(\ell_1)$ with itself. Hence there is a scalar of modulus one $a(\ell_1,\ell_2,\ell_3)$ such that:

$$\mathcal{F}_{\ell_1,\ell_3}\ \mathcal{F}_{\ell_3,\ell_2}\ \mathcal{F}_{\ell_2,\ell_1} = a(\ell_1,\ell_2,\ell_3)\ \mathrm{Id}.$$

(It is easy to see that $a(\ell_1,\ell_2,\ell_3)$ does not depend of e_1,e_2,e_3.)

1.6.1. Theorem: Let ℓ_1, ℓ_2, and ℓ_3 be three Lagrangian planes. Then

$$\mathcal{F}_{1,3}\ \mathcal{F}_{3,2}\ \mathcal{F}_{2,1} = e^{-\frac{i\pi}{4}\tau(\ell_1,\ell_2,\ell_3)}\ \mathrm{Id}_{H_1}.$$

Proof: Let us first compute this for the three dimensional Heisenberg algebra: Let $\ell_1 = \mathbb{R}P$, $\ell_2 = \mathbb{R}Q$, $\ell_3 = \mathbb{R}(P+Q)$. We have $\tau(\ell_1,\ell_2,\ell_3) = -\tau(\ell_1,\ell_3,\ell_2) = -1$ as follows from 1.5.4. Hence we have to prove that in this case

$$\mathcal{F}_{1,3}\ \mathcal{F}_{3,2}\ \mathcal{F}_{2,1} = e^{\frac{i\pi}{4}}\ \mathrm{Id}.$$

Let us identify $H(\ell_1)$, $H(\ell_2)$ and $H(\ell_3)$ with $L^2(\mathbb{R})$ by $(R_1\varphi)(x) = \varphi(\exp xQ)$, $(R_2\varphi)(x) = \varphi(\exp xP)$ and $(R_3\varphi)(x) = \varphi(\exp xP)$. Then on $H = L^2(\mathbb{R})$, we obtain

1) $$(\mathcal{F}_{2,1}\varphi)(\exp xP) = \int \varphi(\exp xP \exp \xi Q)\, d\xi$$

$$= \int e^{-2i\pi x\xi}\, \varphi(\eta)\, d\xi$$

i.e. $\mathcal{F}_{2,1} = \mathcal{F}$

2) $$(\mathcal{F}_{3,2}\varphi)(\exp xP) = \int \varphi(\exp xP \exp \xi(P+Q))\, d\xi.$$

We have: $\exp xP \exp \xi(P+Q) = \exp(x+\xi)P \exp \xi Q \exp \frac{-\xi^2}{2} E.$

Hence

$$(\mathcal{F}_{3,2}\varphi)(x) = \int \varphi(x+\xi)\, e^{i\pi\xi^2}\, d\xi$$

$$= e^{i\pi x^2} \int \varphi(\xi) e^{-2i\pi x\xi}\, e^{i\pi\xi^2}\, d\xi, \text{ changing } \xi \text{ to } \xi$$

i.e. $$\mathcal{F}_{3,2} = e^{i\pi x^2} \mathcal{F} e^{i\pi x^2}$$

where $e^{i\pi x^2}$ denotes the multiplication operator.

3) $$(\mathcal{F}_{1,3}\varphi)(\exp xQ) = \int \varphi(\exp xQ \exp \xi P)\, d\eta.$$

We have:

$$\exp xQ \exp \xi P = \exp(\eta - x)P \exp x(P+Q) \exp(-x\eta + \frac{x^2}{2})\, E.$$

Hence

$$(\mathcal{F}_{1,3}\varphi)(x) = \int \varphi(\xi - x)\, e^{2i\pi x\eta}\, e^{-i\pi x^2}\, d\eta$$

$$= e^{i\pi x^2} \int \varphi(\xi)\, e^{2i\pi x\xi}\, d\xi, \text{ changing } \xi \text{ to } \xi + x$$

i.e. $$\mathcal{F}_{1,3} = e^{i\pi x^2} \mathcal{F}^{-1}.$$

Let us consider the function $\psi_z(x) = e^{i\pi z x^2}$ for $\operatorname{Im} z > 0$. This function is in $L^2(\mathbb{R})$.

1.6.2. <u>Lemma</u>.

$$\int e^{-2i\pi x\eta}\, e^{i\pi z\xi^2}\, d\xi = \left(\tfrac{z}{i}\right)^{-1/2} e^{-i\pi z^{-1} x^2}$$

where the determination of the function $z \to \left(\tfrac{z}{i}\right)^{-1/2}$ on the simply-connected domain $\operatorname{Im} z > 0$ is 1 for $z = i$.

Proof: We define for $z \in D = \{z;\ \operatorname{Im} z > 0\}$, $x \in \mathbb{C}$

$$\delta(z,x) = \int e^{-2i\pi x\xi}\, e^{i\pi z\eta^2}\, d\eta.$$

It is easy to see that $\delta(z,x)$ is a holomorphic function of (z,x) on $D \times \mathbb{C}$, and for $x \in \mathbb{R}$, $\delta(z,x) = (\mathcal{F}\psi_z)(x)$. It is hence sufficient to prove $\delta(z,x) = \left(\tfrac{z}{i}\right)^{-1/2} e^{-i\pi z^{-1} x^2}$ for $z = iy$, $y > 0$, $x = iu$. We have then

$$\delta(iy, iu) = \int e^{2\pi u\xi}\, e^{-\pi y\xi^2}\, d\eta\ ,$$

we change ξ to $y^{-1/2}\eta$

$$= y^{-1/2} \int e^{2\pi u y^{-1/2}\xi}\, e^{-\pi\xi^2}\, d\xi$$

$$= y^{-1/2} \int e^{-\pi(\xi - y^{-1/2}u)^2}\, e^{\pi y^{-1} u^2}\, d\xi$$

$$= y^{-1/2}\, e^{\pi y^{-1} u^2} \int e^{-\pi\xi^2}\, d\xi$$

$$= y^{-1/2}\, e^{\pi y^{-1} u^2}\ ,$$