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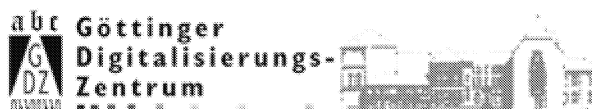
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# An arithmetical property of quadratic forms

By WALTER LEDERMANN, Manchester

In their paper [1] F. HIRZEBRUCH and H. HOPF have encountered an interesting arithmetical property possessed by certain symmetric bilinear forms

$$f(x, y) = \sum_{i, j=1}^n a_{ij} x_i y_j \quad (1)$$

that arise in algebraic topology. In the forms which they consider, the coefficients  $a_{ij}$  and the variables are integers and  $\det a_{ij} = \pm 1$ ; and it is known that there exists an integral vector  $w$  such that

$$f(x, x) \equiv f(x, w) \pmod{2} \quad (2)$$

for all  $x$ . If  $\tau$  is the signature of  $f$ , then it is a corollary of their topological investigations that

$$\tau \equiv f(w, w) \pmod{4}. \quad (3)$$

It is desirable to give a purely algebraic proof of (3), and I am greatly indebted to Professor HOPF for having drawn my attention to this question, which will be discussed in this note.

In fact, it will be shown that (3) is a special case of a result concerning forms (1) in which the coefficients and variables are rational numbers with odd denominators. This subset,  $\Omega$ , of all rationals forms a ring, whose elements may be grouped into residue classes modulo any power of 2 by stipulating that

$$\frac{c_1}{d_1} \equiv \frac{c_2}{d_2} \pmod{2^\alpha}$$

whenever  $c_1 d_2 - d_1 c_2 \equiv 0 \pmod{2^\alpha}$ ; since only odd denominators are allowed, this definition evidently does not depend on the representation of the fractions involved. In particular, a fraction is termed even or odd according as its numerator is even or odd; and we note that, if  $r$  is odd,  $r^2 \equiv 1 \pmod{4}$ .

The set,  $V$ , of  $n$ -tuples or "row-vectors"  $x = (x_1, x_2, \dots, x_n)$  ( $x_i \in \Omega$ ) is a  $\Omega$ -module. A change of basis of  $V$  amounts to replacing  $x$  by the  $n$ -tuple  $\tilde{x} = xP$ , where  $P$  is a fixed  $n$ -rowed matrix in  $\Omega$  with odd determinant.

Let  $f$  be a symmetric bilinear form which relative to the original basis is expressed as  $xAy'$ , where  $A = (a_{ij})$ . After the change of basis,  $f$  becomes  $\tilde{x}B\tilde{y}'$ , where

$$B = PAP' . \quad (4)$$

We write  $\Delta = \Delta, = \det A$ , and throughout this paper we restrict ourselves to forms with odd determinants, a property which is clearly preserved by the transformation (4).

For a given form  $f$  we can in many ways determine a constant vector  $w$  such that (2) holds for all  $x$  in  $\Omega$ . Indeed,  $w$  may be taken as the solution of the vector equation

$$wA = (a_{11}, a_{22}, \dots, a_{nn}),$$

this solution being in  $\Omega$ , because  $\det A$  is odd. For since

$$f(x, x) \equiv \sum_i a_{ii} x_i^2 \equiv \sum_i a_{ii} x_i \pmod{2},$$

we have that

$$f(x, w) = wAx' = \sum a_{ii} x_i,$$

and (2) is satisfied. If  $\tilde{w}$  is another vector satisfying (2), then  $f(x, \tilde{w} - w) \equiv 0 \pmod{2}$  for all  $x$ , so that  $(\tilde{w} - w)A \equiv 0 \pmod{2}$ . It follows that

$$\tilde{w} = w + 2z, \tag{5}$$

where  $z$  is a suitable vector in  $\Omega$ . Conversely, any vector of the form (5) satisfies (2). We have that

$$f(\tilde{w}, \tilde{w}) = f(w, w) + 4f(w, z) + 4f(z, z).$$

Thus

$$f(\tilde{w}, \tilde{w}) \equiv f(w, w) \pmod{4},$$

that is,  $f(w, w)$  (though not  $w$  itself) is an *invariant modulo 4* of  $f$ .

Our aim is to prove the following

**Theorem.** *Let  $f$  be a quadratic form in  $n$  variables in  $\Omega$  with odd determinant  $\Delta$  and with signature  $\tau$ . Then<sup>1)</sup>*

$$f(w, w) - \tau \equiv \Delta - \operatorname{sgn} \Delta \pmod{4}, \tag{6}$$

where  $w$  is a solution of (2).

We remark that, whilst  $\Delta$  is not an invariant of  $f$ , both  $\operatorname{sgn} \Delta$  and  $\Delta$  are invariants mod 4. For in a transformation of the type (4),  $\Delta$  is multiplied by  $(\det P)^2$ , which is congruent with 1 mod 4, since  $\det P$  is odd.

In particular, when  $f$  is unimodular, whether integral or not, we have that  $\Delta = \operatorname{sgn} \Delta$ , so that (6) reduces to (3).

The theorem is proved by an induction with respect to  $n$  which is based on the following simple matrix formula. Consider a partitioning of  $A$ , say

$$A = \begin{pmatrix} K & L' \\ L & M \end{pmatrix},$$

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<sup>1)</sup> As usual, we define  $\operatorname{sgn} \Delta$  to be +1 or -1 according as  $\Delta > 0$  or  $\Delta < 0$ .

where  $K$  is non-singular and of dimension less than  $n$ . Put

$$P = \begin{pmatrix} I & O \\ -LK^{-1} & I \end{pmatrix}$$

where the identity matrices on the diagonal are of dimensions (in general distinct) equal to those of  $K$  and  $M$  respectively. Then

$$PAP' = \begin{pmatrix} K & O \\ O & M - LK^{-1}L' \end{pmatrix}. \quad (7)$$

When  $\det K$  is odd, this transformation is admissible, since  $P$  then lies in  $\mathfrak{Q}$ . Now if not all diagonal elements of  $A$  are even, we may, without loss of generality, assume that  $a_{11}$  is odd and then put  $K = (a_{11})$ . If, on the other hand, all diagonal elements are even, then each row of  $A$  must contain at least one odd element, or else  $\det A$  could not be odd. We may then assume that  $a_{12}$  is odd and that  $K$  is the leading 2-rowed submatrix; for in that case  $\det K = a_{11}a_{22} - a_{12}^2 \equiv -1 \pmod{4}$ , which is certainly odd. Thus, when  $n > 2$ , we can always apply a transformation of the type (7), in which the dimension of  $K$  is either 1 or 2.

When  $V$  is referred to the new basis,  $f$  splits and we write

$$f(x, x) = g(x^{(1)}, x^{(1)}) + h(x^{(2)}, x^{(2)}),$$

where  $x = (x^{(1)}, x^{(2)})$  and the dimensions of the vectors  $x^{(1)}$  and  $x^{(2)}$  are those of  $K$  and  $M$  respectively<sup>2</sup>). Evidently

$$\Delta_f = \Delta_g \Delta_h, \quad \tau_f = \tau_g + \tau_h,$$

where suffixes are used to distinguish quantities corresponding to different forms. Also, if  $w^{(1)}$  and  $w^{(2)}$  are such that

$$g(x^{(1)}, x^{(1)}) \equiv g(x^{(1)}, w^{(1)}) \pmod{2}$$

for all  $x^{(1)}$  and

$$h(x^{(2)}, x^{(2)}) \equiv h(x^{(2)}, w^{(2)}) \pmod{2}$$

for all  $x^{(2)}$ , then  $w = (w^{(1)}, w^{(2)})$  satisfies (2).

Leaving aside for the present the cases in which  $n = 1$  or  $n = 2$ , we may assume, by induction, that the theorem holds for the forms  $g$  and  $h$ . Then, since

$$f(w, w) - \tau_f = (g(w^{(1)}, w^{(1)}) - \tau_g) + (h(w^{(2)}, w^{(2)}) - \tau_h),$$

we have that

$$f(w, w) - \tau_f \equiv \Delta_g - \operatorname{sgn} \Delta_g + \Delta_h - \operatorname{sgn} \Delta_h, \quad (8)$$

<sup>2</sup>) A somewhat similar method of reduction, but in a different context, has been employed by MINKOWSKI ([2], 16-20).

with the convention that henceforth all congruences are mod 4. Now, if  $r$  and  $s$  are odd,  $(1 - r)(1 - s)$  is divisible by 4, so that

$$r + s \equiv 1 + rs.$$

Hence, in particular,

$$\Delta_g + \Delta_h \equiv 1 + \Delta_g \Delta_h = 1 + \Delta_f,$$

and

$$\operatorname{sgn} \Delta_g + \operatorname{sgn} \Delta_h \equiv 1 + \operatorname{sgn}(\Delta_g \Delta_h) = 1 + \operatorname{sgn} \Delta_f.$$

Substituting in (8) we immediately obtain (6).

It only remains to verify the theorem for the two lowest dimensions. When  $n = 1$ ,  $f = a_{11}x_1^2$ , where  $a_{11}$  is odd. We may then put  $w_1 = 1$  to satisfy (2). Thus  $f(w, w) = a_{11} = \Delta$ . Since  $\tau = \operatorname{sgn} a_{11} = \operatorname{sgn} \Delta$ , the relation (6) is certainly true. When  $n = 2$ , that is when  $f = a_{11}x_1^2 + a_{22}x_2^2 + 2a_{12}x_1x_2$ , we have to distinguish two cases.

(i) Assume that  $a_{11}$  and  $a_{22}$  are not both even, so that we may assume that  $a_{11}$  is odd. The transformation (7) can then be applied with  $K = (a_{11})$ , and  $f$  splits into two unary forms. The induction argument is therefore available as before.

(ii) If  $a_{11}$  and  $a_{22}$  are both even,  $a_{12}$  is necessarily odd and  $\Delta = a_{11}a_{22} - a_{12}^2 \equiv -1$ . Evidently,  $f(x, x)$  is even for all  $x$ , so that the vector  $w = 0$  satisfies (2). We have therefore to show that

$$-\tau \equiv -1 - \operatorname{sgn} \Delta. \quad (9)$$

When  $\operatorname{sgn} \Delta = -1$ , the form is indefinite, that is  $\tau = 0$ , and (9) is true. On the other hand, when  $\operatorname{sgn} \Delta = 1$ , then  $\tau = 2$  or  $\tau = -2$  according as  $a_{11} > 0$  or  $a_{11} < 0$ . But  $2 \equiv -2$ , and again (9) holds in each case.

#### REFERENCE

- [1] F. HIRZEBRUCH and H. HOPF, *Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten*. Math. Annalen 136 (1958).  
 [2] H. MINKOWSKI, *Gesammelte Abhandlungen I* (Leipzig 1911).

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